Techniques in Equivariant Ehrhart Theory

Sophia Elia, Donghyun Kim and Mariel Supina

Abstract. Equivariant Ehrhart theory generalizes the study of lattice point enumeration to also account for the symmetries of a polytope under a linear group action. We present a catalogue of techniques with applications in this field, including zonotopal decompositions, symmetric triangulations, combinatorial interpretation of the h^*-polynomial, and certificates for the (non)existence of invariant nondegenerate hypersurfaces. We apply these methods to several families of examples including hypersimplices, orbit polytopes, and graphic zonotopes, expanding the library of polytopes for which their equivariant Ehrhart theory is known.

Keywords. Ehrhart theory, Polyhedra, Triangulations, Zonotopes.

1. Introduction

Ehrhart theory studies the enumeration of lattice points in polytopes via the Ehrhart counting function $\mathrm{L}(P; t) := |tP \cap M'|$ for positive integers t and a lattice M'. If P is a lattice polytope, then $\mathrm{L}(P; t)$ agrees with a polynomial in t which is known as the *Ehrhart polynomial* [1]. When a lattice polytope P is invariant under the linear action of a group G on M' with representation $\rho : G \to \mathrm{GL}(M')$, then we may study its *equivariant Ehrhart theory*. This concept was introduced by Stapledon [2] with motivation from toric geometry, representation theory, and mirror symmetry. The equivariant analogue of the Ehrhart polynomial is the character χ_{tP}, defined as the character of the complex permutation representation on the lattice points in tP. The *equivariant Ehrhart series* is then given as follows:

$$\mathrm{EE}(P; z) = \sum_{t \geq 0} \chi_{tP} z^t = \frac{\mathrm{H}^*(P; z)}{\det(I - z \cdot \rho)}$$

Evaluating this series at the identity element of the group returns the usual Ehrhart series of P. The numerator $\mathrm{H}^*(P; z)$ is a formal power series with coefficients in $R(G)$, the character ring of G. It is an open problem to determine

for which polytopes and group actions $\mathrm{H}^*(P; z)$ is a polynomial and effective. Stapledon provided many conjectures around this topic, including:

Conjecture 1.1. [2, Conjecture 12.1] *The following are equivalent:*

(i) *The toric variety of P admits a G-invariant nondegenerate hypersurface with Newton polytope P.*
(ii) *The H^*-series is effective.*
(iii) *The H^*-series is a polynomial.*

Stapledon further showed that (i) $\implies$ (ii) $\implies$ (iii); see Sects. 2.4 and 2.5 for details. In email correspondence that they shared with the authors in 2019, Stapledon and Francisco Santos gave a counterexample to (iii) $\implies$ (i), which we include here with their permission. The proof will be given in Sect. 3.4.3.

Theorem 1.2. (Santos–Stapledon, Counterexample to Conjecture 1.1) *Let the polytope $P = [0, 1]^3 \subset \mathbb{R}^3$ be the 3-cube and let $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{\mathrm{id}, \sigma, \tau, \sigma\tau = \tau\sigma\}$ act on P by 180 degree rotations as described in Fig. 12. Then, $\mathrm{H}^*(P; z)$ is polynomial and effective, but the toric variety of P does not admit a G-invariant nondegenerate hypersurface.*

In this article, we present a variety of techniques for computing and studying the equivariant Ehrhart series. Our aim is to develop equivariant Ehrhart theory as its own branch of discrete geometry, and to investigate Stapledon's conjectures for a number of families of polytopes. We demonstrate how to use several tools to describe the equivariant Ehrhart series, including zonotopal decompositions, symmetric triangulations, combinatorial interpretations of the h^*-polynomial, and computational methods using `Sagemath`. These techniques serve as a guidebook for future progress toward resolving the many open questions and conjectures in this field.

We first provide more details about our setup in Sect. 2, as well as presenting the necessary background in equivariant Ehrhart theory and representation theory. We then arrive to our catalog of techniques in Sect. 3.

Section 3.1 investigates the case when P is a graphic zonotope and G is the automorphism group of the graph. This section generalizes past work on the permutahedron [3,4], and serves as a blueprint for studying the equivariant Ehrhart theory of general zonotopes. The main results of this section are Corollary 3.17 which describes the equivariant Ehrhart series of a graphic zonotope, and Theorem 3.18 in which we characterize the conditions under which the H^*-series of a graphic zonotope is a polynomial.

In Sect. 3.2, we use symmetric triangulations to calculate the equivariant Ehrhart series. The symmetric triangulations in question are called *G-invariant half-open decompositions* and are closely related to partitionability of posets. In Theorem 3.30, we generalize a theorem of Stapledon for the equivariant Ehrhart series of a simplex to lattice polytopes with G-invariant half-open decompositions. In Theorem 3.32, we show that χ_{tP} is polynomial in t for polytopes with G-invariant half-open decompositions that satisfy an additional

property on the group action, and give a formula for the equivariant Ehrhart series. This section appears in [5].

Section 3.3 deals with instances where we can break down the H*-series using a combinatorial interpretation of its coefficients. In particular, we study hypersimplices under the action of the cyclic group, as well as permutahedra in prime dimensions. Theorem 3.36 uses decorated ordered set partitions to describe the H*-series of the hypersimplex, building off of [6]. In Theorem 3.50, we give an explicit formula for the H*-series of a polytope under a cyclic group action with trivial fixed subpolytopes. This is specified to the case of prime permutahedra in Corollary 3.51, which also shows the H*-series is polynomial and effective.

In Sect. 3.4, we focus on criteria (i) from Conjecture 1.1, demonstrating techniques for proving the existence or non-existence of G-invariant nondegenerate hypersurfaces. The main results of this section are Theorem 3.56 and Corollary 3.59 which characterize a family of S_n-orbit polytopes for which no such hypersurface can exist, and Theorem 3.60 regarding hypersimplices, another subfamily of orbit polytopes which do exhibit these hypersurfaces.

Finally, in Sect. 4, we collect the many open questions that arose during our work on this project, and in Appendix A, we demonstrate the calculation of the equivariant Ehrhart series using our implemented code which is staged for release with `Sagemath` version 9.6.

2. Background

We give relevant background on Ehrhart theory in Sect. 2.1. In Sect. 2.2, we present the general setup for equivariant Ehrhart theory that is used throughout. Section 2.3 presents background on representation theory, and Sect. 2.4 presents background on the equivariant Ehrhart series, in particular providing rational generating functions. In Sect. 2.5, we discuss G-invariant nondegenerate hypersurfaces and a useful corollary for their detection. In Sect. 2.6, we provide a framework for computing equivariant Ehrhart theory with restricted representations.

2.1. Ehrhart Theory

The main reference we follow for Ehrhart theory is [7]. Fix a lattice $M \subset \mathbb{R}^n$. The *Ehrhart counting function* of a polytope $P \subset \mathbb{R}^n$, written $\mathrm{L}(P;t)$, gives the number of lattice points in the t-th dilate of P for $t \in \mathbb{Z}_{\geq 1}$:

$$\mathrm{L}(P;t) = |tP \cap M|.$$

Ehrhart's theorem [1] says that if P is a lattice polytope (i.e., the vertices of P are contained in M), then for positive integers, $\mathrm{L}(P;t)$ agrees with a polynomial in t of degree equal to the dimension of P. Furthermore, the constant term of this polynomial is equal to 1 and the coefficient of the leading term is equal to the Euclidean volume of P within its affine span. The interpretation of other coefficients of the Ehrhart polynomial is an active direction of research, see for example [8].
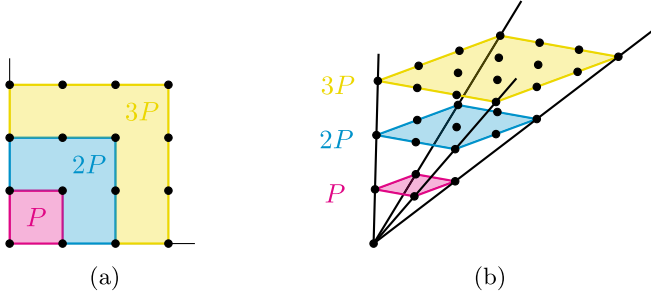
FIGURE 1. **a** The first three dilates of P and **b** the cone over P

Generating functions play a central role in Ehrhart theory. The *Ehrhart series* $\mathrm{Ehr}(P; z)$ of a polytope P is the formal power series given by

$$\mathrm{Ehr}(P; z) = 1 + \sum_{t \geq 1} \mathrm{L}(P; t) z^t.$$

Let $\mathrm{Cone}(P) := \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x} = \lambda(\mathbf{y}, 1), \text{ where } \mathbf{y} \in P, \lambda \geq 0\}$ as in Fig. 1. We may view the coefficient of z^t in the Ehrhart series as counting the number of lattice points in $\mathrm{Cone}(P)$ at height t. This viewpoint allows us to show that if P is a d-dimensional lattice polytope, then the Ehrhart series has the rational generating function

$$\mathrm{Ehr}(P; z) = 1 + \sum_{t \geq 1} \mathrm{L}(P; t) z^t = \frac{h^*(P; z)}{(1 - z)^{d+1}},$$

where $h^*(P; z) = \sum_{i=0}^{d} h_i^* z^i$ is a polynomial in z of degree at most d, called the h^*-*polynomial*. Furthermore, each h_i^* is a nonnegative integer [9]. The coefficients of the h^*-polynomial form the h^*-*vector*: $(h_0^*, h_1^*, \ldots, h_d^*)$. The Ehrhart polynomial may be recovered from the h^*-vector through the transformation

$$\mathrm{L}_P(t) = \sum_{i=0}^{d} h_i^* \binom{t + d - i}{d}. \tag{1}$$

Let $P \subseteq \mathbb{R}^d$ be a rational d-polytope with *denominator* k, i.e., k is the smallest positive integer such that kP is a lattice polytope. Then, $\mathrm{L}(P; t)$ is a *quasipolynomial* with period dividing k, i.e., of the form $\mathrm{L}(P; t) = c_d(t)t^d + \cdots + c_1(t)t + c_0(t)$ where $c_0(t), c_1(t), \ldots, c_d(t)$ are periodic functions. In this case, the Ehrhart series has the rational generating function

$$\mathrm{Ehr}(P; z) = \sum_{t \in \mathbb{Z}_{\geq 0}} \mathrm{L}(P; t) z^t = \frac{h^*(P; z)}{(1 - z^k)^{d+1}}, \tag{2}$$

where $h^*(P; z) \in \mathbb{Z}[z]$ has degree $< k(d + 1)$. The choice of denominator is no longer canonical.

2.2. The Equivariant Setup

For a lattice and $\mathbb{Z}$-module M, we write $M_{\mathbb{R}}$ for $M \otimes_{\mathbb{Z}} \mathbb{R}$.

Setup 2.1. Let G be a finite group acting linearly on a lattice $M' \cong M \times \mathbb{Z}$ of rank $d + 1$ such that the $\mathbb{Z}$-coordinate of each lattice point in M' is fixed under the action of G. Let $P \subset M'_{\mathbb{R}}$ be a d-dimensional G-invariant polytope with vertices in $M \times \{1\}$, where G-*invariant* means that as a set, $g(P) = P$, for all $g \in G$.

We assume $M = \mathbb{Z}^d$ when convenient. The *exponent* of a finite group G is the smallest positive integer N such that $g^N = \mathrm{id}_G$ for all $g \in G$. For an element $g \in G$, the subset of P fixed by g, denoted P^g and called the *fixed subpolytope*, is the convex hull of the barycenters of orbits of vertices of P under the action of g [2, Lemma 5.4]. This implies that any fixed subpolytope P^g for $g \in G$ is a rational polytope with denominator dividing N.

2.3. Representation Theory

For a nice introduction to representation theory of finite groups, see [10] or [11]. Let V be an n-dimensional vector space over a field K of characteristic 0. A *representation* ρ of a group G on V is a group homomorphism $\rho : G \to GL(V)$ from G to the group of invertible K-linear transformations of V. Equivalently ρ may be seen as a group homomorphism from G to the group $GL_n(V)$ of $n \times n$ invertible matrices with entries in K. A subspace $W \subseteq V$ is called G-*invariant* if $W = g(W)$ as a set, for all $g \in G$. A representation is *irreducible* if there are no nontrivial, proper invariant subspaces of V under the action of G.

Let $\rho : G \to GL_n(V)$ be a representation of G. The *character* of ρ, written χ_ρ, is the function $G \to \mathbb{C}$ such that $\chi_\rho(g) := \mathrm{trace}(\rho(g))$; the trace of a matrix is the sum of its diagonal entries. The characters of irreducible representations are referred to as *irreducible characters*. A group G always has a *trivial representation* on a vector space V which sends each group element to the identity map on V. The character of the trivial representation is denoted by χ_{triv} throughout. Characters are *class functions*, functions from the group to the complex numbers that take the same value on every conjugacy class. In fact, a class function is a character of a representation if and only if it can be written as a nonnegative integral linear combination of the irreducible characters of G. We write $R^+(G)$ for the set of the class functions that are characters and refer to them as *effective characters*. The linear span of $R^+(G)$ is denoted $R(G)$ and called the *character ring*; the multiplication of two characters in $R^+(G)$ is given by a trace of a tensor product of the corresponding representations, therefore, $R(G)$ has a ring structure. Elements of $R(G)$ are referred to as *virtual characters*. The character ring $R(G)$ is a subring of the $\mathbb{C}$-vector space $F_{\mathbb{C}}(G)$ of class functions on G with values in $\mathbb{C}$, called the *ring of class functions*. There is an inner product on $F_{\mathbb{C}}(G)$ such that for two class functions ϕ, χ of G, $\langle \phi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\chi(g)}$, where $\overline{\cdot}$ denotes the complex conjugate.

Let a group G act on a finite (ordered) set S. The *permutation representation* of G on S is the group homomorphism that sends every element of G to

the corresponding permutation matrix for the action of G on S. The character of the permutation representation is called the *permutation character*.

2.4. Equivariant Ehrhart Theory

Let χ_{tP} be the complex permutation character induced by the action of G on the lattice points in $tP \cap M'$.

Definition 2.2. The *equivariant Ehrhart series*, $\mathrm{EE}(P; z)$, is the formal power series in $R(G)[[z]]$, such that the coefficient of z^t for $t \in \mathbb{Z}_{\geq 0}$ is the character χ_{tP}:

$$\mathrm{EE}(P; z) = \sum_{t \geq 0} \chi_{tP} z^t.$$

Evaluating the series at $g \in G$, $\mathrm{EE}(P; z)(g)$, gives the Ehrhart series of the fixed subpolytope P^g [2, Lemma 5.2]. Using the rationality of these fixed subpolytopes, Stapledon showed that the characters χ_{tP} are quasipolynomials when considered as functions of t. Here, we provide another proof that includes an explicit quasipolynomial expression for χ_{tP}.

Theorem 2.3. [2, Theorem 5.7] *Let P be a lattice d-polytope invariant under the action of a group G as in the Setup 2.1 and exponent $N \geq 1$. As a function of t, χ_{tP} is quasipolynomial, that is,*

$$\chi_{tP} = f_0(t)t^0 + f_1(t)t^1 + \cdots + f_d(t)t^d,$$

where $f_0(t), f_1(t), \ldots, f_d(t) \in F_{\mathbb{C}}(G)$ are periodic functions in t with period dividing N.

Proof. For all $g \in G$, the Ehrhart counting function $\mathrm{L}(P^g; z)$ can be expressed as a quasipolynomial of period N and degree equal to $\dim(P^g) \leq d$. Therefore,

$$
\begin{aligned}
\mathrm{EE}(P; z)(g) = \sum_{t \geq 0} \chi_{tP}(g) z^t &= \sum_{a \geq 0} \sum_{j=0}^{N-1} \chi_{(aN+j)P}(g) z^{aN+j} \\
&= \sum_{a \geq 0} \sum_{j=0}^{N-1} \mathrm{L}(P^g; aN+j) z^{aN+j} \\
&= \sum_{a \geq 0} \sum_{j=0}^{N-1} \left(\sum_{i=0}^{d} c_{j,i}^g (aN+j)^i \right) z^{aN+j},
\end{aligned}
$$

where $c_{j,i}^g \in \mathbb{Q}$ is the coefficient of the degree i term in the j-th constituent of the Ehrhart quasipolynomial of P^g. Let $\{g_1, \ldots, g_k\}$ be conjugacy class representatives of G. Define class functions $f_{j,i}[g_\ell] := c_{j,i}^{g_\ell}$ for all $j \in [N]$, $\ell \in [k]$, and $i \in \{0, 1, \ldots, d\}$. Then, for $j \in [N]$, and $t \equiv j \mod N$, $\chi_{tP} = f_{j,0}t^0 + \cdots + f_{j,d}t^d$. $\square$

Corollary 2.4. *Fix $j \in [N]$. For all $t \equiv j \mod N$, χ_{tP} may be expressed as a linear combination of the irreducible characters of G with coefficients in $\mathbb{Q}[t]$.*

Corollary 2.5. *Let P be a rational polytope with denominator $S \in \mathbb{Z}_{>0}$ such that P is invariant under the linear action of a group G. Then, χ_{tP} is quasipolynomial in t with period dividing SN.*

Theorem 2.6. *Let P be a G-invariant lattice polytope of dimension d as in the Setup 2.1, where G has exponent N and irreducible characters $\{\chi_1, \ldots, \chi_k\}$. The equivariant Ehrhart series has the following rational expression:*

$$\mathrm{EE}(P; z) = \sum_{t \geq 0} \chi_{tP} z^t = \frac{\tilde{\mathrm{H}}(P; z)}{(1 - z^N)^{d+1}},$$

where $\tilde{\mathrm{H}}(P; z)$ is a polynomial with coefficients in $R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ of degree at most $N(d+1) - 1$.

We, therefore, have two different rational expressions for the equivariant Ehrhart series:

$$\mathrm{EE}(P; z) = \sum_{t \geq 0} \chi_{tP} z^t = \frac{\tilde{\mathrm{H}}(P; z)}{(1 - z^N)^{d+1}} = \frac{\mathrm{H}^*(P; z)}{\det(I - z \cdot \rho)},$$

where $\mathrm{H}^*(P; z) \in R(G)[[z]]$. The H*-series is *effective* if the coefficient of z^i is an effective character for all i.

Remark 2.7. In [2], the H*-series is denoted by ϕ, and ρ is the representation of G acting on $M_{\mathbb{R}}$. Accordingly, the denominator in [2] is $(1 - z) \det(I - z \cdot \rho)$.

Understanding the H*-series is the main goal of Stapledon's Conjecture 1.1, and he stated that (ii) implies (iii). We provide a short proof:

Lemma 2.8. *If $\mathrm{H}^*(P; z)$ is effective, then $\mathrm{H}^*(P; z)$ is a polynomial.*

Proof. The H*-series is a priori an infinite formal sum: $\mathrm{H}^*(P; z) = \sum_{i \geq 0} \mathrm{H}_i^* z^i$. On the level of series, we have:

$$\frac{\mathrm{H}^*(P; z)(\mathrm{id}_G)}{\det(I - z \cdot \rho(\mathrm{id}_G))} = \mathrm{EE}(P; z)(\mathrm{id}_G) = \mathrm{Ehr}(P; z) = \frac{h^*(P; z)}{(1 - z)^{d+1}}.$$

As $\det(I - z \cdot \rho(\mathrm{id}_G)) = (1 - z)^{d+1}$, $\mathrm{H}^*(P; z)(\mathrm{id}_G) = h^*(P; z)$. Let $\{\chi_1, \ldots, \chi_k\}$ be the irreducible characters of G. Since $\mathrm{H}^*(P; z)$ is effective, $\mathrm{H}_i^* = \sum_{j=1}^{k} c_{i,j} \chi_j$, with $c_{i,j} \in \mathbb{Z}_{\geq 0}$ for all i, j. As $\chi_j(\mathrm{id}_G) > 0$ for all j, no cancellation can occur and $\mathrm{H}^*(P; z)$ must be polynomial. $\square$

2.5. Nondegenerate Hypersurfaces

One motivation for considering the H*-series is its connection to toric geometry, see [2, Sects. 7 and 8]. A lattice polytope $P \subset \mathbb{R}^d$ defines a projective toric variety X_P. A hypersurface in X_P is given by the vanishing set of $f = \sum_{\mathbf{v} \in P \cap \mathbb{Z}^d} c_{\mathbf{v}} x^{\mathbf{v}}$, where $c_{\mathbf{v}} \in \mathbb{C}$ and $x^{\mathbf{v}}$ denotes the monomial $x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$. The Newton polytope of f is the convex hull of the lattice points corresponding to monomials of f that have nonzero coefficients $c_{\mathbf{v}}$. A hypersurface has Newton polytope P if $c_{\mathbf{v}} \neq 0$ for all vertices $\mathbf{v}$ of P. Suppose that a lattice polytope P is invariant under the linear action of a group G. Then, a hypersurface in X_P is said to be G-*invariant* if $c_{\mathbf{v}} = c_{\mathbf{w}}$ for all lattice points $\mathbf{v}, \mathbf{w}$

in the same G-orbit of $P \cap \mathbb{Z}^d$. The hypersurface is *smooth* if the gradient vector $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is never zero when $x_1, \ldots, x_n \in \mathbb{C}^*$ and *nondegenerate* if $f|_Q = \sum_{\mathbf{v} \in Q \cap \mathbb{Z}^d} c_{\mathbf{v}} x^{\mathbf{v}}$ is smooth for all faces Q of P.

Theorem 2.9. [2, Theorem 7.7] *If there exists a G-invariant nondegenerate hypersurface with Newton polytope P, then $\mathrm{H}^*(P; z)$ is effective. In particular, this assumption holds if the linear system $\Gamma(X_P, L)^G$ of G-invariant global sections on the corresponding line bundle L is base point free.*

Recall that a subspace $W \subseteq \Gamma(X, L)$ is *basepoint free* if for every $p \in X$, there exists $s \in W$ with $s(p) \neq 0$ [12, Sect. 6.0]. For a face $Q \subset P$ let G_Q denote the stabilizer of Q. By [2, Remark 7.8], the linear system of G-invariant global sections $\Gamma(X_P, L)^G$ on X_P is basepoint free if and only if for each face $Q \subset P$, the linear system

$$\left\{ \sum_{\mathbf{u} \in Q \cap M} c_{\mathbf{u}} x^{\mathbf{u}} \quad \middle| \quad c_{\mathbf{u}} = c_{\mathbf{u}'} \text{ if } \mathbf{u}, \mathbf{u}' \text{ lie in the same } G_Q\text{-orbit} \right\} \quad (3)$$

on the torus T is basepoint free. In addition, this condition is automatically satisfied for faces of dimension ≤ 1.

Suppose that every face Q of P with $\dim(Q) > 1$ contains a lattice point $\mathbf{u}_Q$ that is G_Q-fixed. To show that the linear system (3) is basepoint free on T, we need to show that for every $x \in T$, there exists a polynomial in the system that does not vanish at x. In fact, we have something stronger, since the polynomial $x^{\mathbf{u}_Q}$ is an element of the linear system and does not vanish at *any* $x \in T$. Thus, we recover the following criterion certifying the existence of a G-invariant nondegenerate hypersurface with Newton polytope P.

Corollary 2.10. ([2, Corollary 7.10], reworded) *If every face Q of P with $\dim(Q) > 1$ contains a lattice point that is G_Q-fixed, where G_Q denotes the stabilizer of Q, then the toric variety of P admits a G-invariant nondegenerate hypersurface with Newton polytope P.*

2.6. Restricted Representations

Let P be a polytope invariant under the action of G as in the Setup 2.1. In the case that the H^*-series for the action of G on P does not exhibit polynomiality and/or effectiveness, we may wish to find the largest subgroup of G that does exhibit nice behavior.

For example, consider the *standard permutahedron* Π_n, which we define to be the convex hull of all permutations of the set $\{1, 2, \ldots, n\}$ viewed as points in $\mathbb{R}^n$. Although it seems natural to take the symmetric group S_n acting on Π_n, it was shown in [4] that the H^*-series is neither polynomial nor effective when $n \geq 4$. This is why in Example 2.14 and Sect. 3.3.2, we consider the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ instead.

Let $\mathcal{H}$ be a subgroup of G. The action of G on P induces an action of $\mathcal{H}$ on P by restriction. Let $\mathrm{H}^*_G(P; z)$ and $\mathrm{H}^*_{\mathcal{H}}(P; z)$ be their respective equivariant H^*-series (Table 1).

TABLE 1. Character tables of S_4 and $\mathbb{Z}/4\mathbb{Z}$

S_4	()	(12)	(12)(34)	(123)	(1234)
χ_0	1	-1	1	1	-1
χ_1	3	-1	-1	0	1
χ_2	2	0	2	-1	0
χ_3	3	1	-1	0	-1
χ_4	1	1	1	1	1

$\mathcal{H}$	()	(1234)	(12)(34)	(1432)
μ_0	1	1	1	1
μ_1	1	-1	1	-1
μ_2	1	i	-1	$-i$
μ_3	1	$-i$	-1	i

Theorem 2.11. *If the toric variety of P admits a G-invariant nondegenerate hypersurface with Newton polytope P, then the same hypersurface is a nondegenerate $\mathcal{H}$-invariant hypersurface.*

Proof. Let $\sum_{\mathbf{v} \in P \cap \mathbb{Z}^d} c_{\mathbf{v}} x^{\mathbf{v}} = 0$ be a G-invariant nondegenerate hypersurface. Every $\mathcal{H}$-orbit of lattice points in P is contained in a G-orbit of lattice points. Thus, $c_{\mathbf{v}} = c_{\mathbf{w}}$ for all $\mathbf{v}, \mathbf{w}$ in an orbit of $\mathcal{H}$, and the hypersurface is $\mathcal{H}$-invariant. The non-degeneracy condition is independent of the group. $\square$

Theorem 2.12. *If $\mathrm{H}^*_G(P; z)$ is effective, then $\mathrm{H}^*_{\mathcal{H}}(P; z)$ is effective.*

Proof. The coefficient of each z^i in $\mathrm{H}^*_G(P; z)$ is an effective character. Thus, for each coefficient there is a corresponding representation of G on a vector space which is a direct sum of irreducible representations of G. Restricting to the action of $\mathcal{H}$ decomposes each summand further into a finite number of irreducible representations of $\mathcal{H}$. $\square$

Theorem 2.13. *If $\mathrm{H}^*_G(P; z)$ is polynomial, then $\mathrm{H}^*_{\mathcal{H}}(P; z)$ is polynomial.*

Proof. Let $\{\chi_1, \ldots, \chi_m\}$ be the irreducible representations of G, and let $\{\mu_1, \ldots, \mu_k\}$ be the irreducible representations of $\mathcal{H}$. Suppose $\mathrm{H}^*_G(P; z)$ is polynomial so that

$$\mathrm{H}^*_G(P; z) = \sum_{i=0}^{d} \left(\sum_{j=1}^{m} c_{i,j} \chi_j \right) z^i = \sum_{j=1}^{m} \left(\sum_{i=0}^{d} c_{i,j} z^i \right) \chi_j,$$

where $c_{i,j} \in \mathbb{Z}$ for all $i \in \{0, \ldots, d\}$ and $j \in [m]$. Then, the coefficient of μ_k in $\mathrm{H}^*_{\mathcal{H}}(P; z)$ is

$$\sum_{j=0}^{m} (c_{0,j} z^0 + \cdots + c_{d,j} z^d) \langle \chi_i |_{\mathcal{H}}, \mu_k \rangle_{\mathcal{H}},$$

where $\langle \Phi, \Psi \rangle_{\mathcal{H}} \in \mathbb{C}$ denotes the inner product between characters Φ, Ψ of $\mathcal{H}$, and $\chi_i |_{\mathcal{H}}$ is the representation restricted to $\mathcal{H}$. $\square$

Example 2.14. The cyclic group $\mathbb{Z}/n\mathbb{Z} \subseteq S_n$ acting on the permutahedron Π_n gives an interesting example of the failure of the other direction. Namely, $\mathrm{H}^*_{S_n}(\Pi_n; z)$ is non-polynomial for $n \geq 4$ [4], but $\mathrm{H}^*_{\mathbb{Z}/n\mathbb{Z}}(\Pi_n; z)$ is polynomial for all n (see Lemma 3.47). Here we show how the rational function collapses to a polynomial for the case $\mathcal{H} = \mathbb{Z}/4\mathbb{Z} \subseteq S_4$. We use our implementation of the H*-series in `Sagemath` (see Appendix A) to calculate $\mathrm{H}^*(\Pi_4, ; z)$ below, with characters given in Table 1.

$$\mathrm{H}^*_{S_4}(\Pi_4; z) = \frac{(\chi_2 + \chi_3 + \chi_4) z^4}{z + 1} + \frac{(\chi_0 + 5\chi_1 + 6\chi_2 + 9\chi_3 + 6\chi_4) z^3}{z + 1}$$
$$+ \frac{(\chi_0 + 7\chi_1 + 8\chi_2 + 14\chi_3 + 9\chi_4) z^2}{z + 1}$$
$$+ \frac{(\chi_0 + 3\chi_1 + 3\chi_2 + 5\chi_3 + 4\chi_4) z}{z + 1} + \frac{\chi_4}{z + 1}$$

We compute $\langle \chi_i|_{\mathcal{H}}, \mu_j \rangle_{\mathcal{H}}$ for all pairs i, j in order to restrict $\mathrm{H}^*{}_{S_4}(\Pi_4; z)$ to $\mathrm{H}^*{}_{\mathcal{H}}(\Pi_4; z)$. A factor of $1 + z$ appears in the numerator:

$$
\mathrm{H}^*{}_{\mathcal{H}}(\Pi_4; z) = \frac{(2\mu_0 + 2\mu_1 + \mu_2 + \mu_3)z^4 + (17\mu_0 + 16\mu_1 + 14\mu_2 + 14\mu_3)z^3}{1 + z}
$$

$$
+ \frac{(24\mu_0 + 23\mu_1 + 21\mu_2 + 21\mu_3)z^2 + (10\mu_0 + 9\mu_1 + 8\mu_2 + 8\mu_3)z + \mu_0}{1 + z}
$$

$$
= (2\mu_0 + 2\mu_1 + \mu_2 + \mu_3)z^3 + (15\mu_0 + 14\mu_1 + 13\mu_2 + 13\mu_3)z^2
$$

$$
+ (9\mu_0 + 9\mu_1 + 8\mu_2 + 8\mu_3)z + \mu_0.
$$

3. Techniques

In this section, we introduce our main results, exploring four approaches toward equivariant Ehrhart theory and applying these methods to a variety of examples.

3.1. Zonotopal Decompositions

Zonotopes are a family of polytopes which decompose into convenient building blocks called parallelotopes, from which their Ehrhart polynomials are easily computed. This approach also lends itself nicely to equivariant Ehrhart theory, and has previously been used to study the equivariant Ehrhart theory of the permutahedron under the action of the symmetric group [3, 4]. Here, we adapt this technique to all zonotopes of the Type A root system, also known as graphic zonotopes. We expect this to generalize further to other root systems (see [13] for progress in this direction), as well as to general families of zonotopes. We begin with a brief overview of zonotopes; for a more in-depth introduction to the subject, see [7, Chapter 9].

Let S be a finite set of vectors. The *zonotope* of S is denoted $Z(S)$ and is the Minkowski sum of the line segments connecting $\mathbf{0}$ to $\mathbf{s}$ for each $\mathbf{s} \in S$, or any translation of this polytope. It was shown by Shephard that any zonotope $Z(S)$ can be partitioned into half-open *parallelotopes*, or zonotopes that are also combinatorial cubes, that are in bijection with the linearly independent subsets of S [14, Theorem 54]. For a linearly independent subset $T \subseteq S$, the corresponding half-open parallelotope $\square T$ is, up to translation, the zonotope $Z(T)$ where each line segment in the Minkowski sum is open at the endpoint $\mathbf{0}$. When S consists of lattice vectors, the resulting half-open lattice parallelotopes have two nice properties: their volumes can be easily computed from matrix minors, and the number of lattice points they contain is exactly equal to their volume. Stanley used this to give a formula for the Ehrhart polynomial of a lattice zonotope.

Theorem 3.1. [15, Theorem 2.2] *The Ehrhart polynomial of the lattice zonotope* $Z(S)$ *is*

$$
\mathrm{L}\left(Z(S); t\right) = \sum_{\substack{T \subseteq S \\ lin.\ indep.}} \mathrm{vol}\square T \cdot t^{|T|}.
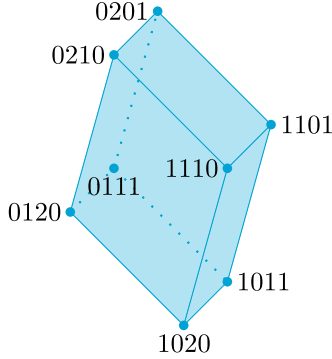$$

FIGURE 2. The graphic zonotope Z_{Path_4} is a parallelopiped

Let $\Gamma = (V, E)$ be a simple graph, where V is a finite set of vertices, and E is the set of undirected edges. We write $E \subseteq \binom{V}{2}$ where $\binom{V}{2}$ is the collection of all 2-element subsets of V, and we denote by $\mathbb{R}V$ a real vector space with a basis $\{\mathbf{e}_v : v \in V\}$ indexed by the elements of V. The *graphic zonotope* Z_Γ is given by taking the Minkowski sum of line segments in $\mathbb{R}V$

$$Z_\Gamma := \sum_{\{u,v\} \in E} [\mathbf{e}_u, \mathbf{e}_v].$$

The dimension of Z_Γ is $|V|$ minus the number of connected components of Γ.

Example 3.2. (*Path graph*) When $V = [n]$ and $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}\}$, we call the graph (V, E) the *path graph* or Path_n. The graphic zonotope Z_{Path_n} is the $(n-1)$-dimensional parallelotope $\sum_{i=1}^{n-1} [\mathbf{e}_i, \mathbf{e}_{i+1}]$; Fig. 2 shows Z_{Path_4}. We include an in-depth analysis of the equivariant H*-series of Z_{Path_n} later in Sect. 3.1.1.

Example 3.3. (*Complete graph*) The graph with $V = [n]$ and $E = \binom{[n]}{2}$ is called the *complete graph*. The corresponding graphic zonotope is $\sum_{1 \le i < j \le n} [\mathbf{e}_i, \mathbf{e}_j]$, which is the permutahedron. The equivariant H*-series of the permutahedron is analyzed in [4].

The following characterizations of the graphic zonotope are well-known; see [16, Section 2] and [17, Section 13] for details.

Proposition 3.4. *The graphic zonotope Z_Γ has....*

- *...the half-space description*

$$Z_\Gamma = \left\{ \mathbf{x} \in \mathbb{R}V : \sum_{v \in V} x_v = |E|, \ \forall S \subset V \sum_{v \in S} x_v \le \mathrm{inc}_\Gamma(S) \right\},$$

where for any subset $S \subseteq V$, we define $\mathrm{inc}_\Gamma(S)$ to be the number of edges of Γ that have an endpoint in S (are "incident to S").

FIGURE 3. The orbits of the vertices of Path_4 under the automorphism $(14)(23)$ **(a)**, and the $(14)(23)$-connectivity graph of Path_4 **(b)**

- *...the vertex description*

$$\mathrm{conv}\left\{\mathbf{v}_{\mathfrak{o}} := \sum_{v \in V} \mathrm{indeg}_{\mathfrak{o}}(v)\mathbf{e}_v : \mathfrak{o} \text{ is an acyclic orientation of } \Gamma\right\},$$

where $\mathrm{indeg}_{\mathfrak{o}}(v)$ *is the number of edges pointing into the vertex v in the acyclic orientation $\mathfrak{o}$ of Γ. Furthermore, $\mathbf{v}_{\mathfrak{o}}$ is a vertex of Z_{Γ} for all acyclic orientations $\mathfrak{o}$ of Γ.*

The symmetric group $\mathrm{Sym}(V)$ acts on the set of vertices V, also inducing an action on $\binom{V}{2}$. The *automorphism group* $\mathrm{Aut}(\Gamma)$ of Γ is the subgroup of $\mathrm{Sym}(V)$ under which the edge set E is invariant. Just as $\mathrm{Aut}(\Gamma)$ acts on V and holds Γ invariant, likewise $\mathrm{Aut}(\Gamma)$ acts linearly on $\mathbb{R}V$ and holds Z_{Γ} invariant. Hence, $\mathbb{R}V$ is a representation of $\mathrm{Aut}(\Gamma)$, and we can consider the equivariant Ehrhart theory of Z_{Γ} under the action of $\mathrm{Aut}(\Gamma)$ (or any subgroup thereof).

For the setup, we will view $\mathrm{Aut}(\Gamma)$ as a subgroup of $\mathrm{Sym}(V)$, so every element $\sigma \in \mathrm{Aut}(\Gamma)$ can be written as a product of $m \leq |V|$ disjoint cycles $\sigma_1, \ldots, \sigma_m$ partitioning V. We denote the cycle lengths by $\ell_1, \ldots, \ell_m$, respectively. For each $\sigma \in \mathrm{Aut}(\Gamma)$, we define the *$\sigma$-connectivity graph* $C_{\Gamma}(\sigma)$ to be the simple graph with vertex set $\{\sigma_1, \ldots, \sigma_m\}$ and an edge between two distinct cycles σ_i and σ_j whenever some element of the cycle σ_i is connected by an edge in Γ to some element of σ_j.

Example 3.5. Let $\Gamma = \mathsf{Path}_4$ and let $\sigma = (14)(23) \in S_4$. Then, $\mathrm{Aut}(\Gamma) = \{\mathrm{id}, \sigma\}$ and $C_{\Gamma}(\sigma)$ has vertex set $\{14, 23\}$ and an edge connecting 14 and 23, as seen in Fig. 3.

Example 3.6. Let Γ be the complete graph on n vertices. Then, $\mathrm{Aut}(\Gamma) = S_n$ and for any $\sigma \in S_n$, the graph $C_{\Gamma}(\sigma)$ is the complete graph whose vertex set is the set of cycles $\{\sigma_1, \ldots, \sigma_m\}$ of σ.

Example 3.7. Let Γ be the graph from Fig. 4a and let $\sigma \in \mathrm{Aut}(\Gamma)$ be $(0)(135)$ $(246)(789)$. Then, the σ-connectivity graph of Γ is given in Fig. 4b.

Let $\sigma \in \mathrm{Aut}(\Gamma)$. Along with the σ-connectivity graph, we also need to keep track of the number of edges of Γ that connect a given vertex v with vertices of Γ appearing in a given cycle σ_j of σ. Denote by $\deg(v, \sigma_j)$ the number of edges of Γ for which one endpoint is v and the other is a vertex in σ_j. If σ_i is the cycle of σ containing v, then for any $u \in \sigma_i$, we necessarily have $\deg(u, \sigma_j) = \deg(v, \sigma_j)$, as shown in the following lemma:
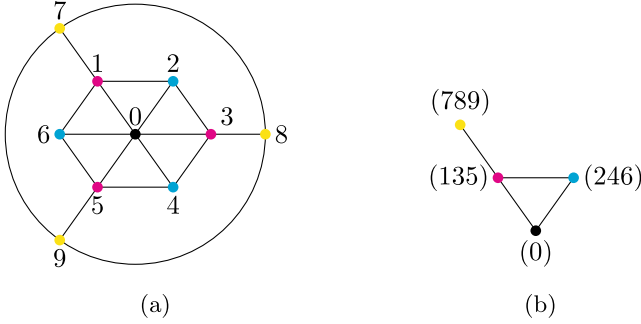
FIGURE 4. A graph Γ with the orbits of its vertices under the automorphism $\sigma = (0)(135)(246)(789)$ colored (**a**), and the corresponding graph $C_\Gamma(\sigma)$ (**b**)

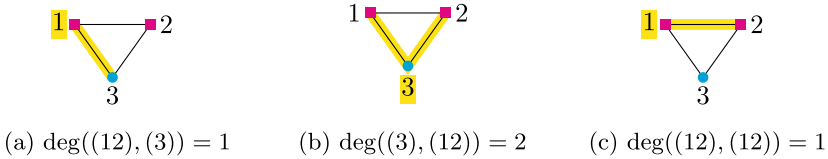(a) $\deg((12), (3)) = 1$ (b) $\deg((3), (12)) = 2$ (c) $\deg((12), (12)) = 1$

FIGURE 5. To determine $\deg(\sigma_i, \sigma_j)$, choose a representative vertex of the cycle σ_i and count how many neighbors it has in the cycle σ_j in the graph Γ

Lemma 3.8. *Let $\sigma \in \mathrm{Aut}(\Gamma)$ and let σ_i, σ_j be (not necessarily distinct) cycles of σ. Then, for vertices $u, v \in \sigma_i$, we have $\deg(u, \sigma_j) = \deg(v, \sigma_j)$.*

Proof. Suppose that $u, v \in \sigma_i$ have different numbers of edges connecting them with elements of σ_j. Since u and v are in the same cycle of σ, there exists r such that $\sigma^r(u) = v$. But σ^r is then not an automorphism of Γ, a contradiction. $\square$

We denote the number described in Lemma 3.8 by $\deg(\sigma_i, \sigma_j)$. (Warning: It is *not* true in general that $\deg(\sigma_i, \sigma_j) = \deg(\sigma_j, \sigma_i)$; see Example 3.9.) We will repeatedly make use of the following identity for distinct cycles σ_i and σ_j:

$$E_{ij} := \ell_i \deg(\sigma_i, \sigma_j) = \ell_j \deg(\sigma_j, \sigma_i) = \#\text{ edges connecting } \sigma_i \text{ and } \sigma_j \text{ in } \Gamma \quad (4)$$

We also write the number of edges within a given cycle σ_i as E_{ii} and note that $E_{ii} = \frac{1}{2}\ell_i \deg(\sigma_i, \sigma_i)$. Observe that unlike $\deg(\sigma_i, \sigma_j)$, the notation E_{ij} *is symmetric in i and j*.

Example 3.9. Let Γ be the complete graph on $\{1, 2, 3\}$ and let $\sigma = (12)(3) \in \mathrm{Aut}(\Gamma)$ as in Fig. 5. Then, $\deg((12), (3)) = 1$ since the vertices 1 and 2 are each connected to 3 by one edge, but $\deg((3), (12)) = 2$ since the vertex 3 has two edges connecting it to vertices in the cycle (12). Moreover, $\deg((12), (12)) = 1$ and $\deg((3), (3)) = 0$.

TABLE 2. Values of $\deg(\sigma_i, \sigma_j)$ for Γ and σ from Fig. 4

$\deg(\sigma_i, \sigma_j)$		σ_j			
		(0)	(135)	(246)	(789)
σ_i	(0)	0	3	3	0
	(135)	1	0	2	1
	(246)	1	2	0	0
	(789)	0	1	0	2

Example 3.10. Let Γ and σ be as in Fig. 4. Then, for cycles σ_i, σ_j of σ, the values $\deg(\sigma_i, \sigma_j)$ are given in Table 2.

We use $\mathbf{e}_{\sigma_i}$ to denote the vector $\sum_{v \in \sigma_i} \mathbf{e}_v \in \mathbb{R}V$. Theorem 3.11 and its proof closely follow the description of the fixed polytopes of the permutahedron from Theorem 2.12 of [3]. Our result generalizes theirs: where they used permutations, we use acyclic orientations of Γ; we also make use of identity (4).

Theorem 3.11. (Description of graphic zonotope fixed polytopes) *Let $\sigma \in \mathrm{Aut}(\Gamma)$ have cycles $\sigma_1, \sigma_2, \ldots, \sigma_m$. Then, the following descriptions of the fixed polytope Z_Γ^σ are equivalent:*

(i) *It is the subset of Z_Γ that is fixed by the action of σ.*
(ii) *It is the convex hull of the points $\mathbf{w}_\mathfrak{o}$, where $\mathfrak{o}$ ranges over acyclic orientations of $C_\Gamma(\sigma)$, and*

$$\mathbf{w}_\mathfrak{o} := \sum_{i=1}^m \left(\frac{1}{2} \deg(\sigma_i, \sigma_i) + \sum_{\substack{j:\sigma_j \to \sigma_i \\ in \ \mathfrak{o}}} \deg(\sigma_i, \sigma_j) \right) \mathbf{e}_{\sigma_i}.$$

Furthermore, this gives a bijection between acyclic orientations of $C_\Gamma(\sigma)$ and vertices of Z_Γ^σ.
(iii) *It is the Minkowski sum*

$$\sum_{\substack{\{\sigma_i, \sigma_j\} \ is \ an \\ edge \ of \ C_\Gamma(\sigma) \\ i < j}} [\deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}, \deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i}] + \sum_{j=1}^m \frac{1}{2} \deg(\sigma_j, \sigma_j)\mathbf{e}_{\sigma_j}. \quad (5)$$

Proof. (ii) $\subseteq$ (i): Let $\hat{\mathfrak{o}}$ be any orientation of Γ "extending" $\mathfrak{o}$, meaning that if $\sigma_i \to \sigma_j$ in $\mathfrak{o}$, and $v \in \sigma_i$ and $v' \in \sigma_j$ are joined by an edge of Γ, then $v \to v'$ in $\hat{\mathfrak{o}}$. Then, $\mathbf{w}_\mathfrak{o}$ is the average of the σ-orbit of the vertex of Z_Γ corresponding to $\hat{\mathfrak{o}}$, so $\mathbf{w}_\mathfrak{o} \in Z_\Gamma$. By definition, the coordinates of $\mathbf{w}_\mathfrak{o}$ are constant over the cycles of σ, so $\mathbf{w}_\mathfrak{o} \in Z_\Gamma^\sigma$ for all acyclic orientations $\mathfrak{o}$.

(iii) $\subseteq$ (ii): Let $\mathbf{w}$ be a vertex of (5) and let $\mathbf{y} = (y_1, \ldots, y_n) \in (\mathbb{R}^n)^*$ be such that $\mathbf{w}$ is the $\mathbf{y}$-maximal face of (5). For $1 \le i \le m$, define y_{σ_i} to be the

average of y_k for $k \in \sigma_i$; so $y_{\sigma_i} = \frac{1}{\ell_i} \sum_{k \in \sigma_i} y_k$. Now, since **y** is maximal at a vertex, we know that for each edge $\{\sigma_i, \sigma_j\}$ of $C_\Gamma(\sigma)$, we must have

$$\mathbf{y}\left(\deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}\right) \neq \mathbf{y}\left(\deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i}\right).$$

Therefore, it holds that

$$\deg(\sigma_j, \sigma_i) \sum_{k \in \sigma_j} y_k \neq \deg(\sigma_i, \sigma_j) \sum_{k \in \sigma_i} y_k$$

$$\ell_j \deg(\sigma_j, \sigma_i) y_{\sigma_j} \neq \ell_i \deg(\sigma_i, \sigma_j) y_{\sigma_i}$$

Using identity (4), we find that $y_{\sigma_j} \neq y_{\sigma_i}$. For each edge $\{\sigma_i, \sigma_j\}$ in $C_\Gamma(\sigma)$, if $y_{\sigma_i} < y_{\sigma_j}$, assign the direction pointing from σ_i to σ_j. This gives an acyclic orientation $\mathfrak{o}$ of $C_\Gamma(\sigma)$, and $\mathbf{w} = \mathbf{w}_\mathfrak{o}$.

(i) $\subseteq$ (iii): According to [2, Lemma 5.4], the fixed polytope Z_Γ^σ is the convex hull of the averages over the σ-orbits of the vertices of Z_Γ. Therefore, we need only show that for any vertex $\mathbf{v}_\mathfrak{o}$ of Z_Γ corresponding to the acyclic orientation $\mathfrak{o}$ of Γ, the average $\overline{\mathbf{v}_\mathfrak{o}}$ of the σ-orbit of $\mathbf{v}_\mathfrak{o}$ is in the Minkowski sum (5). To do this, we will fix an orientation $\mathfrak{o}'$ of Γ, show that $\overline{\mathbf{v}_{\mathfrak{o}'}}$ is in the Minkowski sum, and then show that we can find a path from $\overline{\mathbf{v}_{\mathfrak{o}'}}$ to $\overline{\mathbf{v}_\mathfrak{o}}$ within the sum.

We construct $\mathfrak{o}'$ as follows: choose any ordering of the n vertices of Γ such that all the vertices in the cycle σ_i are less than all the vertices in σ_{i+1} for $1 \leq i < m$, and let $\mathfrak{o}'$ be the acyclic orientation of Γ induced by this ordering of the vertices. Then, the vertex $\mathbf{v}_{\mathfrak{o}'}$ of Z_Γ is defined as in Proposition 3.4, and the average of the σ-orbit of $\mathbf{v}_{\mathfrak{o}'}$ is

$$\overline{\mathbf{v}_{\mathfrak{o}'}} = \sum_{\substack{\{\sigma_i, \sigma_j\} \text{ is an} \\ \text{edge of } C_\Gamma(\sigma) \\ i < j}} \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j} + \sum_{j=1}^m \frac{1}{2} \deg(\sigma_j, \sigma_j)\mathbf{e}_{\sigma_j},$$

which is clearly seen to be in the sum (5).

Now consider any acyclic orientation $\mathfrak{o}$ of Γ. Let $\mathfrak{o}_0 := \mathfrak{o}', \mathfrak{o}_1, \ldots, \mathfrak{o}_\ell := \mathfrak{o}$ be a sequence of acyclic orientations of Γ where $\mathfrak{o}_k$ is obtained from $\mathfrak{o}_{k-1}$ by reversing the orientation of one edge for $1 \leq k \leq \ell$. For each k, if the edge of $\mathfrak{o}_{k-1}$ that is reversed to obtain $\mathfrak{o}_k$ points from a vertex in σ_i to a vertex in σ_j, then we have that

$$\overline{\mathbf{v}_{\mathfrak{o}_k}} - \overline{\mathbf{v}_{\mathfrak{o}_{k-1}}} = \frac{1}{\ell_i}\mathbf{e}_{\sigma_i} - \frac{1}{\ell_j}\mathbf{e}_{\sigma_j} = \frac{1}{E_{ij}}\left(\deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}\right).$$

Note that if the edge that is reversed connects two vertices of Γ that are in the same cycle of σ, then there is no change in the σ-average. Now for each pair of distinct cycles σ_i and σ_j, let D_{ij} be the number of edges between σ_i and σ_j that have different orientations in $\mathfrak{o}'$ and $\mathfrak{o}$. Then, we get

$$\overline{\mathbf{v}_\mathfrak{o}} - \overline{\mathbf{v}_{\mathfrak{o}'}} = \left(\overline{\mathbf{v}_{\mathfrak{o}_\ell}} - \overline{\mathbf{v}_{\mathfrak{o}_{\ell-1}}}\right) + \cdots + \left(\overline{\mathbf{v}_{\mathfrak{o}_2}} - \overline{\mathbf{v}_{\mathfrak{o}_1}}\right) + \left(\overline{\mathbf{v}_{\mathfrak{o}_1}} - \overline{\mathbf{v}_{\mathfrak{o}_0}}\right)$$

$$= \sum_{1 \leq i < j \leq m} \frac{D_{ij}}{E_{ij}}\left(\deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}\right)$$

If there are no edges between σ_i and σ_j in Γ, then D_{ij} must be 0, so $\deg(\sigma_i, \sigma_j)$ $\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}$ may only contribute to this sum if there is an edge between σ_i and σ_j in $C_\Gamma(\sigma)$. Furthermore, the maximum possible number of edges between σ_i and σ_j is E_{ij}, so $0 \leq \frac{D_{ij}}{E_{ij}} \leq 1$. Hence, adding our earlier expression for $\overline{\mathbf{v}_{\mathfrak{o}'}}$ produces a point in the zonotope (5).

Finally, it remains to prove that the acyclic orientations of $C_\Gamma(\sigma)$ are in bijection with the vertices of Z_Γ^σ. By definition, each acyclic orientation $\mathfrak{o}$ results in a distinct point $\mathbf{w}_\mathfrak{o}$. We already showed that every vertex of Z_Γ^σ has the form $\mathbf{w}_\mathfrak{o}$ for some acyclic orientation $\mathfrak{o}$. For any $\mathbf{w}_\mathfrak{o}$, let $\mathbf{y} = \sum_{i=1}^m y_i (\sum_{k \in \sigma_i} \mathbf{e}_k^*)$ be some linear functional in $(\mathbb{R}^n)^*$ such that $y_i < y_j$ whenever $\sigma_i \to \sigma_j$ in $\mathfrak{o}$. Then, $\mathbf{y}(\deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}) = E_{ij}y_j$, so we can see that exactly one endpoint of each of the line segments comprising (5) is maximized by $\mathbf{y}$, and the resulting sum gives exactly $\mathbf{w}_\mathfrak{o}$. Therefore, $\mathbf{w}_\mathfrak{o}$ is the unique point in Z_Γ^σ maximizing $\mathbf{y}$, and is hence a vertex. $\qquad\square$

In general, a zonotope generated by line segments with direction vectors $\mathbf{s}_1, \ldots, \mathbf{s}_k$ can be tiled by half-open parallelotopes that are in bijection with linearly independent subsets of $\{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$. In the case of Z_Γ^σ, the generating vectors are

$$\left\{ \deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j} : \{\sigma_i, \sigma_j\} \text{ is an edge of } C_\Gamma(\sigma),\ i < j \right\}. \quad (6)$$

We define a *subforest* of $C_\Gamma(\sigma)$ to be any graph on the same vertex set $\{\sigma_1, \ldots, \sigma_m\}$ as $C_\Gamma(\sigma)$ whose edge set is a subset of the edges of $C_\Gamma(\sigma)$, and that does not contain any cycles.

Proposition 3.12. *The linearly independent subsets of the vectors in (6) are in bijection with subforests of the σ-connectivity graph $C_\Gamma(\sigma)$.*

Proof. The bijection is as follows: Given a subgraph of $C_\Gamma(\sigma)$, for each edge $\{i, j\}$ where $i < j$, take the vector $\deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}$ to be in the subset; similarly, given a subset of the generating set, take the corresponding edges of $C_\Gamma(\sigma)$. It remains to show that linearly independent subsets of the generators correspond exactly to subforests of $C_\Gamma(\sigma)$. By identity (4), the generating vector corresponding to the edge $\{i, j\}$ is simply a positive multiple of the vector $\frac{\mathbf{e}_{\sigma_i}}{\ell_i} - \frac{\mathbf{e}_{\sigma_j}}{\ell_j}$. From here we can follow exactly the proof of [3, Lemma 3.2], noting that our generators are a subset of theirs and, therefore, inherit the same linear independence properties. $\qquad\square$

For a given subforest F of $C_\Gamma(\sigma)$, we denote by $\square F$ the corresponding half-open parallelotope in the tiling of Z_Γ^σ.

Proposition 3.13. *The volume of $\square F$ is given by the formula*

$$\mathrm{vol}\,\square F := \left(\prod_{\substack{edge\ \{\sigma_i, \sigma_j\} \\ of\ F}} E_{ij} \right) \cdot \left(\prod_{i=1}^m \frac{1}{\ell_i} \right) \cdot \left(\prod_{\substack{conn.\ comp. \\ T\ of\ F}} \gcd(\ell_i : \sigma_i \in T) \right).$$

$$(7)$$

Proof. This follows directly from [3, Lemma 3.3]. □

The 2-*valuation* of a positive integer k, denoted $\mathrm{val}_2(k)$, is the highest power of 2 dividing k. For example, $\mathrm{val}_2(24) = 3$.

Definition 3.14. A subforest F of $C_\Gamma(\sigma)$ is (σ, Γ)-*compatible* if for all connected components T of F, we have

$$\min_{i \in T} \mathrm{val}_2(\ell_i) \leq \mathrm{val}_2 \left(\sum_{j \in T} E_{jj} \right). \tag{8}$$

Proposition 3.15. *Let* $\sigma \in \mathrm{Aut}(\Gamma)$ *have cycle type* $(\ell_1, \ldots, \ell_m)$, *and let* F *be a subforest of* $C_\Gamma(\sigma)$. *Then,* $\mathrm{aff}(\square F)$ *contains lattice points if and only if* F *is* (σ, Γ)-*compatible.*

Proof. The affine span of $\square F$ is the span of the vectors $\deg(\sigma_i, \sigma_j)\mathbf{e}_{\sigma_i} - \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}$ corresponding to the edges $\{\sigma_i, \sigma_j\}$ of F, translated by the vector

$$\sum_{j=1}^m \frac{1}{2} \deg(\sigma_j, \sigma_j)\mathbf{e}_{\sigma_j} + \sum_{\substack{\text{edge } \{\sigma_i, \sigma_j\} \\ \text{of } F, i<j}} \deg(\sigma_j, \sigma_i)\mathbf{e}_{\sigma_j}.$$

Let $\mathbf{y} = \sum_{i=1}^m y_i \mathbf{e}_{\sigma_i}$ be a point in $\mathrm{aff}(\square F)$. Then, the coordinates y_i satisfy, for all connected components T of F,

$$\sum_{\sigma_j \in T} \ell_j y_j = \sum_{\sigma_j \in T} E_{jj} + \sum_{\substack{\text{edge } \{\sigma_i, \sigma_j\} \\ \text{of } T}} E_{ij}. \tag{9}$$

We want to determine when (9) has integer solutions $\mathbf{y}$. It is a fact from elementary number theory that this occurs exactly when $\gcd(\ell_j : \sigma_j \in T)$ divides the right side of (9). Clearly each E_{ij} is divisible by $\gcd(\ell_j : \sigma_j \in T)$, so we may focus our attention on the term $\sum_{\sigma_j \in T} E_{jj}$. If we multiply this term by 2, then we get $\sum_{\sigma_j \in T} \ell_j \deg(\sigma_j, \sigma_j)$ which is clearly divisible by $\gcd(\ell_j : \sigma_j \in T)$. Hence, we need only consider 2-valuations to determine whether divisibility holds. Finally, noticing that $\mathrm{val}_2(\gcd(\ell_j : \sigma_j \in T)) = \min_{\sigma_j \in T} \mathrm{val}_2(\ell_j)$ allows us to conclude that the condition of (σ, Γ)-compatibility introduced in Definition 3.14 is exactly what we need to guarantee integer solutions. □

Let $c(F)$ be the number of connected components of a graph F, and let m be the number of cycles of $\sigma \in \mathrm{Aut}(\Gamma)$. Note that for a subforest F of $C_\Gamma(\sigma)$, the number of edges of F is $m - c(F)$. We omit proofs for the following two corollaries as they mirror the proofs of Theorem 1.1 and Proposition 5.1 in [4].

Corollary 3.16. *The Ehrhart quasipolynomial of the fixed polytope* Z_Γ^σ *is*

$$L(Z_\Gamma^\sigma; t) = \begin{cases} \displaystyle\sum_{\substack{subforest \\ F \ of \ C_\Gamma(\sigma)}} \mathrm{vol}\square F \cdot t^{m-c(F)}, & t \ even \\[2em] \displaystyle\sum_{\substack{(\sigma, \Gamma)\text{-}compatible \\ subforest \\ F \ of \ C_\Gamma(\sigma)}} \mathrm{vol}\square F \cdot t^{m-c(F)}, & t \ odd. \end{cases}$$

Eulerian polynomials commonly arise when computing the Ehrhart series of polytopes. The Eulerian polynomial $A_n(z)$ is defined by the identity

$$\sum_{t \geq 0} t^n z^t = \frac{A_n(z)}{(1-z)^{n+1}}.$$

Notice that $A_n(z)$ has no constant term; the coefficients of $\frac{1}{z} A_n(z)$ are called the *Eulerian numbers*. We can describe them using ascents of permutations, where the position $i \in \{1, 2, \ldots, n-1\}$ is an *ascent* of $\sigma \in S_n$ if $\sigma(i+1) > \sigma(i)$. The coefficient of z^k in $\frac{1}{z} A_n(z)$ is the number of permutations in S_n with exactly k ascents, denoted $A_{n,k}$.

Corollary 3.17. *The Ehrhart series of the fixed polytope Z_Γ^σ has the following form:*

$$\mathrm{Ehr}(Z_\Gamma^\sigma; z) = \sum_{\substack{(\sigma,\Gamma)\text{-}compatible \\ subforests \\ F \text{ of } C_\Gamma(\sigma)}} \frac{\mathrm{vol}\square F \cdot A_{m-c(F)}(z)}{(1-z)^{m-c(F)+1}}$$

$$+ \sum_{\substack{(\sigma,\Gamma)\text{-}incompatible \\ subforests \\ F \text{ of } C_\Gamma(\sigma)}} \frac{2^{m-c(F)} \cdot \mathrm{vol}\square F \cdot A_{m-c(F)}(z^2)}{(1-z^2)^{m-c(F)+1}}$$

To assess whether $\mathrm{H}^*(Z_\Gamma; z)$ is a polynomial, we need to examine for which $\sigma \in \mathrm{Aut}(\Gamma)$ the poles of the Ehrhart series $\mathrm{Ehr}(Z_\Gamma^\sigma; z)$ cancel with the zeros of $\det(I - \rho(\sigma)z) = \prod_{i=1}^m (1 - z^{\ell_i})$, where m is the number of cycles of σ and $\ell_1, \ldots, \ell_m$ are the cycle lengths. The series $\mathrm{Ehr}(Z_\Gamma^\sigma; z)$ always has a pole at $z = 1$, may have a pole at $z = -1$, and never has a pole at any other value of z. We can see from Corollary 3.17 that the pole at $z = 1$ has order at most m, and since $\prod_{i=1}^m (1 - z^{\ell_i})$ has a zero of order m at $z = 1$, the pole at $z = 1$ will always cancel. Hence, we may focus on the pole at $z = -1$. The following theorem tells us when this pole cancels.

Theorem 3.18. *Let $\sigma \in \mathrm{Aut}(\Gamma)$ have cycle type $\ell_1, \ldots, \ell_m$. The series $\mathrm{H}^*(Z_\Gamma; z)$ (σ) is a polynomial if and only if one of the following conditions holds:*

(a) *All ℓ_i are odd.*
(b) *All even ℓ_i have $\deg(\sigma_i, \sigma_i)$ even.*
(c) *There exists an even ℓ_i with $\deg(\sigma_i, \sigma_i)$ odd, and*

$$\# \text{ even } \ell_i > \max\{\# \text{ of edges in a } (\sigma, \Gamma)\text{-incompatible subforest of } C_\Gamma(\sigma)\}.$$

Proof. In case (a), all subforests of $C_\Gamma(\sigma)$ are (σ, Γ)-compatible since $\mathrm{val}_2(\ell_i) = 0$ for all i. Therefore, the Ehrhart series of Z_Γ^σ does not have a pole at $z = -1$. Likewise in case (b); here we can also see that the condition of (σ, Γ)-compatibility from Definition 3.14 is always satisfied.

Now assume (a) and (b) both fail. Then, we must have some cycle σ_i with even length ℓ_i where $\deg(\sigma_i, \sigma_i)$ is odd. In this case, we are guaranteed to have at least one (σ, Γ)-incompatible subforest F of $C_\Gamma(\sigma)$, since every forest where σ_i is alone in a connected component will be (σ, Γ)-incompatible.

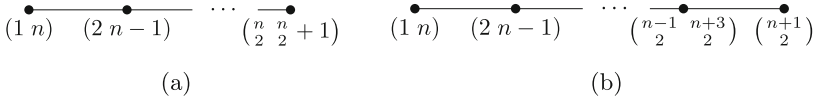$$\underset{(1\ n)}{\bullet\!\!-\!\!\bullet}\quad\underset{(2\ n-1)}{\bullet\!\!-\!\!\bullet}\quad\cdots\quad\underset{(\frac{n}{2}\ \frac{n}{2}+1)}{\bullet\!\!-\!\!\bullet}$$

$$\underset{(1\ n)}{\bullet\!\!-\!\!\bullet}\quad\underset{(2\ n-1)}{\bullet\!\!-\!\!\bullet}\quad\cdots\quad\underset{(\frac{n-1}{2}\ \frac{n+3}{2})}{\bullet\!\!-\!\!\bullet}\quad\underset{(\frac{n+1}{2})}{\bullet}$$

(a)   (b)

FIGURE 6. The connectivity graphs $C_{\mathsf{Path}_n}(\sigma)$ where σ is the generator of $\mathbb{Z}/2\mathbb{Z}$ in the case where n is even (**a**) and odd (**b**)

From Corollary 3.17 we see that the pole of $\mathrm{Ehr}(Z_\Gamma^\sigma;z)$ at $z=-1$ has order $\max_{(\sigma,\Gamma)\text{-incomp. }F}\{m-c(F)+1\}$, which is one more than the maximum number of edges in a (σ,Γ)-incompatible subforest of $C_\Gamma(\sigma)$. Since $\prod_{i=1}^m(1-z^{\ell_i})$ has a zero at $z=-1$ of order equal to the number of even ℓ_i, condition (c) is exactly what is needed for cancellation to occur. $\qquad\square$

Corollary 3.19. $H^*(Z_\Gamma;z)$ *is a polynomial if and only if every $\sigma\in\mathrm{Aut}(\Gamma)$ satisfies one of the conditions of Theorem 3.18.*

3.1.1. Path Graphs. We conclude this section by focusing in on a particularly simple family of graphs. Consider the path graph Path_n as introduced in Example 3.2. Then, $\mathrm{Aut}(\mathsf{Path}_n)=\mathbb{Z}/2\mathbb{Z}$ for all n, but the cycle type of the group generator depends on whether n is even or odd. When n is even, the generator of $\mathrm{Aut}(\mathsf{Path}_n)$ written in cycle notation is $\sigma=(1\ n)(2\ n-1)\ldots(\frac{n}{2}\ \frac{n}{2}+1)$, and when n is odd, the generator is $\sigma=(1\ n)(2\ n-1)\ldots(\frac{n-1}{2}\ \frac{n+3}{2})(\frac{n+1}{2})$. The corresponding connectivity graphs $C_{\mathsf{Path}_n}(\sigma)$ are shown in Fig. 6.

We first consider the case when n is even. Let F be any subforest of $C_{\mathsf{Path}_n}(\sigma)$. Using formula (7), we get

$$\mathrm{vol}\square F=\left(2^{n/2-c(F)}\right)\left(\frac{1}{2^{n/2}}\right)\left(2^{c(F)}\right)=1.$$

We next assess (σ,Path_n)-compatibility of F. Since $\ell_i=2$ for all cycles σ_i, the left side of inequality (8) will always be 1. However, whenever T is the connected component of F containing the cycle $(\frac{n}{2}\ \frac{n}{2}+1)$, the right side of inequality (8) will be 0 since this particular cycle contains one edge of Path_n and all other cycles have no edges within them. This means that no subforest F of $C_{\mathsf{Path}_n}(\sigma)$ is (σ,Path_n)-compatible. Hence, when n is even, the Ehrhart quasipolynomial of $Z_{\mathsf{Path}_n}^\sigma$ is 0 for odd dilations and $(t+1)^{n/2-1}$ (the Ehrhart polynomial of an integral parallelotope) for even dilations t:

$$L_{Z_{\mathsf{Path}_n}^\sigma}(t)=\begin{cases}(t+1)^{n/2-1}, & t\text{ even}\\ 0, & t\text{ odd}\end{cases}.$$

From a geometric viewpoint, this 0 in the odd component of $L_{Z_{\mathsf{Path}_n}^\sigma}(t)$ occurs because $Z_{\mathsf{Path}_n}^\sigma$ is a rational polytope that is not full dimensional. When it is dilated by an odd factor, it misses the lattice completely.

Next, using Corollary 3.17 we get

$$\mathrm{Ehr}_{Z_{\mathsf{Path}_n}^\sigma}(z)=\sum_{t=0}^\infty(2t+1)^{n/2-1}\cdot z^{2t}$$

$$= \sum_{\substack{\text{subforest } F \\ \text{of } C_{\mathsf{Path}_n}(\sigma)}} \frac{2^{n/2-c(F)} \cdot A_{n/2-c(F)}(z^2)}{(1-z^2)^{n/2-c(F)+1}}$$

$$= \frac{\sum_{k=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{k} \cdot 2^k \cdot (1-z^2)^{n/2-k-1} \cdot A_k(z^2)}{(1-z^2)^{n/2}},$$

and we can see directly from this Ehrhart series that $\mathrm{H}^*(Z_{\mathsf{Path}_n}; z)$ is a polynomial when n is even.

Now suppose n is odd. As in the even case, we can use (7) to find that $\mathrm{vol}\,\square\, F$ is again 1 for all subforests F of $C_{\mathsf{Path}_n}(\sigma)$. However, unlike the even case, we have $\deg(\sigma_j, \sigma_j) = 0$ for all cycles σ_j of σ. This means that the half-integral shift of equation (5) disappears in this case, and we are left with an integral polytope $Z_{\mathsf{Path}_n}^\sigma$. Since all the fixed polytopes are integral in this case, the H^*-series is guaranteed to be a polynomial.

Let us go further and show that the H^*-polynomial is effective when n is odd. The fixed polytopes $Z_{\mathsf{Path}_n}^{\mathrm{id}}$ and $Z_{\mathsf{Path}_n}^\sigma$ are both lattice parallelotopes with relative volume 1 and dimensions $n-1$ and $\frac{n-1}{2}$, respectively. Therefore, we get

$$\mathrm{EE}(Z_{\mathsf{Path}_n}; z)(\mathrm{id}) = \frac{\frac{1}{z} A_{n-1}(z)}{(1-z)^n}$$

$$\mathrm{EE}(Z_{\mathsf{Path}_n}; z)(\sigma) = \frac{\frac{1}{z} A_{(n-1)/2}(z)}{(1-z)^{(n+1)/2}} = \frac{(1+z)^{(n-1)/2} \frac{1}{z} A_{(n-1)/2}(z)}{(1-z^2)^{(n-1)/2}(1-z)},$$

where we have arranged both series to have the denominator $\det(I - \rho \cdot z)$. Notice that the numerators of both rational functions are polynomials of degree $n-2$ with positive integer coefficients, and they are $\mathrm{H}^*(Z_{\mathsf{Path}_n}; z)$ evaluated at the elements of $\mathbb{Z}/2\mathbb{Z}$. Write $\mathrm{H}^*(Z_{\mathsf{Path}_n}; z) = \sum_{k=0}^{n-2} H_k^* z^k$ for virtual characters $H_k^* \in R(\mathbb{Z}/2\mathbb{Z})$. For each k, there exist integer coefficients a_k, b_k such that $H_k^* = a_k \chi_{\mathrm{triv}} + b_k \chi_{\mathrm{alt}}$. For a polynomial $f(z)$, let $[z^k] f(z)$ denote the coefficient of z^k in $f(z)$. Plugging in id and σ to $\mathrm{H}^*(Z_{\mathsf{Path}_n}; z)$ gives

$$a_k + b_k = [z^k] \left(\frac{1}{z} A_{n-1}(z) \right)$$

$$a_k - b_k = [z^k] \left((1+z)^{(n-1)/2} \frac{1}{z} A_{\frac{n-1}{2}}(z) \right).$$

Summing these two equations, it is clear to see that $a_k > 0$, which supports [2, Conjecture 12.4]. Subtracting the second equation from the first, we see that $b_k \geq 0$ if and only if the right side of the first equation is greater than or equal to the right side of the second equation. This is indeed true, and we prove it combinatorially.

Lemma 3.20. *For $0 \leq k \leq n-2$, the coefficient of z^k in $\frac{1}{z} A_{n-1}(z)$ is greater than or equal to the coefficient of z^k in $(1+z)^{(n-1)/2} \frac{1}{z} A_{(n-1)/2}(z)$.*

Proof. The coefficient of z^k in $\frac{1}{z}A_{n-1}(z)$ is the Eulerian number $A_{n-1,k}$, the number of permutations of the set $\{1, 2, \ldots, n-1\}$ with exactly k ascents. On the other hand, we have

$$[z^k]\left((1+z)^{(n-1)/2}\frac{1}{z}A_{\frac{n-1}{2}}(z)\right) = [z^k]\left(\left(\sum_{i=0}^{(n-1)/2}\binom{\frac{n-1}{2}}{i}z^i\right)\left(\sum_{j=0}^{(n-3)/2}A_{\frac{n-1}{2},j}z^j\right)\right)$$

$$= \sum_{i=0}^{k}\binom{\frac{n-1}{2}}{k-i}A_{\frac{n-1}{2},i}$$

$$= \sum_{i=0}^{k}A_{\frac{n-1}{2},i}\binom{\frac{n-1}{2}-1}{k-i-1} + \sum_{i=0}^{k}A_{\frac{n-1}{2},i}\binom{\frac{n-1}{2}-1}{k-i}$$

$$\leq A_{n-1,k}. \tag{10}$$

To see that $(10) \leq A_{n-1,k}$, we observe that (10) counts some, but not all, of the permutations in S_{n-1} with exactly k ascents. The first summand of (10) deals with the case where the elements $1, 2, \ldots, \frac{n-1}{2}$ come before the remaining elements in the permutation. The Eulerian number $A_{(n-1)/2,i}$ counts the number of permutations of $\{1, 2, \ldots, \frac{n-1}{2}\}$ with exactly i ascents; we are then guaranteed an ascent at position $\frac{n-1}{2}$, and then the binomial coefficient $\binom{(n-1)/2-1}{k-i-1}$ gives the number of choices for the *positions* of the remaining $k - i - 1$ ascents from $\{(n+1)/2, (n+3)/2, \ldots, n-2\}$ (which is less than or equal to the number of ways to complete the permutation with ascents in those positions). Likewise, the second summand of (10) deals with the case where the elements $\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1$ come before the remaining elements in the permutation; in this case we are guaranteed to have no ascent at position $\frac{n-1}{2}$. Using this reasoning, it is clear that (10) is less than or equal to the number of permutations in S_{n-1} with exactly k ascents where the elements $\{1, 2, \ldots, \frac{n-1}{2}\}$ occur either in the first $\frac{n-1}{2}$ positions or in the last $\frac{n-1}{2}$ positions. This in turn is clearly less than or equal to $A_{n-1,k}$. □

Since $a_k, b_k \geq 0$ for all k, the H*-polynomial of Z_{Path_n} is effective with respect to the action of $\mathbb{Z}/2\mathbb{Z}$ when n is odd.

3.2. Invariant Half-Open Decompositions

In this section, we use G-invariant triangulations to calculate the equivariant Ehrhart series. More precisely, the triangulations are half-open decompositions. For background definitions in discrete geometry, see [18].

Definition 3.21. A *half-open decomposition* of a lattice polytope P with lattice triangulation $\mathcal{T}$ is a partition of the face poset of $\mathcal{T}$ into intervals such that the empty set is not in its own class. We use $\mathcal{T}^{[)}$ to refer to the half-open decomposition, i.e., the triangulation together with the partition of the face poset. We identify a simplex S in $\mathcal{T}$ with its set of vertices when convenient.

Definition 3.21 is closely related to partitionability; the face poset of a pure simplicial complex is *partitionable* if it can be partitioned into disjoint
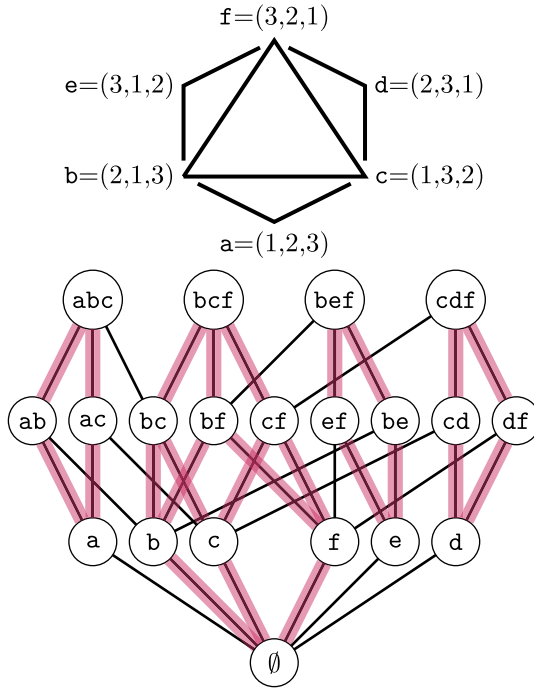
FIGURE 7. A half-open decomposition of the permutahedron Π_3 determined by a triangulation (top image) and a partition of the face poset of the triangulation into intervals shown via its poset diagram (bottom image)

intervals such that each maximal element is a facet of the complex, see [19, Chapter 2]. In this case, it is easy to read off the h-vector of the simplicial complex by recording the heights of the minimal simplices in the intervals. Definition 3.21 generalizes the typical method for creating a half-open decomposition of (the cone over) a triangulated polytope by choosing a generic vector in the interior of one simplex and then taking "sunny-side" of each face; one asks if it is possible to remain in a facet of the triangulation when walking in the direction of the vector, see [20, Sect. 5.3] or [21, Sect. 3.4] for details. Using a generic vector in the typical method creates a shelling (and more weakly a partition) of the simplicial complex. However, this typical method can fail to create symmetric half-open decompositions with respect to the group action, and does not easily allow for simplices of varying dimension in the triangulation, highlighting the usefulness of the more general half-open decompositions from Definition 3.21.

Example 3.22. The two-dimensional permutahedron $\Pi_3 \subset \mathbb{R}^3$ obtained as the convex hull of the permutations of the coordinates $(1, 2, 3)$ admits a half-open decomposition. Figure 7 shows a triangulation of Π_3 and a partition of the face poset of the triangulation. The broken edges in the triangulation indicate
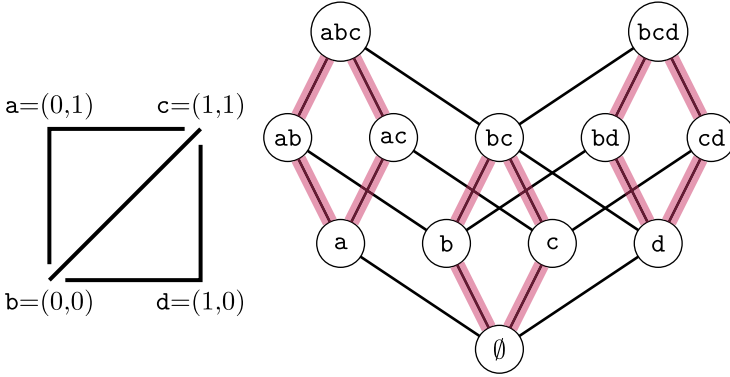
FIGURE 8. A half-open decomposition of the 0/1 square determined by a triangulation (left image) and a partition of the face poset of the triangulation into intervals shown via its poset diagram (right image)

which faces are open and which are closed in the cone over Π_3 according to the partition.

Example 3.23. The 0/1 square admits a half-open decomposition. Triangulate the square into two triangles: $\Delta(abc) = \mathrm{conv}(\{(0,1),(0,0),(1,1)\})$ and $\Delta(bcd) = \mathrm{conv}(\{(0,0),(1,1),(1,0)\})$, as shown on the left in Fig. 8. Partition the face poset into three intervals: $[a, abc], [\emptyset, bc], [d, bcd]$ to create the half-open decomposition, as shown on the right in Fig. 8. The broken edges in the triangulation shown in Fig. 8 indicate which faces in the cone over the triangulated square are open according to the partition of the face poset. The maximal simplex bc in the interval $[\emptyset, bc]$ is not full-dimensional, which emphasizes the additional flexibility offered by these half-open decompositions.

Definition 3.24. Let $P \subset \mathbb{R}^d$ be a lattice polytope, and let $I = [\underline{S}, \overline{S}]$ be an interval in a half-open decomposition $\mathcal{T}^{[)}$ of P. The *half-open cone over the interval I*, denoted $\mathrm{Cone}(I)$, is the set of points $\mathbf{x} \in \mathbb{R}^{d+1}$ such that:

$$\mathbf{x} = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \text{ where } \lambda_i \in \mathbb{R}_{>0} \text{ and } \mu_j \in \mathbb{R}_{\geq 0}.$$

Propositions 3.25 and 3.27 below are similar in spirit to [21, Propositions 3.22, 3.27].

Proposition 3.25. *The cone over a lattice polytope P with half-open decomposition $\mathcal{T}^{[)}$ is equal to the disjoint union of half-open cones over the intervals of $\mathcal{T}^{[)}$:*

$$\mathrm{Cone}(P) = \bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I).$$

Proof. The inclusion $\mathrm{Cone}(P) \supseteq \bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I)$ is clear. The origin is contained in the half-open cone, $\mathrm{Cone}(I)$, for the unique interval $I = [\emptyset, \overline{S}]$ that contains the empty set. Let $\mathbf{z} \in \mathrm{Cone}(P)$, $\mathbf{z} \neq 0$. There exists a minimal simplex $S \in \mathcal{T}^{[)}$ (with respect to inclusion) such that $\mathbf{z} \in \mathrm{Cone}(S)$ and $\mathbf{z} = \sum_{\mathbf{v}_i \in S} \lambda_i(\mathbf{v}_i, 1)$, $\lambda_i \in \mathbb{R}_{>0}$. The simplex S is contained in a unique interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$. Thus,

$$\mathbf{z} = \sum_{\mathbf{v}_i \in S} \lambda_i(\mathbf{v}_i, 1) = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_i \in S \setminus \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus S} 0(\mathbf{v}_j, 1).$$

This implies that all vertices of $\underline{S}$ have a coefficient in $\mathbb{R}_{>0}$, and all vertices of $\mathbf{v} \in \overline{S} \setminus \underline{S}$ have a coefficient in $\mathbb{R}_{\geq 0}$. Thus, $\mathbf{z} \in \mathrm{Cone}(I)$. It remains to show that the union $\bigsqcup_{I \in \mathcal{T}^{[)}} \mathrm{Cone}(I)$ is disjoint. Suppose $\mathbf{z} \neq 0$, $\mathbf{z} \in \mathrm{Cone}(I)$, and $\mathbf{z} \in \mathrm{Cone}(I')$ for two intervals $I = [\underline{S}, \overline{S}]$, $I' = [\underline{S}', \overline{S}']$ of $\mathcal{T}^{[)}$. Then,

$$\mathbf{z} = \sum_{\mathbf{v}_i \in T} \lambda_i(\mathbf{v}_i, 1), \quad \lambda_i \in \mathbb{R}_{>0}, \tag{11}$$

for a *unique* face T of the simplicial complex. As $\mathbf{z}$ is in $\mathrm{Cone}(I)$,

$$\mathbf{z} = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1),$$

where $\lambda_i \in \mathbb{R}_{>0}$ and $\mu_j \in \mathbb{R}_{\geq 0}$. Restricting to summands with strictly positive coefficients expresses $\mathbf{z}$ as a conical combination of vertices of T, since it is an expression of the form (11). This also shows that $\underline{S} \subset T$. Likewise, $\underline{S}' \subseteq T \subseteq \overline{S}'$. Thus, T is contained in two intervals in the half-open decomposition, and they must be the same. $\qquad\square$

Definition 3.26. Let $P \subseteq \mathbb{R}^d$ be a lattice polytope, and let $I = [\underline{S}, \overline{S}]$ be an interval in a half-open decomposition $\mathcal{T}^{[)}$ of P. The *half-open fundamental parallelepiped of the interval* I, denoted $\boxdot(I)$, is the set of points $\mathbf{x} \in \mathbb{R}^{d+1}$ such that:

$$\mathbf{x} = \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{v_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \text{ where } \lambda_i \in (0, 1] \text{ and } \mu_j \in [0, 1).$$

For an interval I of a half-open decomposition, $\mathrm{Box}(I)$ is defined to be the set of lattice points in the half-open fundamental parallelepiped $\boxdot(I)$:

$$\mathrm{Box}(I) = \boxdot(I) \cap \mathbb{Z}^{d+1}.$$

Throughout, we use $\mathrm{Box}(I)_k$ to denote the set of lattice points in the half-open fundamental parallelepiped of I at height k, i.e., with last coordinate equal to k.

Proposition 3.27. *Let P be a lattice polytope with half-open decomposition $\mathcal{T}^{[)}$. Every lattice point $\mathbf{x} \in \mathrm{Cone}(P)$ can be expressed uniquely as $\mathbf{x} = \mathbf{w} + \mathbf{y}$ for some lattice point $\mathbf{w} = \sum_{\mathbf{v}_i \in \overline{S}} \lambda_i(\mathbf{v}_i, 1)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\mathbf{y} \in \mathrm{Box}(I)$ for some interval $I = [\underline{S}, \overline{S}] \in \mathcal{T}^{[)}$.*

Proof. Let $\mathbf{x} \in \mathrm{Cone}(P)$. Then, by Proposition 3.25, $\mathbf{x} \in \mathrm{Cone}(I)$ for a unique interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$, and

$$
\begin{aligned}
\mathbf{x} &= \sum_{\mathbf{v}_i \in \underline{S}} \lambda_i(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \mu_j(\mathbf{v}_j, 1), \quad \text{where } \lambda_i \in \mathbb{R}_{>0} \text{ and } \mu_j \in \mathbb{R}_{\geq 0}, \\
&= \sum_{\mathbf{v}_i \in \underline{S}} \left(\lceil \lambda_i \rceil - 1 + \lambda_i - (\lceil \lambda_i \rceil - 1) \right)(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \left(\lfloor \mu_j \rfloor + (\mu_j - \lfloor \mu_j \rfloor) \right)(\mathbf{v}_j, 1) \\
&= \left(\sum_{\mathbf{v}_i \in \underline{S}} (\lceil \lambda_i \rceil - 1)(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} \lfloor \mu_j \rfloor (\mathbf{v}_j, 1) \right) \\
&\quad + \left(\sum_{\mathbf{v}_i \in \underline{S}} (1 - \lceil \lambda_i \rceil + \lambda_i)(\mathbf{v}_i, 1) + \sum_{\mathbf{v}_j \in \overline{S} \setminus \underline{S}} (\mu_j - \lfloor \mu_j \rfloor)(\mathbf{v}_j, 1) \right).
\end{aligned}
$$

$\square$

Definition 3.28. Let P be a lattice polytope invariant under the action of a group G. A half-open decomposition $\mathcal{T}^{[)}$ of P is called *G-invariant* if

1. Simplices are sent to simplices (the triangulation is G-invariant), and
2. The action of G induces an automorphism of the face poset of $\mathcal{T}^{[)}$ such that intervals of $\mathcal{T}^{[)}$ are sent to intervals.

We present a lemma on the representation theory of the symmetric algebra, which is useful in Theorem 3.30, to follow. Through it, we can see one motivation for considering the denominator $\det(I - z \cdot \rho)$ to express the equivariant Ehrhart series. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for a vector space V. The set $\{\mathbf{e}_{i_1} \odot \cdots \odot \mathbf{e}_{i_t} \mid i_\ell \leq i_{\ell+1}\}$ is a basis for $\mathrm{Sym}^t(V)$, the *t-th symmetric power*. A basis for the *t-th exterior power*, $\Lambda^t(V)$, is the set $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_t} \mid i_\ell < i_{\ell+1}\}$. If ρ is a representation of G on V, then G also acts (diagonally) on $\bigotimes^t V$ by

$$
g(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_t) = g\mathbf{v}_1 \otimes \cdots \otimes g\mathbf{v}_t.
$$

The group G acts diagonally on $\mathrm{Sym}^t(V)$ and $\Lambda^t(V)$ in the same manner. It is useful to look at the characters of these actions, which we denote by $\chi_{\mathrm{Sym}^t(V)}$ and $\chi_{\Lambda^t(V)}$ respectively. For $g \in G$, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal eigenbasis of the action of g on V, with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. This is possible as $\rho(g)$ can be written as a unitary matrix, see for example [11, Chapter 1]. Then,

$$
g(\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_t}) = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_t} \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_t},
$$

where $i_1 \leq i_2 \leq \cdots \leq i_t$. This shows that an eigenvector of $\mathrm{Sym}^t(V)$ has an eigenvalue that is a monomial in the eigenvalues of $\rho(g)$ of degree t. Therefore, the sum of the eigenvalues of g acting on $\mathrm{Sym}^t(V)$ is the sum of all monomials in the eigenvalues of degree t. With respect to the eigenbasis of the action of g on V,

$$\det(I - z \cdot \rho(g)) = \det\left(\begin{bmatrix} 1 - z\lambda_1 & & \\ & \ddots & \\ & & 1 - z\lambda_n \end{bmatrix}\right) = \Pi_{i=1}^n (1 - z\lambda_i).$$

In fact, $\det(I - z \cdot \rho(g))$ is independent of the choice of basis to express ρ. Thus, the generating function of characters of the symmetric powers has the rational form:

$$\sum_{t \geq 0} \chi_{\mathrm{Sym}^t(V)} z^t = \frac{1}{\det(I - z \cdot \rho)}. \tag{12}$$

In $\Lambda^t(V)$, we get $g(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_t}) = \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_t} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_t}$, where $i_1 < i_2 < \cdots < i_t$. The character of the t-th exterior product evaluated at g is the sum of all square-free homogeneous monomials in λ_i of degree t. As such, we can rewrite

$$\det(I - z \cdot \rho) = \sum_{i=0}^n (-1)^i \chi_{\Lambda^i(V)} z^i. \tag{13}$$

We have now recovered Lemma 3.1 in [2], which states:

Lemma 3.29. [2, Lemma 3.1] *Let G be a finite group and let V be an n-dimensional representation. Then,*

$$\sum_{t \geq 0} \mathrm{Sym}^t V z^t = \frac{1}{1 - Vz + \Lambda^2 V z^2 - \cdots + (-1)^n \Lambda^n V z^n}. \tag{14}$$

Moreover, if an element $g \in G$ acts on V via a matrix A, and if I denotes the identity $n \times n$ matrix, then both sides equal $\frac{1}{\det(I - z \cdot A)}$ when the associated characters are evaluated at g.

For a G-invariant simplex S with vertices $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\} \in M \times \{1\}$ as in the Setup 2.1, the infinite-dimensional $\mathbb{C}$-vector space $\mathbb{C}[\mathbb{Z}_{\geq 0} \mathrm{vert}(S)]$ with basis:

$$\left\{ \mathbf{x} = \sum_{i=0}^d c_i \mathbf{v}_i \Big| c_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in \{0, \ldots, d\} \right\}$$

is isomorphic as a graded G-module to the vector space $\sum_{t \geq 0} \mathrm{Sym}^t(M' \bigotimes_{\mathbb{Z}} \mathbb{C})$, and their characters coincide. If S is a unimodular simplex, then (12) expresses the equivariant Ehrhart series of S. This idea is generalized to all simplices in [2, Proposition 6.1], and we now generalize it further to polytopes admitting G-invariant half-open decompositions.

As in Setup 2.1, let P be a lattice polytope with vertices in $M \times \{1\}$, invariant under the linear action of a finite group G. For a half-open decomposition $\mathcal{T}^{[)}$ of P, let $\mathrm{Box}(\mathcal{T}^{[)})_i$ denote the union of the sets $\mathrm{Box}(I)_i$ for intervals I of $\mathcal{T}^{[)}$. The group G permutes the lattice points in $\mathrm{Box}(\mathcal{T}^{[)})_i$, and the corresponding permutation character is written $\chi_{\mathrm{Box}(\mathcal{T}^{[)})_i}$.

Theorem 3.30. *Let $\mathcal{T}^{[)}$ be a G-invariant half-open decomposition of a d-dimensional polytope P as in the Setup 2.1 such that $\dim(\overline{S}) = d$ for all*

intervals $[\underline{S}, \overline{S}]$ of $T^{[)}$ and such that all orbits of intervals of $T^{[)}$ have order $|G|$ except for a unique G-invariant interval. Then, the H^*-polynomial is the permutation representation on the union of box points in $T^{[)}$, graded by height:

$$\mathrm{H}^*(P; z) = \sum_{i=0}^{d} \chi_{\mathrm{Box}\left(T^{[)}\right)_i} z^i.$$

Proof. Let q be the number of orbits of intervals of $T^{[)}$. There necessarily exists a G-invariant simplex $\overline{S_0}$ contained in an interval $[\underline{S_0}, \overline{S_0}]$ of $T^{[)}$ that contains the G-fixed barycenter of the vertices of P. Label its vertices by $\{\mathbf{v}_{0,0}, \ldots, \mathbf{v}_{0,d}\}$ and label the orbit of the interval $[\underline{S_0}, \overline{S_0}]$ by $\mathcal{O}_0$. Order the other orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{q-1}$ of intervals of $T^{[)}$. For each $i \in [q-1]$, label representative simplices $\underline{S_i}$ and $\overline{S_i}$ contained in an interval $[\underline{S_i}, \overline{S_i}]$ of $\mathcal{O}_i$ and label the vertices of $\overline{S_i}$ as $\mathbf{v}_{i,0}, \ldots, \mathbf{v}_{i,d}$. Identify the G-module $\mathbb{C}[\mathbb{Z}_{\geq 0} \operatorname{vert}(\overline{S_0})]$ with the polynomial ring and graded G-module, $\mathbb{C}[X_0, \ldots, X_d]$ where $g(X_i) := X_j$ if $g(\mathbf{v}_{0,i}) = \mathbf{v}_{0,j}$. Viewing $g \in G$ as a permutation on $\{0, 1, \ldots, d\}$, we write, $g(X_0^{c_0} \cdots X_d^{c_d}) = X_{g(0)}^{c_0} \cdots X_{g(d)}^d$. We define a map f from

$$\mathbb{C}[X_0, \ldots, X_d] \bigotimes_{\mathbb{C}} \left(\mathbb{C}\left[\mathrm{Box}\left([\underline{S_0}, \overline{S_0}]\right)\right] \bigoplus_{\substack{i : |\mathcal{O}_i| = |G| \\ g \in G}} \mathbb{C}\left[\mathrm{Box}\left([g\underline{S_i}, \overline{gS_i}]\right)\right] \right) \quad (15)$$

to $\mathbb{C}[\mathrm{Cone}(P) \cap M']$ by defining f on a basis and extending linearly. The map f makes use of the expression for lattice points in $\mathrm{Cone}(P)$ given in Proposition 3.27. To shorten notation, for a lattice point $\mathbf{y}_0$ in $\mathrm{Box}([\underline{S_0}, \overline{S_0}])$ or $g\mathbf{y}_i$ in $\mathrm{Box}([g\underline{S_i}, \overline{gS_i}])$, we denote the corresponding basis vector of

$$\left(\mathbb{C}\left[\mathrm{Box}\left([\underline{S_0}, \overline{S_0}]\right)\right] \bigoplus_{\substack{i : |\mathcal{O}_i| = |G| \\ g \in G}} \mathbb{C}\left[\mathrm{Box}\left([g\underline{S_i}, \overline{gS_i}]\right)\right] \right) \quad (16)$$

as $\mathbf{y}_0$ or $g\mathbf{y}_i$ respectively. The direct sum (16) is also a graded G-module. A basis for the tensor product (15) is the set

$$\left\{ \begin{array}{l} (X_0^{c_0} \cdots X_d^{c_d}) \otimes \mathbf{y}_0 : \{c_0, \ldots, c_d\} \in \mathbb{Z}_{\geq 0}, \mathbf{y}_0 \in \mathrm{Box}([\underline{S_0}, \overline{S_0}]), \\ g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{y}_i : \{c_0, \ldots, c_d\} \in \mathbb{Z}_{\geq 0}, \mathbf{y}_i \in \mathrm{Box}([\underline{S_i}, \overline{S_i}]), g \in G, |\mathcal{O}_i| = |G| \end{array} \right\}.$$

Define

$$f(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{y}_i) = f((X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d}) \otimes g\mathbf{y}_i) := \sum_{j=0}^{d} c_j g(\mathbf{v}_{i,j}) + g\mathbf{y}_i.$$

Suppose $g\mathbf{y}_0 = \mathbf{w}_0$ for box points $\mathbf{y}_0$ and $\mathbf{w}_0$ in $\mathrm{Box}([\underline{S_0}, \overline{S_0}])$ and, therefore,

$$X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d} \otimes g\mathbf{y}_0 = X_0^{c_{g^{-1}(0)}} \cdots X_d^{c_{g^{-1}(d)}} \otimes \mathbf{w}_0.$$

To check f is well-defined, we compute:

$$f\left(X_{g(0)}^{c_0} \cdots X_{g(d)}^{c_d} \otimes g\mathbf{y}_0\right) = \sum_{j=0}^{d} c_j g(\mathbf{v}_{i,j}) + g(\mathbf{y}_0)$$

$$= \sum_{j=0}^{d} c_j \mathbf{v}_{i,g(j)} + \mathbf{w}_0$$

$$= \sum_{\gamma=0}^{d} c_{g^{-1}(\gamma)} \mathbf{v}_{i,\gamma} + \mathbf{w}_0$$

$$= f(X_0^{c_{g^{-1}(0)}} \cdots X_d^{c_{g^{-1}(d)}} \otimes \mathbf{w}_0).$$

The map f is an isomorphism of vector spaces (Proposition 3.27) and a G-module isomorphism, as we verify on the basis. For $h \in G$,

$$f\left(h\left(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{y}_i\right)\right) = f\left((X_{hg(0)}^{c_0} \cdots X_{hg(d)}^{c_d}) \otimes hg\mathbf{y}_i\right)$$

$$= \sum_j c_j hg(\mathbf{v}_{i,j}) + hg\mathbf{y}_i$$

$$= h\left(\sum_j c_j g(\mathbf{v}_{i,j}) + g\mathbf{y}_i\right)$$

$$= h(f(g(X_0^{c_0} \cdots X_d^{c_d}) \otimes g\mathbf{y}_i)).$$

The G-module isomorphism also respects the grading and yields an equality among characters. Identify $\mathbb{C}[X_0, \ldots, X_d]$ and $\sum_{t \geq 0} \mathrm{Sym}^t(M' \bigotimes_{\mathbb{Z}} \mathbb{C})$ as graded G modules as in the discussion around Lemma 3.29. This yields

$$\sum_{t \geq 0} \chi_{tP} z^t = \frac{1}{\det(I - z \cdot \rho)} \sum_{i=0}^{d} \chi_{\mathrm{Box}(\mathcal{T}^{\flat})_i} z^i.$$

$\square$

Example 3.31. [Example 3.22 continued] The two-dimensional permutahedron $\Pi_3 \subseteq \mathbb{R}^3$ under the action of the group $\mathbb{Z}/3\mathbb{Z}$ cyclically permuting the standard basis vectors admits a G-invariant half-open decomposition as dictated by Theorem 3.30. This half-open decomposition is described in Fig. 7 through a triangulation $\mathcal{T}$ of Π_3 and a partition of the face poset of $\mathcal{T}$ into intervals. In this partition, each maximal simplex in an interval is a triangle and every orbit of triangles has order 1 or 3. With this half-open decomposition, the lattice points in the union of the fundamental parallelepipeds are

$$\{(0,0,0)\} \qquad\qquad \text{at height 0,}$$
$$\{(1,2,3),(2,3,1),(3,1,2),(2,2,2)\} \qquad \text{at height 1,}$$
$$\{(4,4,4)\} \qquad\qquad \text{at height 2.}$$

At height 0 and 2, there is a unique $\mathbb{Z}/3\mathbb{Z}$-invariant box point, each giving a copy of the trivial representation. At height 1, the character of the permutation representation on the 4 lattice points evaluates to 4 at the identity

element, and 1 at the other two group elements. Taking the inner product to express this representation in the basis of irreducible representations for $\mathbb{Z}/3\mathbb{Z}$: $\chi_{triv}, \chi_g, \chi_{g^2}$, yields the H*-polynomial

$$\mathrm{H}^*(\Pi_3; z) = \chi_{triv} + (2\chi_{triv} + \chi_g + \chi_{g^2})z + \chi_{triv}z^2.$$

The next theorem introduces a second way to compute the equivariant Ehrhart series using G-invariant half-open decompositions. The conditions required by the theorem ensure that the quasipolynomial χ_{tP} is actually polynomial in t. For a G-invariant half-open decomposition $\mathcal{T}^{[)}$ of a polytope, let $\mathcal{O}$ denote an orbit of intervals of $\mathcal{T}^{[)}$, $\dim(\mathcal{O}):=\dim(\overline{S})$ for any interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$, and $\mathrm{Box}(\mathcal{O})_i := \bigcup_{I \in \mathcal{O}} \mathrm{Box}(I)_i$. Let $\chi_{\mathrm{Box}(\mathcal{O})_i}$ denote the permutation character on the lattice points in $\mathrm{Box}(\mathcal{O})_i$.

Theorem 3.32. *Let P be a G-invariant d-polytope as in the Setup 2.1 with a G-invariant half-open decomposition $\mathcal{T}^{[)}$ such that no interval consists of a single d-face of the triangulation. Furthermore, suppose that for each interval $I = [\underline{S}, \overline{S}]$ of $\mathcal{T}^{[)}$ and each $g \in G$, if a box point $\mathbf{y} \in \mathrm{Box}(I)$ is fixed by g, then the simplex $\overline{S}$ is fixed pointwise by g. Then, the equivariant Ehrhart series has the following rational generating function:*

$$\sum_{t \geq 0} \chi_{tP} z^t = \frac{\sum_{\mathcal{O} \in \mathcal{T}^{[)}} (1-z)^{d-\dim(\mathcal{O})} \sum_{i=0}^{\dim(\mathcal{O})} \chi_{\mathrm{Box}(\mathcal{O})_i} z^i}{(1-z)^{d+1}}.$$

In this case, χ_{tP} is a polynomial in t with coefficients in $R(G)$.

Proof. We break $\mathrm{Cone}(P)$ into orbits and describe the permutation representation on each piece. Let $\mathcal{O}$ be an orbit of intervals of $\mathcal{T}^{[)}$, and let $g \in G$. Furthermore, let $\mathrm{Cone}(\mathcal{O}):=\bigcup_{I \in \mathcal{O}} \mathrm{Cone}(I)$.

Claim 3.33.

$$\sum_{\mathbf{u} \in \mathrm{Cone}(\mathcal{O})^g \cap \mathbb{Z}^{d+1}} z^{\mathbf{u}} = \sum_{I=[\underline{S},\overline{S}] \in \mathcal{O}} \frac{\sum_{\mathbf{m} \in \mathrm{Box}(I)^g} z^{\mathbf{m}}}{\prod_{\mathbf{s} \in \overline{S}} (1 - z^{\mathbf{s}})} \tag{17}$$

Proof of Claim. We first show that the left hand side of Eq. (17) is a subset of the right in terms of the lattice points appearing in the exponents of the series expansions. Suppose $\mathbf{u} \in \mathrm{Cone}(\mathcal{O})^g \cap \mathbb{Z}^{d+1}$. By Proposition 3.25, there exists a unique interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$ such that $\mathbf{u} \in \mathrm{Cone}(I)$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ be the vertices of $\overline{S}$. Then, we may write $\mathbf{u}$ uniquely as $\mathbf{u} = \sum_{i=1}^{n+1}(c_i \mathbf{v}_i) + \mathbf{y}$, with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n+1]$, and $\mathbf{y} \in \mathrm{Box}(I)$ by Proposition 3.27. As the expression of $\mathbf{u}$ is unique, $\mathbf{u} = g(\mathbf{u}) = \sum_{i=1}^{n+1}(c_i g(\mathbf{v}_i)) + g(\mathbf{y})$ implies that $\mathbf{y} = g(\mathbf{y})$. This implies that the coefficient of $z^{\mathbf{u}}$ in the series expansion of the right side of Eq. (17) is 1.

We now show the right hand side of Eq. (17) is a subset of the left. By Proposition 3.27, every lattice point $\mathbf{u}$ in the series expansion of the right side of Eq. (17) as $z^{\mathbf{u}}$ has coefficient one. Furthermore, there exists a unique interval $I = [\underline{S}, \overline{S}] \in \mathcal{O}$ with $\mathrm{vert}(\overline{S}) = \{\mathbf{v}_1, \ldots \mathbf{v}_{n+1}\}$ such that $\mathbf{u} = \sum_{i=1}^{n+1}(c_i \mathbf{v}_i) + \mathbf{y}$, with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n+1]$ and $\mathbf{y} \in \mathrm{Box}(I)^g$. Then,

$$g\left(\sum_{i=1}^{n+1} c_i\mathbf{v}_i + \mathbf{y}\right) = \sum_{i=1}^{n+1}(c_i g(\mathbf{v}_i)) + g(\mathbf{y})$$

$$= \sum_{i=1}^{n+1}(c_i g(\mathbf{v}_i)) + \mathbf{y}$$

$$= \sum_{i=1}^{n+1}(c_i\mathbf{v}_i) + \mathbf{y} \text{ (by assumption).}$$

Thus, $\mathbf{u} \in \mathrm{Cone}(\mathcal{O})^g$ and Equation (17) holds. $\qquad\square$

Homogenize, sending $z \to (\mathbf{1}, z_{d+1})$. Let $n = \dim(\mathcal{O})$. Then, Eq. (17) becomes

$$\sum_{t\geq 0}\left|\mathrm{Cone}(\mathcal{O})^g \cap (\mathbb{Z}^d \times t)\right| z^t = \frac{\sum_{k=0}^{n}|\mathrm{Box}(\mathcal{O})_k^g|\, z^k}{(1-z)^{n+1}}.$$

For all $t \in \mathbb{Z}_{\geq 0}$, $|\mathrm{Cone}(\mathcal{O})^g \cap (\mathbb{Z}^d \times t)|$ is equal to the permutation character on the lattice points at height t in $\mathrm{Cone}(\mathcal{O})$ evaluated at g, which we denote by $\chi_{t\mathcal{O}}(g)$. Likewise, $|\mathrm{Box}(\mathcal{O})_k^g|$ is equal to $\chi_{\mathrm{Box}(\mathcal{O})_k}(g)$ for all k such that $0 \leq k \leq n$. Therefore, in general,

$$\sum_{t\geq 0}\chi_{t\mathcal{O}}z^t = \frac{\sum_{k=0}^{n}\chi_{\mathrm{Box}(\mathcal{O})_k}z^k}{(1-z)^{n+1}}.$$

Summing over all the orbits yields

$$\sum_{t\geq 0}\chi_{tP}z^t = \sum_{\mathcal{O}\in T^{[)}}\frac{\sum_{k=0}^{\dim(\mathcal{O})}\chi_{\mathrm{Box}(\mathcal{O})_k}z^k}{(1-z)^{\dim(\mathcal{O})+1}}$$

$$= \frac{\sum_{\mathcal{O}\in T^{[)}}(1-z)^{d-\dim(\mathcal{O})}\chi_{\mathrm{Box}(\mathcal{O})_k}z^k}{(1-z)^{d+1}}.$$

$\qquad\square$

Example 3.34. (*Example 3.31 continued*) Again consider the two-dimensional permutahedron $\Pi_3 \subseteq \mathbb{R}^3$ under the action of $\mathbb{Z}/3\mathbb{Z}$ cyclically permuting the standard basis vectors. It is necessary to use a different half-open decomposition from that in Example 3.31 to compute the equivariant Ehrhart series.

As shown in Fig. 9, we triangulate Π_3 using a barycentric subdivision. The intervals we use for a G-invariant half-open decomposition are

$$[\emptyset, g], [a, abg], [b, beg], [c, acg], [d, cdg], [e, efg], [f, dfg].$$

The only lattice points that can be box points must have a coordinate sum that is a multiple of 6. The box points for each of the respective intervals are

$$\{(0,0,0)\}, \{(1,2,3)\}, \{(2,1,3)\}, \{(1,3,2)\}, \{(2,3,1)\}, \{(3,1,2)\}, \{(3,2,1)\}$$

There are 3 orbits of box points. The character table of $\mathbb{Z}/3\mathbb{Z}$ is as follows, with ζ a cube root of unity.

We calculate the following rational generating function for the equivariant

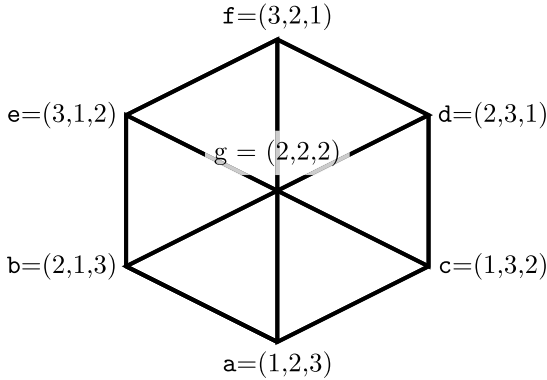| $\mathbb{Z}/3\mathbb{Z}$	id	g	g^2
χ_{triv}	1	1	1
χ_g	1	ζ	ζ^2
χ_{g^2}	1	ζ^2	ζ

FIGURE 9. A triangulation of the permutahedron Π_3 into six triangles

Ehrhart series using the formula from Theorem 3.32:

$$\sum_{t \geq 0} \chi_{t\Pi_3} z^t = \frac{\chi_{triv} + 2(\chi_g + \chi_{g^2})z + \chi_{triv}z^2}{(1-z)^3}$$

Evaluating at the group elements yields the Ehrhart series of the fixed subpolytopes. Transforming to χ_{tP} using the usual transformation gives

$$\chi_{t\Pi_3} = \chi_{triv}\binom{t+2}{2} + 2(\chi_g + \chi_{g^2})\binom{t+1}{2} + \chi_{triv}\binom{t}{2}$$
$$= (\chi_{triv} + \chi_g + \chi_{g^2})t^2 + (\chi_{triv} + \chi_g + \chi_{g^2})t + \chi_{triv}.$$

Evaluating at the identity and g yields the Ehrhart polynomials of the fixed subpolytopes, $3t^2 + 3t + 1$ and 1 respectively. Theorem 3.32 would need to be modified to allow for rational triangulations in order to be applied to Π_3 under the action of S_3 permuting the standard basis vectors.

3.3. Breaking Down the H*-Series

In this section, we consider the hypersimplex and the (prime) permutahedra under the cyclic group action and provide an explicit formulas for the equivariant Ehrhart series.

3.3.1. Cyclic Group Action on the Hypersimplex. We give a combinatorial interpretation (Theorem 3.36) for the H*-series of the hypersimplex $\Delta_{k,n}$ under the group $\mathbb{Z}/n\mathbb{Z}$ where the action is given by cyclically shifting coordinates. We build upon [22] to compute the H*-series and then follow the arguments

in [6] to give an equivariant generalization. Before stating the main theorem, we give several definitions.

For two integers, $0 < k < n$, the (k, n)-*th hypersimplex* is defined to be

$$\Delta_{k,n} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1, x_1 + \cdots + x_n = k\}.$$

It is an $(n-1)$-dimensional polytope inside $\mathbb{R}^n$ whose vertices are $(0,1)$-vectors with exactly k 1's. In particular, it has $\binom{n}{k}$ many vertices.

A *decorated ordered set partition* $((L_1)_{l_1}, \ldots, (L_m)_{l_m})$ of type (k, n) consists of an ordered partition $(L_1, \ldots, L_m)$ of $\{1, 2, \ldots, n\}$ and an m-tuple $(l_1, \ldots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \ge 1$. We call each L_i a *block* and we place them on a circle in a clockwise fashion and then think of l_i as the clockwise distance between adjacent blocks L_i and L_{i+1} where indices are considered modulo m (see Fig. 10). Therefore, the circumference of the circle is $l_1 + \cdots + l_m = k$. We regard decorated ordered set partitions up to cyclic rotation of blocks (together with corresponding l_i). For example, the decorated ordered set partition $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$ is the same as $(\{3, 5\}_3, \{4, 6\}_1, \{1, 2, 7\}_2)$. A decorated ordered set partition is called *hypersimplicial* if it satisfies $1 \le l_i \le |L_i| - 1$ for all i.

Given a decorated ordered set partition, we define the *winding vector* and the *winding number*. To define the winding vector, let w_i be the distance of the path starting from the block containing i to the block containing $(i + 1)$ moving clockwise (where i and $(i + 1)$ are considered modulo n). If i and $(i + 1)$ are in the same block, then $w_i = 0$. In Fig. 10, the winding vector is $w = (0, 2, 3, 3, 3, 1, 0)$. The total length of the path is $(w_1 + \cdots + w_n)$, which should be a multiple of k as we started from 1 and came back to 1 moving clockwise. If $(w_1 + \cdots + w_n) = kd$, then we define the winding number to be d. In Fig. 10, the winding number is 2.

We define a cyclic group $\mathbb{Z}/n\mathbb{Z}$ action on the set of decorated ordered set partitions of type (k, n) by cyclically shifting elements in their parts. It is straightforward to check that the $\mathbb{Z}/n\mathbb{Z}$ action cyclically shifts the winding vectors so the action preserves the winding number.

For some $\sigma \in \mathbb{Z}/n\mathbb{Z}$, a decorated ordered set partition is called σ-fixed if it is invariant under the action of σ. Note that a decorated ordered set partition is σ-fixed if and only if it has σ-fixed winding vector. For example, consider $\sigma \in \mathbb{Z}/4\mathbb{Z}$ defined by $\sigma(i) = i + 1$. Then, σ sends $(\{1, 3\}_1, \{2, 4\}_1)$ to $(\{2, 4\}_1, \{1, 3\}_1)$ which is identified with $(\{1, 3\}_1, \{2, 4\}_1)$. Therefore, $(\{1, 3\}_1, \{2, 4\}_1)$ is σ-fixed. However, $(\{1, 3\}_2, \{2, 4\}_1)$ is not σ-fixed as it goes to $(\{1, 3\}_1, \{2, 4\}_2)$. Note that $(\{1, 3\}_2, \{2, 4\}_1)$ is σ^2-fixed.

Example 3.35. Consider $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$, a decorated ordered set partition of type $(6, 7)$, and $\sigma \in \mathbb{Z}/7\mathbb{Z}$ defined by $\sigma(i) = i + 1$. Then, σ sends $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$ to $(\{1, 2, 3\}_2, \{4, 6\}_3, \{5, 7\}_1)$. Both are not hypersimplicial as $3 > |\{3, 5\}| - 1$.

The winding vector of $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$ is $(0, 2, 3, 3, 3, 1, 0)$ and the winding vector of $(\{1, 2, 3\}_2, \{4, 6\}_3, \{5, 7\}_1)$ is $(0, 0, 2, 3, 3, 3, 1)$ which is obtained by applying σ to $(0, 2, 3, 3, 3, 1, 0)$.

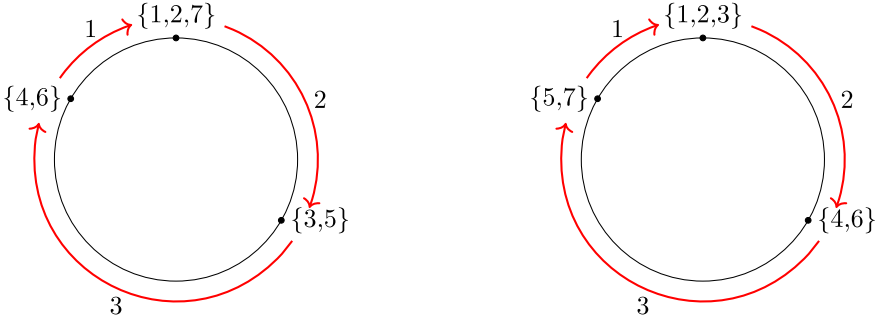Now we state our main theorem in this section.

FIGURE 10. The figure on the left is the picture associated to the decorated ordered set partition $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$ and the figure on the right is the picture after acting by σ

Theorem 3.36. *The character* $\mathrm{H}^*_d(\Delta_{k,n}; z)$ *is the permutation character on the set of decorated ordered set partitions of type* (k, n) *with winding number* d.

The proof of Theorem 3.36 will be given as follows. For a fixed element σ in $\mathbb{Z}/n\mathbb{Z}$, we will first give an explicit formula for $\mathrm{H}^*(\Delta_{k,n}; z)(\sigma)$ (see (22)). Then, we enumerate σ-fixed hypersimplicial decorated ordered set partitions with a certain winding number.

From now on, we fix $\sigma \in \mathbb{Z}/n\mathbb{Z}$. We denote the order of σ with b and letting $a = \frac{n}{b}$, any σ-fixed vector in $\mathbb{R}^n$ is of the form

$$(x_1, \ldots, x_a, x_1, \ldots, x_a, \ldots, x_1, \ldots, x_a).$$

Note that

$$d\Delta_{k,n} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le d, x_1 + \cdots + x_n = dk\},$$

so the number of σ-fixed lattice points of $d\Delta_{k,n}$ equals the number of integer solutions of

$$0 \le x_i \le d, \qquad b\left(\sum_{i=1}^a x_i\right) = dk. \tag{18}$$

We denote

$$h = gcd(b, k)$$
$$b = hb_1, \qquad k = hk_1. \tag{19}$$

Then, (18) has an integer solution if and only if d is a multiple of b_1. Suppose so, and let $d = d'b_1$, then (18) becomes

$$0 \le x_i \le d'b_1, \qquad \sum_{i=1}^a x_i = d'k_1.$$

By [22, Theorem 2.8], the equivariant Ehrhart series of $\Delta_{k,n}$ evaluated at σ is

$$\frac{\sum_{i\geq0}(-1)^i\binom{a}{i}\left(\sum_{j\geq0}\binom{i}{j}(z^{b_1}-1)^j\left(\sum_{\ell\geq0}\binom{a-j}{\ell(k_1-b_1i)}_{k_1-b_1i}(z^{b_1})^\ell\right)\right)}{(1-z^{b_1})^a} \tag{20}$$

where the notation $\binom{n}{k}_p$ means the coefficient of x^k in $(1+x+\cdots+x^{p-1})^n$. In [6, Sect. 2.1], the second author showed that (20) can be simplified as follows:

$$\mathrm{EE}(\Delta_{k,n};z)(\sigma)=\frac{\sum_{i\geq0}(-1)^i\binom{a}{i}\left(\sum_{\ell\geq0}\binom{a}{\ell(k_1-b_1i)-i)}_{k_1-b_1i}(z^{b_1})^\ell\right)}{(1-z^{b_1})^a}.$$

Since $\det(I-z\cdot\rho(\sigma))=(1-z^b)^a$, we conclude

$$\mathrm{H}^*(\Delta_{k,n};z)(\sigma)=\det(I-z\cdot\rho(\sigma))\,\mathrm{EE}(\Delta_{k,n};z)(\sigma)$$

$$=\left(\frac{1-z^{hb_1}}{1-z^{b1}}\right)^a\sum_{i\geq0}(-1)^i\binom{a}{i}\left(\sum_{\ell\geq0}\binom{a}{\ell(k_1-b_1i)-i)}_{k_1-b_1i}(z^{b_1})^\ell\right)$$

$$=\left(\sum_{\ell\geq0}\binom{a}{\ell}_h(z^{b_1})^\ell\right)\sum_{i\geq0}(-1)^i\binom{a}{i}\left(\sum_{\ell\geq0}\binom{a}{\ell(k_1-b_1i)-i)}_{k_1-b_1i}(z^{b_1})^\ell\right) \tag{21}$$

Remark 3.37. The sums in (20) and (21) are finite sums since $\binom{n}{k}_p$ is zero if k is greater than $n(p-1)$.

To simplify (21), we need the following lemma.

Lemma 3.38. *For positive integers a,h,i and p, we have*

$$\left(\sum_{\ell\geq0}\binom{a}{\ell}_h z^\ell\right)\left(\sum_{\ell\geq0}\binom{a}{\ell p-i}_p z^\ell\right)=\sum_{\ell\geq0}\binom{a}{\ell p-i}_{hp}z^\ell.$$

Proof. The number $\binom{a}{\ell p-i}_{hp}$ counts the number of integer solutions to equations

$$0\leq x_j\leq hp-1,\qquad x_1+\cdots+x_a=\ell p-i.$$

Given a solution $x_1,\ldots,x_a$, we write each x_j as

$$x_j=py_j+z_j,\qquad 0\leq z_j\leq p-1.$$

Note that $0\leq y_j\leq h-1$. If $y_1+\cdots+y_a=\ell'$, then we have

$$z_1+\cdots+z_a=(\ell-\ell')p-i.$$

For a fixed ℓ', the number of such integer solutions for $(y_1,\ldots,y_a,z_1,\ldots,z_a)$ equals $\binom{a}{\ell'}_h\binom{a}{(\ell-\ell')p-i}_p$. This gives

$$\sum_{\ell'\geq0}\binom{a}{\ell'}_h\binom{a}{(\ell-\ell')p-i}_p=\binom{a}{\ell p-i}_{hp}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Applying Lemma 3.38 to (21), we conclude

$$\mathrm{H}^*(\Delta_{k,n}; z)(\sigma) = \sum_{i \geq 0} (-1)^i \binom{a}{i} \left(\sum_{\ell \geq 0} \binom{a}{\ell(k_1 - b_1 i) - i}_{k-bi} (z^{b_1})^\ell \right). \quad (22)$$

Now we will count σ-fixed hypersimplicial decorated ordered set partitions of type (k, n) with winding number d. The argument follows the structure of the argument in [6, Section 3], but now we must consider σ-fixed decorated ordered set partitions. Here, we briefly summarize the argument. As before, we fix an element $\sigma \in \mathbb{Z}/n\mathbb{Z}$ and denote the order of σ with b and set $a = \frac{n}{b}$.

Definition 3.39. For $L = ((L_1)_{l_1}, (L_2)_{l_2}, ..., (L_m)_{l_m})$ a decorated ordered set partition, a block L_i is *bad* if $l_i \geq |L_i|$. Let $I(L) = \{L_i \mid L_i \text{ is bad}\}$.

Since all blocks of hypersimplicial decorated ordered set partitions must satisfy $1 \leq l_i \leq |L_i| - 1$, a decorated ordered set partition is hypersimplicial if and only if $I(L)$ is empty.

Definition 3.40. For a subset $T \subseteq [n]$ that is σ-fixed, define $\mathrm{UP}(T)$ to be a set of all (unordered) partitions of T that are σ-fixed. For each $S \in \mathrm{UP}(T)$, we define $S(\sigma)$ to be the number of orbits of blocks in S under the action of σ. For example, consider $n = 6$ and $\sigma \in \mathbb{Z}/6\mathbb{Z}$ defined by $\sigma(i) = i+3$. Then, $T = \{1, 2, 4, 5\}$ is σ-fixed and $\mathrm{UP}(T)$ has 7 elements which are $\{\{1, 2, 4, 5\}\}$, $\{\{1, 2\}, \{4, 5\}\}$, $\{\{1, 4\}, \{2, 5\}\}$, $\{\{1, 5\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2\}, \{5\}\}$, $\{\{1\}, \{4\}, \{2, 5\}\}$, and $\{\{1\}, \{2\}, \{4\}, \{5\}\}$. Note that

$$\{\{1\}, \{2\}, \{4\}, \{5\}\}(\sigma) = 2,$$
$$\{\{1, 2\}, \{4, 5\}\}(\sigma) = 1.$$

Definition 3.41. For σ-fixed $T \subseteq [n]$ and $S \in \mathrm{UP}(T)$, define $K_{k,n}^d(S)$ to be the set of all σ-fixed decorated ordered set partitions of type (k, n) with winding number d such that $S \subseteq I(L)$.

Definition 3.42. For σ-fixed $T \subseteq [n]$, let $H_{k,n}^d(T) = \sum\limits_{S \in \mathrm{UP}(T)} (-1)^{S(\sigma)} |K_{k,n}^d(S)|$.

Proposition 3.43. *The number of σ-fixed hypersimplicial decorated ordered set partitions of type (k, n) with winding number d is*

$$\sum_{\substack{T \subseteq \{1,2,...,n\} \\ T : \sigma-fixed}} H_{k,n}^d(T). \quad (23)$$

Proof. We have

$$\sum_{\substack{T \subseteq \{1,2,...,n\} \\ T : \sigma-fixed}} H_{k,n}^d(T) = \sum_{\substack{T \subseteq \{1,2,...,n\} \\ T : \sigma-fixed}} \left(\sum_{S \in \mathrm{UP}(T)} (-1)^{S(\sigma)} \left| K_{k,n}^d(S) \right| \right).$$

A decorated ordered set partition L belongs to $K_{k,n}^d(S)$ if and only if S is a subset of $I(L)$. If $I(L)$ is empty, then L will be counted once when $S = \varnothing$.

Now, assume $I(L)$ is non empty. Note that $I(L)$ is σ-fixed; we denote the number of σ-orbits in $I(L)$ with i. The number of σ-fixed $S \subseteq I(L)$ such that $|S| = j$ equals $\binom{i}{j}$, therefore, L will be counted $\binom{i}{j}$ times with the sign $(-1)^j$ as S ranges over all σ-fixed subsets of $I(L)$ with $S(\sigma) = j$. We conclude that the contribution of L to (23) is $\sum_{j=0}^{m} (-1)^j \binom{i}{j} = 0$. Therefore, (23) equals the number of σ-fixed hypersimplicial decorated ordered set partitions of type (k, n) with winding number d. $\qquad\square$

Proposition 3.44. *Given a σ-fixed subset $T \subseteq [n]$ such that $|T| = bi$, the value $(-1)^i H_{k,n}^d(T)$ equals the number of σ-fixed (integer) vectors $(v_1, v_2, \ldots, v_n)$ satisfying*

$$0 \le v_j \le k - bi - 1 \qquad if \qquad j \notin T, \qquad (24)$$
$$1 \le v_j \le k - bi \qquad if \qquad j \in T, \qquad (25)$$
$$v_1 + \cdots + v_n = (k - bi)d. \qquad (26)$$

Proof. This can be proved by the same method in the proof of [6, Proposition 2.22]. $\qquad\square$

Corollary 3.45. *Given a σ-fixed $T \subseteq \{1, 2, \ldots, n\}$ such that $|T| = bi$, we have*

$$H_{k,n}^d(T) = \begin{cases} (-1)^i \binom{a}{\frac{d}{b_1}(k_1 - b_1 i) - i}_{k - bi} & \text{if } d \text{ is a multiple of } b_1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $b_1 = \frac{b}{gcd(b,k)}$ as in (19).

Proof. For a σ-fixed vector $v = (v_1, \ldots, v_n)$ satisfying (24), (25) and (26), let the vector $v' = (v_1', \ldots, v_n')$ be such that $v_j' = v_j$ if $j \notin T$, and $v_j' = v_j - 1$ if $j \in T$. Then, v' is also σ-fixed so we write

$$v' = (v_1', \ldots, v_a', v_1', \ldots, v_a', \ldots, v_1', \ldots, v_a').$$

Property (24) and (25) becomes $0 \le v_j' \le (k - bi - 1)$ for all j and the property (26) becomes

$$b \left(\sum_{j=1}^{a} v_j' \right) = (k - bi)d - bi \qquad \rightarrow \qquad \sum_{j=1}^{a} v_j' = \frac{k_1 d}{b_1} - i(d + 1).$$

Thus, the solution exists only if d is a multiple of b_1 and in that case the number of such v' is $\binom{a}{\frac{d}{b_1}(k_1 - b_1 i) - i}_{k - bi}$. $\qquad\square$

Proof of Theorem 3.36. By Proposition 3.43 and Corollary 3.45, the number of σ-fixed hypersimplicial decorated ordered set partitions (of type (k, n) with winding number d) is

$$\sum_{\substack{T \subseteq \{1,2,\ldots,n\} \\ T : \sigma - fixed}} H_{k,n}^d(T) = \sum_{i \ge 0} \left(\sum_{\substack{|T| = bi \\ T : \sigma - fixed}} H_{k,n}^d(T) \right)$$

$$= \begin{cases} \sum_{i \geq 0} (-1)^i \binom{a}{i} \binom{a}{\frac{d}{b_1}(k_1 - b_1 i) - i}_{k - bi} & \text{if } d \text{ is a multiple of } b_1, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with (22) completes the proof. $\qquad\square$

Example 3.46. Consider hypersimplicial decorated ordered set partitions of type $(2, 4)$:

winding number $0 : \{\{1, 2, 3, 4\}_2\}$

winding number $1 : \{\{1, 2\}_1, \{3, 4\}_1\}, \{\{1, 4\}_1, \{2, 3\}_1\}$

winding number $2 : \{\{1, 3\}_1, \{2, 4\}_1\}$.

For $\sigma \in \mathbb{Z}/4\mathbb{Z}$ defined by $\sigma(i) = i + 2$, the fixed subpolytope $\Delta_{2,4}^\sigma$ is the line segment with vertices $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. The number of lattice points in $t\Delta_{2,4}^\sigma$ equals $(t + 1)$, thus we conclude

$$\mathrm{Ehr}(\Delta_{2,4}^\sigma; z) = \sum_{t \geq 0} (t + 1) z^t = \frac{1}{(1 - z)^2} = \frac{1 + 2z + z^2}{(1 - z^2)^2}$$

$$\rightarrow \mathrm{H}^*(\Delta_{2,4}; z)(\sigma) = 1 + 2z + z^2.$$

Every hypersimplicial decorated ordered set partition of type $(2, 4)$ is σ-fixed, which gives $\mathrm{H}^*(\Delta_{2,4}; z)(\sigma)$ equals $(1 + 2z + z^2)$ by Theorem 3.36.

3.3.2. Cyclic Group Action on Prime Permutahedra.

Lemma 3.47. *The* H^**-series of the permutahedron* Π_n *under the action of* $\mathbb{Z}/n\mathbb{Z}$ *permuting the standard basis vectors is polynomial.*

Proof. By Lemma 5.2 of [4], for $\sigma \in S_n$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, $\mathrm{H}^*(\Pi_n; z)$ is polynomial if and only if the number of even parts in λ is 0, m, or $m - 1$. For any element σ of $\mathbb{Z}/n\mathbb{Z}$, all cycles of σ have the same length, so $\mathrm{H}^*(\Pi_n; z)$ is polynomial, and thus H^* itself is a polynomial. $\qquad\square$

We now focus on the case when p is prime and $\mathbb{Z}/p\mathbb{Z}$ acts on Π_p.

Theorem 3.48. *The* h^**-polynomial of the permutahedron* Π_p *for prime* p *has coefficients equivalent to* $1 \mod p$.

Proof. If $p = 2$, then $h^*(z) = 1$. Let $p > 2$. Let $\mathcal{I}$ be the set of all nonempty linearly independent subsets of $\{e_j - e_k : 1 \leq j < k \leq p\}$, and let T be a linearly independent subset in $\mathcal{I}$. The elements of $\mathcal{I}$ correspond to forests on p labeled vertices (excluding the graph with no edges) [7, Lemma 9.6]. Associate to each T the half-open parallelepiped $\square T = \sum_{e_j - e_k \in T} (0, e_j - e_k]$. As discussed in [7, Chapter 9], the $(p - 1)$-dimensional permutahedron Π_p is equal to the disjoint union of translates of the parallelotopes:

$$\Pi_p = \{0\} \cup \bigcup_{T \in \mathcal{I}} \left(\sum_{e_j - e_k \in T} (0, e_j - e_k] \right).$$

Let $\sigma = (1\,2\,\ldots\,p)$ be a generator of $\mathbb{Z}/p\mathbb{Z}$. The group $\mathbb{Z}/p\mathbb{Z}$ acts on the independent sets $\mathcal{I}$ by cyclically permuting the forest vertex labels. Each orbit

has size p, as follows. For $1 \leq k < p$, suppose $\sigma^k T = T \in \mathcal{I}$. Then, for each edge (i, j) in the forest corresponding to T, the edge $(|i + k|_p, |j + k|_p)$ is also in the forest. As k is coprime to p, the forest thus contains p edges, and therefore, a cycle, which is a contradiction. Thus, σ^k does not fix any $T \in \mathcal{I}$, and $|\mathbb{Z}/p\mathbb{Z}\, T| = p$. Let $\mathcal{S} \subset \mathcal{I}$ be a transversal containing one representative from each orbit in $\mathcal{I}$. Each half-open parallelepiped in an orbit has the same Ehrhart series, and is translated by an integral vector. Thus,

$$\mathrm{Ehr}_{\Pi_p}(z) = \frac{1}{1 - z} + p \sum_{T \in \mathcal{S}} \mathrm{Ehr}(\square T; z)$$

$$= \frac{(1 - z)^{p-1}}{(1 - z)^p} + \sum_{T \in \mathcal{S}} \frac{p \cdot h^*(\square T; z)(1 - z)^{p - |T| - 1}}{(1 - z)^p}.$$

The coefficients of $(1 - z)^{p-1}$ are equivalent to one mod p, because the coefficient of z^i is $\binom{p-1}{i}(-1)^i$, and

$$\binom{p - 1}{i} = \frac{(p - 1) \cdots (p - i)}{i!} \equiv \frac{(-1) \cdots (-i)}{i!} \quad \mathrm{mod}\ p$$

$$\equiv \frac{(-1)^i i!}{i!} \quad \mathrm{mod}\ p \equiv (-1)^i \quad \mathrm{mod}\ p.$$

Hence, the claim follows. $\qquad\square$

Example 3.49. For the permutahedron Π_3 we have $\mathcal{S} = \{\{e_1 - e_2\}, \{e_1 - e_2, e_2 - e_3\}\}$. Thus,

$$\mathrm{Ehr}_{\Pi_3}(z) = \frac{1 - 2z + z^2}{(1 - z)^3} + \frac{3t(1 - z)}{(1 - z)^3} + \frac{3(z + z^2)}{(1 - z)^3} = \frac{1 + 4z + z^2}{(1 - z)^3}.$$

Theorem 3.50. *Let $P \subset \mathbb{R}^n$ be an $(n-1)$-dimensional lattice polytope invariant under the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ permuting the coordinates of $\mathbb{R}^n$. Furthermore, suppose that for each non-identity element $g \in \mathbb{Z}/n\mathbb{Z}$, P^g is a single lattice point. Then,*

$$\mathrm{H}^*(P; z) = \sum_{i=0}^{n-1} \left(\chi_0 + \frac{h_i^* - 1}{n} \sum_{j=0}^{n-1} \chi_j \right) z^i,$$

where h_i is the coefficient of z^i in the h^*-polynomial of P, and $\{\chi_0, \ldots, \chi_{n-1}\}$ are the irreducible representations of $\mathbb{Z}/n\mathbb{Z}$.

Proof. It is enough to show that our formula for $\mathrm{H}^*(P; z)$ specializes to the fixed Ehrhart series for each conjugacy class. Let ζ be an n-th root of unity. Then,

$$\sum_{i=0}^{n-1} \zeta^i = \frac{1 - \zeta^n}{1 - \zeta} = \begin{cases} n \text{ if } \zeta = 1 \text{ (L'Hospital's)} \\ 0 \text{ else.} \end{cases}$$

For the identity element, we have

$$\mathrm{H}^*(P; z)(\mathrm{id}) = \sum_{i=0}^{n-1} \left(1 + \frac{h_i^* - 1}{n} n \right) z^i = \sum_{i=0}^{n-1} h_i^* z^i.$$

For $g \neq \mathrm{id}$, our formula gives $\mathrm{H}^*(P; z)(g) = \sum_{i=0}^{n-1}(1 + 0)z^i$. Combining this with $\det(\mathrm{id} - \rho(g)z) = (1 - z^n)$ gives

$$\frac{\mathrm{H}^*(P; z)(g)}{\det(\mathrm{id} - \rho(g)z)} = \frac{1 + z + \cdots + z^n}{1 - z^n} = \frac{1}{1 - z}.$$

$\square$

Let Π_n be the permutahedron of dimension $n - 1$ in $\mathbb{R}^n$ for n prime. Let $\mathbb{Z}/n\mathbb{Z}$ act on Π_n by cyclically permuting coordinates of $\mathbb{R}^n$.

Corollary 3.51. *For an odd, prime number n, the H^*-series of Π_n under the action of $\mathbb{Z}/n\mathbb{Z}$ is*

$$\mathrm{H}^*(\Pi_n; z) = \sum_{i=0}^{n-1} \left(\chi_0 + \frac{h_i^* - 1}{n} \sum_{j=0}^{n-1} \chi_j \right) z^i,$$

where h_i^* is the coefficient of z^i in the h^*-polynomial of Π_n.

We conclude our discussion of prime permutahedra by applying Stapledon's Corollary 2.10 to show the existence of a $\mathbb{Z}/p\mathbb{Z}$-invariant nondegenerate hypersurface. In Sect. 3.4 we continue in a similar vein, establishing a criterion that guarantees the non-existence of such a hypersurface under the action of the symmetric group.

Theorem 3.52. *Let p be prime. Then, for the action of $\mathbb{Z}/p\mathbb{Z}$ on Π_p, there exists a $\mathbb{Z}/p\mathbb{Z}$-invariant nondegenerate hypersurface with Newton polytope Π_p. Consequently $\mathrm{H}^*(\Pi_p; z)$ is effective and polynomial.*

Proof. If $p = 2$, then Π_p is a line segment and the condition of Corollary 2.10 is trivially satisfied. Suppose p is an odd prime. Then, every proper face of Π_p with dimension > 1 corresponds to some ordered set partition of p with at least 2 parts, which do not all have the same size and hence cannot possibly be fixed under the action of any $\sigma \in \mathbb{Z}/p\mathbb{Z}$ other than the identity. Therefore, the stabilizer of all proper faces is trivial, and hence, all vertices are fixed under this stabilizer. Now consider the face Π_p. Since n is odd, Π_p contains the lattice point $(\frac{p+1}{2}, \ldots, \frac{p+1}{2})$, which is fixed under the action of $\mathbb{Z}/p\mathbb{Z}$. Hence, by Corollary 2.10, the H^*-series is effective. $\square$

Remark 3.53. Theorem 3.48 combined with Corollary 3.51 gives an alternative proof of effectiveness. Since p is prime, $\frac{h_i^* - 1}{p}$ is a nonnegative integer, so we can explicitly see $\mathrm{H}^*(\Pi_p; z)$ is effective.

3.4. Describing Nondegenerate Hypersurfaces

Up until now, we have mainly focused our attention on criteria (ii) and (iii) from Conjecture 1.1. We now turn to (i), detailing what can be said about the existence of a S_n-invariant nondegenerate hypersurface in the case of orbit polytopes. In Sect. 3.4.3, we exhibit a lattice polytope with a group action that satisfies (ii) and (iii) but not (i), disproving Conjecture 1.1.

3.4.1. Orbit Polytopes. Weight polytopes arise in representation theory of semisimple Lie algebras, and the combinatorics of these polytopes have been studied a great deal in recent years [23–27]. Here we specifically focus on weight polytopes of the Type A root system; we refer to these as *orbit polytopes* (also sometimes known as *permutohedra*).

Definition 3.54. An *orbit polytope* is a polytope of the form $\text{conv}\{\sigma \cdot \mathbf{w} : \sigma \in S_n\}$ where $\mathbf{w} \in \mathbb{Z}^n$ is any integer point, and S_n acts by permuting its coordinates.

Using the machinery of Hopf monoids, it is shown in [17, Proposition 5.4] that the faces of generalized permutahedra are products of generalized permutahedra. It is easily shown as a corollary (see [27, Example 3.5]) that the faces of orbit polytopes are products of orbit polytopes. Here, we are particularly interested in orbit polytopes which have some rectangular 2-dimensional faces. Rectangles themselves are not orbit polytopes, so the only way for a rectangle to arise as a product of orbit polytopes is as the product of exactly two line segments and any number of points.

Example 3.55. Let $\mathbf{w} = (0, 1, 2, 3) \in \mathbb{R}^4$. Then, the orbit polytope P of $\mathbf{w}$ is the 3-dimensional standard permutahedron. Consider $\mathbf{y} = (0, 0, 1, 1) \in (\mathbb{R}^4)^*$. Then, the $\mathbf{y}$-maximal face of P is the convex hull of the points in the S_4-orbit of $\mathbf{w}$ that maximize $\mathbf{y}$. These are exactly the points where the 0 and 1 are in the first two positions and the 2 and 3 are in the last two positions: $(0, 1, 2, 3)$, $(1, 0, 2, 3)$, $(0, 1, 3, 2)$, and $(1, 0, 3, 2)$. The convex hull of these four points gives a square face of P; it can be viewed as the product of a line segment living in a 2-dimensional subspace of $\mathbb{R}^4$ with vertices $(0, 1)$ and $(1, 0)$, and another line segment living in the 2-dimensional orthogonal complement with vertices $(2, 3)$ and $(3, 2)$.

Theorem 3.56. *Let P be an orbit polytope which has a rectangular 2-dimensional face F with odd side lengths, and let G be the symmetric group acting by the standard representation. Then, the toric variety of P does not admit a G-invariant nondegenerate hypersurface.*

Proof. Let F be a rectangular 2-face of P with odd side lengths. We have established that F can be written as the product of two line segments and any number of points. When we restrict a hypersurface in the toric variety of P to the face F, the points will simply contribute a monomial factor to the expression for the hypersurface, which will not affect whether it is smooth at F. Hence, we are free to ignore the points and focus our attention on the two line segments.

By analyzing the linear functionals that are maximized at the face F, one can show that the two line segments must have the form $\text{conv}\{(w_1, w_2), (w_2, w_1)\}$ and $\text{conv}\{(w_3, w_4), (w_4, w_3)\}$ where $w_1 < w_2 \leq w_3 < w_4$ (see Example 3.55). Since the segments must have odd length, we further know that $w_2 - w_1$ and $w_4 - w_3$ are odd. If we look at the orbits of the lattice points in F under the action of S_n, we will find that each orbit contains exactly four
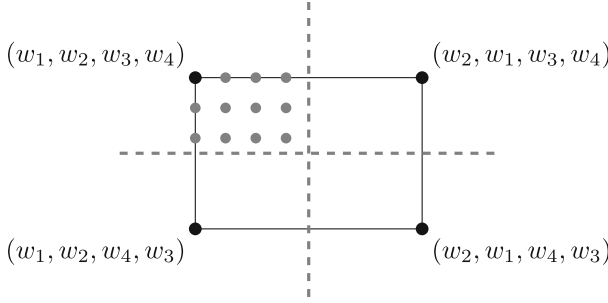
FIGURE 11. A rectangular face arising as the product of two line segments. We are interested in the case where $w_1 < w_2 \leq w_3 < w_4$ and $w_2 - w_1$ and $w_4 - w_3$ are odd. Then, the S_n-orbits of integer points in the rectangle are in correspondence with the integer points in the upper left quadrant of the figure, and each orbit consists of precisely one point in each quadrant

points, with exactly one in each quadrant of F (see Fig. 11). Therefore, the parameters appearing in the equation for our hypersurface restricted to F can be identified with points in one quadrant of F. We will denote them by $c_{i,j}$ where $0 \leq i \leq \lfloor \frac{w_2 - w_1}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{w_4 - w_3}{2} \rfloor$. Using this, a generic S_n-invariant hypersurface restricted to the face F of P has the following equation:

$$
\sum_{j=0}^{\lfloor \frac{w_4 - w_3}{2} \rfloor} \left(\left(x_3^{w_3+j} x_4^{w_4-j} + x_3^{w_4-j} x_4^{w_3+j} \right) \right.
$$
$$
\left. \left(\sum_{i=0}^{\lfloor \frac{w_2 - w_1}{2} \rfloor} c_{i,j} \left(x_1^{w_1+i} x_2^{w_2-i} + x_1^{w_2-i} x_2^{w_1+i} \right) \right) \right) = 0
$$

Since $w_2 - w_1$ is odd, then for each i one of $w_1 + i$ and $w_2 - i$ will be odd and the other even. Likewise, for each j one of $w_3 + j$ and $w_4 - j$ will be odd and the other even. Hence, each term $x_1^{w_1+i} x_2^{w_2-i} + x_1^{w_2-i} x_2^{w_1+i}$ vanishes when $x_2 = -x_1$, regardless of the choices for $c_{i,j}$, and each term $x_3^{w_3+j} x_4^{w_4-j} + x_3^{w_4-j} x_4^{w_3+j}$ vanishes when $x_4 = -x_3$. Since the equation has a zero of multiplicity 2 whenever $x_2 = -x_1$ and $x_4 = -x_3$, the gradient vanishes at this locus, so no S_n-invariant hypersurface in the toric variety of P is smooth at F. □

We will now characterize exactly which orbit polytopes have rectangular 2-dimensional faces with odd side lengths. To do this, we will associate any given S_n-orbit polytope with a composition of n using the following setup.

Setup 3.57. Given $\mathbf{w}' \in \mathbb{Z}_n$, let $\mathbf{w}$ be the representative from the S_n-orbit of $\mathbf{w}'$ with coordinates in increasing order; $\mathbf{w} = (w_1, \ldots, w_1, w_2, \ldots, w_2, \ldots, w_k, \ldots, w_k)$ where $w_1 < w_2 < \cdots < w_k$. From this, construct a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n where α_i is the number of w_i's that appear in the coordinates

of $\mathbf{w}$. For example, if we have $\mathbf{w} = (0, 1, 1, 3, 3, 3, 4, 5)$, then the corresponding composition of 8 would be $(1, 2, 3, 1, 1)$.

Given two compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$, we define their *concatenation* to be $\alpha \mathbin{+\!\!+} \beta := (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$ and their *near-concatenation* to be $\alpha \odot \beta := (\alpha_1, \ldots, \alpha_k + \beta_1, \ldots, \beta_\ell)$.

Lemma 3.58. *An orbit polytope has a rectangular 2-dimensional face if and only if its corresponding composition α satisfies one of the following criteria:*

(i) *α has four or more parts, or*
(i) *α has three parts and the middle part is ≥ 2.*

Proof. Proposition 4.19 of [27] implies that for an orbit polytope with composition α, any face can be described as a product of orbit polytopes with compositions $\beta^{(1)}, \ldots, \beta^{(r)}$ from which α can be obtained by some combination of concatenations and near-concatenations. Moreover, every such collection of compositions $\beta^{(1)}, \ldots, \beta^{(r)}$ gives a face of the orbit polytope. In order for this face to be a rectangle, we need two of the compositions β to be $(1, 1)$ (the composition of a line segment) and the remaining β's to be (1) (the composition of a point). When such a collection of compositions is combined together with any sequence of concatenations and near-concatenations, the resulting α is as described. $\square$

Corollary 3.59. *An orbit polytope with composition α arising from the point $\mathbf{w}$ (as in Setup 3.57), that satisfies the condition of Lemma 3.58, has a rectangular 2-face with odd side lengths if and only if one of the following criteria is satisfied:*

(i) *There exist i and j with $1 \leq i < i+1 < j < j+1 \leq k$ such that the differences $w_{i+1} - w_i$ and $w_{j+1} - w_j$ are both odd.*
(ii) *There exists an i with $1 \leq i \leq k-2$ such that $\alpha_{i+1} \geq 2$ and the differences $w_{i+1} - w_i$ and $w_{i+2} - w_{i+1}$ are both odd.*

3.4.2. Hypersimplices. We let the symmetric group S_n acts on the hypersimplex $\Delta_{k,n}$ by permuting coordinates. The following theorem implies that H*-series of the hypersimplex $\Delta_{k,n}$ is effective.

Theorem 3.60. *The toric variety of the hypersimplex $\Delta_{k,n}$ admits an S_n-invariant nondegenerate hypersurface.*

Proof. Let $e_k(x_1, x_2, \ldots, x_n)$ be the k-th elementary symmetric polynomial in variables $x_1, \ldots, x_n$:

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=k}} \prod_{i \in I} x_i.$$

Consider the hypersurface $H \subset (\mathbb{C}^*)^n$ defined by $e_k(x_1, x_2, \ldots, x_n) = 0$. It is straightforward to check that H is S_n-invariant.

We claim that H is smooth using the induction on k. The base case $k = 1$ is trivial as $\frac{\partial e_1(x_1, x_2, \ldots x_n)}{\partial x_i} = 1 \neq 0$. Assume the claim is true for $k-1$ and

consider the case k. Suppose there exists a non-smooth point $(a_1, a_2, \ldots, a_n) \in H$, then we have

$$e_k(a_1, a_2, \ldots, a_n) = 0$$

$$\frac{\partial e_k}{\partial x_i}(a_1, a_2, \ldots, a_n) = e_{k-1}(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n) = 0 \qquad \text{for } 1 \le i \le n.$$

Since $\sum_{i=1}^{n} e_{k-1}(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n) = (n - k + 1)e_{k-1}(a_1, a_2, \ldots, a_n)$, we obtain

$$e_{k-1}(a_1, a_2, \ldots, a_n) = 0. \tag{27}$$

The following identity on elementary symmetric polynomials

$$e_{k-1}(a_1, a_2, \ldots, a_n) = e_{k-1}(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n) + a_i e_{k-2}(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n)$$

combined with (27) gives

$$e_{k-2}(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n) = 0 \qquad \text{for } 1 \le i \le n. \tag{28}$$

By the induction hypothesis, there is no $(a_1, a_2, \ldots, a_n) \in (\mathbb{C}^*)^n$ satisfying (27) and (28).

It remains to show $e_k(x_1, x_2, \ldots, x_n)|_F$ defines a smooth hypersurface inside a torus for every face F. Let F be the face that maximizes the linear functional $L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$. We partition the set $\{1, 2, \ldots, n\}$ with $S_1, S_2, \ldots, S_\ell$ so that:

- For $i, j \in S_p$, $c_i = c_j$.
- For $p < q$ and $i \in S_p, j \in S_q$, $c_i > c_j$.

Let h be the integer such that $\sum_{i=1}^{h}|S_i| \le k$ and $\sum_{i=1}^{h+1}|S_i| > k$. Then, vertices of $\Delta_{k,n}$ that maximizes L are given by $(x_1, x_2, \ldots, x_n)$ such that

- For $i \in S_p$ with $p \le h$, $x_i = 1$.
- For $i \in S_p$ with $p > h + 1$, $x_i = 0$.
- The number of i's such that $x_i = 1$ for $i \in S_{h+1}$ equals $(k - \sum_{i=1}^{h}|S_i|)$. For other i's, $x_i = 0$.

We conclude that $e_k(x_1, x_2, \ldots, x_n)|_F$ equals monomial times elementary symmetric functions with variables $\{x_i\}_{i \in S_{h+1}}$. Using the same argument, this defines a smooth hypersurface inside a torus. $\qquad \square$

3.4.3. Counterexample to Conjecture 1.1. Although Stapledon initially conjectured that the existence of an invariant nondegenerate hypersurface is equivalent to H*-polynomiality and effectiveness, he later exhibited a counterexample with Francisco Santos. This result was never published, but we include it here with their permission.

Theorem 1.2. (Santos–Stapledon, Counterexample to Conjecture 1.1) *Let the polytope $P = [0,1]^3 \subset \mathbb{R}^3$ be the 3-cube and let $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{\text{id}, \sigma, \tau, \sigma\tau = \tau\sigma\}$ act on P by 180 degree rotations as described in Fig. 12. Then, $H^*(P; z)$ is polynomial and effective, but the toric variety of P does not admit a G-invariant nondegenerate hypersurface.*
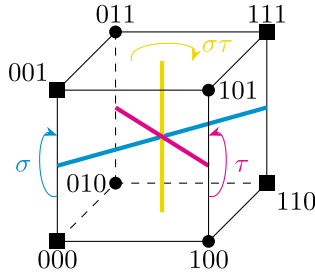
FIGURE 12. Action of $(\mathbb{Z}/2\mathbb{Z})^2$ on the 3-cube. Each of σ, τ, and $\sigma\tau$ acts by a rotation of 180 degrees around a corresponding axis. The vertices are divided into two orbits under this action, which are indicated by their shapes

Proof. By computing the Ehrhart series of the rational fixed polytope of the cube under each of these rotations, and then comparing these to a character table for G, one can find the H*-series of P under this action to be $\chi_{\mathrm{triv}} + \chi_{\mathrm{reg}} z + \chi_{\mathrm{triv}} z^2$ where these characters indicate the trivial and regular representations of G. Therefore, the H*-series is a polynomial and is effective.

However, the toric variety of P does not admit a G-invariant nondegenerate hypersurface. Figure 12 shows the two orbits of the vertices of P under the action of G. Therefore, a G-invariant hypersurface in the toric variety of P has the equation

$$0 = (1 + z + xy + xyz) + c(x + y + xz + yz) = (1 + z)(1 + xy + c(x + y))$$

for some complex parameter c. Regardless of the value of c, such a hypersurface is singular when $z = -1$ and $1 + xy + c(x + y) = 0$. □

4. Further Questions

Question 4.1. Sect. 3.1 uses zonotopal decompositions to compute the equivariant Ehrhart series of zonotopes of the Type A root system. How can this technique be adapted to general root systems? See [13] for some progress in this direction.

Question 4.2. Can we characterize graphs Γ for which every $\sigma \in \mathrm{Aut}(\Gamma)$ satisfies one of the conditions of Theorem 3.18?

Question 4.3. Theorem 1.4 of [28] gives a formula for the h^*-polynomial of a zonotope in terms of the basis activity of its corresponding matroid. Is there an equivariant analogue of this formula that describes the H*-series of a zonotope in terms relating to the G-action on its matroid?

Question 4.4. For which n is there a G-invariant triangulation as in Theorem 3.30 or Theorem 3.32 of the permutahedron $\Pi_n \in \mathbb{R}^n$ under the cyclic group action permuting the standard basis vectors? Can Theorem 3.32 be extended to compute equivariant Ehrhart theory using symmetric zonotopal tilings?

TABLE 3. Conditions of Conjecture 1.1 when S_4 acts on dilates of Π_4

	H^* polynomial?	H^* effective?	S_4-inv. non-deg. hypersurface?
Π_4	No[a]	No[a]	No[b]
$2\Pi_4$	Yes[c]	Yes[d]	?
$3\Pi_4$	No[d]	No[d]	No[b]
$4\Pi_4$	Yes[c]	Yes[d]	?
$5\Pi_4$	No[d]	No[d]	No[b]
Even dilates	Yes[c]	?	?
Odd dilates	?	?	No[b]

[a]Shown in [4].
[b]Theorem 3.56
[c]All fixed polytopes are integral.
[d]Verified in Sagemath

Question 4.5. Can we find large families of polytopes exhibiting the G-invariant half-open triangulations described in Theorems 3.30 and 3.32?

Question 4.6. Theorem 3.36 gives a combinatorial description of the H_i^* characters for the action of $\mathbb{Z}/n\mathbb{Z}$ on the hypersimplex $\Delta_{k,n}$. However, we also know from Theorem 3.60 that the H^*-series for the S_n action on $\Delta_{k,n}$ is effective. Can we find a combinatorial interpretation of these characters H_i^* of S_n?

Question 4.7. Theorem 3.56 shows that any orbit polytope containing a rectangular 2-face with odd side lengths does not admit an S_n-invariant nondegenerate hypersurface. Is the converse true as well? In other words, are rectangular 2-faces with odd length sides the only obstruction to the existence of an S_n-invariant nondegenerate hypersurface in the toric variety of an orbit polytope?

Question 4.8. Suppose that the G-invariant polytope P admits an invariant nondegenerate hypersurface, or has polynomial/effective H^*-series. Must the same then be true for the action of G on positive integer dilations of P? Table 3 summarizes what we know about the action of S_4 on the dilates of Π_4.

Question 4.9. Although condition (i) of Conjecture 1.1 was proven to not be implied by conditions (ii) and (iii) (Theorem 1.2), it may still be the case that (ii) and (iii) are equivalent. Does polynomiality of the H^*-series imply effectiveness in general? Furthermore, what additional hypotheses need to be added to Conjecture 1.1 in order to make it true?

Question 4.10. A topic of great interest in classical Ehrhart theory is determining which polytopes have h^*-vectors that are *unimodal*; i.e., each coefficient

is weakly larger the previous, until a peak is reached, after which point each coefficient is weakly smaller than the previous. In the equivariant setting, we can interpret unimodality by looking at the differences $H_i^* - H_{i-1}^*$ and asking that these differences form a sequence consisting of effective characters up to a certain point, after which all the differences are negatives of effective characters. Which polytopes exhibit this equivariant unimodality property, and what parallels can be drawn with the classical concept of h^*-unimodality?

Question 4.11. The Ehrhart f^*-vector is a change-of-basis transformation of the h^*-vector, and f^*-positivity for polyhedral complexes is a weaker property than h^*-positivity. What is the equivariant analogue of the f^*-vector, and what can we say about its effectiveness? This question was suggested by Katharina Jochemko.

Acknowledgements

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A Calculating the H*-Series in `Sagemath`

Elia implemented `Sagemath` functionality for calculating the equivariant H*-series. For more detailed documentation, see [5]. It is open-source and staged for release with `Sagemath` version 9.6. We show how to compute the H*-series of the permutahedron Π_4 under the action of the symmetric group. First, we create the permutahedron, and set its backend to be `normaliz`, a necessary step for the usage of these methods. Furthermore we create Π_4's `restricted_automorphism_group`, which is the group of linear transformations mapping the polytope to itself and such that d-dimensional faces are mapped to d-dimensional faces. The output of the `restricted_automorphism_group` must be set to `permutation`. This means that every group element is expressed as a permutation of the vertices of the polytope. Later, we will create S_4 as a subgroup of the `restricted_automorphism_group`.

```
sage: Pi4 = polytopes.permutahedron(4, backend='normaliz')
sage: Pi4
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 24 vertices
sage: G = Pi4.restricted_automorphism_group(output='permutation')
sage: G
Permutation Group with generators
[(1,6)(3,12)(4,8)(5,14)(9,18)(11,20)(15,19)(17,22),
(0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)(16,17)(18,19)(20,21)(22,23),
(0,2)(1,4)(3,5)(6,8)(7,10)(9,11)(12,14)(13,16)(15,17)(18,20)(19,22)(21,23),
(0,23)(1,17)(2,21)(3,11)(4,15)(5,9)(6,22)(7,16)(8,19)(10,13)(12,20)(14,18)]
sage: G.order()
48
```

The order of G is 48; we must create $S4$ as a subgroup. Using our method `permutations_to_matrices`, which returns the matrix representation of elements of the restricted automorphism group, we see that the generator $(1,6)(3,12)(4,8)(5,14)(9,18)(11,20)(15,19)(17,22)$ does not correspond to the action of the symmetric group on Π_4:

```
sage: conj_reps = G.conjugacy_classes_representatives()
sage: Pi4.permutations_to_matrices(conj_class_reps=conj_reps, acting_group=G)
{(1,6)(3,12)(4,8)(5,14)(9,18)(11,20)(15,19)(17,22):
[  1/2   1/2   1/2  -1/2      0]
[  1/2   1/2  -1/2   1/2      0]
[  1/2  -1/2   1/2   1/2      0]
[ -1/2   1/2   1/2   1/2      0]
[    0     0     0     0      1],
...
```

We create $S4$ as a subgroup of G and compute the H*-series using the method `Hstar_function`:

```
sage: S4 = G.subgroup(gens=[G.gens()[i] for i in range(1,4)])
sage: S4.order()
24
```

```
sage: Pi4.Hstar_function(acting_group=S4, output='complete')
{'Hstar': ((chi_0 + chi_2 + chi_4)*t^4
 + (6*chi_0 + chi_1 + 6*chi_2 + 5*chi_3 + 9*chi_4)*t^3
 + (9*chi_0 + chi_1 + 8*chi_2 + 7*chi_3 + 14*chi_4)*t^2
 + (4*chi_0 + chi_1 + 3*chi_2 + 3*chi_3 + 5*chi_4)*t + chi_0)/(t + 1),
 'Hstar_as_lin_comb': ((t^4 + 6*t^3 + 9*t^2 + 4*t + 1)/(t + 1),
 (t^3 + t^2 + t)/(t + 1),
 t^3 + 5*t^2 + 3*t, (5*t^3 + 7*t^2 + 3*t)/(t + 1),
 (t^4 + 9*t^3 + 14*t^2 + 5*t)/(t + 1)),
 'conjugacy_class_reps': [(),
 (0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)(16,17)(18,19)(20,21)(22,23),
 (0,3,4)(1,5,2)(6,9,10)(7,11,8)(12,15,16)(13,17,14)(18,21,22)(19,23,20),
 (0,7)(1,6)(2,13)(3,12)(4,19)(5,18)(8,15)(9,14)(10,21)(11,20)(16,23)(17,22),
 (0,9,16,18)(1,11,22,12)(2,15,10,19)(3,17,20,6)(4,21,8,13)(5,23,14,7)],
 'character_table': [ 1  1  1  1  1]
[ 1 -1  1  1 -1]
[ 2  0 -1  2  0]
[ 3 -1  0 -1  1]
[ 3  1  0 -1 -1],
 'is_effective': False}
```

To reiterate, we have found

$$\mathrm{H}^*(\Pi_4; z) = \frac{(\chi_0 + \chi_2 + \chi_4)z^4 + (6\chi_0 + \chi_1 + 6\chi_2 + 5\chi_3 + 9\chi_4)z^3}{1+z}$$

$$+ \frac{(9\chi_0 + \chi_1 + 8\chi_2 + 7\chi_3 + 14\chi_4)z^2 + (4\chi_0 + \chi_1 + 3\chi_2 + 3\chi_3 + 5\chi_4)z + \chi_0}{1+z},$$

where the characters are labeled according to the rows in the character table.

To see the documentation of the `Hstar_function`, or of the related supporting methods, `fixed_subpolytopes`, `permutations_to_matrices`, or indeed of any function in `Sagemath`, one can type `?` after the function:

```
sage: P = polytopes.cube(backend='normaliz')
sage: P.fixed_subpolytopes?
```

To see the both source code and the documentation simultaneously, type `??` after the function:

```
sage: P = polytopes.cube(backend='normaliz')
sage: P.Hstar_function??
```

References

[1] Ehrhart, E.: Sur les polyèdres rationnels homothétiques à n dimensions. C. R. Acad. Sci. Paris 254, 616–618 (1962)

[2] Stapledon, A.: Equivariant Ehrhart theory. Advances in Mathematics 226(4), 3622–3654 (2011)

[3] Ardila, F., Schindler, A., Vindas-Meléndez, A.R.: The equivariant volumes of the permutahedron. Discrete and Computational Geometry, 1–18 (2019)

[4] Ardila, F., Supina, M., Vindas-Meléndez, A.: The equivariant Ehrhart theory of the permutahedron. Proceedings of the American Mathematical Society 148, 5091–5107 (2020). https://doi.org/10.1090/proc/15113

[5] Elia, S.: On three Ehrhart theories & simplicial hyperplane arrangements. PhD thesis, Freie Universität Berlin (2022). https://doi.org/10.17169/refubium-34540

[6] Kim, D.: A combinatorial formula for the Ehrhart h^*-vector of the hypersimplex. J. Comb. Theory Ser. A 173(C), 105213 (2020). https://doi.org/10.1016/j.jcta.2020.105213

[7] Beck, M., Robins, S.: Computing the Continuous Discretely, 2nd edn. Undergraduate Texts in Mathematics, p. 285. Springer, New York (2015). https://doi.org/10.1007/978-1-4939-2969-6

[8] Ferroni, L., Jochemko, K., Schröter, B.: Ehrhart polynomials of rank two matroids. Advances in Applied Mathematics 141, 102410 (2022). https://doi.org/10.1016/j.aam.2022.102410

[9] Stanley, R.P.: Decompositions of rational convex polytopes. Ann. Discrete Math. 6, 333–342 (1980)

[10] Sagan, B.E.: The Symmetric Group, 2nd edn. Graduate Texts in Mathematics, vol. 203, p. 238. Springer, New York (2001). https://doi.org/10.1007/978-1-4757-6804-6

[11] Serre, J.-P.: Linear Representations of Finite Groups, p. 170. Springer, New York-Heidelberg (1977). Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42

[12] Cox, D.A., Little, J.B., Schenck, H.K.: Toric Varieties. American Mathematical Society, Providence, RI (2011)

[13] Ardila, F., Beck, M., McWhirter, J.: The arithmetic of coxeter permutahedra. Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales 44, 1152–1166 (2020). https://doi.org/10.18257/raccefyn.1189

[14] Shephard, G.C.: Combinatorial properties of associated zonotopes. Canad. J. Math. 26, 302–321 (1974). https://doi.org/10.4153/CJM-1974-032-5

[15] Stanley, R.P.: A zonotope associated with graphical degree sequences. In: Applied Geometry and Discrete Mathematics. DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, pp. 555–570. Amer. Math. Soc., Providence, RI (1991)

[16] Stanley, R.: An Introduction to Hyperplane Arrangements, pp. 389–496. IAS/Park City Mathematics Series, Park City (2007). https://doi.org/10.1090/pcms/013/08

[17] Aguiar, M., Ardila, F.: Hopf monoids and generalized permutahedra. arXiv e-prints, 1709–07504 (2017) arXiv:1709.07504 [math.CO]

[18] Ziegler, G.M.: Lectures on Polytopes. Graduate Texts in Mathematics, vol. 152, p. 370. Springer, New York (1995). https://doi.org/10.1007/978-1-4613-8431-1

[19] Stanley, R.P.: Combinatorics and Commutative Algebra, 2nd edn. Progress in Mathematics, vol. 41, p. 164. Birkhäuser Boston, Inc., Boston, MA (1996)

[20] Beck, M., Sanyal, R.: Combinatorial Reciprocity Theorems. GSM, vol. 195, p. 308. American Mathematical Society, Providence, RI (2018). https://doi.org/10.1090/gsm/195

[21] Haase, C., Nill, B., Paffenholz, A.: Lecture Notes on Lattice Polytopes. https://www2.mathematik.tu-darmstadt.de/~paffenholz/daten/preprints/20210628_Lattice_Polytopes.pdf (2021)

[22] Katzman, M.: The Hilbert series of algebras of Veronese type. Communications in Algebra 33, 1141–1146 (2005). https://doi.org/10.1081/AGB-200053828

[23] Postnikov, A.: Permutohedra, associahedra, and beyond. Int. Math. Res. Not. IMRN (6), 1026–1106 (2009). https://doi.org/10.1093/imrn/rnn153

[24] Khare, A., Ridenour, T.: Faces of weight polytopes and a generalization of a theorem of Vinberg. Algebras and Representation Theory 15, 593–611 (2012)

[25] Khare, A.: Standard parabolic subsets of highest weight modules. Transactions of the American Mathematical Society 369, 2363–2394 (2014). https://doi.org/10.1090/tran/6710

[26] Ardila, F., Castillo, F., Eur, C., Postnikov, A.: Coxeter submodular functions and deformations of Coxeter permutahedra. Advances in Mathematics 365, 107039 (2020). https://doi.org/10.1016/j.aim.2020.107039

[27] Supina, M.: The Hopf monoid of orbit polytopes. Journal of Combinatorics 11(4), 575–601 (2020)

[28] Beck, M., Jochemko, K., McCullough, E.: h*-polynomials of zonotopes. Transactions of the American Mathematical Society 371, 2021–2042 (2019). https://doi.org/10.1090/tran/7384

Sophia Elia
Department of Mathematics
Freie Universität Berlin
Arnimallee 2
14195 Berlin
Berlin
Germany
e-mail: sophiasageelia@gmail.com

Donghyun Kim
Department of Mathematics
Seoul National University
Gwanak-ro, Gwanak-gu
Seoul-si 08826
South Korea
e-mail: hyun920310@snu.ac.kr

Mariel Supina
Department of Mathematics
KTH Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
Stockholms län
Sweden
e-mail: mjsupina@gmail.com