



Quadratic Coefficients of Goulden–Rattan Character Polynomials

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Abstract. Goulden–Rattan polynomials give the exact value of the subdominant part of the normalized characters of the symmetric groups in terms of certain quantities (C_i) which describe the macroscopic shape of the Young diagram. The Goulden–Rattan positivity conjecture states that the coefficients of these polynomials are positive rational numbers with small denominators. We prove a special case of this conjecture for the coefficient of the quadratic term C_2^2 by applying certain bijections involving maps (i.e., graphs drawn on surfaces).

Keywords. Characters of the symmetric groups, free cumulants, Kerov polynomials, Goulden–Rattan polynomials, maps.

1. Introduction

1.1. Normalized Characters

Characters are a basic tool of representation theory. After normalization, they are also useful in asymptotic problems.

If $k \leq n$ are natural numbers, then any permutation $\pi \in S_k$ can also be treated as an element of the larger symmetric group S_n by adding $n - k$ additional fixpoints. For any permutation $\pi \in S_k$ and any irreducible representation ρ^λ of the symmetric group S_n which corresponds to the Young diagram λ , we define *the normalized character*

$$\Sigma_\pi(\lambda) = \begin{cases} n(n-1) \cdots (n-k+1) \frac{\text{Tr } \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

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Of particular interest are the character values on the cycles, therefore we will use the shorthand notation

$$\Sigma_k(\lambda) = \Sigma_{(1,2,\dots,k)}(\lambda).$$

1.2. Free Cumulants

Free cumulants are an important tool of free probability theory [9] and random matrix theory [10]. In the context of the representation theory of the symmetric groups, they can be defined as follows, see [1]. For a Young diagram λ , we define its free cumulants $R_2(\lambda), R_3(\lambda), \dots$ as

$$R_k(\lambda) = \lim_{s \rightarrow \infty} \frac{1}{s^k} \Sigma_{k-1}(s\lambda),$$

where the diagram $s\lambda$ is created from the diagram λ by dividing each box of λ into an $s \times s$ square.

The free cumulants have been defined in such a way as to be very helpful for studying asymptotic behaviour of the characters on a cycle of length k when the size of the Young diagram tends to infinity [2].

1.3. Kerov Character Polynomials

Kerov [6] formulated the following result: for each permutation π and any Young diagram λ , the normalized character $\Sigma_\pi(\lambda)$ is equal to the value of some polynomial $K_\pi(R_2(\lambda), R_3(\lambda), \dots)$ (now called *the Kerov character polynomial*) with integer coefficients. The first published proof of this fact was provided by Biane [1]. The Kerov character polynomial is *universal* because it does not depend on the choice of λ . We are interested in the values of the characters on cycles; therefore, for $\pi = (1, 2, \dots, k)$, we use the simplified notation

$$\Sigma_k = K_k(R_2, R_3, \dots) \tag{1.1}$$

for such Kerov polynomials. The first few Kerov polynomials K_k are as follows:

$$K_1 = R_2,$$

$$K_2 = R_3,$$

$$K_3 = R_4 + R_2,$$

$$K_4 = R_5 + 5R_3,$$

$$K_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$

$$K_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3,$$

$$K_7 = R_8 + 180R_2 + 224R_2^2 + 14R_2^3 + 56R_3^2 + 469R_4 + 84R_2R_4 + 70R_6.$$

Kerov conjectured that the coefficients of the polynomial K_k are *non-negative* integers. Goulden and Rattan [5] found an explicit formula for the coefficients of the Kerov polynomial K_k ; unfortunately, their formula was complicated and did not give any combinatorial interpretation to the coefficients. Later, Féray proved positivity [4] and together with Dołęga and Śniady found a combinatorial interpretation of the coefficients [3]. In this paper, we will use the combinatorial interpretation given by them in the special case of linear and square coefficients.

1.4. Goulden–Rattan Conjecture

Goulden and Rattan [5] introduced a family of functions C_2, C_3, \dots on the set of Young diagrams given by $C_0 = 1, C_1 = 0$ and

$$C_k^\lambda = \frac{24}{k(k+1)(k+2)} \lim_{s \rightarrow \infty} \frac{1}{s^k} \left(\Sigma_{k+1}(s\lambda) - R_{k+2}(s\lambda) \right)$$

for $k \geq 2$.

Śniady [7] proved the explicit form of C_k (conjectured by Biane [1]) as a polynomial in the free cumulants R_2, R_3, \dots given by

$$C_k = \sum_{\substack{j_2, j_3, \dots \geq 0 \\ 2j_2 + 3j_3 + \dots = k}} (j_2 + j_3 + \dots)! \prod_{i \geq 2} \frac{((i-1)R_i)^{j_i}}{j_i!} \quad (1.2)$$

for $k \geq 2$. The aforementioned formula of Goulden and Rattan for the Kerov polynomials was naturally expressed in terms of these quantities C_2, C_3, \dots [5]. More specifically, they constructed an explicit polynomial L_k with rational coefficients such that

$$K_k - R_{k+1} = L_k(C_2, C_3, \dots). \quad (1.3)$$

These polynomials are called *the Goulden–Rattan polynomials*. They formulated the following conjecture:

Goulden–Rattan conjecture. *The coefficients of the Goulden–Rattan polynomials are non-negative numbers with small denominators.*

The first few Goulden–Rattan polynomials are as follows [5]:

$$\begin{aligned} K_1 - R_2 &= 0 \\ K_2 - R_3 &= 0 \\ K_3 - R_4 &= C_2, \\ K_4 - R_5 &= \frac{5}{2}C_3, \\ K_5 - R_6 &= 5C_4 + 8C_2, \\ K_6 - R_7 &= \frac{35}{4}C_5 + 42C_3, \\ K_7 - R_8 &= 14C_6 + \frac{469}{3}C_4 + \frac{203}{3}C_2^2 + 180C_2. \end{aligned}$$

Linear coefficients of the Goulden–Rattan polynomials are non-negative, because they are equal to certain scaled coefficients of the Kerov polynomial:

$$[C_j]L_k = \frac{1}{j-1}[R_j]K_k.$$

In this paper, we will prove that the coefficient of C_2^2 is non-negative. We hope that edge sliding we will define in this article will also be a useful tool in proving non-negativity of the square coefficients $[C_i C_j]L_k$. The next step towards the proof of Goulden–Rattan conjecture would be to understand the cubic coefficients $[C_i C_j C_u]L_k$; we hope that our methods will still be applicable

there, nevertheless, there seem to be some difficulties related to the inclusion–exclusion principle.

1.5. Graphs on Surfaces, Maps and Expanders

We will consider graphs drawn on an oriented surface. Each face of such a graph has some number of edges ordered cyclically by going along the boundary of the face and touching it with the right hand. We will call it *the clockwise boundary direction*. If we use the left hand and visit the edges in the opposite order, we will call it the counterclockwise boundary direction.

By a *map* we mean a bipartite graph drawn without intersections on an oriented and connected surface with minimal genus. The maps which we consider have a fixed choice of colouring of the vertices, i.e., each vertex is coloured black or white, with the edges connecting the vertices of the opposite colours. An example of a map is shown in Fig. 1.

An *expander* [8, Appendix A.1] is a map with the following properties.

- It has a distinguished edge (known as the root) and one face.
- Each black vertex is assigned a natural number, known as a weight, such that each non-empty proper subset of the set of black vertices has more white neighbours than the sum of its weights.
- The sum of all weights is equal to the number of white vertices.

The map from Fig. 1 is an expander if each black vertex has weight 1 (any choice of the root is valid).

Using the Euler characteristic we get

$$2 - 2g = \chi = V - k + 1, \quad (1.4)$$

where g denotes the genus of the surface, V denotes the number of vertices and k denotes the number of edges.

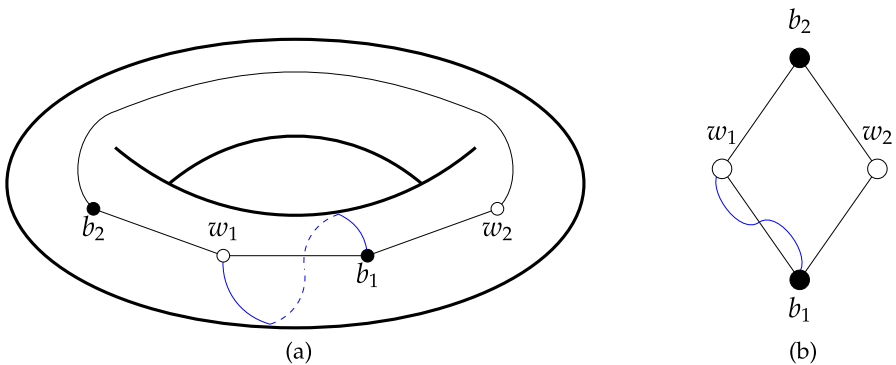


FIGURE 1. **a** An example of a map with 4 vertices and 5 edges drawn on a torus. **b** The same map drawn for simplicity on the plane (Color figure online)

1.6. Combinatorial Interpretation of the Kerov Polynomial Coefficients

The following two theorems Theorem 1 and Theorem 3 give a combinatorial interpretation to the linear and square coefficients of the Kerov character polynomials [3, Theorem 1.2, Theorem 1.3]. The first is as follows.

Theorem 1. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of pairs (σ_1, σ_2) of permutations $\sigma_1, \sigma_2 \in S(k)$ such that $\sigma_1\sigma_2 = (1\ 2\ \cdots\ k)$ and such that σ_2 consists of one cycle and σ_1 consists of $l - 1$ cycles. (We use the convention $\sigma_1\sigma_2 = \sigma_2(\sigma_1) = \sigma_2 \circ \sigma_1$.)*

The expanders are a graphical interpretation of these pairs of permutations. There is a natural bijection between such pairs of permutations (σ_1, σ_2) and the expanders with one face, one black vertex, $l - 1$ white vertices and k edges. Additionally, one edge is selected as the root and the unique black vertex has a weight $l - 1$. More precisely:

- The edges are numbered $1, 2, \dots, k$. The edge with number 1 is selected as the root.
- The counterclockwise angular cyclic order of the edges on a given vertex (i.e., the order of edges ending at this vertex around it) corresponds to a cycle of a permutation depending on the colour of this vertex, i.e., σ_1 for white and σ_2 for black (in this case we have a unique cycle of the permutation σ_2).
- The unique face corresponds to the unique counterclockwise boundary cycle of the permutation $(1\ 2\ \cdots\ k)$.

Since there is only one face, the root determines the numbering of all edges. We can reformulate Theorem 1 as follows. (See Fig. 2a for an example.)

Theorem 2. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of expanders with k edges, $l - 1$ white vertices and 1 black vertex with the weight $l - 1$.*

Similarly, we use the second theorem [3, Theorem 1.3] for square coefficients.

Theorem 3. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1}R_{l_2}]K_k$ is equal to the number of triples (σ_1, σ_2, q) with the following properties.*

- The permutations $\sigma_1, \sigma_2 \in S_k$ fulfill the equality $\sigma_1\sigma_2 = (1\ 2\ \cdots\ k)$.
- The permutation σ_1 consists of two cycles and the permutation σ_2 consists of $l_1 + l_2 - 2$ cycles.
- The function q associates the numbers l_1 and l_2 to the two cycles of σ_1 . Furthermore, for each cycle c of σ_1 there exist at least $q(c)$ cycles of σ_2 which nontrivially intersect c .

Analogously, we can also reformulate Theorem 3. (See Fig. 2b for an example.)

Theorem 4. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1}R_{l_2}]K_k$ is equal to the number of expanders with k edges, $l_1 + l_2 - 2$ white vertices and 2 black vertices with weights $l_1 - 1, l_2 - 1$.*

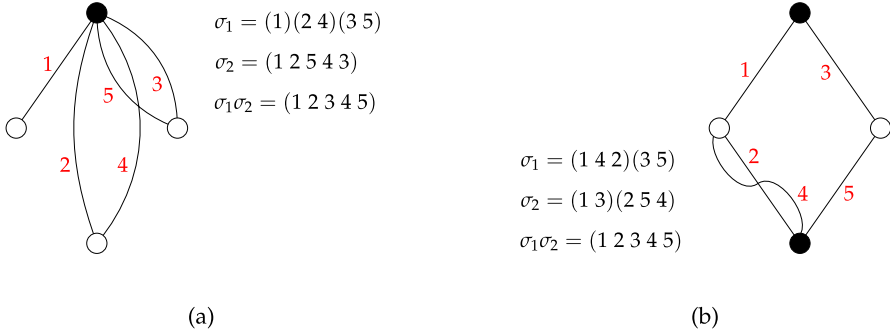


FIGURE 2. Examples of expanders with 5 edges and their corresponding pair of permutations σ_1, σ_2 such that $\sigma_1\sigma_2 = (1\ 2\ 3\ 4\ 5)$. The root is assigned the number 1. **a** The expander with one black vertex and three white vertices. **b** The expander with two black vertices and two white vertices (Color figure online)

1.7. Relationship Between Coefficients of Goulden–Rattan Polynomials and Coefficients of Kerov Polynomials

The formula (1.2) allows us to express (C_i) in terms of free cumulants; we see that the coefficients of the terms $R_i R_j$, R_{i+j} , R_j^2 and R_{2j} in the expressions $C_i C_j$, C_{i+j} , C_j^2 and C_{2j} are given for $i \neq j$ by

$$\begin{aligned} C_i C_j &= (i-1)(j-1)R_i R_j + 0R_{i+j} + (\text{sum of other terms}), \\ C_{i+j} &= 2(i-1)(j-1)R_i R_j + (i+j-1)R_{i+j} + (\text{sum of other terms}), \\ C_j^2 &= (j-1)^2 R_j^2 + 0R_{2j} + (\text{sum of other terms}), \\ C_{2j} &= (j-1)^2 R_j^2 + (2j-1)R_{2j} + (\text{sum of other terms}). \end{aligned}$$

Moreover, any product $C_{i_1} C_{i_2} \cdots C_{i_t}$ of at least $t \geq 3$ factors does not contain any of the terms $C_i C_j$, C_{i+j} , C_j^2 and C_{2j} . It follows that the square coefficients of the Goulden–Rattan polynomial are related to the coefficients of the Kerov polynomial via

$$\begin{aligned} \left. \frac{\partial^2 L_k}{\partial C_i \partial C_j} \right|_{0=C_1=C_2=\dots} &= \frac{1}{(i-1)(j-1)} \left. \frac{\partial^2 K_k}{\partial R_i \partial R_j} \right|_{0=R_1=R_2=\dots} \\ &\quad - 2 \left. \frac{\partial L_k}{\partial C_{i+j}} \right|_{0=C_1=C_2=\dots} \\ &= \frac{1}{(i-1)(j-1)} \left. \frac{\partial^2 K_k}{\partial R_i \partial R_j} \right|_{0=R_1=R_2=\dots} \\ &\quad - \left. \frac{2}{(i+j-1)} \frac{\partial K_k}{\partial R_{i+j}} \right|_{0=R_1=R_2=\dots} \end{aligned}$$

for $i \neq j$. Whereas the quadratic coefficients are related via

$$\begin{aligned} \left. \frac{\partial^2 L_k}{\partial C_j^2} \right|_{0=C_1=C_2=\dots} &= \frac{1}{(j-1)^2} \left. \frac{\partial^2 K_k}{\partial R_j^2} \right|_{0=R_1=R_2=\dots} - 2 \left. \frac{\partial L_k}{\partial C_{2j}} \right|_{0=C_1=C_2=\dots} \\ &= \frac{1}{(j-1)^2} \left. \frac{\partial K_k^2}{\partial R_j^2} \right|_{0=R_1=R_2=\dots} - \frac{2}{(2j-1)} \left. \frac{\partial K_k}{\partial R_{2j}} \right|_{0=R_1=R_2=\dots} \end{aligned}$$

for any natural number j . Thus, we obtain the explicit formula for the square coefficients of the Goulden–Rattan polynomial:

$$[C_j^2]L_k = \frac{1}{(j-1)^2} [R_j^2]K_k - \frac{1}{2j-1} [R_{2j}]K_k \quad (1.5)$$

and

$$[C_i C_j]L_k = \frac{1}{(i-1)(j-1)} [R_i R_j]K_k - \frac{2}{i+j-1} [R_{i+j}]K_k \quad \text{for } i \neq j. \quad (1.6)$$

2. The Main Result

Let $Y_k(u)$ denote the set of expanders with k edges, $u-1$ white vertices and one black vertex. Let $X_k(i, j)$ denote the set of expanders with k edges, $i+j-2$ white vertices and two black vertices with weights $i-1$ and $j-1$. Using Theorems 2 and 4 we can also reformulate the Goulden–Rattan conjecture for the square coefficients in terms of expanders, as follows.

Conjecture 1. *Let $i \neq j$ be natural numbers. Then*

$$(2j-1) \|X_k(j, j)\| \geq (j-1)^2 \|Y_k(2j)\|$$

and

$$(i+j-1) \|X_k(i, j)\| \geq 2(i-1)(j-1) \|Y_k(i+j)\|$$

for any natural number k .

These inequalities are equivalent to the positivity of the coefficients $[C_j^2]L_k$ and $[C_i C_j]L_k$ respectively. In this text we prove only the first inequality in the special case $j=2$. We hope to present a proof of Conjecture 1 in its general form in a future paper.

Using Eq. (1.5), we can calculate several examples of the coefficient of C_2^2 of the Goulden–Rattan polynomials

$$\begin{aligned} [C_2^2]L_4 &= 0 - 0 = 0, \\ [C_2^2]L_5 &= 5 - \frac{1}{3} \cdot 15 = 0, \\ [C_2^2]L_6 &= 0 - 0 = 0, \\ [C_2^2]L_7 &= 224 - \frac{1}{3} \cdot 469 = \frac{203}{3}, \\ [C_2^2]L_8 &= 0 - 0 = 0. \end{aligned}$$

Note that if k is even then $[C_2^2]L_k = 0$ because there does not exist an expander with 4 vertices and an even number of edges, since $2 - 2g = 2j - k + 1$ by Eq. (1.4). Additionally, $[C_2^2]L_1 = 0$ and $[C_2^2]L_3 = 0$. Thus, we can assume that the number of edges k is odd and $k \geq 5$.

Let

$$X_k = X_k(2, 2), \quad (2.1)$$

$$Y_k = Y_k(4). \quad (2.2)$$

The set X_k consists of expanders with 2 black vertices and 2 white vertices such that each black vertex is connected with both white vertices; each black vertex necessarily has weight equal to 1. The set Y_k consists of expanders with one black vertex (which necessarily has weight 3) connected with all 3 white vertices. From now on we will omit the weights of the black vertices.

The main goal of this paper is to prove the following:

Theorem 5. *The inequality*

$$3\|X_k\| \geq \|Y_k\|$$

is true for any natural number k .

3. Edge Sliding

In this section, we provide some necessary background details for the proof of Theorem 5. We will denote a transposition exchanging a and h by $(a\ h)$. For any set $H \subset \{1, 2, \dots\}$ such that $a, h \in H$, we can treat the transposition $(a\ h)$ as a permutation of the set H . (By adding fixed points to the transposition.) Therefore, for any permutation π of the set H , the products $\pi \circ (a\ h)$ and $(a\ h) \circ \pi$ are also permutations of the set H .

Let G be a graph with edges numbered $1, \dots, k$ drawn without intersections on an oriented and connected surface with minimal genus. There is a natural bijection between the graph G and a multiset of cycles M_G (cyclic



FIGURE 3. **a** The graph with the multiset of cycles $\{(2)(1)(5\ 6\ 4)(2\ 3\ 4\ 5\ 6)(3\ 1)\}$. **b** The graph with the multiset of cycles $\{(2)(1)(5\ 6\ 4)(2\ 3)(4\ 5\ 6\ 3\ 1)\}$ (Color figure online)

permutations of a subset of $\{1, \dots, k\}$ such that each number j belongs to exactly two cycles. More precisely, the counterclockwise angular cyclic order of edges at a given vertex of the graph G (i.e., the order of edges ending at this vertex around it) corresponds to a single cycle of M_G . Two examples of a graph with its multiset of cycles are shown in Fig. 3.

In this section, each end of every edge (or equivalently, each element of every cycle) will have one of three values assigned to it: clockwise direction, counterclockwise direction, or no direction. Note that the two ends of the same edge can be assigned different values. By default, we assume that a cycle element to which we have not assigned a direction has an assigned the value no direction.

3.1. Edge Sliding for a Single Number

Let M_G be the multiset of cycles of a graph G . Let a belonging to a cycle $c \in M_G$ be a number with assigned the clockwise direction. (The number “a” from a cycle different from “c” and any number different from “a” from any cycle do not have a direction assigned to them.) Let $c' \neq c$ be the second cycle of the multiset M_G containing the number $h = c^{-1}(a)$. We assume that the number a does not belong to the cycle c' . We define *the single edge sliding for the number a in the clockwise boundary direction from the cycle c to the cycle c' along the number h* as the replacement of the cycles c, c' in the multiset M_G by two new cycles $s(c), s'(c')$ given by

$$\begin{aligned} s(c) &= [c \circ (a \ h)] \setminus (a), \\ s'(c') &= (a \ h) \circ [(a) \ c']. \end{aligned}$$

Note that the product $c \circ (a \ h)$ consists of two cycles: the cycle (a) and the cycle formed from the cycle c by removing the number a . Therefore, the above

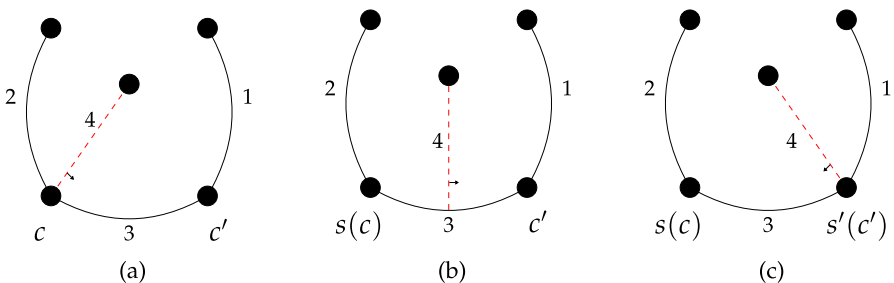


FIGURE 4. An example of the single edge sliding for the number $a = 4$ in the clockwise boundary direction from the cycle $c = (2 \ 3 \ 4)$ to the cycle $c' = (1 \ 3)$ along the number $h = 3$. **a** The initial graph G . The edge labelled a is dashed and coloured red. **b** Visualization of the single edge sliding. **c** The resulting graph with two updated cycles: $s(c) = [(2 \ 3 \ 4) \circ (3 \ 4)] \setminus (4) = (2 \ 3)$ and $s'(c') = (3 \ 4) \circ [(1 \ 3) \ (4)] = (1 \ 4 \ 3)$ (Color figure online)

transformation removes the number a from the cycle c and adds it to the cycle c' before the number h . Finally, we change the direction of the number a to the counterclockwise direction. We can naturally think of such operations as sliding of edges in a graph. An example of the single edge sliding is shown in Fig. 4.

Analogously, we define the single edge sliding for the number a in the *counterclockwise boundary direction* from the cycle c to the cycle c' along the number $h = c(a)$ as the replacement of the cycles c, c' in the multiset M_G by two new cycles $s(c), s'(c')$ given by

$$\begin{aligned} s(c) &= [(a \ h) \circ c] \setminus (a), \\ s'(c') &= [(a \ c') \circ (a \ h)]. \end{aligned}$$

3.2. Edge Sliding for a Sequence of Numbers

Let M_G be the multiset of cycles of a graph G . Let a_1, \dots, a_l be a sequence of successive but not all numbers belonging to a cycle $c \in M_G$, i.e. such that $a_j = c^{j-1}(a_1)$ for each index $j \leq l$. Moreover, to each of the numbers a_1, \dots, a_l is assigned the clockwise direction. Let $c' \neq c$ be the second cycle of the multiset of M_G containing the number $h = c^{-1}(a_1)$. We assume that the numbers a_1, \dots, a_l do not belong to the cycle c' . We define the *package edge sliding for the sequence of the numbers a_1, \dots, a_l in the clockwise boundary direction from the cycle c to the cycle c' along the number h* as the replacement of the cycles c, c' in the multiset M_G by two new cycles $s_l(c), s'_l(c')$ obtained by recursion with initial conditions $s_0(c) = c, s'_0(c') = c'$ and a recursive step given for each

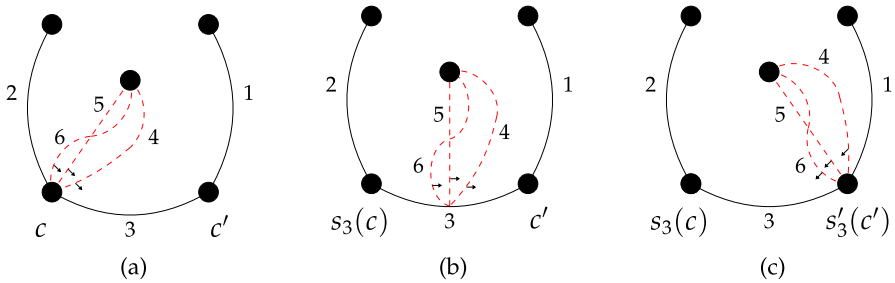


FIGURE 5. An example of the package edge sliding for the numbers $a_1 = 4$, $a_2 = 5$, $a_3 = 6$ in the clockwise boundary direction from the cycle $c = (2 \ 3 \ 4 \ 5 \ 6)$ to the cycle $c' = (1 \ 3)$ along the number $h = 3$. **a** The initial graph G . Edges numbered a_1, a_2, a_3 are dashed and coloured red. **b** Visualization of the package edge sliding. **c** The resulting graph with two updated cycles: $s_3(c) = [(2 \ 3 \ 4 \ 5 \ 6) \circ (3 \ 4) \circ (3 \ 5) \circ (3 \ 6)] \setminus [(4) \ (5) \ (6)] = (2 \ 3)$ and $s'_3(c') = (3 \ 6) \circ (3 \ 5) \circ (3 \ 4) \circ [(1 \ 3) \ (4)] = (1 \ 4 \ 5 \ 6 \ 3)$ (Color figure online)

$1 < j \leq l$ by

$$\begin{aligned} s_j(c) &= [s_{j-1}(c) \circ (a_j \ h)] \setminus (a_j), \\ s'_j(c') &= (a_j \ h) \circ [(a_j \ s'_{j-1}(c'))]. \end{aligned}$$

Note that the package edge sliding for a sequence of numbers a_1, \dots, a_l is actually equivalent to the sequential single edge sliding for numbers a_1, \dots, a_l . Therefore, the above transformation removes the numbers a_1, \dots, a_l from the cycle c and adds them in the same order to the cycle c' before the number h . Finally, we change the directions of the numbers a_1, \dots, a_l to the counterclockwise boundary direction. Note that the single edge sliding is a special case of the package edge sliding. An example of the package edge sliding is shown in Fig. 5.

Analogously, we define the package edge sliding for the numbers a_1, \dots, a_l in the *counterclockwise boundary direction* from the cycle c to the cycle c' along the number $h = c(a)$ as the replacement of the cycles c, c' in the multiset M_G by two new cycles $s_l(c), s_l(c')$ obtained by recursion with initial conditions $s_0(c) = c, s'_0(c') = c'$ and a recursive step given for each $0 < j \leq l$ by

$$\begin{aligned} s_j(c) &= [(a_j \ h) \circ s_{j-1}(c)] \setminus (a_j), \\ s'_j(c') &= [(a_j \ s'_{j-1}(c'))] \circ (a_j \ h). \end{aligned}$$

In other words, if we would like to slide a number a in the clockwise boundary direction from a cycle c along a number h , then $c^{-1}(a) = h$ or before that, a number $c^{-1}(a)$ must be slid in the same direction. (Similarly for the counterclockwise boundary direction and number $c(a)$.) For each number a in a cycle c with a fixed direction, the number h along which it will be slid is uniquely determined. Therefore, for simplicity, we will say that each of the numbers a_1, \dots, a_l from the cycle c is slid in a fixed direction. Of course, we assume that at least one number in the cycle c has no direction assigned.

3.3. Edge Sliding in the General Case

We define *the edge sliding* on a graph as follows. We start from the graph G in which some ends of certain edges are assigned a clockwise or counterclockwise boundary direction. We assume that:

- At each vertex at least one edge has no direction assigned.
- Two consecutive edges do not have conflicting directions, i.e., there is no situation in which a number a from a cycle c has assigned the counterclockwise boundary direction and the number $c(a)$ has assigned the clockwise boundary direction.

Any such set of directions can be decomposed into an package edge sliding system for sequences of numbers:

$$\begin{cases} a_{1,1}, \dots, a_{1,l_1} \text{ in a direction } d_1 \text{ from the cycle } c_1 \text{ to the cycle } c'_1 \text{ along the number } h_1, \\ \vdots \\ a_{t,1}, \dots, a_{t,l_t} \text{ in a direction } d_t \text{ from the cycle } c_t \text{ to the cycle } c'_t \text{ along the number } h_t. \end{cases}$$

Furthermore, we require that:

- The numbers, along which other numbers are sliding, are not themselves sliding, i.e.,

$$\{a_{1,1}, \dots, a_{t,l_t}\} \cap \{h_1, \dots, h_t\} = \emptyset.$$

- Two ends of the same edge will not appear in the same vertex, i.e., if $a_{i_1,j_1} = a_{i_2,j_2}$, then

$$\{c_{i_1}, c'_{i_1}\} \cap \{c_{i_2}, c'_{i_2}\} = \emptyset.$$

(This condition can be weakened to $c'_{i_1} \neq c'_{i_2}$, but it is not necessary in this paper.)

- On one side of the edge along which we slide, the numbers do not slide in opposite directions, i.e., for each number h_j belonging to the cycles $c \neq c'$, there is no situation in which a number $c(h_j)$ has assigned the clockwise direction and the number $c'^{-1}(h_j)$ has assigned the counter-clockwise direction.

We will call the selection of directions that satisfy the above conditions as *correct*.

We define the edge sliding as applying sequentially the package edge sliding for all sequences in any order. Such an action is well-defined, since permutation multiplication is associative and for any distinct numbers a_1, a_2, h_1, h_2 holds

$$(a_1 h_1) \circ (a_2 h_2) = (a_2 h_2) \circ (a_1 h_1).$$

An example of the edge sliding is shown in Fig. 6.

The edge sliding is an involution on the set of graphs drawn on an oriented and connected surface with a correct selection of directions. Edge sliding is an invertible transformation, with the inverse also given by edge sliding.

In addition, it is easy to see that the edge sliding on a graph does not change the number of faces of this graph.

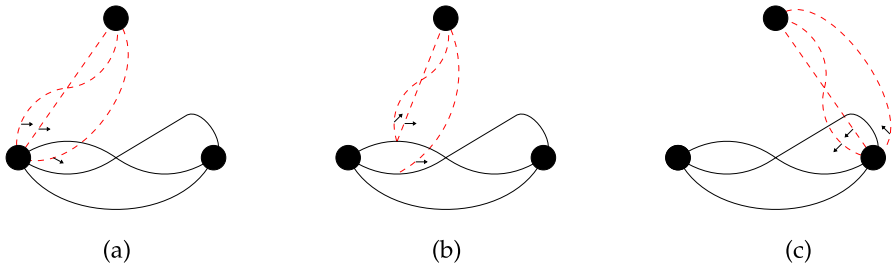


FIGURE 6. An example of edge sliding. **a** A graph with the slid edges dashed and coloured red. The directions of the edge ends are indicated by arrows. **b** The graph during the edge sliding in the clockwise boundary direction. **c** The graph after the edge sliding. Directions have already been reversed (Color figure online)

In the rest of this article, we will treat the edge sliding as transformation on a graph.

3.4. The Set X_k of Maps

We consider any map from the set X_k . This map has one face and an odd number of edges $k \geq 5$. We denote the black vertices by b_1, b_2 and the white vertices by w_1, w_2 . There is at least one edge between each pair of the vertices of different colours. Of course, $\deg(b_1) + \deg(b_2) = k$ is an odd number. Without loss of generality we may assume that $\deg(b_1) > 0$ is an odd number and $\deg(b_2) > 0$ is an even number. Let $k_1, k_2 > 0$ denote the numbers of edges which connect the vertex b_1 with the vertices w_1, w_2 , respectively. As $\deg(b_1) = k_1 + k_2$ is an odd number, without loss of generality we may assume that k_1 is even and k_2 is odd. For example, the unique (up to choice of the root) map from the set X_5 is shown in Fig. 7a.

3.5. The Set Y_k of Maps

Let σ be the cycle that encodes the clockwise boundary cyclic order of the corners on the unique face of G . We will say that *the vertex w_j is a descendant of the vertex w_i* (we denote it by $w_i \rightarrow w_j$) if using the clockwise boundary order of the corners on the unique face of the map we can move (by walking along the edges and holding them with the right hand) in two steps from a certain corner c_i of the vertex w_i to a certain corner c_j of the vertex w_j , i.e., $\sigma^2(c_i) = c_j$.

We consider any map from the set Y_k . Any such map has one face and an odd number of edges $k \geq 5$. We denote the black vertex by b and the white vertices by w_1, w_2, w_3 . We will write the set Y_k as a union of three sets which will be defined below.

Let $Y_k^{\text{odd}} \subseteq Y_k$ be the set of maps for which there exists an *odd degree* white vertex (let us say it is w_3) which has the other two white vertices as

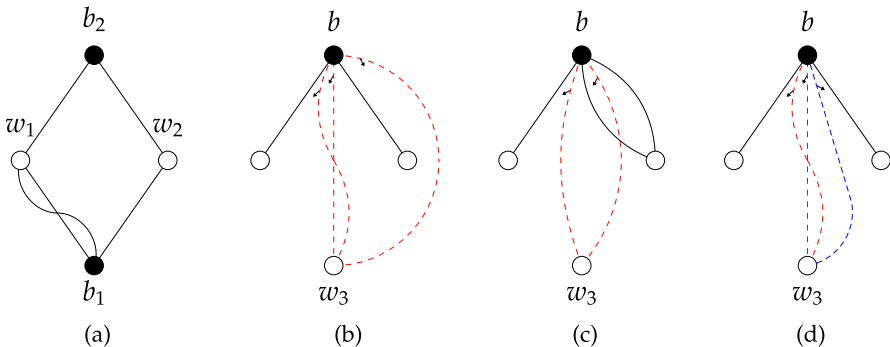


FIGURE 7. Examples of maps from a the set X_5 , b the set T_5^{odd} , c the set T_5^{even} , d the set T_5^{rest} . Slid edges are dashed and coloured. The directions are indicated by arrows, and the root is not marked (Color figure online)

descendants, i.e., $w_3 \rightarrow w_1$ and $w_3 \rightarrow w_2$. Let T_k^{odd} be the set of all maps from the set Y_k^{odd} with a distinguished vertex w_3 with this property. Moreover, each edge between the vertex w_3 and the vertex b , has a clockwise boundary direction assigned at the vertex b . The unique (up to choice of the root) map from the set T_5^{odd} is shown in Fig. 7b. Clearly

$$|T_k^{\text{odd}}| \geq |Y_k^{\text{odd}}|. \quad (3.1)$$

Let $Y_k^{\text{even}} \subseteq Y_k$ be the set of maps such that there exists an *even degree* white vertex (let us say it is w_3) which has the other two white vertices as descendants, i.e., $w_3 \rightarrow w_1$ and $w_3 \rightarrow w_2$. Let T_k^{even} be the set of all the maps from the set Y_k^{even} with a distinguished vertex w_3 with this property. Moreover, each edge between the vertex w_3 and the vertex b , has a clockwise boundary direction assigned at the vertex b . The unique (up to choice of the root) map from the set T_5^{even} is shown in Fig. 7c. Clearly

$$|T_k^{\text{even}}| \geq |Y_k^{\text{even}}|. \quad (3.2)$$

Let $Y_k^{\text{rest}} \subseteq Y_k$ be the set of maps not included in the sets Y_k^{odd} and Y_k^{even} , i.e.,

$$Y_k^{\text{rest}} = Y_k \setminus (Y_k^{\text{odd}} \cup Y_k^{\text{even}}). \quad (3.3)$$

Consider some map $m \in Y_k^{\text{rest}}$. Obviously $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_1$ or the other way around. Without loss of generality we may assume that $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_1$ and as a consequence $w_1 \leftarrow w_2 \leftarrow w_3 \leftarrow w_1$.

Lemma. *The map m has a white vertex of odd degree, greater than 1.*

Proof. By contradiction, suppose this is not the case. The map m has at least one odd degree white vertex, because $\deg(w_1) + \deg(w_2) + \deg(w_3) = k$ is odd. Without loss of generality we may assume that $\deg(w_1)$ is odd. Since

$$\deg(w_1) + \deg(w_2) + \deg(w_3) = k > 3 = 1 + 1 + 1,$$

it follows that $\deg(w_1) = 1$ and $\deg(w_2), \deg(w_3)$ are even, because m does not have a white vertex with odd degree greater than 1. The vertex w_1 is a leaf and thus has a unique corner which we denote by c_1 .

Naturally $\sigma^2(c_1)$ is a corner of the vertex w_2 . Note that σ^2 is a permutation of the corners of the white vertices which has only one cycle, because the map m has only one face. The corners of the white vertices can be labelled 1, 2, 3 according to the names of the vertices they are in. If a corner c has the label a , its descendant $\sigma^2(c)$ has either the label a or $1 + a \bmod 3$. There is only one corner which has the label 1, so the corner labels of m (arranged in the cyclic order according to the unique cycle of σ^2) are $(1, 2, \dots, 2, 3, \dots, 3)$. Since there exists only one corner c_2 of the white vertex w_2 such that $\sigma^2(c_2)$ is a corner of the vertex w_3 , then there exists a unique corner $c_0 = \sigma(c_2)$ of the vertex b such that $\sigma(c_0)$ is the corner of the vertex w_3 and $\sigma^{-1}(c_0)$ is the corner of the vertex w_2 . Thus the clockwise angular cyclic order of the edges around the black vertex b is as follows: one edge connected to the vertex w_1 , a certain number of edges connected to the vertex w_2 , a certain number of edges connected to the vertex w_3 . Figure 8a visualizes this situation.

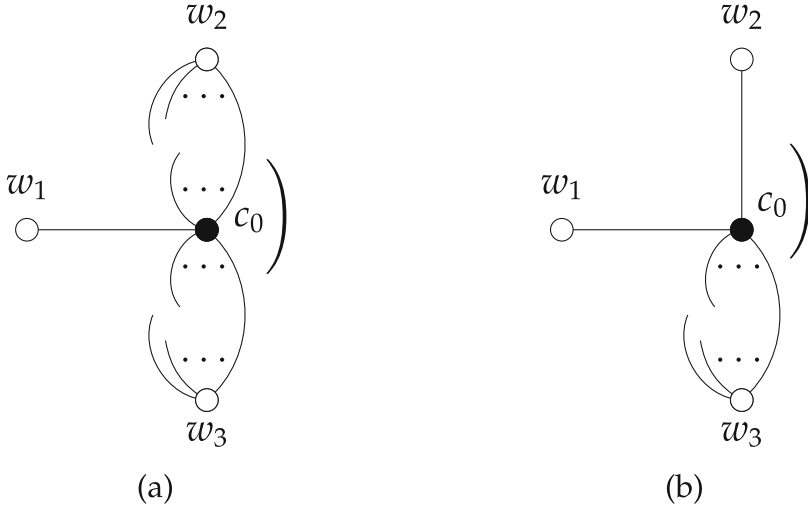


FIGURE 8. **a** The map m before replacement including the marked corner c_0 . **b** A non-existent hypothetical map with four vertices and an even number of edges (Color figure online)

Let $k' = k - (\deg(w_2) - 1)$. The number k' is even because k is odd and $\deg(w_2)$ is even. We remove all edges except one of the white vertex w_2 obtaining a new graph m' with k' edges. Obviously, the clockwise angular cyclic order of the edges around the black vertex b is as follows: one edge connected to the vertex w_1 , one edge connected to the vertex w_2 , a certain number of edges connected to the vertex w_3 . Whereas the corner labels of m' are $(1, 2, 3, \dots, 3)$. Therefore, m' has one face. Figure 8b visualizes this situation. If m' does not have selected a root, we choose any edge of m' as the root. If the genus of the surface on which the map m' is drawn is not minimal, we draw the map m' on a surface with minimal genus. As a result, we obtain an expander from the set $Y_{k'}$ with 4 vertices, one face and an even number of edges k' . We get a contradiction because such a map does not exist (see Eq. (1.4)). Therefore, the map m has a white vertex with an odd degree greater than 1. \square

Now, we fix the directions. To all ends in the vertex b of the edges between the vertex w_3 and b , we assign the direction in such way that among them there is an even number with the clockwise boundary direction and an odd number with the counterclockwise boundary direction. This is always possible, e.g. for a single edge with the counterclockwise boundary direction.

Let T_k^{rest} be the set of all the maps from the set Y_k^{rest} with a distinguished white vertex denoted by w_3 with a fixed choice of the set of special edges together with the directions of their ends satisfying the conditions just mentioned above. The unique (up to choice of the root) example of the map from the set T_5^{rest} is shown in Fig. 7d. Clearly

$$|T_k^{\text{rest}}| \geq |Y_k^{\text{rest}}|. \quad (3.4)$$

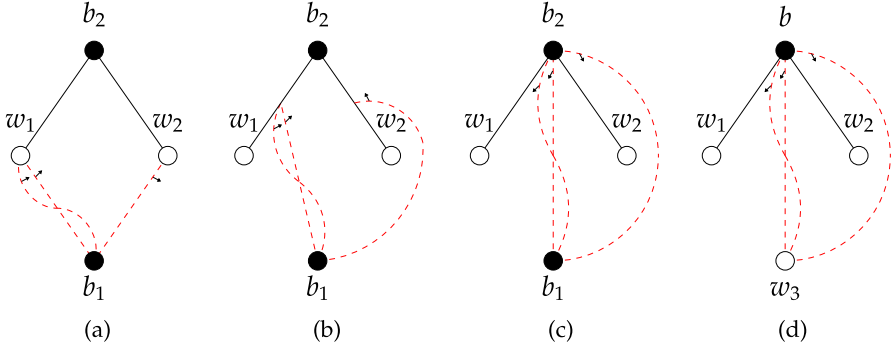


FIGURE 9. The example of the first bijection for the 5-edged map. **a** The map from the set X_5 . **b** The map during the edge sliding. **c** The map after the edge sliding. **d** The map from the set Y_5^{odd} (Color figure online)

4. Proof of Main Result

In this section, we will construct three bijections which show the cardinalities of the corresponding sets are equal. Using these equalities and the definitions of these sets we will prove Theorem 5.

4.1. Three Bijections

The first bijection between X_k and T_k^{odd} . We start from a map $m \in X_k$. Recall that we have assumed that $\deg(b_1)$ is odd. All edges connecting a vertex b_1 to white vertices, have the counterclockwise boundary direction assigned at the vertices w_1, w_2 . The choice of directions is correct because m is a bipartite graph. We apply the edge sliding to the map m . Then we change the colour of the black vertex b_1 to white and its name to w_3 , and the name of the vertex b_2 to b . Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \rightarrow w_1$ and $w_3 \rightarrow w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{odd} . (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{odd} can be produced in this way. Such a transformation is a bijection between the set X_k and the set T_k^{odd} , since the edge sliding is reversible. Figure 9 shows an example of this bijection for $k = 5$. Thus,

$$|X_k| = |T_k^{\text{odd}}|. \quad (4.1)$$

The second bijection between X_k and T_k^{even} . We start from a map $m \in X_k$. Recall that we have assumed that $\deg(b_2)$ is even. All edges connecting a vertex b_1 to white vertices, have the counterclockwise boundary direction assigned at the vertices w_1, w_2 . The choice of directions is correct because m is a bipartite graph. We apply the edge sliding to the map m . Then we change the colour of the black vertex b_2 to white and its name to w_3 , and the name of the vertex b_1 to b . Of course, the degree of the vertex w_3 does not change and is even.

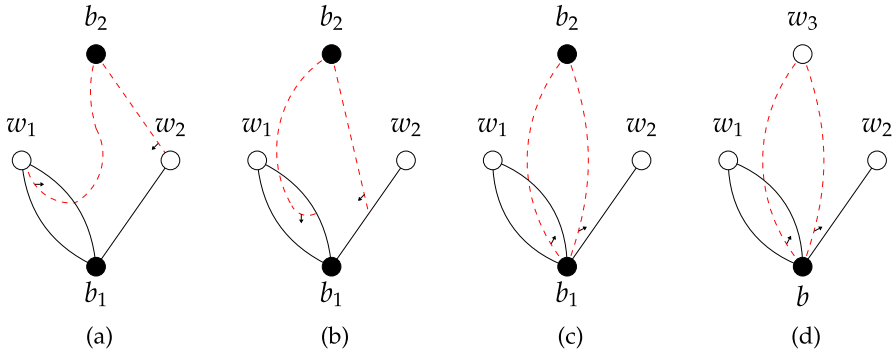


FIGURE 10. **a–d** The example of the second bijection for the 5-edged map (Color figure online)

In addition, $w_3 \rightarrow w_1$ and $w_3 \rightarrow w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{even} . (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{even} can be produced. Such a transformation is a bijection between the set X_k and the set T_k^{even} , since the edge sliding is reversible. Figure 10 shows an example of this bijection for $k = 5$. Thus,

$$|X_k| = |T_k^{\text{even}}|. \quad (4.2)$$

The third bijection. We start from a map $m \in X_k$. Recall that we have assumed that $\deg(b_1)$ is odd. All edges connecting a vertex b_1 to white vertices, have the counterclockwise boundary direction assigned at the vertex w_1 and the clockwise boundary direction assigned at the vertex w_2 . The choice of directions is correct because m is a bipartite graph. We apply the edge sliding to the map m . Then we change the colour of the black vertex b_1 to white and its name to w_3 , and the name of the vertex b_2 to b . Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \rightarrow w_1$ and $w_2 \rightarrow w_3$ (and $w_1 \rightarrow w_2$), because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We do not necessarily obtain a map from the set T_k^{rest} (it may be that we obtain a map from set T_k^{odd}), but it can be seen that each map from the set T_k^{rest} can be produced. (At all times one of the edges is selected as the root.) Such a transformation is a bijection between the set X_k and some superset of the set T_k^{odd} , since the edge sliding is reversible. Figure 11 shows an example of this bijection for $k = 5$. Thus

$$|X_k| \geq |T_k^{\text{rest}}|. \quad (4.3)$$

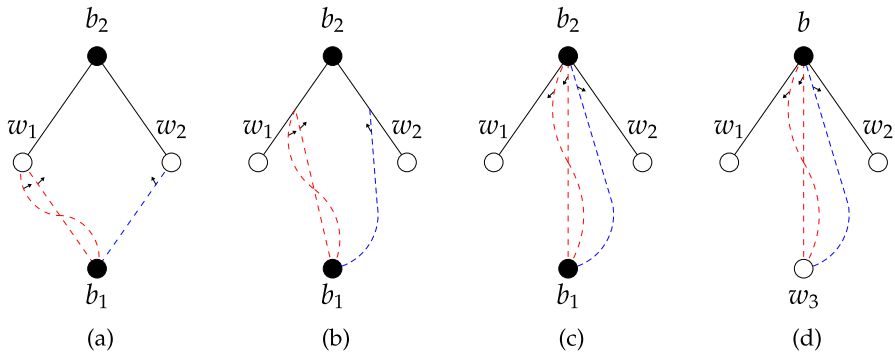


FIGURE 11. **a–d** The example of the third bijection for the 5-edged map (Color figure online)

4.2. The Conclusion of the Proof

We can now proceed to the proof of Theorem 5, we have:

$$\begin{aligned}
 3[C_2^2]L_k &= 3[R_2^2]K_k - [R_4]K_k && \text{by (1.5)} \\
 &= 3|X_k| - |Y_k| && \text{by (2.1), (2.2)} \\
 &\geq |T_k^{\text{odd}}| + |T_k^{\text{even}}| + |T_k^{\text{rest}}| - |Y_k| && \text{by (4.1), (4.2), (4.3)} \\
 &\geq |Y_k^{\text{odd}}| + |Y_k^{\text{even}}| + |Y_k^{\text{rest}}| - |Y_k| && \text{by (3.1), (3.2), (3.4)} \\
 &= |Y_k^{\text{odd}} \cap Y_k^{\text{even}}| && \text{by (3.3)} \\
 &\geq 0.
 \end{aligned}$$

□

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