Polynomization of the Bessenrodt–Ono Inequality

Bernhard Heim , Markus Neuhauser and Robert Tröger

Abstract. In this paper, we investigate a generalization of the Bessenrodt–Ono inequality by following Gian–Carlo Rota’s advice in studying problems in combinatorics and number theory in terms of roots of polynomials. We consider the number of k -colored partitions of n as special values of polynomials $P_n(x)$. We prove for all real numbers $x > 2$ and $a, b \in \mathbb{N}$ with $a + b > 2$ the inequality:

$$P_a(x) \cdot P_b(x) > P_{a+b}(x).$$

We show that $P_n(x) < P_{n+1}(x)$ for $x \geq 1$, which generalizes $p(n) < p(n+1)$, where $p(n)$ denotes the partition function. Finally, we observe for small values, the opposite can be true, since, for example: $P_2(-3 + \sqrt{10}) = P_3(-3 + \sqrt{10})$.

Mathematics Subject Classification. Primary 05A17, 11P82;
 Secondary 05A20.

Keywords. Partition, Polynomial, Partition inequality.

1. Introduction and Main Results

Let $p(n)$ be the number of partitions of n [2, 3, 17]. It is well known that this arithmetic function increases strictly:

$$p(1)p(n) < p(n+1) \tag{1.1}$$

for all $n \in \mathbb{N}$, since every partition of n can be lifted to a partition of $n+1$. Bessenrodt and Ono [5] discovered that (1.1) is actually an exception in the context of more general products $p(a)p(b)$. Recently, DeSalvo and Pak [8] proved that the sequence $\{p(n)\}$ is log-concave for $n > 25$.

Theorem 1.1. (Bessenrodt–Ono 2016) *Let a, b be natural numbers. Let $a + b > 9$, then*

$$p(a)p(b) > p(a+b). \tag{1.2}$$

Bessenrodt and Ono provided proof, based on a theorem of Rademacher [18] and Lehmer [15]. They speculated at the end of their paper that a combinatorial proof could be possible. Shortly after their paper was published, Alanazi, Gagola, and Munagi [1] found such a proof. Chern, Fu, and Tang [6] generalized Bessenrodt and Ono's theorem to k -colored partitions $p_{-k}(n)$ of n .

Theorem 1.2. (Chern, Fu, Tang 2018) *Let a, b, k be natural numbers. Let $k > 1$, then*

$$p_{-k}(a)p_{-k}(b) > p_{-k}(a+b), \quad (1.3)$$

except for $(a, b, k) \in \{(1, 1, 2), (1, 2, 2), (2, 1, 2), (1, 3, 2), (3, 1, 2), (1, 1, 3)\}$.

Let $P_n(x)$ be the unique polynomial of degree n satisfying $P_n(k) = p_{-k}(n)$ for all $k \in \mathbb{N}$ (see [9]). In this paper we prove the following results:

Theorem 1.3. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq 1$. Then*

$$P_n(x) < P_{n+1}(x) \quad \text{and} \quad 1 \leq P'_n(x) < P'_{n+1}(x). \quad (1.4)$$

Moreover, let $n+1$ be an odd prime number. Then, there exists $x_n \in (0, 1)$, such that

$$P_{n+1}(x_n) < P_n(x_n). \quad (1.5)$$

Our main result is the following extension of the Bessenrodt–Ono type inequality.

Theorem 1.4. *Let $a, b \in \mathbb{N}$, $a+b > 2$, and $x > 2$. Then*

$$P_a(x)P_b(x) > P_{a+b}(x). \quad (1.6)$$

The case $x = 2$ is true for $a+b > 4$.

This gives a precise answer to a conjecture stated in [11]. The theorem is proven by induction. The proof uses a special formula for the derivative of $P_n(x)$, the inequality (1.3) for $k = 2$, a bound of Lehmer (5.2), and Theorem 1.3. The inequality (1.3) for $k = 2$ is proven by Chern, Fu, and Tang [6].

The values in Table 1 for $1 \leq a, b \leq 10$ imply that inequality (1.6) can also hold for values of x smaller than 2. If we assume $a \geq b$ then except for $b = 1$ and $a = 1, 2, 3$ all values are less than 2. For values of x larger than the numbers (rounded to two decimal places) in position (a, b) inequality (1.6) holds.

Bessenrodt–Ono-type inequalities also appeared in works by Beckwith and Bessenrodt [4] on k -regular partitions and by Hou and Jagadeesan [12] on the numbers of partitions with ranks in a given residue class modulo $t = 3$. Recently, Males [16] obtained results for general t . Dawsey and Masri [7] obtained new results for Andrews spt -function. It is very likely that some of these recent results can be extended to an inequality of certain polynomials.

This paper is organized as follows. In Sect. 2, we introduce the polynomials $P_n(x)$ and provide basic properties. In Sect. 3, we give numerical evidence for Theorem 1.3 and Theorem 1.4. In Sect. 4, we prove Theorem 1.3 and in Sect. 5, we prove Theorem 1.4.

2. Partitions and Polynomials

A partition λ of a positive integer n is any non-increasing sequence $\lambda_1, \lambda_2, \dots, \lambda_d$ of positive integers, whose sum is n . The λ_i denote the parts of the partition. The number of partitions of n is denoted by $p(n)$ (see [2, 17]).

Example. The partitions of $n = 4$ and $n = 5$ are

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

Hence, $p(4) = 5$ and $p(5) = 7$. Note that $p(200)$ is already equal to 3972999029388.

A partition is called a k -colored partition of n if each part can appear in k colors. Let $p_{-k}(n)$ denote the number of k -colored partitions of n (see [6]). Note that $p_{-1}(n) = p(n)$ and $p_{-k}(n) < p_{-(k+1)}(n)$. The generating function of $p_{-k}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{-k}(n) q^n = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^k} = \frac{1}{(q; q)_{\infty}^k} \quad (k \in \mathbb{N}). \quad (2.1)$$

Here, $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - a q^n)$ is the q -Pochhammer symbol.

Definition. We define recursively a family of polynomials $P_n(x)$. Let $\sigma(n) := \sum_{d|n} d$ denote the sum of divisors of n . Then let $P_0(x) := 1$ and

$$P_n(x) := \frac{x}{n} \sum_{k=1}^n \sigma(k) P_{n-k}(x). \quad (2.2)$$

It is known that $P_n(k) = p_{-k}(n)$ for $k \in \mathbb{N}$ (see [9] and compare the generating functions of $P_n(k)$ and $p_{-k}(n)$). Let $p_{-k}(0) := 1$. If we put $q := e^{2\pi i \tau}$ with τ in the upper complex half-plane, then

$$\sum_{n=0}^{\infty} P_n(z) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-z} \quad (z \in \mathbb{C}).$$

We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x/2(x+3)$, $P_3(x) = x/6(x^2+9x+8)$. In addition, we have $P_n(x) = x/n! \cdot \tilde{P}_n(x)$, where $\tilde{P}_n(x)$ is a monic polynomial of degree $n-1$ with positive integer coefficients. There is a useful formula for the derivatives $P'_n(x)$ for all $n \in \mathbb{N}$ [10]:

$$P'_n(x) = \sum_{k=1}^n \frac{\sigma(k)}{k} P_{n-k}(x). \quad (2.3)$$

3. Numerical Evidence for Theorems 1.3 and 1.4

3.1. The Difference Functions

In this section, we study the difference function

$$\Delta_n(x) := P_{n+1}(x) - P_n(x) \quad (3.1)$$

FIGURE 1. Roots of $\Delta_n(x)$ with positive real part. The real part is displayed

for $n \in \mathbb{N}$. The results will be utilized to prove Theorem 1.3. We first observe that

$$\lim_{x \rightarrow \infty} \Delta_n(x) = +\infty. \quad (3.2)$$

This is true since $P_n(x)$ is a polynomial of degree n of leading coefficient $1/n!$. We also deduce that $\Delta_n(0) = 0$ for $n \geq 1$. We are especially interested in the largest non-negative real root x_n of $\Delta_n(x)$ (compare Fig. 1), since $P_{n+1}(x) > P_n(x)$ for $x > x_n$.

By (3.2) and continuity for all values of x larger than this root the difference function is positive. The real roots of $\Delta_1(x)$ are $\{-1, 0\}$ and the real roots of $\Delta_2(x)$ are

$$\{-3 - \sqrt{10}, 0, -3 + \sqrt{10}\}.$$

We can see already that $\Delta_n(x)$ is not always positive, here $\Delta_2(x) < 0$ for

$$0 < x < -3 + \sqrt{10}.$$

3.2. The Multiplicative Differences Functions

Let $a, b \in \mathbb{N}$. The Bessenrodt–Ono inequality $p(a)p(b) > p(a+b)$ is always satisfied for all $a, b \geq 2$ and $a+b > 9$. Since the inequality is symmetric in a and b we assume $a \geq b$. It was also shown [5] that there is equality for

$(a, b) \in \{(6, 2), (7, 2), (4, 3)\}$. The inequality fails completely for $b = 1$ and

$$(a, b) \in \{(2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (5, 3)\},$$

while it is true for the remaining cases $(a, b) \in \{(4, 4), (5, 4), (6, 3)\}$.

The BO for 2-colored partitions is true for all $a, b \in \mathbb{N}$ except for $(a, b) = (1, 1)$, where $p_{-2}(1)p_{-2}(1) < p_{-2}(2)$. Let $a \geq b$. Then, we have equality for $(a, b) \in \{(2, 1), (3, 1)\}$. The BO for 3-colored partitions holds for all $a, b \in \mathbb{N}$ except for $(a, b) = (1, 1)$, where we have equality. If $k \geq 4$, then BO is fulfilled for all $a, b \in \mathbb{N}$ (see [6]).

Let $P_{a,b}(x) := P_a(x)P_b(x) - P_{a+b}(x)$. Then $P_{a,b}(0) = 0$, $P'_{a,b}(0) = -\sigma(a+b)/(a+b)$ and

$$\lim_{x \rightarrow \infty} P_{a,b}(x) = \infty.$$

In contrast to $\Delta_n(x)$, the polynomials $P_{a,b}(x)$ appear to have only one root $x_{a,b}$ with a positive real part. Table 1 records these roots for $1 \leq a, b \leq 10$. The root distribution in Table 1 explains all exceptions in the papers [5] and [6]. In Fig. 2 we have displayed the single positive root $x_{a,1}$ of $P_{a,1}(x)$ for $1 \leq a \leq 100$.

Conjecture. *It seems that, in general, $P_{a,1}(x)$ has exactly one positive real root (and no non-real roots) and that the limit exists and is equal to 1.*

4. Proof of Theorem 1.3

Proof of Theorem 1.3. We prove the first part of Theorem 1.3 by induction.

Claim: $P_{n+1}(x) > P_n(x)$ for all $n \in \mathbb{N}$ and $x \geq 1$. Let $n = 1$. We have $\Delta_1(x) > 0$ for all $x > 0$. Suppose that $\Delta_m(x) > 0$ is true for all real numbers $x \geq 1$ and integers $1 \leq m \leq n-1$. Using (2.3) and the induction hypothesis, we obtain for $P'_{n+1}(x)$ the strict lower bound:

$$\sum_{k=1}^n \frac{\sigma(k)}{k} P_{n+1-k}(x) \geq \sum_{k=1}^n \frac{\sigma(k)}{k} P_{n-k}(x). \quad (4.1)$$

Hence, $P'_{n+1}(x) > P'_n(x)$. Property (1.1) for the partition function provides

$$P_{n+1}(1) = p(n+1) > p(n) = P_n(1). \quad (4.2)$$

As a corollary, we obtain that $\Delta'_n(x) > 0$.

We finally prove (1.5). We have already observed $\Delta_2(x) < 0$ for $0 < x < \sqrt{10} - 3$ in Sect. 3.1. Since $\Delta_n(0) = 0$ and

$$\lim_{x \rightarrow \infty} \Delta_n(x) = +\infty$$

it is sufficient to show that $\Delta'_n(0) < 0$ for $n+1$ an odd prime. We have

$$\Delta'_n(0) = \frac{\sigma(n+1)}{n+1} - \frac{\sigma(n)}{n} = \frac{n+2}{n+1} - \frac{\sigma(n)}{n} < 0 \quad (4.3)$$

using that $\sigma(n) > n+1$.

TABLE 1. Positive real roots of $P_{a,b}(x)$

$x_{a,b}$	1	2	3	4	5	6	7	8	9	10
1	3.00	2.00	2.00	1.69	1.74	1.57	1.59	1.50	1.51	1.45
2	2.00	1.40	1.25	1.13	1.09	1.00	1.00	0.95	0.92	0.91
3	2.00	1.25	1.24	1.00	1.05	0.90	0.94	0.85	0.87	0.81
4	1.69	1.13	1.00	0.87	0.86	0.76	0.76	0.72	0.69	0.67
5	1.74	1.09	1.05	0.86	0.88	0.75	0.79	0.70	0.71	0.67
6	1.57	1.00	0.90	0.76	0.75	0.66	0.66	0.60	0.60	0.57
7	1.59	1.00	0.94	0.76	0.79	0.66	0.69	0.62	0.63	0.58
8	1.50	0.95	0.85	0.72	0.70	0.60	0.62	0.56	0.55	0.53
9	1.51	0.92	0.87	0.69	0.71	0.60	0.63	0.55	0.56	0.52
10	1.45	0.91	0.81	0.67	0.67	0.57	0.58	0.53	0.52	0.49

FIGURE 2. Positive real roots of $P_{a,1}(x)$

5. Proof of Theorem 1.4

5.1. Lehmer’s Bound

Let $\mu(n) := \frac{\pi}{6} \sqrt{24n - 1}$. Rademacher [18] proved the formula:

$$p(n) = \frac{\sqrt{12}}{24n - 1} \sum_{k=1}^N A_k^*(n) \left(\left(1 - \frac{k}{\mu}\right) e^{\mu/k} + \left(1 + \frac{k}{\mu}\right) e^{-\mu/k} \right) + R_2(n, N).$$

Here, $A_k^*(n) = \frac{1}{\sqrt{k}} A_k(n)$ are real numbers, where $A_k(n)$ is a complicated sum of $24k$ th roots of unity. Lehmer [13–15] obtained the estimate

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left(\left(\frac{N}{\mu}\right)^3 \sinh\left(\frac{\mu}{N}\right) + \frac{1}{6} - \left(\frac{N}{\mu}\right)^2 \right). \tag{5.1}$$

for all $n, N \in \mathbb{N}$. DeSalvo and Pak [8] recently utilized the case $N = 2$ and proved that $p(n)$ is log-concave for all $n > 25$. They proved two of Chen’s conjectures. For our purpose the case $N = 1$, which was studied by Bessenrodt and Ono [5], is more convenient. They obtained:

$$\frac{\sqrt{3}}{12m} \left(1 - \frac{1}{\sqrt{m}}\right) e^{\frac{\pi}{6} \sqrt{24m-1}} < P_m(1) < \frac{\sqrt{3}}{12m} \left(1 + \frac{1}{\sqrt{m}}\right) e^{\frac{\pi}{6} \sqrt{24m-1}}. \tag{5.2}$$

This is the effective estimate for $P_m(1)$ that we use throughout our proof.

5.2. Proof of Theorem 1.4 When $b = 1$

In the following, we utilize the well-known upper bound:

$$\sigma(m) \leq m(1 + \ln(m)). \quad (5.3)$$

Proposition 5.1. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. The inequality*

$$x P_n(x) > P_{n+1}(x) \quad (5.4)$$

holds for all $x > 2$ and $n \geq 2$. In the case $n = 1$ it holds for $x > 3$.

Proof. The case $n = 1$ is easy to see, since $P_{1,1}(x) = x/2(x-3)$. We prove the proposition by induction on n . Let $n = 2, 3$. Then $P_{2,1}(x) = x/3(x^2-4) > 0$ for $x > 2$ and $P_{3,1}(x) = \frac{x}{8}(x-2)(x+1)(x+7) > 0$ for $x > 2$. Suppose that $n \geq 4$ and that $P_{m,1}(x) > 0$ for $2 \leq m \leq n-1$ and $x > 2$. It is sufficient to prove that $\frac{d}{dx}P_{n,1}(x) > 0$, since we already know that $P_{n,1}(2) \geq 0$ ([6], see Theorem 1.2). The derivative of $P_{n,1}(x)$ is equal to

$$\begin{aligned} P_n(x) + \sum_{k=1}^n \frac{\sigma(k)}{k} x P_{n-k}(x) - \sum_{k=1}^{n+1} \frac{\sigma(k)}{k} P_{n+1-k}(x) \\ > P_n(x) + \frac{\sigma(n-1)}{n-1} ((x-3)x/2) - \frac{\sigma(n+1)}{n+1} \\ &\geq P_n(x) - (1 + \ln(n-1)) \frac{9}{8} - (1 + \ln(n+1)) \\ &\geq P_n(x) - \frac{17}{8} (1 + \ln(2n)). \end{aligned}$$

We recall that $P_n(x)$ has non-negative coefficients. It is, therefore, sufficient to show that

$$P_n(2) \geq \frac{17}{8} (1 + \ln(2n)) \quad (5.5)$$

for $n \geq 4$. Since $P_n(2) > P_n(1)$, we can now use (5.2). Suppose $n \geq 49$. Then

$$\begin{aligned} P_n(1) &> \frac{\sqrt{3}}{12n} \left(1 - \frac{1}{\sqrt{n}}\right) e^{\frac{\pi}{6}\sqrt{24n-1}} \\ &> \frac{\sqrt{3}}{4!12n} \left(1 - \frac{1}{\sqrt{n}}\right) \left(\frac{\pi}{6}\sqrt{24n-1}\right)^4 \\ &= \frac{\sqrt{3}}{4!12} \frac{24n-1}{n} \left(1 - \frac{1}{\sqrt{n}}\right) \left(\frac{\pi}{6}\right)^4 (24n-1) \\ &> \frac{23\sqrt{3}}{4!12} \frac{6}{7} \left(\frac{\pi}{6}\right)^4 (24n-1) =: f(n). \end{aligned}$$

For

$$n \geq 58 > \left(\frac{23\sqrt{3}}{4!12} \frac{6}{7} \left(\frac{\pi}{6}\right)^4 + \frac{17}{8} \ln(98) \right) \bigg/ \left(\frac{23\sqrt{3}}{12} \frac{6}{7} \left(\frac{\pi}{6}\right)^4 - \frac{17}{392} \right)$$

we see that $17/8(1 + \ln(2n)) \leq 17/8(\ln(98) + \frac{n}{49}) \leq f(n)$. Thus for $n \geq 58$ it holds that $P_n(x) > P_n(1) > 17/8(\ln(98) + \frac{n}{49}) \geq 17/8(1 + \ln(2n))$. It remains to check $P_n(2) > \frac{17}{8}(1 + \ln(2n))$ for $4 \leq n \leq 57$. In this case we have

$(1 + \ln(2n)) < 6$. Since $P_n(2)$ increases monotonously in n by Theorem 1.3 we also have $\frac{17}{8}(1 + \ln(2n)) < 20 = P_4(2) \leq P_n(2)$ for $4 \leq n \leq 57$. $\square$

We deduce from the proof of Proposition 5.1 the following property.

Corollary 5.2. *Let $n \geq 2$ and $x > 2$ or $n \geq 4$ and $x \geq 1$. Then*

$$P_n(x) - (1 + \ln(2n)) > 0. \quad (5.6)$$

5.3. Proof of Theorem 1.4 When $b > 1$

Proof of Theorem 1.4. We show (1.6) by induction on $n = a + b$. For $n = a + b = 3$ we have $P_1(x)P_2(x) - P_3(x) = \frac{x}{3}(x^2 - 4) > 0$ for all $x > 2$.

Suppose $n \geq 4$ and $P_A(x)P_B(x) > P_{A+B}(x)$ for all $3 \leq A+B \leq n-1 = a+b-1$ and $x > 2$. Without loss of generality we assume $a \geq b \geq 2$. (The case $b = 1$ was proved in Proposition 5.1.)

We have $P_a(2)P_b(2) \geq P_{a+b}(2)$ for $a+b \geq 3$ by Theorem 1.2 of [6]. If we can now show that

$$\frac{d}{dx}(P_a(x)P_b(x)) > P'_{a+b}(x) \quad (5.7)$$

the proof is completed, as this implies $P_a(x)P_b(x) > P_{a+b}(x)$ for all $x > 2$.

Note that $P_A(x)P_0(x) = P_A(x)$. Thus,

$$\begin{aligned} & P'_a(x)P_b(x) + P_a(x)P'_b(x) - P'_{a+b}(x) \\ &= \sum_{k=1}^a \frac{\sigma(k)}{k} P_{a-k}(x)P_b(x) + P_a(x) \sum_{k=1}^b \frac{\sigma(k)}{k} P_{b-k}(x) - \sum_{k=1}^{a+b} \frac{\sigma(k)}{k} P_{a+b-k}(x) \\ &> \sum_{k=1}^a \frac{\sigma(k)}{k} P_{a+b-k}(x) + \sum_{k=1}^b \frac{\sigma(k)}{k} P_{a+b-k}(x) - \sum_{k=1}^{a+b} \frac{\sigma(k)}{k} P_{a+b-k}(x) \\ &= \sum_{k=1}^b \frac{\sigma(k)}{k} P_{a+b-k}(x) - \frac{\sigma(k+a)}{k+a} P_{b-k}(x) \\ &\geq \sum_{k=1}^b P_{a+b-k}(x) - (1 + \ln(2a)) P_{b-k}(x). \end{aligned}$$

We now consider

$$P_{a+b-k}(x) - (1 + \ln(2a)) P_{b-k}(x) \quad (5.8)$$

for each k separately. From Theorem 1.3, we know that $P_{a+b-k}(x)$ increases faster than $P_{b-k}(x)$ for $x \geq 1$. Hence, to show that (5.8) is positive, it is enough to show this for $1 \leq x \leq 2$.

Using (5.2) for $k < b$

$$\begin{aligned} & P_{a+b-k}(1) - (1 + \ln(2a)) P_{b-k}(1) \\ &> \frac{\sqrt{3}}{12(a+b-k)} \left(1 - \frac{1}{\sqrt{a+b-k}} \right) e^{\frac{\pi}{6}\sqrt{24(a+b-k)-1}} \\ &\quad - (1 + \ln(2a)) \frac{\sqrt{3}}{12(b-k)} \left(1 + \frac{1}{\sqrt{b-k}} \right) e^{\frac{\pi}{6}\sqrt{24(b-k)-1}} \end{aligned}$$

and the last is positive if and only if

$$e^{\frac{\pi}{6}(\sqrt{24(a+b-k)-1}-\sqrt{24(b-k)-1})} > (1 + \ln(2a)) \frac{a+b-k}{b-k} \frac{1 + \frac{1}{\sqrt{b-k}}}{1 - \frac{1}{\sqrt{a+b-k}}}. \quad (5.9)$$

Now

$$\begin{aligned} & \frac{\sqrt{24(a+b-k)-1} - \sqrt{24(b-k)-1}}{24a} \\ &= \frac{24a}{\sqrt{24(a+b-k)-1} + \sqrt{24(b-k)-1}} \\ &\geq \frac{24a}{\sqrt{24(2a-1)-1} + \sqrt{24(a-1)-1}} \\ &> \frac{24a}{7\sqrt{a} + 5\sqrt{a}} = 2\sqrt{a}. \end{aligned}$$

On the other hand

$$(1 + \ln(2a)) \frac{a+b-k}{b-k} \frac{1 + \frac{1}{\sqrt{b-k}}}{1 - \frac{1}{\sqrt{a+b-k}}} < (1 + \ln(2a)) (1+a) \frac{2}{1 - \frac{1}{\sqrt{a}}}.$$

If a fulfills

$$e^{\pi\sqrt{a}/3} > (1 + \ln(2a)) (1+a) \frac{2}{1 - \frac{1}{\sqrt{a}}} \quad (5.10)$$

then a also fulfills (5.9). We show (5.10) for $a \geq 94$ in Lemma 5.3 below. This implies that (5.8) is positive for $x = 1$. Theorem 1.3 then implies that (5.8) is positive for $x \geq 1$.

We used PARI/GP to check that Eq. (5.8) with $x = 2$ is positive for $1 \leq k < b \leq a \leq 93$. Again Theorem 1.3 implies (5.8) is positive for $x \geq 2$.

What remains to be considered is (5.8) for $k = b$. This follows from Corollary 5.2 setting $n = a$.

Thus, for all $a \geq b \geq k \geq 1$ the value of (5.8) is positive. Hence, also

$$P'_a(x) P_b(x) + P_a(x) P'_b(x) > P'_{a+b}(x)$$

for $x > 2$. This completes the induction step, as explained in the beginning of the proof. $\square$

In the following, we provide a proof of (5.10) for $a \geq 94$. With more work and tighter bounds, it is possible to show (5.10) for $a \geq 34$.

Lemma 5.3. *For $a \geq 94$, Eq. (5.10) holds, that is*

$$e^{\pi\sqrt{a}/3} > (1 + \ln(2a)) (1+a) \frac{2}{1 - \frac{1}{\sqrt{a}}}.$$

Proof. Since $a > 0$

$$e^{\pi\sqrt{a}/3} > \frac{1}{6!} \left(\frac{\pi}{3} \sqrt{a} \right)^6 = \frac{1}{6!} \left(\frac{\pi}{3} \right)^6 a^3.$$

Since $a \geq 81$, we have

$$(1 + \ln(2a)) (1+a) \frac{2}{1 - \frac{1}{\sqrt{a}}} \leq \left(\frac{a}{81} + \ln(162) \right) (1+a) \frac{18}{8}.$$

Thus, (5.10) is fulfilled if

$$\frac{1}{6!} \left(\frac{\pi}{3}\right)^6 a \geq \left(\frac{1}{81} + \frac{\ln(162)}{81}\right) \left(\frac{1}{81} + 1\right) \frac{18}{8} \geq \left(\frac{1}{81} + \frac{\ln(162)}{a}\right) \left(\frac{1}{a} + 1\right) \frac{9}{4}$$

and this is fulfilled for $a \geq 94 > \left(\left(\frac{1}{81} + \frac{\ln(162)}{81}\right) \left(\frac{1}{81} + 1\right) \frac{9}{4}\right) / \left(\frac{1}{6!} \left(\frac{\pi}{3}\right)^6\right)$. $\square$

Acknowledgements

Open access funding provided by Projekt DEAL. We thank the RWTH Aachen University and the Graduate School: Experimental and constructive algebra for their support and the two referees for carefully reading the manuscript and their excellent comments.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] A. A. Alanazi, S. M. Gagola III, A. O. Munagi: *Combinatorial proof of a partition inequality of Bessenrodt–Ono*. Ann. Comb. **21** (2017), 331–337.
- [2] G. E. Andrews: *The Theory of Partitions*. Cambridge University Press, Cambridge (1998).
- [3] G. E. Andrews, K. Eriksson: *Integer Partitions*. Cambridge University Press, Cambridge (2004).
- [4] O. Beckwith, C. Bessenrodt: *Multiplicative properties of the number of k -regular partitions*. Ann. Comb. **20** No. 2 (2016), 231–250.
- [5] C. Bessenrodt, K. Ono: *Maximal multiplicative properties of partitions*. Ann. Comb. **20** No. 1 (2016), 59–64.
- [6] S. Chern, S. Fu, D. Tang: *Some inequalities for k -colored partition functions*. Ramanujan J. **46** (2018), 713–725.

- [7] M. Locus Dawsey, R. Masri: *Effective bounds for the Andrews spt-function*. Forum Math. **31**(3), 743–767 (2019)
- [8] S. DeSalvo, I. Pak: *Log-concavity of the partition function*. Ramanujan J. **38** (2015), 61–73.
- [9] B. Heim, F. Luca, M. Neuhauser: *On cyclotomic factors of polynomials related to modular forms*. Ramanujan J. **48** (2019), 445–458.
- [10] B. Heim, M. Neuhauser: *Polynomials related to powers of the Dedekind eta function*. Integers **18** Paper No. A97 (2018).
- [11] B. Heim, M. Neuhauser: *Variants of a partition inequality of Bessenrodt–Ono*. Research in Number Theory (2019), 5: 32.
- [12] E. Hou, M. Jagadeesan: *Dyson’s partition ranks and their multiplicative extension*. Ramanujan J. **45** No. 3 (2018), 817–839.
- [13] D. H. Lehmer: *On the Hardy–Ramanujan series for the partition function*. J. Lond. Math. Soc. **12** (1937), 171–176.
- [14] D. H. Lehmer: *On the series for the partition function*. Trans. Amer. Math. Soc. **43** (1938), 271–295.
- [15] D. H. Lehmer: *On the remainders and convergence of the series for the partition functions*. Trans. Amer. Math. Soc. **46** (1939), 362–373.
- [16] J. Males: *Asymptotic equidistribution and convexity for partition ranks*. Ramanujan J. (2020). <https://doi.org/10.1007/s11139-019-00202-8>
- [17] K. Ono: *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -series*. Conference Board of Mathematical Sciences **102** (2003).
- [18] H. Rademacher: *A convergent series for the partition function $p(n)$* . Proc. Natl. Acad. Sci. USA **23** (1937), 78–84.

Bernhard Heim, Markus Neuhauser and Robert Tröger
 Faculty of Science
 German University of Technology in Oman (GUtech)
 PO Box 1816
 Athaibah PC 130
 Sultanate of Oman
 e-mail: markus.neuhauser@mathA.rwth-aachen.de

Robert Tröger
 e-mail: robert@silva-troeger.de

and

Faculty of Mathematics, Computer Science, and Natural Sciences
RWTH Aachen University
52056 Aachen
Germany
e-mail: bernhard.heim@rwth-aachen.de

Received: 23 October 2019.

Accepted: 11 August 2020.