Ovals in $\mathbb{Z}_{2p}^2$

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Abstract. By an oval in $\mathbb{Z}_{2p}^2$, p odd prime, we mean a set of $2p+2$ points, such that no three of them are on a line. It is shown that ovals in $\mathbb{Z}_{2p}^2$ only exist for $p = 3, 5$ and they are unique up to an isomorphism.

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1. Introduction

A k -arc is a set of k points, such that no three of them are collinear. Arcs are well studied in projective geometry, see, e.g., [1, 9] and [2] for further references. Recall that any non-singular conic of $\text{PG}(2, q)$ is a $(q+1)$ -arc. If K is any k -arc of $\text{PG}(2, q)$ with q odd, then $k \leq q+1$. A $(q+1)$ -arc is called an *oval*. A famous theorem of Segre [9] tells us that for q odd, every oval of $\text{PG}(2, p)$ is a non-singular conic.

Many authors continue to study the classical problem in the context of Hjelmslev geometry, see, e.g., [3, 4, 8] and [6] for the definition of the abstract Hjelmslev plane. In this article, we consider similar questions in $\mathbb{Z}_n^2$. This modified problem is still interesting and it was investigated in [5, 7] and [10]. If $n = p$ is prime, the resulting space is just $\text{AG}(2, p)$. The correspondence between projective and affine planes leads to the following facts: The maximum size of an arc in $\mathbb{Z}_p^2$ is $p+1$; every $(p+1)$ -arc in $\mathbb{Z}_p^2$ is a non-singular conic.

The maximum size of a cap in $\mathbb{Z}_{2p}^2$, p odd prime, is $2p+2$ (see Lemma 4.1 in [7] and Theorem 3.1 in [10]). We will call a $(2p+2)$ -arc in $\mathbb{Z}_{2p}^2$ an *oval* in $\mathbb{Z}_{2p}^2$. In this article, we completely solve the problem of the existence and the uniqueness of ovals in $\mathbb{Z}_{2p}^2$.

We define a *line* in $\mathbb{Z}_n^2$ to be a subset of $\mathbb{Z}_n^2$ of the form:

$$\{(x; y) : ax + by + c = 0\},$$

where $\gcd(a; b; n) = 1$. This line is denoted by $\ell_{[a; b; c]}$.

By an automorphism of $\mathbb{Z}_n^2$, we mean a mapping from $\mathbb{Z}_n^2$ to $\mathbb{Z}_n^2$ which preserves arcs. Let X, Y be arcs in $\mathbb{Z}_n^2$. We say that X and Y are isomorphic if there exists an automorphism of $\mathbb{Z}_n^2$ mapping X to Y .

2. The General Case

Due to the Chinese remainder theorem, we have the following.

Lemma 2.1. *Let p be an odd prime, $\phi_2 : \mathbb{Z}_{2p}^2 \rightarrow \mathbb{Z}_2^2$, $\phi_p : \mathbb{Z}_{2p}^2 \rightarrow \mathbb{Z}_p^2$ be reduction maps. Then, any three points $x, y, z \in \mathbb{Z}_{2p}^2$ are collinear if and only if $\phi_2(x), \phi_2(y), \phi_2(z) \in \mathbb{Z}_2^2$ and $\phi_p(x), \phi_p(y), \phi_p(z) \in \mathbb{Z}_p^2$ are collinear.*

From now on, we will use the notation of the previous Lemma.

Lemma 2.2. *Let p be an odd prime. Let σ be an arbitrary permutation of $\mathbb{Z}_2^2$. Then, there exists an automorphism F of $\mathbb{Z}_{2p}^2$, such that $\phi_2 \circ F = \sigma \circ \phi_2$ and $\phi_p \circ F = Id \circ \phi_p$.*

Proof. Let F_A and F_B be linear transformations determined by matrices $A = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$ and $B = \begin{pmatrix} p+1 & p \\ p & p+1 \end{pmatrix}$, respectively. Denote by $t_{[0,p]}$ the translation by a vector $[0, p]$. Note that the group of permutations of the set $\mathbb{Z}_2^2$ is generated by transpositions $\sigma_1 = ((0,0), (0,1))$, $\sigma_2 = ((0,1), (1,0))$, $\sigma_3 = ((1,0), (1,1))$.

One can verify by a straightforward calculation that $\phi_2 \circ F_A \circ t_{[0,p]} = \sigma_1 \circ \phi_2$, $\phi_2 \circ F_B = \sigma_2 \circ \phi_2$ and $\phi_2 \circ F_A = \sigma_3 \circ \phi_2$. $\square$

Lemma 2.3. *Let X be an oval in $\mathbb{Z}_{2p}$. Then, the following holds.*

- (1) $|\phi_2(X)| = 4$.
- (2) $|\phi_2^{-1}(a) \cap X| = \frac{p+1}{2}$ for every $a \in \mathbb{Z}_2^2$.
- (3) $|\phi_p(X)| = 2p + 2$.

Proof. To prove (1), suppose that $|\phi_2(X)| < 4$. Note that in this case, there exists a line l in $\mathbb{Z}_2^2$, such that $|\phi_2^{-1}(l) \cap X| > p + 1$. As a consequence of Lemma 2.1 together with the fact that the maximum size of an arc in $\mathbb{Z}_p$ is $p + 1$, we obtain a contradiction with the assumption that X is an arc. (2) follows immediately from (1) and Lemma 2.2. To prove (3), suppose, to the contrary, that $\phi_p(a) = \phi_p(b)$ for some $a, b \in X$. By (2), there exists a $c \in X$, such that $\phi_2(a) = \phi_2(c)$, so there exists a c , such that $\phi_2(a), \phi_2(b)$ and $\phi_2(c)$ are collinear. Moreover, it is clear that $\phi_p(a), \phi_p(b)$, and $\phi_p(c)$ are collinear. Therefore, a, b, c is a collinear triple by Lemma 2.1, a contradiction. $\square$

Theorem 2.4. *There is no oval in $\mathbb{Z}_{2p}$ for $p \geq 7$.*

Proof. It is shown in [10] by computer calculations that the maximum size of an arc in $\mathbb{Z}_{14}$ is 12. Assume, to the contrary, that there is an oval X in $\mathbb{Z}_{2p}$, where $p > 7$ is an odd prime. By Lemma 2.3 (1), we have $\phi_2(X) = \mathbb{Z}_2^2$. Let $a \in \mathbb{Z}_2^2$. By Lemma 2.3 (2), (3), $|\phi_p(\phi_2^{-1}(a) \cap X)| > 4$. Denote by ℓ a line through a in $\mathbb{Z}_2^2$. By Lemma 2.3 (2) and (3) and Lemma 2.1, $\phi_p(\phi_2^{-1}(\ell) \cap X)$ is a $(p + 1)$ -arc in $\mathbb{Z}_p^2$, i.e., $\phi_p(\phi_2^{-1}(\ell) \cap X)$ is a non-singular conic. Note that there are exactly three distinct lines through a in $\mathbb{Z}_2^2$. Therefore, we have three

FIGURE 1. An oval in $\mathbb{Z}_6^2$

FIGURE 2. An oval in $\mathbb{Z}_{10}^2$

distinct non-singular conics through $\phi_p(\phi_2^{-1}(a) \cap X)$ in $\mathbb{Z}_p$. However, this is a contradiction to the fact that there exists a unique non-singular conic through five points, no three collinear. $\square$

3. The Cases $p = 3, 5$

The existence of ovals in $\mathbb{Z}_6^2$ and $\mathbb{Z}_{10}^2$ was proved in [10], where appropriate examples were given, see also Figs. 1 and 2. Therefore, it remains to show the uniqueness.

First, we introduce some notation. Let $X \subset \mathbb{Z}_{2p}^2$ be an oval. For $(i, j) \in \mathbb{Z}_2^2$, let $X_{ij} = \phi_p(\phi_2^{-1}(i, j) \cap X)$. A d -secant of K is a line l , such that $|l \cap K| = d$. Denote by $\mathcal{S}_K^d$ the set of d -secants of K .

The following Lemma is an immediate consequence of Lemma 2.1.

Lemma 3.1. *Let X be an oval in $\mathbb{Z}_{2p}^2$, $a, b \in X$. If $\phi_2(a) = \phi_2(b)$, then the line through $\phi_p(a)$ and $\phi_p(b)$ is a 2-secant of $\phi_p(X)$.*

Theorem 3.2. *An oval in $\mathbb{Z}_6$ is unique up to an isomorphism.*

Proof. Let Y be an oval in $\mathbb{Z}_6$ given in Fig. 1. This means that:

$$Y = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 1), (2, 2), (3, 5), (5, 3)\}.$$

Then, $|\phi_3(Y)| = 8$, $Y_{00} = \{(0, 0), (2, 2)\}$, $Y_{01} = \{(0, 1), (2, 1)\}$, $Y_{10} = \{(1, 0), (1, 2)\}$, $Y_{11} = \{(0, 2), (2, 0)\}$. Let X be another oval in $\mathbb{Z}_6^2$. We will show that X is isomorphic to Y . By Lemma 2.3 (3), $|\phi_3(X)| = 8$. Let $(i, j) \in \mathbb{Z}_2^2$. Using Lemma 2.3(2) and (3), we obtain $|X_{ij}| = 2$. From Lemma 3.1, it follows that X_{ij} is a 2-secant of $\phi_3(X)$. Since the translations are automorphisms of $\mathbb{Z}_6^2$, we can assume that $(1, 1) \notin \phi_3(X)$. This means that the points of X_{ij} lie on one of the 4 lines through $(1, 1)$. Thus, there exists $(i', j') \in \mathbb{Z}_2^2$, such that $X_{ij} = Y_{i'j'}$. Consequently, we obtain the permutation σ of $\mathbb{Z}_2^2$ given by $\sigma(i, j) = (i', j')$. The proof is completed by Lemma 2.2. $\square$

The proof of Theorem 3.6 is straightforward but tedious. Therefore, we collect some useful Lemmas.

Lemma 3.3. *Let X, Y be ovals in $\mathbb{Z}_{10}^2$. If $\phi_5(X)$ and $\phi_5(Y)$ are isomorphic, then X and Y are isomorphic.*

Proof. First, we will show that for every automorphism f of $\mathbb{Z}_5^2$, there exists an automorphism F of $\mathbb{Z}_{10}^2$, such that $\phi_5 \circ F = f \circ \phi_5$. If f is a translation by a vector, then we may take $F = f$. Obviously, F satisfies the required condition. Let f be an automorphism arising from multiplication by $A \in GL(2, \mathbb{Z}_5)$. It can be verified that at least one of $A : A + \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, A + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}, A + \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}, A + \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}, A + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is irreducible in $\mathbb{Z}_{10}$, call it $\tilde{A}$. Then, F given by multiplication by $\tilde{A}$ satisfies the required condition.

Next, we claim that $f(X_{ij}) = Y_{i'j'}$. Suppose, to the contrary, that $f(X_{ij})$ has a nonempty intersection with at least two of $Y_{00}, Y_{01}, Y_{10}, Y_{11}$. Recall that by Lemma 2.1, there exist six conics $C_i, i = 1, \dots, 6$, in Y corresponding to six lines in $\mathbb{Z}_2^2$. Consider a conic C containing X_{ij} . Then, $f(C)$ is the conic different from $C_i, i = 1, \dots, 6$. Moreover, there exists C_i for some i , such that $|C \cup C_i| \geq 5$. However, this contradicts the fact that there exists a unique non-singular conic through five points, no three collinear.

Finally, consider the permutation σ of $\mathbb{Z}_2^2$ given by $\sigma(i, j) = (i', j')$. Denote by F_1 an automorphism described in Lemma 2.2. Then, the composition of F and F_1 maps X onto Y . $\square$

Lemma 3.4. *Let $X \subset \mathbb{Z}_{10}^2$ be an oval, $F(x, y) = (-x, y)$. If $F(X_{00}) = X_{01}$ and $F(X_{10}) = X_{00}$, then $\ell_{[1,0,0]} \notin \mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2$.*

Proof. Since X_{10} is an arc, we have $\ell_{[1,0,0]} \in \mathcal{S}_{X_{10}}^k$ for $k = 0, 1, 2$. Suppose, to the contrary, that $\ell_{[1,0,0]} \in \mathcal{S}_{X_{10}}^2$, i.e., $|X_{10} \cap \ell_{[1,0,0]}| = 2$. Obviously $|X_{10} \cap F(X_{10})| = 2$ and $X_{10} \neq F(X_{10})$. We obtain ovals, i.e., non-singular conics, $C_1 = X_{00} \cup X_{10}$ and $C_2 = F(X_{01} \cup X_{10}) = X_{00} \cup F(X_{10})$ with $|C_1 \cap C_2| = 5$. However, this contradicts the fact that there exists a unique non-singular conic through five points, no three collinear. Similarly, $\ell_{[1,0,0]} \notin \mathcal{S}_{X_{11}}^k$. $\square$

Lemma 3.5. *Let $X \subset \mathbb{Z}_{10}^2$ be an oval. Then, the following holds:*

- (1) $\mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2 \subset \mathcal{S}_{X_{00} \cup X_{01}}^0$;
- (2) $X_{ij} = \{l_1 \cap l_2, l_1 \cap l_3, l_2 \cap l_3\}$, where $\mathcal{S}_{X_{ij}}^2 = \{l_1, l_2, l_3\}$.

Proof. (1) is a direct consequence of Lemma 2.1. (2) follows from the definition of $\mathcal{S}_{X_{ij}}^2$ together with the fact that both X_{ij} and $\mathcal{S}_{X_{ij}}^2$ have size 3. $\square$

We are now ready for the proof of Theorem 3.6.

Theorem 3.6. *An oval in $\mathbb{Z}_{10}$ is unique up to an isomorphism.*

Proof. Let Y be an oval in $\mathbb{Z}_{10}$ given in Fig. 2. This means that:

$$Y = \{(0, 0), (1, 0), (0, 1), (1, 1), (4, 2), (7, 2), (5, 4), (6, 4), (5, 7), (6, 7), (4, 9), (7, 9)\}.$$

Let X be another arc of size 12 in $\mathbb{Z}_{10}^2$. We will show that X is isomorphic to Y . Recall that $X_{00} \cup X_{01}$ is a non-singular conic C in $\mathbb{Z}_5^2$. Thus, C can be transformed by some automorphism of $\mathbb{Z}_5^2$ to $C_1 = \{(x, y) : x^2 + 2y^2 = 1\}$. By Lemma 3.3, we can write $X_{00} \cup X_{01} = \{(1, 0), (2, 1), (2, 4), (4, 0), (3, 1), (3, 4)\}$.

It is straightforward to check that in this case, we have:

$$\mathcal{S}_{X_{00} \cup X_{01}}^0 = \{\ell_{[1;0;0]}, \ell_{[1;1;0]}, \ell_{[4;1;0]}, \ell_{[0;1;2]}, \ell_{[0;1;3]}, \ell_{[3;1;1]}, \ell_{[3;1;4]}, \ell_{[2;1;1]}, \ell_{[2;1;4]}\}. \quad (3.0.1)$$

We need to split $X_{00} \cup X_{01}$ into two subsets X_{00} , X_{01} . By Lemma 2.2, we may assume that $(1, 0) \in X_{00}$. There are ten cases for X_{00} . To reduce the number of cases, we use Lemma 3.3 and the automorphism G of $\mathbb{Z}_5^2$ given by $G(x, y) = (3x + 3y, x + 3y)$. A straightforward computation shows that it remains to consider the following three cases: (1) $X_{00} = \{(1, 0), (3, 1), (3, 4)\}$, (2) $X_{00} = \{(1, 0), (2, 1), (2, 4)\}$, (3) $X_{00} = \{(1, 0), (2, 1), (3, 4)\}$.

We shall consider the cases (1) and (2) separately, and then, we will show that cases 1 and 3 give isomorphic ovals.

Case 1 $X_{00} = \{(1, 0), (3, 1), (3, 4)\}$. Then, $X_{01} = \{(4, 0), (2, 1), (2, 4)\}$. It is straightforward to check that $\mathcal{S}_{X_{00}}^2 = \{\ell_{[2;1;3]}, \ell_{[3;1;2]}, \ell_{[1;0;2]}\}$ and $\mathcal{S}_{X_{01}}^2 = \{\ell_{[3;1;3]}, \ell_{[2;1;2]}, \ell_{[1;0;3]}\}$. Let $\langle \mathcal{S}_{X_{ij}}^2 \rangle = \{p \in \ell \mid \ell \in \mathcal{S}_{X_{ij}}^2\}$. Denote by $\overline{\langle \mathcal{S}_{X_{ij}}^2 \rangle}$ the complement of $\langle \mathcal{S}_{X_{ij}}^2 \rangle$. Next, one can verify that $\overline{\langle \mathcal{S}_{X_{00}}^2 \rangle} \cap \overline{\langle \mathcal{S}_{X_{01}}^2 \rangle} = \{(0, 0), (0, 1), (0, 4), (1, 2), (1, 3), (4, 2), (4, 3)\}$. It follows from Lemma 3.1 that $X_{10} \cup X_{11} \subset \overline{\langle \mathcal{S}_{X_{00}}^2 \rangle} \cap \overline{\langle \mathcal{S}_{X_{01}}^2 \rangle}$. Furthermore, note that $(0, 0)$ is only one point of $\ell_{[4;1;0]}$ in $\overline{\langle \mathcal{S}_{X_{00}}^2 \rangle} \cap \overline{\langle \mathcal{S}_{X_{01}}^2 \rangle}$. Thus, $\ell_{[4;1;0]} \notin \mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2$. By the same argument, $\ell_{[1;1;0]} \notin \mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2$. Lemma 3.4 implies that $\ell_{[1;0;0]} \notin \mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2$. The above considerations together with Lemma 3.5 (1) and Eq. (3.0.1) imply that $\mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2 = \{\ell_{[0;1;2]}, \ell_{[0;1;3]}, \ell_{[3;1;1]}, \ell_{[3;1;4]}, \ell_{[2;1;1]}, \ell_{[2;1;4]}\}$. (We use the fact that $|\mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2| = 6$.) Lemma 3.5 (2) and simple calculations show that $(0, 0) \notin X_{10} \cup X_{11}$. Since $|X_{10} \cup X_{11}| = 6$, $X_{10} \cup X_{11} = \{(0, 1), (0, 4), (1, 2), (1, 3), (4, 2), (4, 3)\}$. Now, we need to split $X_{10} \cup X_{11}$ into two subsets X_{10} , X_{11} . By Lemma 2.2, we may assume that $(0, 1) \in X_{10}$. It can be verified that $\{(0, 1)\} = \ell_{[3;1;4]} \cap \ell_{[2;1;4]}$. By Lemma 3.5 (2), $\ell_{[3;1;4]}, \ell_{[2;1;4]} \in \mathcal{S}_{X_{10}}^2$. One can verify that $\ell_{[3;1;4]} \cap (X_{10} \cup X_{11}) = \{(0, 1), (1, 3)\}$ and $\ell_{[2;1;4]} \cap (X_{10} \cup X_{11}) = \{(0, 1), (4, 3)\}$. Hence, $X_{10} = \{(0, 1), (1, 3), (4, 3)\}$, and so, $X_{11} = \{(0, 4), (1, 2), (4, 2)\}$.

Thus:

$$X = \{(6, 0), (8, 6), (8, 4), (4, 5), (2, 1), (2, 9), (5, 6), (1, 8), (9, 8), (5, 9), (1, 7), (9, 7)\}.$$

Finally, one can check that $H(X) = Y$, where $H(x, y) = (3x + 7y + 8, 4x + 9y + 3)$.

Case 2 $X_{00} = \{(1, 0), (2, 1), (2, 4)\}$. Then, $X_{01} = \{(4, 0), (3, 1), (3, 4)\}$. In this case, $\mathcal{S}_{X_{00}}^2 = \{\ell_{[4;1;1]}, \ell_{[1;0;3]}, \ell_{[1;1;4]}\}$ and $\mathcal{S}_{X_{01}}^2 = \{\ell_{[4;1;4]}, \ell_{[1;0;2]}, \ell_{[1;1;1]}\}$. With the notation of the above case, it can be verified that:

$$\overline{\langle \mathcal{S}_{X_{00}}^2 \rangle} \cap \overline{\langle \mathcal{S}_{X_{01}}^2 \rangle} = \{(0, 0), (0, 2), (0, 3), (1, 1), (1, 4), (4, 1), (4, 4)\}.$$

Hence, $X_{10} \cup X_{11} \subset \{(0, 0), (0, 2), (0, 3), (1, 1), (1, 4), (4, 1), (4, 4)\}$, see Lemma 3.1. By the same arguments as in the previous case, $\ell_{[1;0;0]}, \ell_{[0;1;3]}, \ell_{[0;1;2]} \notin \mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2$ and, therefore, $\mathcal{S}_{X_{10}}^2 \cup \mathcal{S}_{X_{11}}^2 = \{\ell_{[1;1;0]}, \ell_{[4;1;0]}, \ell_{[3;1;1]}, \ell_{[3;1;4]}, \ell_{[2;1;1]}, \ell_{[2;1;4]}\}$. It can be verified that $\ell_{[3;1;4]} \cap \ell_{[2;1;4]} = \{(0, 1)\}$. However, $(0, 1) \notin X_{10} \cup X_{11}$. This contradicts Lemma 3.5 (2). Hence, this case does not occur.

Finally, we will show that cases 1 and 3 give isomorphic ovals. For $(i, j) \in \mathbb{Z}_2^2$ and $k = 1, 3$, denote by X_{ij}^k the set X_{ij} from Case k . With this notation, $X_{00}^3 = \{(1, 0), (2, 1), (3, 4)\}$ and $X_{01}^3 = \{(4, 0), (2, 4), (3, 1)\}$. Consider f given by $f(x, y) = (3x + 2y, 3x + 3y + 2)$. One easily verifies that $f(X_{11}^1) = X_{00}^3$ and $f(X_{01}^1) = X_{01}^3$. The result now follows from Lemma 3.3. The proof is complete. $\square$

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