Matching Numbers and the Regularity of the Rees Algebra of an Edge Ideal

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Abstract. The regularity $\text{reg } R(I(G))$ of the Rees ring $R(I(G))$ of the edge ideal $I(G)$ of a finite simple graph G is studied. It is shown that, if $R(I(G))$ is normal, one has $\text{mat}(G) \leq \text{reg } R(I(G)) \leq \text{mat}(G) + 1$, where $\text{mat}(G)$ is the matching number of G . In general, the induced matching number is a lower bound for the regularity, which can be shown by applying the squarefree divisor complex.

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Introduction

In the study of powers of monomial ideals, the Rees algebra plays an important role. In the present paper, we focus on the Rees algebra of the edge ideal of a finite simple graph.

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K in the variables $x_1, \dots, x_n$. Furthermore, let G be a finite simple graph on the vertex set $V(G) = [n]$ and edge set $E(G)$. We assume that G has no isolated vertex. The edge ideal $I = I(G)$ of G is the monomial ideal in S generated by the monomials $x_i x_j$ with $\{i, j\} \in E(G)$, and the edge ring $K[G]$ is the K -algebra generated by the monomials $x_i x_j \in I$. The Rees algebra $R(I) = \bigoplus_{s \geq 0} I^s$ of I may be viewed as the edge ring of the graph G^* with $V(G^*) = [n + 1]$ and $E(G^*) = E(G) \cup \{\{i, n + 1\} : i \in V(G)\}$, see [5, p. 192]. In other words, G^* is just the original graph G with a cone vertex appended. We are interested in bounding the regularity of $R(I)$, because this provides information about the regularity of the powers of I . In our situation, computing regularity of $R(I)$ amounts to computing the regularity of the edge ring of graphs of the form G^* . This class of algebras are particular classes of toric rings. To our knowledge,

there are essentially two methods available to compute the regularity of a toric ring A . The first method, which can always be applied, is to compute the multigraded Betti numbers of A using the *squarefree divisor complex* (see [2, Proposition 1.1] and [6, Theorem 3.28]). In concrete cases, as discussed in Sect. 1, this allows one to give lower bounds for the regularity, but, in general, it is hard to use. The second method can be applied, if A is Cohen–Macaulay, in which case one needs to compute the a -invariant of A . If in addition, A is normal, then following Danilov and Stanley [1, Theorem 6.3.5], the canonical module can be computed which in particular gives us the a -invariant of A .

Let A be a toric ring generated by monomials in S . A K -subalgebra B of A is called a *combinatorial pure* subalgebra of A if there exists a subset $T \subset [n]$ such that $B = A \cap K[x_i : i \in T]$. In Sect. 1, we recall squarefree divisor complexes and use them to prove that if $B \subset A$ is a combinatorial pure subring of A , then $\beta_{i,j}(B) \leq \beta_{i,j}(A)$ for all i and j . Here the $\beta_{i,j}(M)$ denote the graded Betti numbers of a graded module M . This result implies in particular that if I is a monomial ideal generated in a single degree, then $\operatorname{reg} R(I) \geq \operatorname{reg} F(I)$, where $F(I)$ is the fiber cone of I . We also use squarefree divisor complexes to give lower bounds on the regularity of the Rees algebra of $I(G)$, when G is a disjoint union of edges. These results will be used in the next section.

Now, we devote Sect. 2 to finding an upper bound and a lower bound for the regularity of the Rees algebra $R(I(G))$ of the edge ideal $I(G)$ of a finite simple graph G in terms of the matching number of G and the induced matching number of G . Recall that a *matching* of G is a subset $M \subset E(G)$ such that $e \cap e' = \emptyset$ for all e and e' belonging to M with $e \neq e'$. A matching M of G is called *perfect* if, for each $i \in [n]$, there is $e \in M$ with $i \in e$. An *induced matching* of G is a matching M of G such that if e and e' belong to M with $e \neq e'$, then there is no edge $f \in E(G)$ with $e \cap f \neq \emptyset$ and $e' \cap f \neq \emptyset$. The maximal cardinality of matchings of G is called the *matching number* of G and is denoted by $\operatorname{mat}(G)$. The *induced matching number* of G , denoted by $\operatorname{indmat}(G)$, is the maximal cardinality of induced matchings of G .

It is known [7, Corollary 2.3] that $K[G]$ is normal if and only if each connected component G' of G satisfies the *odd cycle condition*, which says that if C and C' are odd cycles of G' with $V(C) \cap V(C') = \emptyset$, then there are $i \in V(C)$ and $j \in V(C')$ with $\{i, j\} \in E(G')$. In particular, if G is bipartite, then $K[G]$ is normal.

In [4, Proposition 4.5], the a -invariant of $R(I(G))$ is computed in terms of the independence number of the graph G , if G is bipartite. This result can be rephrased by saying that $\operatorname{reg} R(I(G)) = \operatorname{mat}(G)$ if G is bipartite, as observed by Cid-Ruiz in [3, Theorem 4.2].

Our main result (Theorem 2.2) says that, if $R(I(G))$ is normal, then

$$\operatorname{mat}(G) \leq \operatorname{reg} R(I(G)) \leq \operatorname{mat}(G) + 1.$$

Furthermore, if G has a perfect matching, then $\operatorname{reg} R(I(G)) = \operatorname{mat}(G)$. Our proof heavily depends on the theory of normal edge polytopes created in [7] as well as on the result [8, Theorem 3.3] which says that if G' is a subgraph of G

and if each of $K[G']$ and $K[G]$ is normal, then one has $\text{reg } K[G'] \leq \text{reg } K[G]$. Even though $K[G]$ is normal, the above upper bound may not be satisfied if $R(I(G))$ is not normal (Remarks 1.6).

On the other hand, it follows from Proposition 1.2 that, for any finite simple graph G , one has $\text{indmat}(G) \leq \text{reg } R(I(G))$. We, however, very much believe that the inequality $\text{mat}(G) \leq \text{reg } R(I(G))$ is valid for any finite simple graph G .

1. Combinatorial Pure Subrings and Regularity

Let K be an infinite field and $A = K[u_1, \dots, u_m] \subset S = K[x_1, \dots, x_n]$ be the K -algebra minimally generated by the monomials $u_i = \mathbf{x}^{\mathbf{a}_i}$ in the polynomial ring S . Here for $\mathbf{a} = (a_1, \dots, a_n)$, we denote by $\mathbf{x}^{\mathbf{a}}$ the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

The K -algebra A has a K -basis consisting of monomials $\mathbf{x}^{\mathbf{a}}$. The set of exponents $\mathbf{a}$ appearing as exponents of the basis elements of A together with addition form a positive affine semigroup $H \subset \mathbb{N}^n$ which is generated by $\mathbf{a}_1, \dots, \mathbf{a}_m$.

Given an element $\mathbf{a} \in H$, we define the simplicial complex

$$\Delta_{\mathbf{a}}(A) = \{F \subset [m] : u^F \text{ divides } \mathbf{x}^{\mathbf{a}} \text{ in } A\},$$

where $u^F = \prod_{j \in F} u_j$.

The simplicial complex $\Delta_{\mathbf{a}}(A)$ is called the *squarefree divisor complex* of H (or of A) with respect to $\mathbf{a}$.

We recall the following theorem from [2, Proposition 1.1] (see also [6, Theorem 3.28]).

Theorem 1.1. *For the multigraded Betti numbers of A with respect to a minimal presentation, one has*

$$\beta_{i,\mathbf{a}}(A) = \dim_K \tilde{H}_{i-1}(\Delta_{\mathbf{a}}; K).$$

Here $\tilde{H}_i(\Gamma; K)$ denotes the i th reduced simplicial homology of a simplicial complex Γ .

We demonstrate this theorem by a simple example. Let $A = K[x_1^2, x_1 x_2, x_2^2] \subset K[x_1, x_2]$. Then $u_1 = x_1^2$, $u_2 = x_1 x_2$ and $u_3 = x_2^2$ and $H \subset \mathbb{Z}^2$ is generated by $(2, 0)$, $(1, 1)$ and $(0, 2)$. We want to compute $\beta_{1,(2,2)}(A)$. Squarefree divisors of $x_1^2 x_2^2$ are $u_1 u_3$ and u_2 . Therefore, the facets of $\Delta_{(2,2)}$ are $\{1, 3\}$ and $\{2\}$. Thus, $\Delta_{(2,2)}$ has two connected components and so $\dim_K \tilde{H}_0(\Delta_{(2,2)}; K) = 1$. In the total standard grading, this contributes 1 to $\beta_{1,2}$, see the following diagram.

0	1	

0:	1	-
1:	-	1

Tot:	1	1

Betti diagram of A

Proposition 1.2. *Let $B \subset A$ a combinatorial pure subring of the toric K -algebra A . Then $\beta_{i,\mathbf{a}}(B) = \beta_{i,\mathbf{a}}(A)$ for all $\mathbf{a}$ with $\mathbf{x}^{\mathbf{a}} \in B$.*

Proof. It is easily seen that $\Delta_{\mathbf{a}}(B) = \Delta_{\mathbf{a}}(A)$.

Corollary 1.3. *Let A be a toric ring with all generators of same degree d , and let B be a combinatorial pure subring. Then A and B are naturally standard graded, and we have $\beta_{i,j}(B) \leq \beta_{i,j}(A)$ for all i and j .*

Proof. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set $|\mathbf{a}| = \sum_{i=1, \dots, n} a_i$. Let $H = \{\mathbf{a}: \mathbf{x}^{\mathbf{a}} \in B\}$ and $H' = \{\mathbf{a}: \mathbf{x}^{\mathbf{a}} \in A\}$. Then

$$\beta_{i,j}(B) = \sum_{\mathbf{a} \in H, |\mathbf{a}|=dj} \beta_{i,\mathbf{a}}(B) \leq \sum_{\mathbf{a} \in H', |\mathbf{a}|=dj} \beta_{i,\mathbf{a}}(A) = \beta_{i,j}(A).$$

Corollary 1.4. *Let G be a finite simple graph and $I = I(G)$ its edge ideal. Then $\beta_{i,j}(R(I)) \geq \beta_{i,j}(K[G])$ for all i and j . In particular, $\text{reg}(R(I)) \geq \text{reg}(K[G])$.*

Proof. The Rees ring $R(I)$ is isomorphic to $K[G^*]$. Since $K[G]$ is a combinatorial pure subring of $K[G^*]$, the assertion follows from Corollary 1.3. $\square$

In the next two propositions, we consider the Rees algebra of a special graph which plays a role in the next section.

Proposition 1.5. *Let G be the graph consisting of m disjoint edges, and let $I = I(G)$ be the edge ideal of G . Then $\text{reg } R(I) = 0$ if $m = 1$ and $\text{reg } R(I) \geq m$ if $m > 1$.*

Proof. If $m = 1$, then $E(G) = \{1, 2\}$ and $E(G^*) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and hence G^* is a 3-cycle. Hence, $R(I)$ which is isomorphic to $K[G^*]$ is a polynomial ring since the generators of $K[G^*]$ are algebraically independent.

Now let $m > 1$, and for $i = 1, \dots, m$ let $e_i = \{2i - 1, 2i\}$ be the edges of G . Then G^* has the additional edges $f_i = \{i, 2m + 1\}$ ($i = 1, \dots, 2m$). Let the $u_i = x_{2i-1}x_{2i}$ be the monomial generators of $K[G^*]$ corresponding to the edges e_i and the $v_i = x_i x_{2m+1}$ the generators of $K[G^*]$ corresponding to the edges f_i . In $K[G^*]$, we consider the element $\mathbf{x}^{\mathbf{a}} = u_1 \prod_{i=3, \dots, 2m} v_i = x_1 x_2 \dots x_{2m} x_{2m+1}^{2m-2}$, and claim that

$$\dim_K \tilde{H}_{m-2}(\Delta_{\mathbf{a}}) \neq 0.$$

Thus, by Theorem 1.1, the claim implies that $\beta_{m-1,\mathbf{a}}(R(I)) \neq 0$. Since $|\mathbf{a}| = 2(2m - 1)$, it follows that $\beta_{m-1,2m-1}(R(I)) \geq 1$, and this implies that $\text{reg}(R(I)) \geq m$.

Proof of the claim.: we first notice that $\Delta_{\mathbf{a}}$ has the facets

$$F_i = \{u_i, v_1, \dots, v_{2m}\} \setminus \{v_{2i-1}, v_{2i}\} \quad \text{for } i = 1, \dots, m,$$

where the vertices of $\Delta_{\mathbf{a}}$ are identified with the monomials u_i and v_j .

Indeed, since $u_i \prod_{\substack{j=1, \dots, 2m \\ j \neq 2i-1, 2i}} v_j = \mathbf{x}^{\mathbf{a}}$ for $i = 1, \dots, m$, it follows that $F_1, \dots, F_m$ are facets of $\Delta_{\mathbf{a}}$.

To prove that these are all the facets of $\Delta_{\mathbf{a}}$, we show that if F is a face of $\Delta_{\mathbf{a}}$, then $F \subset F_i$ for some i . Suppose $\{u_i, u_j\} \subset F$ for some $i \neq j$. By symmetry, we may assume that $\{u_1, u_2\} \subset F$. Then $\mathbf{x}^{\mathbf{a}}/x_1x_2x_3x_4 = x_5 \cdots x_{2m}x_{2m+1}^{2m-2} \in A$ which is impossible because $x_5 \cdots x_{2m}x_{2m+1}^{2m-2}$ must contain $2m - 2$ factors of the v_j . Next suppose that $F \cap \{u_1, \dots, u_m\} = \emptyset$. If there exists i such that $F \cap \{v_{2i-1}, v_{2i}\} = \emptyset$, then $F \subset F_i$. Otherwise, $F \cap \{v_{2i-1}, v_{2i}\} \neq \emptyset$ for all i , and by symmetry we may assume that $v_{2i-1} \in F$ for $i = 1, \dots, m$. Since u^F divides $\mathbf{x}^{\mathbf{a}}$ it follows that $|F| \leq 2m - 2$. By symmetry we may assume that v_2 and v_4 do not belong to F . Then $\mathbf{x}^{\mathbf{a}}/u^F = x_2x_4 \prod_{i: v_{2i} \notin F} x_{2i} \notin A$. Hence, $F \notin \Delta_{\mathbf{a}}$. It remains to consider the case that $F \cap \{u_1, \dots, u_m\} = \{u_i\}$ for some i . By symmetry we may assume that $i = 1$. Suppose that $v_1 \in F$. Then $u_1v_1 = x_1^2x_2x_{2m+1}$ divides $\mathbf{x}^{\mathbf{a}}$, a contradiction. Thus, $v_1 \notin F$. Similarly, $v_2 \notin F$. This shows that $F \subset F_1$.

Next we notice that the geometric realization $|\Delta_{\mathbf{a}}|$ of $\Delta_{\mathbf{a}}$ is homotopic to the geometric realization of the simplicial complex Γ whose facets are

$$G_i = \{v_1, v_3, \dots, v_{2m-1}\} \setminus \{v_{2i-1}\}, \quad i = 1, \dots, m.$$

We choose the standard geometric realization by identifying the vertices

$$u_1, \dots, u_m, v_1, \dots, v_{2m}$$

of $\Delta_{\mathbf{a}}$ with the standard unit vectors in $\mathbb{R}^{3m}$.

Indeed, the homotopy is given by the affine maps φ_t induced by $u_i \mapsto u'_i = tu_i + (1-t)v_1$ if $i > 1$ and $u_1 \mapsto u'_1 = tu_1 + (1-t)v_3$. Moreover, $v_i \mapsto v'_i = v_i$ if i is odd and $v_i \mapsto v'_i = tv_i + (1-t)v_{i-1}$ if i is even. Here $0 \leq t \leq 1$. We have $\varphi_1 = \text{id}$ and $|\Gamma| = \varphi_0(|\Delta_{\mathbf{a}}|)$, as desired.

Now since $|\Gamma|$ is homotopic to $|\Delta_{\mathbf{a}}|$, we see that $\tilde{H}_{m-2}(\Gamma) \cong \tilde{H}_{m-2}(\Delta_{\mathbf{a}})$. Observe that $|\Gamma|$ is homotopic to an $(m-2)$ -sphere, so that $\tilde{H}_{m-2}(\Gamma) \neq 0$. This concludes the proof of the proposition. $\square$

Remarks 1.6. Let G be the disjoint union of the graphs G_1 and G_2 . Assume that G has no isolated vertices and G_1 or G_2 has at least 2 edges. Considering several examples, we come up with the following question: Is it true that $\text{reg } R(I(G)) \geq \text{reg } R(I(G_1)) + \text{reg } R(I(G_2))$?

If this question has a positive answer, then Proposition 1.5 is just a very special case of this statement. Of course, it is also a simple consequence of the theorem of Cid-Ruiz [3]. However, to keep this paper as self-contained as possible and also to demonstrate the use of squarefree divisor complexes, we included Proposition 1.5 in this paper.

With CoCoA (<http://cocoa.dima.unige.it>), we considered the case that G is the sum of two 3-cycles, and found $\text{reg } R(I(G)) = 4$, as expected. In the next section, we show that $\text{reg } R(I(G)) \leq \text{mat}(G) + 1$ if $R(I(G))$ is normal. In our example, with the two 3-cycles, $R(I(G))$ is not normal and $\text{mat}(G) = 2$. Therefore, the inequality $\text{reg } R(I(G)) \leq \text{mat}(G) + 1$ is in general not valid if $R(I(G))$ is not normal.

If our question has a positive answer, then one has $\text{reg } R(I(G)) \geq 2m$ if G is the sum of m 3-cycles. On the other hand, for this graph, $\text{mat}(G) = m$.

This then gives a family of graphs for which $\text{reg } R(I(G)) - \text{mat}(G)$ exceeds any positive integer.

2. Bounds for the Regularity of the Rees Algebra of an Edge Ideal

Let, as before, G be a finite simple graph on the vertex set $V(G) = [n]$ and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K . Let $\mathcal{P}_G \subset \mathbb{R}^n$ be the *edge polytope* of G which is the convex hull of $\{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G)\}$, where $\mathbf{e}_i$ is the i th unit coordinate vector of $\mathbb{R}^n$. Let $A_{\mathcal{P}_G}$ denote the *Ehrhart ring* of G , which is the toric ring in the $n + 1$ variables $x_1, \dots, x_n, t$ whose K -basis consists of those monomials $x_1^{a_1} \dots x_n^{a_n} t^q$, where $1 \leq q \in \mathbb{Z}$, with $(a_1, \dots, a_n) \in q\mathcal{P}_G \cap \mathbb{Z}_{\geq 0}^n$. We refer the reader to [7] for basic materials and fundamental results on edge polytopes and their Ehrhart rings. The Ehrhart ring $A_{\mathcal{P}_G}$ is normal and its canonical module is spanned by those monomials $x_1^{a_1} \dots x_n^{a_n} t^q$ with $(a_1, \dots, a_n) \in q(\mathcal{P}_G \setminus \partial\mathcal{P}_G) \cap \mathbb{Z}_{\geq 0}^n$. The edge ring $K[G]$ is normal if and only if $A_{\mathcal{P}_G}$ is standard graded, i.e., $A_{\mathcal{P}_G}$ is generated by those monomials $x_1^{a_1} \dots x_n^{a_n} t$ with $(a_1, \dots, a_n) \in \mathcal{P}_G \cap \mathbb{Z}_{\geq 0}^n$ as an algebra over K . Thus, in particular, $K[G]$ is normal if and only if $K[G]$ is isomorphic to $A_{\mathcal{P}_G}$. Furthermore, it is shown [7, Corollary 2.3] that $K[G]$ is normal if and only if each connected component G' of G satisfies the *odd cycle condition*, that is, if C and C' are odd cycles of G' with $V(C) \cap V(C') = \emptyset$, then there exist $i \in V(C)$ and $j \in V(C')$ with $\{i, j\} \in E(G')$. In particular, if G is bipartite, then $K[G]$ is normal.

We say that a finite subset $L \subset E(G)$ is an *edge cover* of G if $\cup_{e \in L} e = [n]$. Let $\mu(G)$ denote the minimal cardinality of edge covers of G .

Lemma 2.1. *Let G be a finite simple graph on $V(G) = [n]$. Then*

$$\mu(G) + \text{mat}(G) = n.$$

Proof. Let $L \subset E(G)$ be an edge cover with $|L| = \mu(G)$ and M' a matching of G which is maximal among those matchings M with $M \subset L$. Then for each edge $e \in L \setminus M'$, there is $f \in M'$ with $e \cap f \neq \emptyset$. Hence, $2|M'| + (\mu(G) - |M'|) \geq n$. Thus, $\text{mat}(G) \geq |M'| \geq n - \mu(G)$. Hence, $\mu(G) \geq n - \text{mat}(G)$. However, clearly, one has $\mu(G) \leq n - \text{mat}(G)$. Thus, $\mu(G) = n - \text{mat}(G)$, as desired. $\square$

To achieve the proof of Theorem 2.2, the information of the facets of edge polytopes is indispensable. Let G be a connected non-bipartite graph on $V(G) = [n]$. We say that $i \in [n]$ is *regular* [7, p. 414] if each connected component of the induced subgraph $G_{[n] \setminus \{i\}}$ is non-bipartite. When $i \in [n]$ is regular, the hyperplane $\mathcal{H}_i$ of $\mathbb{R}^n$ defined by the equation $z_i = 0$ is a supporting hyperplane of $\mathcal{P}_G$ and $\mathcal{H}_i \cap \mathcal{P}_G$ is a facet of $\mathcal{P}_G$ [7, Theorem 1.7]. A nonempty subset $T \subset [n]$ is called *independent* if $\{i, j\} \notin E(G)$ for i and j belonging to T with $i \neq j$. When T is independent, we write $N_G(T) \subset [n]$ for the set of those vertices $i \in [n] \setminus T$ for which there is an edge $e \in E(G)$ with $i \in e$ and $e \cap T \neq \emptyset$. When T is independent, we write $T^\sharp$ for the bipartite graph on the vertex set $T \cup N_G(T)$ whose edges are those $\{i, j\} \in E(G)$ with $i \in T$ and $j \in N_G(T)$.

When T is independent, we say that T is *fundamental* [7, p. 415] if (i) $T^\sharp$ is connected and (ii) either $T \cup N_G(T) = [n]$ or each connected component of the induced subgraph $G_{[n] \setminus (T \cup N_G(T))}$ is non-bipartite. When T is fundamental, the hyperplane $\mathcal{H}_T$ of $\mathbb{R}^n$ defined by the equation $\sum_{i \in T} z_i = \sum_{j \in N_G(T)} z_j$ is a supporting hyperplane of $\mathcal{P}_G$ and $\mathcal{H}_T \cap \mathcal{P}_G$ is a facet of $\mathcal{P}_G$ [7, Theorem 1.7].

We now come to the main result of the present paper. It is known by [4, Proposition 4.5] and [3, Theorem 4.2] that, when G is bipartite, one has $\text{reg } R(I(G)) = \text{mat}(G)$.

Theorem 2.2. (a) Let G be a finite simple graph with $G_1, \dots, G_c$ its connected components. Then the Rees algebra $R(I(G)) = K[G^*]$ is normal if and only if each $K[G_i]$ is normal and at most one of $G_1, \dots, G_c$ is non-bipartite. (b) Let $|E(G)| \geq 2$. Suppose that $R(I(G))$ is normal. Then

$$\text{mat}(G) \leq \text{reg } R(I(G)) \leq \text{mat}(G) + 1.$$

Proof. Result (a) follows immediately from [5, Corollary 2.3], see also [9, Proposition 10.5.8]. However, for the sake of the reader, we give a quick proof: the “If” part is clear. We show now the “Only If” part. If, say, G_1 fails to satisfy the odd cycle condition, then G^* also fails to satisfy the odd cycle condition. If, say, G_1 and G_2 are non-bipartite and if C_i is an odd cycle of G_i for $i \in \{1, 2\}$, then, even though G^* is connected, there is no edge $e \in E(G^*)$ with $e \cap V(C_i) \neq \emptyset$ for $i \in \{1, 2\}$, as desired.

(b) We first prove the lower bound. In the case $\text{mat}(G) = 1$, the graph G is a star graph which by assumption has at least 2 edges. Then $R(I(G))$ is not polynomial and, therefore, $\text{reg } R(I(G)) \geq 1 = \text{mat}(G)$.

Now let $m = \text{mat}(G)$ and assume that $m \geq 2$ and $\{\{i_1, j_1\}, \dots, \{i_m, j_m\}\}$ is a matching of G . Write H for the subgraph of G with $E(H) = \{\{i_1, j_1\}, \dots, \{i_m, j_m\}\}$. Now, $R(I(H)) = K[H^*]$ is normal with $\text{reg } R(I(H)) \geq m$, by Proposition 1.5, and H^* is a subgraph of G^* . Thus, since both $R(I(H))$ and $R(I(G))$ are Cohen–Macaulay, it follows from [8, Theorem 3.3] that

$$\text{mat}(G) = m \leq \text{reg } R(I(H)) \leq \text{reg } R(I(G))$$

as desired.

We now prove the upper bound. The highlight of the proof is to estimate the positive integer

$$q_0 = \min\{q \geq 1 : q(\mathcal{P}_{G^*} \setminus \partial \mathcal{P}_{G^*}) \cap \mathbb{Z}^{n+1} \neq \emptyset\}.$$

Case I Suppose that G is connected and non-bipartite. Each vertex $i \in [n+1]$ of $\mathcal{P}_{G^*}$ is regular. It then follows from [7, Theorem 1.7] that, if $(a_1, \dots, a_n, a_{n+1})$ belongs to $q(\mathcal{P}_{G^*} \setminus \partial \mathcal{P}_{G^*}) \cap \mathbb{Z}^{n+1}$, then each $a_i > 0$. Furthermore, since $\mathcal{P}_{G^*}$ possesses the integer decomposition property, there exist edges $e_1, e_2, \dots, e_q$ (possibly, $e_i = e_j$ with $i \neq j$) of G^* with

$$(a_1, \dots, a_n, a_{n+1}) = \sum_{i=1}^q \rho(e_i),$$

where $\rho(e_i) = \mathbf{e}_j + \mathbf{e}_{j'}$ if $e_i = \{j, j'\}$. The fact that each $a_i > 0$ says that $\{e_1, e_2, \dots, e_q\}$ must be an edge cover of G^* . Using Lemma 2.1, one has

$$q_0 \geq \mu(G^*) = (n+1) - \text{mat}(G^*) \geq (n+1) - (\text{mat}(G) + 1).$$

Now, since q_0 is the smallest degree of generators of the canonical module of the Ehrhart ring $A_{\mathcal{P}_{G^*}}$ and since $A_{\mathcal{P}_{G^*}}$ is Cohen–Macaulay with $\dim A_{\mathcal{P}_{G^*}} = \dim \mathcal{P}_{G^*} + 1 = n+1$, it follows that $\text{reg } A_{\mathcal{P}_{G^*}} = (n+1) - q_0$. Since $R(I(G)) = K[G^*]$ is normal, one has $A_{\mathcal{P}_{G^*}} = K[G^*]$. Thus,

$$\text{reg } R(I(G)) = (n+1) - q_0 \leq \text{mat}(G) + 1$$

as desired.

Case II Suppose that G is disconnected and non-bipartite. Let $G_1, \dots, G_c$ be the connected components of G , where G_1 is non-bipartite and where each of $G_2, \dots, G_c$ is bipartite. For $i > 1$, let $V(G_i) = V_i \cup V'_i$ be the decomposition of $V(G_i)$. Let $(a_1, \dots, a_n, a_{n+1})$ belong to $q(\mathcal{P}_{G^*} \setminus \partial \mathcal{P}_{G^*}) \cap \mathbb{Z}^{n+1}$. Since each $i \in [n]$ is regular, one has $a_i > 0$ for each $i \in [n]$. Since $T = V_2 \cup \dots \cup V_c$ is fundamental with $N_G(T) = V'_2 \cup \dots \cup V'_c \cup \{n+1\}$ and since $T' = V'_2 \cup \dots \cup V'_c$ is fundamental with $N_G(T') = V_2 \cup \dots \cup V_c \cup \{n+1\}$ it follows that

$$\sum_{i \in T} a_i < a_{n+1} + \sum_{j \in T'} a_j, \quad \sum_{j \in T'} a_j < a_{n+1} + \sum_{i \in T} a_i.$$

Thus, $2a_{n+1} > 0$ and $a_{n+1} > 0$. Hence, as in Case I, one has $\text{reg } R(I(G)) \leq \text{mat}(G) + 1$, as required. $\square$

In the proof of (b) of Theorem 2.2, one has $\text{mat}(G^*) = \text{mat}(G)$ if and only if G has a perfect matching. As a result,

Corollary 2.3. *If G has a perfect matching and if $R(I(G))$ is normal, then*

$$\text{reg } R(I(G)) = \text{mat}(G).$$

The converse of Corollary 2.3 is false. In fact,

Example 2.4. Let G be a finite simple connected non-bipartite graph on $[6]$ whose edges are

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{2, 6\}.$$

Even though G has no perfect matching, one has $\text{reg } R(I(G)) = \text{mat}(G) = 2$. The lattice points $(a_1, \dots, a_7)$ belonging to $4\mathcal{P}_{G^*} \cap \mathbb{Z}^7$ with each $a_i > 0$ are

$$(1, 1, 1, 1, 1, 1, 2), (1, 1, 1, 2, 1, 1, 1).$$

Since $T = \{1, 3, 5, 6\}$ is fundamental, neither of these lattice points cannot belong to $4(\mathcal{P}_{G^*} \setminus \partial \mathcal{P}_{G^*})$. Thus, $q_0 \geq 5$.

It would, of course, be of interest to characterize finite simple graphs G with $\text{reg } R(I(G)) = \text{mat}(G)$. On the other hand, if G is a 5-cycle, then $R(I(G))$ is normal, $\text{reg } R(I(G)) = 3$ and $\text{mat}(G) = 2$.

When $K[G]$ is non-normal, instead of [8, Theorem 3.3], we can enjoy the merit of combinatorial pure subrings (Corollary 1.3).

Proposition 2.5. *Let G be an arbitrary finite simple graph. Then one has*

$$\text{indmat}(G) \leq \text{reg } R(I(G)).$$

We, however, believe that the lower bound inequality $\text{mat}(G) \leq \text{reg } R(I(G))$ is valid for any finite simple graph G .

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