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Existence Theorems for Parameter Dependent Weakly Continuous Operators with Applications

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Abstract. The paper presents results on the solvability and parameter dependence for problems driven by weakly continuous potential operators with continuously differentiable and coercive potential. We provide a parametric version on the existence result to nonlinear equations involving coercive and weakly continuous operators. Applications address a variant of elastic beam equation.

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1. Introduction

In [4] the author gives a very general theorem about the existence of solutions to nonlinear problems which involve some partial linearity:

Theorem 1. *Let E be a separable reflexive Banach space. Assume that the operator $A : E \to E^*$ is (i) weakly continuous, i.e. $u_n \rightharpoonup u_0$ in E implies $A(u_n) \rightharpoonup A(u_0)$ in E^*, and (ii) coercive, i.e.*

$$\lim_{\|v\|\to\infty} \frac{\langle A(v), v\rangle}{\|v\|} = +\infty.$$

Then for any $b \in E^*$ the equation

$$A(u) = b$$

has a solution.

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Remark 2. It follows from the proof contained in [4] that instead of assuming (i) one can impose a slightly relaxed condition, namely given a weakly convergent sequence (u_n) with a weak limit u_0 it follows for some subsequence (u_{n_k}) that $A(u_{n_k}) \rightharpoonup A(u_0)$ in E^*. Moreover, we can replace (u_n) with some bounded sequence. In fact we need to choose another subsequence in Step 4 of the proof of Theorem 1.2 in [4]. Such a remark applies to all other subsequent results and we will not repeat it so that not to complicate the formulation of results.

In this work we aim at putting some further insight into the solvability of nonlinear equations inspired by Theorem 1 by providing:

a) some information about the convergence of Galerkin schemes in Theorem 1;
b) the parametric version of Theorem 1 considering the situation of equation

$$A(u,v) = b,$$

with v from a parameter space Y, and where the dependence of solutions as parameter varies is investigated;
c) the variational counterpart of Theorem 1 (together with its parametric version) in case where A is potential with lack of coercivity but its potential is coercive;
d) applications of the above mentioned results to a fourth order problem being a variant of the elastic beam equation.

Theorem 1 has been revived as of late as an important abstract tool, see for example: [15] and also [14] with some additions in [12].

We would like to mention that problems involving partial linearity of the problem under consideration are very common in the literature and pertain to both second order, see for example [7], and higher order problems among which there appears the boundary value problem connected to the fourth order elastic beam equation with either simply supported or rigidly fastened ends. In this direction there exist a vast research by variational methods pertaining to the use of various multiplicity criteria, like the Ricceri three critical point theorem and also fixed point arguments pertaining to Krasnosel'skiĭ-Guo and the Leggett-Williams fixed point theorems, see [1,2,10,11,13].

For the background on nonlinear analysis tools uses here we refer to [5] and [8].

The paper is organized as follows. In Sect. 2 we gather abstract results developed in this paper connected to the parametric and variational versions of Theorem 1 as well as some comments and additions. In Sect. 3 we give applications to both non-variational and variational parametric version of Theorem 1 to the fourth order boundary value problem related to the beam equation and containing an unbounded perturbation.

2. Abstract Results

Let us recall that operator $A : E \to E^*$, where E is a reflexive and separable Banach space, satisfies condition (S) if $u_n \rightharpoonup u_0$ *in* E *and* $\langle A(u_n) - A(u_0), u_n - u_0 \rangle \to 0$ *imply* $u_n \to u_0$ *in* E.

Since E is separable it contains a dense and countable set $\{h_1, ..., h_n, ...\}$. Define E_n for $n \in \mathbb{N}$ as a linear hull of $\{h_1, ..., h_n\}$. The sequence of subspaces E_n has the approximation property: for each $u \in E$ there is a sequence $(u_n)_{n=1}^{\infty}$ such that $u_n \in E_n$ for $n \in \mathbb{N}$ and $u_n \to u$. Let $b \in E^*$ be fixed. By b_n we denote the restriction of functional b to space E_n. Similarly by A_n we understand the restriction of A to space E_n. We call the sequence (u_n) with $u_n \in E_n$ of solutions to

$$A_n(u) = b_n$$

the Galerkin scheme connected with equation

$$A(u) = b.$$

We begin with remarking on the convergence of Galerkin type schemes in Theorem 1:

Corollary 3. *Let E be a separable reflexive Banach space. Assume that the operator $A : E \to E^*$ is*

(i) weakly continuous,

and

(ii) coercive

Then for any $b \in E^*$ the equation

$$A(u) = b$$

has a solution $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E, where (u_n) stands for the Galerkin scheme. In case A satisfies additionally condition (S), we have that $u_n \to u_0$ in E.

Proof. The assertions about the weak convergence of Galerkin type scheme follows directly from the proof contained in [4], while the remark about the norm convergence under condition (S) then follows as in [5, Chapter 6.2]. □

Now we proceed to the parametric version of Theorem 1. This relies on a type of uniform coercivity subject to a parameter.

Theorem 4. *Assume that Y is a normed space and E is a reflexive and separable Banach space. Assume that $A : E \times Y \longrightarrow E^*$ is an operator satisfying the following conditions: (i) A is (weakly,norm)→weakly continuous, i.e. $u_n \rightharpoonup u_0$ in E and $v_n \to v_0$ in Y imply $A(u_n, v_n) \rightharpoonup A(u_0, v_0)$ in E^*; (ii) there exists a function $\rho : [0, \infty)^2 \longrightarrow \mathbb{R}$ such that*

$$\langle A(u, v), u \rangle \geq \rho(\|u\|, \|v\|)\|u\| \quad \text{for all } v \in Y \text{ and } u \in E,$$

and that

$$\lim_{x\to\infty} \rho(x,y)=\infty$$

uniformly with respect to y from every bounded interval. Let $b \in E^*$ be fixed. Then for every $v \in Y$ there exists an element $u_v \in E$ solving the equation

$$A(u,v)=b. \tag{1}$$

Moreover $v_n \to v_0$ in Y implies up to a subsequence that $u_{v_n} \rightharpoonup u_0$ in E with $A(u_0,v_0)=b$.

Proof. For any fixed $v \in Y$ it follows by Theorem 1 that there exists an element $u_v \in E$ such that $A(u_v,v)=b$. Now let us consider a sequence $(v_n) \subset Y$ norm convergent to some $v_0 \in Y$. Then for any $n \in \mathbb{N}$ it holds

$$\rho(\|u_{v_n}\|,\|v_n\|)\|u_{v_n}\| \leq \langle A(u_{v_n},v_n),u_{v_n}\rangle=\langle b,u_{v_n}\rangle \leq \|b\|_*\|u_{v_n}\|.$$

It follows by assumption (ii) that the sequence (u_{v_n}) is bounded and therefore up to a subsequence which we do not renumber, weakly convergent to some $u_0 \in E$. By assumption (i) we see passing to the limit in $A(u_{v_n},v_n)=b$ that $A(u_{v_0},v_0)=b$, so the assertion follows. □

According to Remark 2 the assumptions may slightly be relaxed. We note that we can rephrase the above result as follows: Given a sequence $(v_n) \subset Y$ norm convergent to some $v_0 \in Y$ we can find a sequence (S_n) of sets where each S_n consists of solutions to (1) corresponding to y_n. Denote by S_0 the set of solutions to (1) corresponding to v_0. Then any sequence (u_n) such that $u_n \in S_n$ contains a weak cluster point in S_0. Thus we would obtain the upper limit of the sequence of sets (S_n) in the Painleve-Kuratowski sense should we would be able to demonstrate that $u_{v_n} \to u_{v_0}$ in E. In the following result - in which we impose a parametric version of condition (S)- we consider such a situation:

Corollary 5. *If in addition to the assumptions of Theorem 4 the following condition about operator A is satisfied: (iii) $u_n \rightharpoonup u_0$ in E, $v_n \to v_0$ in Y, and $\langle A(u_n,v_n)-A(u_0,v_0),u_n-u_0\rangle \to 0$ imply $u_n \to u_0$ in E, then the conclusion in Theorem 4 is that $v_n \to v_0$ in Y implies $u_{v_n} \to u_{v_0}$ in E.*

Proof. From Theorem 4 we get that $u_{v_n} \rightharpoonup u_{v_0}$ in E. Since $A(u_{v_n},v_n)=b$ and $A(u_{v_0},v_0)=b$, we see that also $\langle A(u_{v_n},v_n)-A(u_{v_0},v_0),u_n-u_{v_0}\rangle \to 0$. Now, using condition (iii) we obtain the assertion. □

Now we proceed to a variational counterpart of Theorem 1 which involves neither monotonicity nor its generalizations. This is why we do not expect to have sequential weak lower semicontinuity of the Euler action functional. For the proof of our next result we will need the celebrated Ekeland Variational Principle in the differential form (see, e.g., [8]) that we recall:

Theorem 6 (Ekeland Variational Principle—differentiable form) *Let $\mathcal{I} : E \to \mathbb{R}$ be a Gâteaux differentiable functional which is bounded from below and lower semicontinuous. Then there exists a minimizing sequence (u_n) of $\mathcal{I}$ consisting of almost critical points, i.e., such that $\mathcal{I}(u_n) \to \inf_{u\in E} \mathcal{I}(u)$ and $\mathcal{I}'(u_n) \to 0$.*

We state the following result.

Theorem 7. *Let E be a separable reflexive Banach space. Let $b \in E^*$ be fixed. Assume that:*

(i) the operator $A : E \to E^*$ is potential with the C^1 potential $\mathcal{A} : E \to \mathbb{R}$, i.e., $\mathcal{A}$ is continuously differentiable with $\mathcal{A}' = A$;

(ii) the operator A is weakly continuous;

(iii) the functional $J : E \to \mathbb{R}$ given by

$$J(u) = \mathcal{A}(u) - \langle b, u\rangle$$

is coercive and bounded from below.

Then the equation

$$A(u) = b \tag{2}$$

has a solution u_0 (equivalently, $J^{'}(u_0) = 0$). Moreover there is a (minimizing) sequence (u_n) with

$$\inf_{u\in E} J(u) = \lim_{n\to\infty} J(u_n) \text{ and } u_n \rightharpoonup u_0.$$

Proof. The functional J is continuously differentiable due to (i). By (iii), J is bounded from below, so Theorem 6 can be applied providing a minimizing sequence (u_n) with $J'(u_n) \to 0$ in E^* as $n \to \infty$.

We show that J has a critical point solving (2). By the coercivity postulated in (iii), the minimizing sequence (u_n) is bounded. Through the reflexivity of the space E, passing to a subsequence it can be assumed to be weakly convergent to some $u_0 \in E$. Since by (ii) the operator A is weakly continuous, we see that

$$J^{'}(u_n) = A(u_n) - b \rightharpoonup A(u_0) - b = J^{'}(u_0),$$

thus $J'(u_0) = 0$, and the proof is completed. □

Remark 8. We emphasize that with the assumptions of Theorem 7 we may not use Theorem 1 since the coercivity of the potential need not imply the coercivity of its differential as seen by the example of the coercive function

$$f(x) = \begin{cases} x^2 - 1, & x \le 2 \\ 4x - 5, & x > 2 \end{cases}$$

whose derivative is not coercive.

Remark 9. Theorem 7 is related to the existence result from [3, Theorem 4], where it is assumed about the operator that it is bounded, coercive and continuous and satisfies some compactness condition related to the one which we employ. Our advantage is again that we do not impose the coercivity on the operator, while we can impose exactly the same compactness condition.

In accordance with Corollary 3 we can also consider the case when the minimizing sequence obtained in Theorem 7 is norm convergent:

Corollary 10. *In addition to the assumptions of Theorem 7 impose that operator A satisfies property (S). Then a (minimizing) sequence (u_n) is norm convergent.*

Next, we proceed to formulate a parameter dependent version of Theorem 7 which is in turn a variational counterpart of Theorem 4.

Theorem 11. *Assume that Y is a normed space and E is a reflexive Banach space. Let $b \in E^*$ be fixed. Assume that $A : E \times Y \longrightarrow E^*$ is an operator satisfying the following conditions:*

(i) for each $v \in Y$ the operator $A(\cdot, v) : E \longrightarrow E^*$ is potential with continuously differentiable potential $\mathcal{A}(\cdot, v)$;

(ii) A is (weakly,norm)$\rightarrow$weakly continuous;

(iii) there exists a function $\rho : [0, \infty)^2 \longrightarrow \mathbb{R}$ such that

$$\mathcal{A}(u, v) - \langle b, u\rangle \geq \rho(\|u\|, \|v\|) \quad \text{for all } v \in Y \text{ and } u \in E,$$

and that

$$\lim_{x\rightarrow+\infty} \rho(x, y) = +\infty \tag{3}$$

uniformly with respect to y from every bounded interval. Then for every $v \in Y$ there exists an element $u_v \in E$ solving the equation

$$A(u_v, v) = b. \tag{4}$$

Moreover, $v_n \rightarrow v_0$ in Y implies $u_{v_n} \rightharpoonup u_{v_0}$ in E with u_{v_0} being a solution to (4) for $v = v_0$. If we additionally assume condition (iii) from Corollary 5, then $u_{v_n} \rightarrow u_{v_0}$ in E.

Proof. For any fixed $v \in Y$, Theorem 7 yields a critical point $u_v \in E$ of the functional $u \mapsto \mathcal{A}(u) - \langle b, u\rangle$. Now let $v_n \rightarrow v_0$ in Y. Then by assumption (iii) it follows that the sequence (u_{v_n}) with u_{v_n} solution to (4) corresponding to v_n is bounded in E. Therefore one has up to a subsequence that $u_{v_n} \rightharpoonup u_0$ in E for some $u_0 \in E$. By assumption (ii) we see passing to the limit that

$$A(u_n, y_n) - b \rightharpoonup A(u_0, y_0) - b.$$

The remaining assertions follow as in Corollary 5. □

Remark 12. From Theorem 7 we know that corresponding to v_0 there is a critical point of the functional $u \mapsto \mathcal{A}(u, v_0) - \langle b, u\rangle$ which solves (4). However, we do not know if the limit solution obtained in Theorem 11 is a minimizer.

We illustrate the insight of our abstract results with the following model problem: given $v \in Y$ find $u \in E$ such that

$$Bu + F(u) + G(v) = b. \tag{5}$$

The data in (5) are required to fulfill the hypotheses below.

(H_1) There are given the Hilbert spaces E, X, and Y and $E \subset X$ compactly and densely (thus, $X^* \subset E^*$).

(H_2) The map $B : E \to E^*$ is continuous, linear, self-adjoint and positive definite, i.e. there is a constant $c_0 > 0$ such that

$$\langle Bu, u\rangle \geq c_0 \|u\|_E^2$$

for all $u \in E$ (thus B is strongly monotone).

(H_3) The map $F : X \to X^*$ is of potential type with $F = \mathcal{F}'$ for some continuously differentiable functional $\mathcal{F} : X \to \mathbb{R}$ satisfying

$$|\mathcal{F}(u)| \leq C(\|u\|_X^\alpha + 1)$$

with constants $C > 0$ and $\alpha \in [0, 2)$.

(H_4) The map $G : Y \to E^*$ is continuous and satisfies $\|G(v)\|_{E^*} \leq \varphi(\|v\|_Y)$ for all $v \in Y$ with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous.

Theorem 13. *Assume that the conditions (H_1)–(H_4) hold. Let $b \in E^*$ be fixed. Then for every $v \in Y$ there exists $u_v \in E$ solving problem (5). Moreover, $v_n \to v_0$ in Y implies $u_{v_n} \to u_{v_0}$ in E with u_{v_0} being solution to (5).*

Proof. We show that Theorem 11 applies. Let us define the map $A : E \times Y \longrightarrow E^*$ by

$$A(u, v) = Bu + F(u) + G(v), \quad \text{for all } (u, v) \in E \times Y. \tag{6}$$

From assumptions (H_1), (H_2), and (H_3), we see for every $v \in Y$ that the map $A(\cdot, v)$ in (6) is potential with the potential $\mathcal{A}(\cdot, v) : E \to \mathbb{R}$ as

$$\mathcal{A}(u, v) = \frac{1}{2}\langle Bu, u\rangle + \mathcal{F}(u) + \langle G(v), u\rangle), \qquad \text{for all } (u, v) \in E \times Y, \tag{7}$$

so condition (i) in Theorem 11 is verified.

In order to check condition (ii) in Theorem 11, let $u_n \rightharpoonup u_0$ in E and $v_n \to v_0$ in Y. The linearity and the continuity of the operator B imply $Bu_n \rightharpoonup Bu_0$ in E^*, while hypothesis (H_1) ensures $u_n \to u_0$ in X, thus $F(u_n) \to F(u_0)$ in E^*. As the continuity of G provides $G(v_n) \to G(v_0)$ in E^*, from (6) we find that $A(u_n, y_n) \rightharpoonup A(u_0, y_0)$ in E^*. Therefore condition (ii) in Theorem 11 is satisfied.

From (7), (H_1), (H_2), (H_3), and (H_4), we get the estimate

$$\begin{aligned}\mathcal{A}(u, v) - \langle b, u\rangle &\geq \frac{c_0}{2}\|u\|_E^2 - C(\|u\|_X^\alpha + 1) - \|G(v)\|_{E^*}\|u\|_E - \|b\|_{E^*}\|u\|_E \\ &\geq \frac{c_0}{2}\|u\|_E^2 - c_1(\|u\|_E^\alpha + 1) - (\varphi(\|v\|_Y) + \|b\|_{E^*})\|u\|_E, \\ &\quad \text{for all } (u, v) \in E \times Y,\end{aligned}$$

with a constant $c_1 > 0$ depending on the embedding and independent of u. Define the function $\rho : [0,\infty)^2 \longrightarrow \mathbb{R}$ by setting

$$\rho(x,y) = \frac{c_0}{2}x^2 - (\varphi(y) + \|b\|_{E^*})\, x - c_1\, (x^\alpha + 1) \quad \text{for all } x, y \geq 0.$$

Since $\alpha < 2$ and since φ is continuous, it follows that (3) holds true. In addition, we have

$$\mathcal{A}(u,v) - \langle b, u\rangle \geq \rho(\|u\|_E, \|v\|)_Y), \qquad \text{for all } (u,v) \in E \times Y.$$

This amounts to saying that condition (iii) in Theorem 11 is fulfilled.

Note that a strongly monotone operator B satisfies condition (S) which is not violated by perturbation F. Indeed, if we assume that $u_n \rightharpoonup u_0$ in E then $F(u_n) \to F(u_0)$ in E^* and

$$\langle Bu_n - Bu_0 + F(u_n) - F(u_0), u_n - u_0\rangle \to 0$$

implies that $\langle Bu_n - Bu_0, u_n - u_0\rangle \to 0$. Since $\langle Bu_n - Bu_0, u_n - u_0\rangle \geq c_0\|u_n - u_0\|_E^2$ we have the assertion.

Consequently, Theorem 11 can be applied to resolve problem (5) obtaining the desired conclusion. □

It easily follows from Theorem 13 that we can consider the case of weakly convergent sequence of parameters under some structure assumptions as shown below:

Corollary 14. *Assume that the conditions* (H_1)-(H_3) *hold and that* $Y = X$*, where* X *is identified with its dual. Then for every* $v \in X$ *there exists* $u_v \in E$ *solving problem*

$$Bu + F(u) = v. \tag{8}$$

Moreover, $v_n \rightharpoonup v_0$ *in* X *implies* $u_{v_n} \to u_{v_0}$ *in* E *with* u_{v_0} *being solution to* (8) *corresponding to* v_0*.*

Without the assumption about the potentiality of operator F we can consider the direct application of Theorem 1 to problem (5) under the following version of (H_3) :

$\left(H_3'\right)$ The map $F : X \to X^*$ is continuous and satisfies that

$$\langle F(u), u\rangle \geq -C(\|u\|_X^\alpha + 1) \text{ for all } u \in X$$

with constants $C > 0$ and $\alpha \in [0,2)$.

Now we proceed to the formulation of relevant result:

Theorem 15. *Assume that the conditions* (H_1)*,* (H_2)*,* $\left(H_3'\right)$*,* (H_4) *hold. Let* $b \in E^*$ *be fixed. Then for every* $v \in Y$ *there exists* $u_v \in E$ *solving problem* (5)*. Moreover,* $v_n \to v_0$ *in* Y *implies* $u_{v_n} \to u_{v_0}$ *in* E *with* u_{v_0} *being solution to* (5)*.*

Proof. We follow the lines of the proof of Theorem 13, so define the map $A : E \times Y \longrightarrow E^*$ by (6).

We check that condition (i) in Theorem 7 is satisfied exactly as in the proof of Theorem 13. In order to see condition (ii) holds, we note that from (H_1), (H_2), $\left(H_3^{'}\right)$, and (H_4), we get the estimate

$$\langle A(u,v),u\rangle \geq c_0\|u\|_E^2 - C(\|u\|_X^\alpha + 1) - \|G(v)\|_{E^*}\|u\|_E - \|b\|_{E^*}\|u\|_E \geq$$
$$c_0\|u\|_E^2 - c_1(\|u\|_E^\alpha + 1) - (\varphi(\|v\|_Y) + \|b\|_{E^*})\|u\|_E, \quad \text{for all } (u,v) \in E \times Y,$$

with a constant $c_1 > 0$ depending on the embedding and independent of u.

Consequently, Theorem 7 can be applied to resolve problem (5) obtaining the desired conclusion. □

3. Applications

We present applications of our abstract results to non-potential and potential versions of the beam equation extending the method developed in [6].

3.1. Results by Theorem 4

We are interested in the following variant of the elastic beam equation expressed as the fourth order problem with perturbation g and a functional parameter $v \in L^2(0,1)$

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\right) = f(t, u(t), \frac{d}{dt}u(t), \\ \quad v(t)) \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{9}$$

where

$$f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ and } g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$$

are functions which are subject to some conditions provided below. We seek weak solutions in the space

$$H_0^2(0,1) = \left\{u \in H_0^1(0,1) : \ddot{u} \in L^2(0,1), \dot{u}(0) = \dot{u}(1) = 0\right\}$$

normed by

$$\|u\|_{H_0^2} = \sqrt{\int_0^1 \left|\frac{d^2}{dt^2}u(t)\right|^2 dt}.$$

As is the case of the well known space $H_0^1(0,1)$, the Sobolev and Poincaré inequalities read as follows: for any $u \in H_0^2(0,1)$ it holds

$$\|u\|_C \leq \|u\|_{H_0^1} \leq \frac{1}{\pi}\|u\|_{H_0^2},$$
$$\|u\|_{L^2} \leq \frac{1}{\pi}\|u\|_{H_0^1} \leq \frac{1}{\pi^2}\|u\|_{H_0^2}.$$

and

$$\|\dot{u}\|_C \leq \|u\|_{H_0^2}.$$

where

$$\|u\|_C := \max_{t\in[0,1]} |u(t)|.$$

Let us recall that $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if the following conditions are satisfied:

(i) $t \mapsto f(t,x,y,z)$ is measurable on $[0,1]$ for each fixed $x,y,z \in \mathbb{R}$,

(ii) $(x,y,z) \mapsto f(t,x,y,z)$ is continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ for a.e. $t \in [0,1]$.

The assumptions are as follows:

A1 *$g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function for which there are a constant $g_0 \geq 0$ and a function $g_1 : \mathbb{R}_+ \to \mathbb{R}$ such that $g(t,x) \geq g_1(x) \geq g_0$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$ and $\lim_{x\to\infty} g_1(x) = +\infty$.*

A2 *$f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exist $a_1, b_1 \in L^2(0,1;\mathbb{R}_+)$, $c_1 \in L^1(0,1)$ for which*

$$|f(t,x,y,z)| \leq a_1(t)|x| + b_1(t)|y| + c_1(t)$$

for a.e. $t \in [0,1]$ and all $x,y,z \in \mathbb{R}$.

A3 *There exist $a, b \in L^\infty(0,1;\mathbb{R}_+)$, $c \in L^1(0,1)$ such that*

$$\pi^4 > \|a\|_{L^\infty} + \pi^2 \|b\|_{L^\infty} \tag{10}$$

and that for a.e. $t \in [0,1]$ and all $x,y,z \in \mathbb{R}$ it holds

$$f(t,x,y,z)x \leq a(t)|x|^2 + b(t)|y|^2 + c(t).$$

We consider weak solutions, namely we say that $u \in H_0^2(0,1)$ solves (9) in the weak sense provided that

$$\int_0^1 \frac{d^2}{dt^2}u(t)\frac{d^2}{dt^2}w(t)\,dt + \int_0^1 g\left(t, |\frac{du}{dt}|\right)\frac{d}{dt}u(t)\frac{d}{dt}w(t)\,dt = \int_0^1 f(t,u(t),\frac{d}{dt}u(t),v(t))w(t)\,dt$$

for all $w \in H_0^2(0,1)$. From **A1**, **A2** we note that the above formula makes sense. Now we demonstrate that any solution to (9) is necessarily bounded.

Lemma 16. *Assume that conditions **A1, A2, A3** are satisfied. Let $v \in L^2(0,1)$ be fixed. Then there is some $R > 0$ such that $\|u\|_{H_0^2} \leq R$ and $\|\dot{u}\|_C \leq R$ for every $u \in H_0^2(0,1)$ which solves problem (9).*

Proof. Assume that $u \in H_0^1(0,1)$ solves problem (9). Testing it with $w = u$ we have

$$\|u\|_{H_0^2}^2 + \int_0^1 g\left(t, \left|\frac{d}{dt}u(t)\right|\right)\left|\frac{d}{dt}u(t)\right|^2 dt = \int_0^1 f(t,u(t),\frac{d}{dt}u(t),v(t))u(t)\,dt. \tag{11}$$

Then we obtain concerning the left hand side of (11) that

$$\begin{aligned}&\|u\|_{H_0^2}^2+\int_0^1 g\left(t,\left|\frac{d}{dt}u\left(t\right)\right|\right)\left|\frac{d}{dt}u\left(t\right)\right|^2dt\\&\geq\|u\|_{H_0^2}^2+g_0\|u\|_{H_0^1}^2\geq\|u\|_{H_0^2}^2.\end{aligned}$$

Estimating the right hand side of (11) we have by **A3** what follows

$$\begin{aligned}&\int_0^1 f(t,u(t),\frac{d}{dt}u\left(t\right),v\left(t\right))u\left(t\right)dt\\&\leq\int_0^1 a\left(t\right)\left|u\left(t\right)\right|^2dt+\int_0^1 b\left(t\right)\left|\dot{u}\left(t\right)\right|^2dt+\int_0^1 c\left(t\right)dt\\&\leq\tfrac{1}{\pi^4}\|a\|_{L^\infty}\|u\|_{H_0^2}^2+\tfrac{1}{\pi^2}\|b\|_{L^\infty}\|u\|_{H_0^2}^2+\|c\|_{L^1}\end{aligned}$$

Summing up we arrive at

$$\left(1-\left(\frac{\|a\|_{L^\infty}}{\pi^4}+\frac{\|b\|_{L^\infty}}{\pi^2}\right)\right)\|u\|_{H_0^2}^2-\|c\|_{L^1}\leq 0\tag{12}$$

which implies the assertion $\|u\|_{H_0^2}\leq R$ since (10) holds. We see that we can take

$$R^2:=\frac{\|c\|_{L^1}}{1-\left(\frac{\|a\|_{L^\infty}}{\pi^4}+\frac{\|b\|_{L^\infty}}{\pi^2}\right)}.$$

The remaining assertion follows by the Sobolev inequality. □

With $R>0$ in Lemma 16, we introduce the continuous function $g_R:[0,1]\times\mathbb{R}_+\to\mathbb{R}$

$$g_R\left(t,x\right)=\left\{\begin{array}{ll}g\left(t,x\right), & 0\leq x\leq R\\ g\left(t,R\right), & x>R.\end{array}\right\}\tag{13}$$

We see that for all $t\in[0,1]$ and $x\in\mathbb{R}_+$,

$$g_0\leq g_R\left(t,x\right)\leq\max_{t\in[0,1],0\leq x\leq R}g\left(t,x\right).$$

Consider the following truncated problem with a functional parameter

$$\begin{cases}\frac{d^4}{dt^4}u\left(t\right)-\frac{d}{dt}\left(g_R\left(t,\left|\frac{d}{dt}u\left(t\right)\right|\right)\frac{d}{dt}u\left(t\right)\right)=f(t,u(t),\frac{d}{dt}u\left(t\right),v\left(t\right))\\ u\left(0\right)=u\left(1\right)=0,\ \dot{u}\left(0\right)=\dot{u}\left(1\right)=0,\end{cases}\tag{14}$$

Before looking for weak solutions to problem (14) we set forth the regularity of the weak solution by means of the higher order regularity in du Bois-Reymond Lemma. We follow the pattern in [6] but rewritten to fit our problem. By [9, Proposition 4.5], we have the following result.

Lemma 17. *If $h\in L^2\left(0,1\right)$ satisfies*

$$\int_0^1 h\left(t\right)\frac{d^2}{dt^2}w\left(t\right)dt=0$$

for all $w \in H_0^2(0,1)$*, then there exist constants* $c_0, c_1 \in \mathbb{R}$ *such that* $h(t) = c_0 + c_1 t$ *a.e. on* $[0,1]$.

The following regularity result regarding problem (14) is available.

Proposition 18. *Assume that conditions* ***A1, A2, A3*** *are satisfied. Let* $v \in L^2(0,1)$ *be fixed. Then any* $u \in H_0^2(0,1)$ *which is a weak solution to* (14) *is such that* u, $\frac{d}{dt}u$, $\frac{d^2}{dt^2}u$, $\frac{d^3}{dt^3}u$ *are absolutely continuous and* $\frac{d^4}{dt^4}u \in L^2(0,1)$*, and* u *satisfies* (14) *a.e. on* $[0,1]$.

Proof. Since $u \in H_0^2(0,1)$ is a weak solution to (14), we see that u, $\frac{d}{dt}u$ are absolutely continuous. Next, using the definition of the weak solution to (14) and integrating by parts twice, which makes sense due to (13), we see that the following holds for any $w \in H_0^2(0,1)$

$$\int_0^1 \left(\frac{d^2}{dt^2}u(t) + \int_0^t \left(g_R\left(s, |\frac{d}{ds}u(s)|\right)\frac{d}{ds}u(s)\right)ds\right)\frac{d^2}{dt^2}w(t)\,dt$$
$$-\int_0^1\int_0^t\left(\int_0^s f(\tau, u(\tau), v(\tau))d\tau\right)ds\frac{d^2}{dt^2}w(t)\,dt = 0.$$

Now using Lemma 17 and differentiating twice, we obtain the assertion. □

We say that a function $u \in H_0^2(0,1)$ is a **classical solution** to (14), if it is a weak solution, if it satisfies (14) a.e. on $[0,1]$ and if u, $\frac{d}{dt}u$, $\frac{d^2}{dt^2}u$, $\frac{d^3}{dt^3}u$ are absolutely continuous and $\frac{d^4}{dt^4}u \in L^2(0,1)$.

Remark 19. Proposition 18 implies that under conditions **A1, A2, A3** any weak solution is a classical one.

In order to proceed further we define the operator $A : H_0^2(0,1) \times L^2(0,1) \to \left(H_0^2(0,1)\right)^*$ by

$$\langle A(u,v), w\rangle = \int_0^1 \frac{d^2}{dt^2}u(t)\,\frac{d^2}{dt^2}w(t)\,dt + \int_0^1 g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\,\frac{d}{dt}w(t)\,dt$$
$$-\int_0^1 f(t, u(t), \frac{d}{dt}u(t), v(t))w(t)\,dt. \tag{15}$$

Lemma 20. *Under conditions* ***A1, A2, A3****, the operator* A *satisfies the assumptions of Corollary* 5.

Proof. Let us define $A_1, A_2 : H_0^2(0,1) \to \left(H_0^2(0,1)\right)^*$ by

$$\langle A_1(u), w\rangle = \int_0^1 \frac{d^2}{dt^2}u(t)\,\frac{d^2}{dt^2}w(t)\,dt,$$
$$\langle A_2(u), w\rangle = \int_0^1 g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\,\frac{d}{dt}w(t)\,dt,$$

and $A_3 : H_0^2(0,1) \times L^2(0,1) \to \left(H_0^2(0,1)\right)^*$ by

$$\langle A_3(u,v), w\rangle = -\int_0^1 f(t, u(t), \frac{d}{dt}u(t), v(t))w(t)\,dt.$$

Then (15) results in

$$A(\cdot, v) = A_1 + A_2 + A_3(\cdot, v) \text{ for } v \in L^2(0,1).$$

The weak continuity of A_1 follows from its linearity and continuity. As shown in [6], A_1 is coercive, bounded and satisfies condition (S). Concerning the operator A_2, from [6] we know that it is bounded and strongly continuous, thus it is also weakly continuous. The operator A_3 is bounded due to assumption **A2**. We prove that it is (weakly,norm)→weakly continuous. Take a sequence $(u_n)_{n=1}^{\infty}$ which is weakly convergent to some u_0 in $H_0^2(0,1)$ and a sequence $(v_n)_{n=1}^{\infty}$ norm convergent to some $v_0 \in L^2(0,1)$. Then both $(u_n)_{n=1}^{\infty}$ and $(\dot{u}_n)_{n=1}^{\infty}$ are norm convergent in $L^2(0,1)$. Therefore by assumption **A2** it holds for a.e. $t \in [0,1]$ that

$$\left| f(t, u_n(t), \frac{d}{dt}u_n(t), v_n(t))w(t)\right| \le g(t), \qquad \text{for all } n \in \mathbb{N},$$

for some function $g \in L^1(0,1)$. Hence we can apply the Lebesgue Dominated Convergence Theorem reaching the continuity claim.

Now we focus on the uniform coercivity. Arguing as for (12) we get for all $u \in H_0^2(0,1)$ and $v \in L^2(0,1)$ that

$$\frac{\langle A(u), u\rangle}{\|u\|_{H_0^2}} \ge \left(1 - \left(\frac{\|a\|_{L^\infty}}{\pi^4} + \frac{\|b\|_{L^\infty}}{\pi^2}\right)\right)\|u\|_{H_0^2} - \frac{\|c\|_{L^1}}{\|u\|_{H_0^2}},$$

which proves the thesis. □

From Lemma 20, Remark 19 and Corollary 5 it follows that:

Theorem 21. *Assume that conditions* ***A1, A2, A3*** *are satisfied. Let $(v_n)_{n=1}^{\infty}$ be a sequence of parameters which is norm convergent to some v_0 in $L^2(0,1)$. Then for each $n \in \mathbb{N} \cup \{0\}$ there is at least one classical solution u_n to problem* (14) *corresponding to v_n. Moreover, there is subsequence of (u_n) convergent weakly to u_0 in $H_0^2(0,1)$.*

We observe that any solution to problem (14) solves in fact (9). This is true because of the choice of $R > 0$ in (13) complying with Lemma 16.

We can state the main existence result of this subsection.

Theorem 22. *Assume that conditions* ***A1, A2, A3*** *are satisfied. Let $(v_n)_{n=1}^{\infty}$ be a sequence of parameters which is norm convergent to some v_0 in $L^2(0,1)$. Then for each $n \in \mathbb{N} \cup \{0\}$ there is at least one classical solution u_n to problem (9) corresponding to v_n. Moreover, there is subsequence of (u_n) convergent weakly to u_0 in $H_0^2(0,1)$.*

3.2. Results by Theorem 11

Now we consider a potential version of (9), specifically the Dirichlet problem

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g\left(t, 2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right)\frac{d}{dt}u(t)\right) = f(t, u(t), \\ \quad v(t)) \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{16}$$

with a functional parameter $v \in L^2(0,1)$ and functions $g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ satisfying **A1** and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is subject to the following conditions where $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t,x,z) = \int_0^x f(t,s,z)\, ds \ \text{ for a.e. } t \in [0,1] \text{ and all } x, z \in \mathbb{R}.$$

A4 $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function such that there exist* $a_1 \in L^\infty(0,1)$ and $b_1 \in L^2(0,1)$ *for which*

$$|f(t,x,z)| \leq a_1(t)|x| + b_1(t)$$

for a.e. $t \in [0,1]$ *and all* $x, z \in \mathbb{R}$.

A5 *There exist* $a \in L^\infty(0,1)$ *and* $b \in L^1(0,1)$ *such that*

$$\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4} > 0, \tag{17}$$

and for a.e. $t \in [0,1]$ *and all* $x, z \in \mathbb{R}$ *it holds*

$$F(t,x,z) \leq a(t)|x|^2 + b(t).$$

We will follow the same pattern as in the results of the previous subsection with the necessary changes arising from the fact that now the associated nonlinear operator is not coercive. For the same reason we cannot invoke [6].

Our goal is to develop a variational approach for problem (16). Let us fix parameter $v \in L^2(0,1)$. Notice that assumption **A1** does not guarantee that the term

$$\int_0^1 g\left(t, 2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right)\frac{d}{dt}u(t)\frac{d}{dt}v(t)\, dt \tag{18}$$

can define a potential operator on $H_0^2(0,1)$. To overcome this difficulty we set

$$R = \left(\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4}\right)^{-1}\frac{1}{\pi^2}\|b\|_{L^2}, \tag{19}$$

which is a positive number due to (17), and define the cut-off function g_R by formula (13). It follows from [6] that (18) with $g_R(t,\cdot)$ in place of $g(t,\cdot)$ defines a potential operator with the potential

$$\int_0^{\frac{1}{2}\left|\frac{d}{dt}u(t)\right|^2} g_R(t,s)s ds.$$

Instead of problem (16) we consider the problem

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g_R\left(t, 2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right)\frac{d}{dt}u(t)\right) = f(t, u(t), v(t)) \\ u(0) = u(1) = 0,\ \dot{u}(0) = \dot{u}(1) = 0 \end{cases} \tag{20}$$

whose weak solutions are exactly the critical points of the Euler action integral $J_v : H_0^2(0,1) \to \mathbb{R}$ given by

$$J_v(u) = \frac{1}{2}\int_0^1 \left|\frac{d^2}{dt^2}u(t)\right|^2 dt + \int_0^1 G\left(2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right) dt - \int_0^1 F(t, u(t), v(t))\, dt, \tag{21}$$

with

$$G(t, x) = \int_0^x g_R(t, s)\, ds.$$

The next results establish properties of the functional J_v in (21).

Lemma 23. *Assume that conditions* ***A1, A4*** *are satisfied. Let* $v \in L^2(0,1)$ *be fixed. The functional* $J_v : H_0^2(0,1) \to \mathbb{R}$ *is continuously differentiable with the differential at any* $u \in H_0^2(0,1)$ *given by*

$$\langle J_v'(u), w\rangle = \int_0^1 \frac{d^2}{dt^2}u(t)\frac{d^2}{dt^2}w(t)dt + \int_0^1 g_R\left(t, 2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right)\frac{d}{dt}u(t)\frac{d}{dt}w(t)dt - \int_0^1 f(t,u(t),v(t))w(t)dt \text{ for all } w \in H_0^2(0,1). \tag{22}$$

Proof. The conclusion is achieved by standard arguments that we omit here. □

Lemma 24. *Under assumptions* ***A1, A4*** *the operator* $J_v' : H_0^2(0,1) \to \left(H_0^2(0,1)\right)^*$ *expressed in* (22) *is weakly continuous.*

Proof. This is the consequence of the results in [6] (see also the proof of Lemma 20). □

Lemma 25. *Assume that conditions* ***A1, A4, A5*** *are satisfied. Let* $v \in L^2(0,1)$ *be fixed. Then the functional* J_v *in* (21) *is coercive and bounded from below. Moreover, for the number* $R > 0$ *introduced in* (19)*, we have that* $\|u\|_{H_0^2} \le R$ *and* $\|\dot{u}\|_C \le R$ *whenever* $u \in H_0^2(0,1)$ *is a global minimizer of* J_v *in* (21).

Proof. We see by hypothesis **A1** that

$$\int_0^1 G\left(2^{-1}\left|\frac{d}{dt}u(t)\right|^2\right) dt = \int_0^1 \int_0^{\frac{1}{2}\left|\frac{d}{dt}u(t)\right|^2} g_R(t,s)dsdt \ge \frac{1}{2}g_0\left\|\frac{d}{dt}u\right\|_{L^2}^2 \ge 0, \text{ for all } u \in H_0^2(\Omega).$$

Using hypothesis **A5** we obtain

$$\int_0^1 F(t,u(t),v(t))dt \le \|a\|_{L^\infty} \int_0^1 |u(t)|^2 dt + \|b\|_{L^1} \le \frac{\|a\|_{L^\infty}}{\pi^4} \|u\|_{H_0^2}^2 + \|b\|_{L^1}$$

for all $u \in H_0^2(\Omega)$. The preceding estimates show for any $u \in H_0^2(\Omega)$ that

$$J_v(u) \ge \left(\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4} \right) \|u\|_{H_0^2}^2 - \|b\|_{L^1}.$$

By (17) we have $\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4} > 0$, so the functional J_v is coercive.

Let now $u \in H_0^1(0,1)$ be a solution to problem (16) which is a global minimizer of J_v in (21). Then we obtain

$$0 = J_v(0) \ge J_v(u) \ge \left(\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4} \right) \|u\|_{H_0^2}^2 - \|b\|_{L^1}$$

which implies that

$$\|u\|_{H_0^2} \le \|b\|_{L^1} \left(\frac{1}{2} - \frac{\|a\|_{L^\infty}}{\pi^4} \right)^{-1}$$

and therefore we can take R indicated in (19). □

We are led to the following existence result.

Proposition 26. *Assume that conditions* ***A1, A4, A5*** *are satisfied. Then for every $v \in L^2(0,1)$ there exists a classical solution to problem* (16).

Proof. Lemmas 23 and 24 enable us to apply Theorem 7 ensuring the existence of a classical solution u_R to problem (20). This is valid because the operator J_v' in (22) satisfies condition (S) (see [6, Theorem 2]). By Lemma 25 we know that u_R is a classical solution to problem (16), too. □

We can state the main existence result regarding the dependence on parameters.

Theorem 27. *Assume that conditions* ***A1, A4, A5*** *are satisfied. Let $(v_n)_{n=1}^\infty$ be a sequence of parameters which is norm convergent to some $v_0 \in L^2(0,1)$. Then for each $n \in \mathbb{N} \cup \{0\}$ there is at least one classical solution u_n to problem* (16) *corresponding to v_n. Moreover, there is a subsequence of (u_n) norm convergent to u_0.*

Proof. We show that the assumptions of Theorem 11 are satisfied. Condition (i) has been verified in Lemma 23, while condition (iii) follows from Lemma 24. The proof of condition (ii) can be done arguing with the operator $A : H_0^2(0,1) \times L^2(0,1) \to \left(H_0^2(0,1)\right)^*$ defined in (15) as in the proof of Lemma 20. Note that operator $J_v' : H_0^2(0,1) \to \left(H_0^2(0,1)\right)^*$ defined in (22) satisfies condition (S). Application of Theorem 11 completes the proof. □

3.3. Final Comments and Examples

Concerning the concrete models about the fourth order boundary value problem connected with the beam equation, in the sources mentioned in the Introduction the authors mainly considered, as we do here, rigidly fastened beams, i.e. fourth order equation

$$\frac{d^4}{dt^4}u = f\left(t, u\right) \tag{23}$$

pertaining to boundary conditions

$$u\left(0\right) = x\left(1\right) = \dot{u}\left(0\right) = \dot{u}\left(1\right) = 0 \tag{24}$$

or simply supported beams, i.e. the Eq. (23) with conditions

$$u\left(0\right) = u\left(1\right) = \ddot{u}\left(0\right) = \ddot{u}\left(1\right) = 0$$

are considered. Equation (23) is a simplified version of the following one

$$\frac{d^2}{dt^2}\left(E\left(t\right)I\left(t\right)\frac{d^2}{dt^2}u\left(t\right)\right) + w\left(t\right)u\left(t\right) = f\left(t, u\left(t\right)\right)$$

with suitable assumptions placed on f and where $E : [0,1] \to R$ is Young's modulus of elasticity for the beam, $I : [0,1] \to R$ is the moment of inertia of cross section of the beam and w is the load density (force per unit length of a beam). It is usually assumed that that $w\left(t\right) > 0$, $E\left(t\right) \geq E_0 > 0$, $I\left(t\right) \geq I_0 > 0$ for $t \in [0,1]$ and that E, $I, w \in L^\infty\left(0,1\right)$. Connected to this model we have the following direct result about the existence and continuous dependence on parameters:

Theorem 28. *Assume that conditions **A4**, **A5** are satisfied. Let $\left(v_n\right)_{n=1}^\infty$ be a sequence of parameters which is norm convergent to some $v_0 \in L^2\left(0,1\right)$. Then for each $n \in \mathbb{N} \cup \{0\}$ there is at least one classical solution u_n to problem*

$$\begin{array}{l} \frac{d^2}{dt^2}\left(E\left(t\right)I\left(t\right)\frac{d^2}{dt^2}u\left(t\right)\right) + w\left(t\right)u\left(t\right) = f\left(t, u\left(t\right), v\left(t\right)\right) \\ u\left(0\right) = x\left(1\right) = \dot{u}\left(0\right) = \dot{u}\left(1\right) = 0 \end{array} \tag{25}$$

corresponding to v_n. Moreover, there is subsequence of $\left(u_n\right)$ norm which is convergent to u_0.

We note that in case functions E, I are not constant we cannot apply Theorem 13 here.

Now we provide some examples of nonlinear terms which satisfy our assumptions.

Example 29. Concerning the nonlinear perturbation $g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ we may consider the following unbounded from above function

$$g\left(t, x\right) = e^{x^2}\left(2 + \arctan\left(t\right)\right)$$

which is bounded from below.

Example 30. Concerning the nonlinear term $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying **A2**, **A3** we may consider the following function (where we drop the dependence on t for clarity)

$$f(x, y, z) = a \ln\left(x^2 + 1\right) + 2axe^{-z^2} + \frac{2y}{\pi} \arctan z$$

which satisfies the required growth conditions with $0 < a < \pi^4 - \pi^2$.

Example 31. Concerning the nonlinear term $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying conditions **A4**, **A5** we propose the function related to Remark 8, namely we put

$$F(x, z) = \begin{cases} a\left(x^2 - 1\right) e^{-z^2}, & x \leq 2 \\ a(4x - 5) e^{-z^2}, & x > 2 \end{cases}$$

and therefore

$$f(x, z) = \begin{cases} 2axe^{-z^2}, & x \leq 2 \\ 4xae^{-z^2}, & x > 2 \end{cases}$$

where $a \in \left(0, \frac{\pi^4}{2}\right)$.

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