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Relativity of Reductive Chain Complexes of Non-abelian Simplexes

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Abstract. Chain total double complexes with reductive differentials for non-abelian simplexes with associated spaces are considered. It is conjectured that corresponding relative cohomology is equivalent to the coset space of vanishing functionals over non-vanishing functionals related to differentials of complexes. The conjecture is supported by the theorem for the case of spaces of correlation functions and generalized connections on vertex operator algebra bundles.

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1. Introduction

It is natural to consider non-abelian simplexes with associated spaces [1, 8, 40] and corresponding cohomology. In [8] spectral sequences for ordinary simplexes with associated functional spaces were studied. In [10] it was shown that the Gelfand–Fuks cohomology of vector fields on a smooth compact manifold M is isomorphic to the singular cohomology of the space of continuous cross sections of a certain fiber bundle over M. Passing to a non-abelian simplex setup, one would be interested in construction of explicit examples of chain complexes, spectral sequences, and relations to geometrical structures of associated manifolds. In [10, 18, 22, 27, 40, 50, 53–55, 58, 62] cohomology of non-commutative structures with associated manifolds was studied.

In this paper we consider chain total double complexes of non-abelian simplexes with associated spaces and reductive differentials. The reductivity

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property explained in the text allows to prove the relativity of corresponding cohomology, as well as its equivalence to coset spaces of functionals associated to differentials of chain complexes. The main conjecture 1 is illustrated by the explicit proof of Theorem 1 describing a particular case of the simplex, the total chain double complex of associated spaces of correlation functions, and intrinsic invariant bundle for a vertex operator algebra [9,14,19,20,20,25,26, 30,56] considered on Riemann surfaces [16] of various genus. The geometrical meaning of the theorem provides a vertex operator algebra description of Bott–Segal relation [10] for Lie algebras.

1.1. Double Complex Families with Reductive Differentials

Let X be (non necessary commutative) space of simplexes endowed with a double filtration $X=\bigcup_{\kappa,n\geq 0} X_{\kappa,n}$, with an associated functional space $\mathcal{C}^{\kappa,n}(X_{\kappa,n})$. Let us define reductive differentials $\mathcal{D}^\kappa=\mathcal{D}^\kappa(X)$, $\mathcal{D}^n=\mathcal{D}^n(X)$ such that

$$\left(X_{\kappa+1,n}, \mathcal{C}^{\kappa+1,n}\right) = \mathcal{D}^\kappa. \left(X_{\kappa,n}, \mathcal{C}^{\kappa,n}\right), \quad \left(X_{\kappa,n+1}, \mathcal{C}^{\kappa,n+1}\right) = \mathcal{D}^n. \left(X_{\kappa,n}, \mathcal{C}^{\kappa,n}\right). \tag{1.1}$$

Requiring single chain complex property with respect to each of the differentials

$$\mathcal{D}^{\kappa+1}\circ\mathcal{D}^\kappa. \left(\mathcal{C}^{\kappa,n}\right)=0, \qquad \mathcal{D}^{\mathrm{n}+1}\circ\mathcal{D}^n. \left(\mathcal{C}^{\kappa,n}\right)=0, \tag{1.2}$$

and the double complex property

$$\left(\mathcal{D}^\kappa\circ\mathcal{D}^n-\mathcal{D}^n\circ\mathcal{D}^\kappa\right). \left(\mathcal{C}^{\kappa,n}\right)=0, \tag{1.3}$$

the diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow \mathcal{D}^{\kappa-1} & & \downarrow \mathcal{D}^{\kappa-1} & & \\
\cdots & \longrightarrow & \mathcal{C}^{\kappa,n} & \stackrel{\mathcal{D}^{\kappa,n}}{\longrightarrow} & \mathcal{C}^{\kappa,n+1} & \longrightarrow & \cdots \\
 & & \downarrow \mathcal{D}^{\kappa} & & \downarrow \mathcal{D}^{\kappa} & & \\
\cdots & \longrightarrow & \mathcal{C}^{\kappa+1,n} & \stackrel{\mathcal{D}^{\kappa+1,n}}{\longrightarrow} & \mathcal{C}^{\kappa+1,n+1} & \rightarrow & \cdots \\
 & & \downarrow \mathcal{D}^{\kappa+1} & & \downarrow \mathcal{D}^{\kappa+1} & & \\
 & & \vdots & & \vdots & &
\end{array} \tag{1.4}$$

is then commutative. For countable direct sums of $C^{\kappa,n}(X_{\kappa,n})$, one introduces the total complex $\mathcal{C}^m=\bigoplus_{m=\kappa+n}\mathcal{C}^{\kappa,n}(X_{\kappa,n})$, Corresponding differential is given by

$$d^m=\mathcal{D}^\kappa+(-1)^\kappa\mathcal{D}^n, \qquad d^m\circ d^{m-1}. \left(\mathcal{C}^m\right)=0, \tag{1.5}$$

with the cohomology of the total complex $(d^m, \mathcal{C}^m)$ defined in the standard way. We call the single part of a complex $\mathcal{C}^{\kappa,n}$ reductive if $\mathcal{C}^k=\mathcal{D}\circ\cdots\circ\mathcal{D}.\mathcal{C}^0=P(k).\mathcal{C}^0$ with some operators $P(k)$, $k\geq 0$, and $\mathcal{D}$ is a finite combination of $\mathcal{D}^\kappa$ and $\mathcal{D}^n$.

Introduce now the maps Φ, $\Psi : X \to Y$; F, $\mathcal{G} : Y \to W$, the action $\Psi.\Phi : Y \times Y \to Y$, for spaces Y, W, and a map $G : Y \to W \times W$, $x \in X$, of the form

$$G(\Psi, \Phi) = F(\Psi(x')).\mathcal{G}(\Phi(x)) + F(\Phi(x)).\mathcal{G}(\Psi(x')) + \sum_{\substack{x'_0, x_0 \\ \subset X}} \mathcal{G}\left(F(\Psi(x'_0)).\Phi(x_0)\right). \tag{1.6}$$

For a double-filtered X denote by $G^m = \left\{\bigoplus_{m=\kappa+n} G(x), x \in X_{\kappa,n}\right\}$, the space of functionals $G(\Psi, \Phi)$ satisfying 1.6, and by Con^m the space of vanishing $G(\Psi, \Phi)$. Let us fix the maps $\mathcal{G}$, Φ, Ψ. $G(\Psi, \Phi)$ depends on the map F as a functional. Suppose that the differential of the total complex $d = d(F)$ is also a functional on F. If we fix a subspace $\mathfrak{F}$ of maps F, then the cohomology $H^m(C^m, \mathfrak{F})$ of the total complex is a relative cohomology with respect to $\mathfrak{F}$.

We call $G(\Psi, \Phi)$ covariant with respect to the differentials if (1.6) remains of the same general form under arbitrary combinations of $\mathcal{D}^\kappa$ and $\mathcal{D}^n$. We then formulate the following conjecture which is a counterpart of a proposition of [10,50], i.e., the Bott-Segal theorem.

Conjecture 1. The relative cohomology of the reductive chain total complex $(d^m(F),\ C^m)$ is equivalent to the coset space Con^m/G^m for some m. For G covariant with respect to d^m, the equivalence extends to all $m \geq 0$.

Note that the vanishing (1.6) represents a version of Leibniz rule. Thus, the cohomology relation above measures the inclination of F and $\mathcal{G}$ from that rule.

Our main example of the construction above is provided by the space X of n-simplexes of pairs $x = (v, z)$ of a vertex operator algebra elements $v \in V$ and a formal parameter z, and the space $\mathcal{C} = C^{\kappa,n}(V)$ of vertex operator algebra V-module W n-point correlation functions [5,6,12,13,15,21,28,42,43,52,57]. considered on a Riemann surface of genus $\kappa = g$. Due to the structure of correlations functions and reduction relations [11,23,32–38,44–49,52] one can form chain complexes of converging n-point functions. In this paper we assume that all n-point correlation functions are reductive to corresponding zero-point correlation function at any genus of Riemann surfaces.

Recall the notion of a vertex operator algebra bundle given in "Appendix 5". Motivated by the definition of a holomorphic connection for a vertex operator algebra bundle (cf. Sect. 6, [6] and [24]) over a smooth complex curve, we introduce the definition of the multiple point connection of the vertex operator algebra bundle (see also [43]) over a direct product $\bigoplus_{g\geq 0} \Sigma^{(g)}$. With $G(\Psi, \Phi) = 0$, the map $\mathcal{G}$ provides a generalization of the classical holomorphic connection over a smooth variety. We call the functional $G(\Psi, \Phi)$ (1.6) the form of connection. The main results of this paper is the following theorem.

Theorem 1. *The relative cohomology $H^m(W, \mathfrak{F})$ of the chain total D^n-reductive complex $(d^m\ (F),\ C^m)$ of a vertex operator algebra V-module W correlation* *functions on the direct product of Riemann surfaces is isomorphic to the factor space of D^g- and D^n-covariant connections $\mathcal{G}$ over the space of G^{m-1} of $(m-1)$-forms on corresponding V-module W-bundle $\mathcal{W}$.*

The plan of the paper is the following. In Sect. 2 we recall the notion of vertex operator algebra V-module correlation functions, construct single chain complexes, and reductive differentials. In Sect. 3 we describe the total chain complex. Section 4 contains a proof of the main result of this paper, Theorem 1. In "Appendix 5" we recall the notion of a vertex operator algebra and its properties. "Appendix 6" is devoted to a formalism of composing a genus $g+1$ Riemann surface starting from a genus g Riemann surface. "Appendix 7" reviews classical and generalized elliptic functions. In "Appendix 8" examples of spaces of correlation functions and reduction formulas are provided.

The results of this paper may be interesting in various fields of mathematics including mathematical physics [13,21,42], Riemann surface theory [16,24,27], theta-functions [17,39], cosimplisial geometry of manifolds [8,18,50], non-commutative geometry, modular forms [7,29,38,41], and the theory of foliations [2–4,8,59–61].

2. The Families of Chain Complexes for Vertex Operator Algebra Correlation Functions

2.1. Spaces of Correlation Functions

In this Section we introduce a family of chain complexes of correlation functions a vertex operator algebra V-module W on a genus g Riemann surface. Let us fix a vertex operator algebra V. Depending on its commutation relations and configuration of a genus g Riemann surface $\Sigma^{(g)}$, the space of all V-module W multipoint functions may represent various forms of complex functions defined on $\Sigma^{(g)}$. Consider by $\mathbf{v}_{n,g} = (v_{1,g}, \ldots, v_{n,g}) \in V^{\otimes n}$ a tuple of vertex operator algebra elements. Pick n points on a Riemann surface $\Sigma^{(g)}$. Denote by $\mathbf{z}_{n,g} = (z_{1,g}, \ldots, z_{n,g})$ local coordinates around that points. Let us introduce our standard notation: $\mathbf{x}_{n,g} = (\mathbf{v}_{n,g}, \mathbf{z}_{n,g})$. Note that we use such notations to emphasize that the elements $\mathbf{x}_{n,g}$ may be chosen different for different genuses.

Let $B^{(g)} \subset \mathcal{B}^{(g)}$ be moduli parameters describing $\Sigma^{(g)}$. Here $\mathcal{B}^{(g)}$ is the set of moduli parameters for all genus g Riemann surfaces. In particular, $\mathcal{B}^{(g)}$ characterizes a geometrical way if $\Sigma^{(g)}$ was constructed in a sewing procedure [51]. As we mentioned in Introduction, for each genus an $n \geq 0$-point vertex operator algebra V-module W correlation function $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}; B^{(g)}\right)$ on $\Sigma^{(g)}$ has a certain specific form. It depends on g, $B^{(g)}$, the way a Riemann surface $\Sigma^{(g)}$ was formed, the type of conformal field theory model used for definitions of multipoint functions, and the type of commutation relations for V-elements. We assume that, for a fixed Riemann surface set of parameters

$B^{(g)}$, multiple point functions are completely determined by all choices of $\mathbf{x}_{n,g} \in V^{\otimes n} \times \left(\Sigma^{(g)}\right)^n$. Thus, in the ρ-sewing procedure described in "Appendix 6", the reduction cohomology can be treated as depending on the set of $\mathbf{x}_{n,g}$ only with appropriate action of endomorphisms generated by $x_{n+1,g}$.

For $\Sigma^{(g)}$, a V-module W, and $n \geq 0$, $\mathbf{x}_{n,g} \in V^{\otimes n} \times \left(\Sigma^{(g)}\right)^n$, we consider the spaces of all multipoint correlation functions $C^{g,n}(W) = \left\{\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}; B^{(g)}\right)\right\}$. Note that we choose elements of $\mathbf{v}_n$ belong to the same V-module W. A construction with different V-modules W_i will be considered elsewhere. Since we fix a vertex operator algebra module W and $B^{(g)}$ we will omit them in what follows where it is possible.

2.2. Single Chain Complexes for Vertex Operator Algebra Multipoint Functions

In this subsection we recall the definition of a single chain complex with respect to the number of points, and introduce a single complex with respect to the raise of genus of corresponding Riemann surface. The differentials for corresponding complexes are constructed according to the previous experience [11,23,32,37,45,49] in applying the reduction procedure to vertex operator algebra n-point functions. For $g \geq 0$, $n \geq 0$, define

$$D^n(x_{n+1,g}, g) : C^{g,n} \to C^{g,n+1},$$

$$D^n(x_{n+1,g}, g) = D_1^n(x_{n+1,g}, g) + D_2^n(x_{n+1,g}, g), \tag{2.1}$$

$$\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n+1,g}\right) = D^n(x_{n+1,g}, g)\, \mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right), \tag{2.2}$$

with differentials $D_1^n(x_{n+1,g}, g)$, $D_2^n(x_{n+1,g}, g)$ given by

$$\begin{aligned} D_1^n(x_{n+1,g}, g).\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right) &= \sum_{l=1}^{l(g)} f_1^{(g)} \left(x_{n+1,g}, l\right)\ T_l^{(g)}(x_{n+1,g}).\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right), \\ D_2^n(x_{n+1,g}, g).\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right) &= \sum_{k=1}^{n} \sum_{m \geq 0} f_2^{(g)}(x_{n+1,g}, k, m)\ T_k^{(g)}(v_{n+1,g}(m)). \\ &\quad \mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right), \end{aligned} \tag{2.3}$$

where $l(g) \geq 0$ is a constant depending on g, and the meaning of indices $1 \leq k \leq n$, $1 \leq l \leq l(g)$, $m \geq 0$ explained below. Then the operator $T_l^{(g)}(x_{n+1}).\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right)$ gives a function of $F^{(g)} \left(\mathbf{x}_{n,g}\right)$ depending on $x_{n+1,g}$. The operator

$$T_k^{(g)}(v_{n+1,g}(m)).\mathcal{F}_W^{(g)} \left(\mathbf{x}_{n,g}\right) = \mathcal{F}_W^{(g)} \left(T_k(v_{n+1,g}(m)).\mathbf{x}_{n,g}\right),$$

is the insertion of the m-th mode $v_{n+1,g}(m)$ (or $v_{n+1,g}[m]$-mode depending on g), $m \geq 0$. of vertex operator algebra elements $v_{n+1,g}$, in front of the k-th argument

$v_{k,g}$ of $x_{k,g}$ inside the k-th vertex operator in the functional $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right)$. Here we use the notation

$$T_k^{(g)}(\gamma).\ f(\mathbf{x}_{n,g}) = f\left(x_{1,g}, \ldots, \gamma.x_{k,g}, \ldots, x_{n,g}\right),$$

for an operator γ acting on k-th argument of a functions f. Note that commutation properties of $D_1^n(x_{n+1,g}, g)$ and $D_2^n(x_{n+1,g}, g)$ depend on genus g. Operator-valued functions $f_1^{(g)}\left(x_{n+1,g}, l\right)\ T_l^{(g)}(v_{n+1,g})$, $f_2^{(g)}(x_{n+1,g}, k, m)$. $T_k^{(g)}(v_{n+1,g}(m))$ depend on genus of a Riemann surface $\Sigma^{(g)}$. For $n \geq 0$, let us denote by $\mathfrak{V}_n$ the subsets of all $x_{n+1,g} \in V \times \Sigma^{(g)}$, such that the chain condition

$$D^{n+1}(x_{n+2,g}, g) \circ D^n(x_{n+1,g}, g).\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right) = 0, \tag{2.4}$$

for the differentials (2.3) for complexes $C^{g,n}$ is satisfied.

Next, consider the differentials

$$D^g : C^{g,n} \to C^{g+1,n}, \quad \mathcal{F}_W^{(g+1)}\left(\mathbf{x}_{n,g+1}; B^{(g+1)}\right) = D^g.\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}; B^{(g)}\right), \tag{2.5}$$

for V-module W on a genus g Riemann surface. There exist [7,23,28,32,34–36,45,46,49,51] various geometrical ways how to increase the genus of a Riemann surface, and, therefore, ways how to introduce corresponding differential D^g. In this paper we will use the ρ-formalism of attaching a handle to a genus g Riemann surface to form a genus $g+1$ Riemann surface (see "Appendix 6"). In this geometric setup the differential D^g is given by

$$\begin{aligned}
\mathcal{F}_W^{(g+1)}\left(\mathbf{x}_{n,g+1}; B^{(g+1)}\right) &= D^g \cdot \mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}; B^{(g)}\right), \\
\mathcal{F}_W^{(g+1)}\left(\mathbf{x}_{n,g+1}; B^{(g+1)}\right) &= \sum_{k\geq 0} \sum_{w_k \in W_{(k)}} \rho_g^k\ \mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}, \overline{w}_k, \zeta_1, w_k, \zeta_2; B^{(g)}\right) \\
&= \sum_{k\geq 0} \sum_{w_k \in W_{(k)}} \rho_g^k T(\overline{w}_k, \zeta_1, w_k, \zeta_2).\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}; B^{(g)}\right),
\end{aligned} \tag{2.6}$$

Note that in this formulation the differential D^g does not depend on n. It is assumed that 2.6 converges in ρ for W. The resulting expression for $\mathcal{F}_W^{(g+1)}$ $(\mathbf{x}_{n,g+1})$ depends on the positions of $(\overline{w}_k,\ \zeta_1)$, $(w_k,\ \zeta_2)$-insertions into $\mathcal{F}_W^{(g)}$ $(\mathbf{x}_{n,g})$ and their permutation properties with $\mathbf{x}_{n,g}$. Here we fix the position of insertion right agter the element $x_{n,g}$ as it was done in [44,45].

In this paper we consider $C^{g,n}$ as spaces of arbitrary $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right)$ not necessary obtained as a result of ρ-procedure from some $\mathcal{F}_W^{(g-1)}\left(\mathbf{x}_{n',g-1}\right)$ considered on a genus $g-1$ Riemann surface. At the same time we act on $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right)$ by the differential D^g which involves the ρ_g-sewing procedure. We assume also that $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n+1,g}\right)$ can be obtained from $\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right)$ via reduction formulas. The single chain complex condition for the differentials D^g (2.6) has the form

$$D^{g+1} \circ D^g.\mathcal{F}_W^{(g)}\left(\mathbf{x}_{n,g}\right) = 0. \tag{2.7}$$

For $g \geq 0$, let us denote by $\mathfrak{V}_g$ the subsets of all $x_{n,g} \in V \times \Sigma^{(g)}$, such that the single chain condition (2.7) for the differentials (2.6) of complexes $C^{g,n}$ is satisfied.

For the single chain complexes with differentials D^g and $D^n(x_{n+1,g}, g)$ given in this Section one defines corresponding partial cohomology in the standard way. Combining the formulations for two single complexes, we introduce the family of the single chain complexes $C^{g,n}$, $g \geq 0$, $n \geq 0$ for a vertex operator algebra V on Riemann surfaces. Using the differentials (2.3) and (2.5) one can compose the functional $G(\Psi, \Phi)$ (1.6). For each g and various types of vertex operator algebras, there exists standard sets of operators F [11,23,32–37,44–49, 52], i.e., $T(\overline{w}_k, \zeta_1, w_k, \zeta_2)$, $f_1^{(g)}(x_{n+1,g}, l)$ $T_l^{(g)}(x_{n+1,g})$, and $f_2^{(g)}(x_{n+1,g}, k, m)$ $T_k^{(g)}(v_{n+1,g}(m))$. determining the differentials D^g, $D^n(x_{n+1,g}, g)$.

3. The Chain Total Complex

In order to turn the family $C^{g,n}$ of single chain complexes into a double chain complex we have to apply further requirement of commutation on the differentials D^g and $D^n(x_{n+1,g}, g)$. In addition to the conditions (2.4) and (2.7) we require for the differentials D^g and $D^n(x_{n+1,g}, g)$ to satisfy

$$\left(D^g \circ D^n(x_{n+1,g}, g) - D^n(x_{n+1,g}, g) \circ D^g\right).\mathcal{F}_W^{(g)}(\mathbf{x}_{n,g}) = 0. \tag{3.1}$$

Then the family $C^{g,n}$ turns into a chain double complex. We denote by $\mathfrak{V}_{g,n}$ the subsets of all $x_{n+1,g} \in V \times \Sigma^{(g)}$, such that (3.1) is satisfied.

3.1. The Total Complex

For the double complex $C^{g,n}$ the associated total complex is given by $\mathrm{Tot}^m(C^{g,n}) = \bigoplus_{m=g+n} C^{g,n} = C^m$, for $m \geq 0$, with the differential

$$d^m : \mathrm{Tot}^m(C^{g,n}) \to \mathrm{Tot}^{m+1}(C^{g,n}), \quad d^m = \sum_{m=g+n} \left(D^g + (-1)^g D^n(x_{n+1,g}, g)\right). \tag{3.2}$$

Note that the differential $D^n(x_{n+1,g}, g)$ in (3.2) is defined for all choices of $x_{n+1,g} \in V \times \Sigma^{(g)}$, chosen separately for all possible combinations of g and n such that $m = g + n$. For $n \geq 0$, $g \geq 0$, $m = g + n$, let us denote by $\mathfrak{V}_d$ the subsets of all $x_{n+1,g} \in V \times \Sigma^{(g)}$, such that the chain conditions (2.4), (2.7), (3.1) and

$$d^{m+1} \circ d^m.(C^m) = 0, \tag{3.3}$$

for the differentials (2.3) for complexes $C^{g,n}$ are satisfied. Note that $\mathfrak{V}_g$, $\mathfrak{V}_n$, and $\mathfrak{V}_{g,n}$ are subsets of $\mathfrak{V}_d$. The spaces with conditions (2.4), (2.7), (3.1), and (3.3) constitute a semi-infinite chain double complex with the commutative diagram

$$
\begin{array}{llllll}
 & \vdots & & \vdots & & \\
 & \downarrow D^{g-1} & & \downarrow D^{g-1} \ldots & & \\
0 \longrightarrow & C^{g,0} & \xrightarrow{D^0(x_{1,g},g)} & C^{g,1} & \xrightarrow{D^1(x_{2,g},g)} \ldots \longrightarrow C^{g,n-1} \xrightarrow{D^{n-1}(x_{n,g},g)} & C^{g,n} \longrightarrow \\
 & \downarrow D^{g} & & \downarrow D^{g} \quad \ldots & & \\
0 \longrightarrow & C^{g+1,0} & \xrightarrow{D^0(x_{1,g+1},g+1)} & C^{g+1,1} & \xrightarrow{D^1(x_{2,g+1},g+1)} \ldots \xrightarrow{D^{n-1}(x_{n,g+1},g+1)} & C^{g+1,n} \longrightarrow \\
 & \downarrow D^{g+1} & & \downarrow D^{g+1} \quad \ldots & & \\
 & \vdots & & \vdots & &
\end{array}
$$

and standardly defined the reduction cohomology involving d^m of the total complex (3.2).

Due to vertex operator algebra properties, the conditions (2.4), (2.7), (3.1), and (3.3) result in expressions containing finite series of vertex operator algebra modes and coefficient functions. That conditions narrow the space of compatible elements $x_{n+1,g}$, and, therefore, corresponding multipoint functions $\mathcal{F}^{(g)}(\mathbf{x}_{n,g})$. Nevertheless, the subspaces of $C^{g,n}(W)$, $g \geq 0$, $n \geq 0$, of multipoint functions such that the conditions above are fulfilled for reduction cohomology complexes are non-empty. For all g, the conditions mentioned represent an infinite $n \geq 0$, $g \geq 0$ set of functional-differential equations (with finite number of summands) on converging complex functions $\mathcal{F}^{(g)}_W(\mathbf{x}_{n,g})$ defined for n local complex variables on Riemann surfaces of genus g with extra action of the operator $T(\overline{w}_k, \zeta_1, w_k, \zeta_2)$, with functional coefficients $f^{(g)}_1(x_{n+1,g}, l)$, $f^{(g)}_2(x_{n+1,g}, k, m)$. In examples given in "Appendix 8" the functional coefficients $f^{(g)}_1(x_{n+1,g}, l)$, $f^{(g)}_2(x_{n+1,g}, k, m)$ are genus g generalizations of elliptic functions on $\Sigma^{(g)}$. Note that all vertex operator algebra elements of $\mathbf{v}_n \in V^{\otimes n}$, as non-commutative parameters are not present in final form of functional-differential equations since they incorporated into either matrix elements, traces, and other forms in corresponding genus g multipoint functions. According to the theory of such equations, equations resulting from (2.4) (2.7), (3.1), and (3.3) always have non-vanishing solutions in the domains they are defined. Applying the reduction procedure by differentials $D^n(x_{n+1,g}, g)$ we reduce the functions $\mathcal{F}^{(g)}_W(\mathbf{x}_{n,g})$ to corresponding zero-point functions $\mathcal{F}^{(g)}_{W,0}$, i.e., we obtain in general $\mathcal{F}^{(g)}_W(\mathbf{x}_{n,g}) = P_n(F; g, n, \mathbf{x}_{n,g})\ \mathcal{F}^{(g)}_{W,0}$, where $P_n(F; g, n, \mathbf{x}_{n,g})$ are explicitly computable functions containing genus $g \geq 0$ generalized elliptic functions (see "Appendix 7"). For non-zero zero-point functions, equations (2.4), (2.7), (3.1), and (3.3) expressed for $P_n(F; g, n, \mathbf{x}_{n,g})$ can be solved by methods of the analytic number theory.

4. Proof of the Main Theorem

In this Section we provide a proof of Theorem 1. First, recall that the form of $\mathcal{F}^{(g)}_W(\mathbf{x}_{n,g})$ is specific for each g. The differential D^g (2.5) makes the genus transitions from $\mathcal{F}_W^{(g)}(\mathbf{x}_{n,g})$ to $\mathcal{F}_W^{(g+1)}(\mathbf{x}_{n,g+1})$. Recall the notion of a vertex operator algebra bundle given in "Appendix 5". The definition (5.2) corresponds to the case $g=0$ of the $\mathcal{W}^*$-section of the vertex operator algebra V bundle. One can see that (5.2) combined with the differential D^g (2.5) extends (5.2) to $g>0$ cases. Namely, we define $\mathcal{F}_{\mathcal{W},n}^{(g)}(\mathbf{x}_n)$ as follows

$$\begin{aligned}\mathcal{F}_{\mathcal{W}}^{(g)}(\mathbf{x}_n) &= \sum_{\mathbf{k}_g\in\mathbb{Z}^g,\ u_k\in V_{(k)}} \langle(\overline{u}_k, \mathbf{z}_n; \overline{w}_{k_j}, \zeta_{1,j}), \mathcal{Y}_{\mathcal{W}}(\mathbf{v}_n)\cdot(u_k, \mathbf{z}_n; w_{k_j}, \zeta_{2,j})\rangle \\ &= D^g\circ\cdots\circ D^0.\mathcal{F}_{\mathcal{W}}^{(0)}(\mathbf{x}_n),\end{aligned} \tag{4.1}$$

for $\zeta_{a,j}$, $a=1,2$, $u_k\in W_{(k)}$, $w_{k_j}\in W_{(k_j)}$, $1\le j\le g$, $\overline{u}_k$, $\overline{w}_{k_j}$ their corresponding duals, and where the action of D^g, …, D^0 is realized via (2.5). Note that (4.1) preserves the linearity properties in $\overline{u}$, u, and $\mathcal{O}_x$-linearity in $\mathbf{v}_n$. Let us mention that due to the relation (4.1) we could formulate all the material of Sects. 2–3 in terms of complexes constituted by $\mathcal{F}_{\mathcal{W},m}$.

Let us denote by $\mathcal{SW}$ the space of sections of the vertex operator algebra V-module W bundle $\mathcal{W}$. In our setup, by identifying Ψ and Φ with sections $\psi(x')$ and $\phi(x)$ of $\mathcal{W}$ correspondingly, the map $G(\Psi,\Phi)$ (1.6) with a $\mathbb{C}$-multilinear map $\mathcal{G}:\mathcal{SW}=\bigoplus_{m=g+n}\mathcal{W}^{\otimes n}\times\left(\Sigma^{(g)}\right)^n\to\mathbb{C}$, and for any operator F, turns into

$$G(\psi(\mathbf{x}'),\phi(\mathbf{x})) = F(\psi(\mathbf{x}')).\mathcal{G}\left(\phi(\mathbf{x})\right) + F(\phi(\mathbf{x})).\mathcal{G}\left(\psi(\mathbf{x}')\right) + \mathcal{G}\left(F(\psi(\mathbf{x}')).\phi(\mathbf{x})\right). \tag{4.2}$$

The vanishing map (4.2) gives raise to a generalization $\mathcal{G}$ of the holomorphic connection on $\mathcal{W}$. Geometrically, for a vector bundle $\mathcal{W}$ defined over $\Sigma^{(g)}$, the vanishing generalized connection (4.2) relates two sections $\psi(\mathbf{x}')$ and $\phi(\mathbf{x})$. Now we are ready to give a proof of Theorem 1

Proof. Let us denote $\mathcal{F}_{\mathcal{W},m}=\sum_{m=g+n}\mathcal{F}_{\mathcal{W}}^{(g)}(\mathbf{x}_{n,g})$. We assume that operators F satisfy (2.4), (2.7), (3.1), and (3.3). The definition (4.1) provides the coordinateless expression for $\mathcal{F}_{\mathcal{W},m}$, i.e.,

$$\mathcal{F}_{\mathcal{W},m} = \sum_{m=g+n}\ \sum_{\mathbf{k}_g\in\mathbb{Z}^g}\ \sum_{u_k\in V_{(k)}} \langle(\overline{u}_k,\mathbf{z}_n;\overline{w}_{k_j},\zeta_{1,j}),\mathcal{Y}_{\mathcal{W}}(i_z(\mathbf{v}_n))\cdot(u_k,\mathbf{z}_n;w_{k_j},\zeta_{2,j},)\rangle. \tag{4.3}$$

Using (2.3) and (2.6) we set for $m=g+n$, $m+1=g'+n'$,

$$\mathcal{G}\left(\phi(\mathbf{x})\right)=\mathcal{F}_{\mathcal{W},m},\ \psi(\mathbf{x}')=\mathcal{Y}_{\mathcal{W}}(i_z(\mathbf{v}_{n',g'}))\cdot(.,\mathbf{z}_{n',g'}),\ \phi(\mathbf{x})=\mathcal{Y}_{\mathcal{W}}(i_z(\mathbf{v}_{n,g}))\cdot(.,\mathbf{z}_{n,g}),$$

$$-F\left(\psi(\mathbf{x}')\right).\mathcal{G}\left(\phi(\mathbf{x})\right) = \left(\sum_{k\ge1}\ \sum_{w_k\in W_{(k)}} \rho_g^k\, T\left(\overline{w}_k,\zeta_1,w_k,\zeta_2\right)\right).\mathcal{F}_{\mathcal{W},m}, \tag{4.4}$$

$$-\mathcal{G}\left(F(\psi(\mathbf{x}')).\phi(\mathbf{x})\right) = (-1)^g \left[\sum_{l=1}^{l(g)} f_1^{(g)}\left(x_{n+1,g}, l\right)\ T_l^{(g)}(x_{n+1,g}) \right.$$
$$\left. + \sum_{k=1}^{n} \sum_{r\geq 0} f_2^{(g)}(x_{n+1,g}, k, r)\ T_k^{(g)}(v_{n+1,g}(r)) \right] .\mathcal{F}_{\mathcal{W},m}.$$

Now, let us assume that the form of the generalized connection $\mathcal{G} = \mathcal{F}_{\mathcal{W},m}$ remains the same for the vanishing $G^m \in Con^m$ and non-vanishing $G^m \in G^m$ given by (4.2), but the operator F, and the differentials $(D^n(x_{n+1,g}, g))'$ may differ from (4.4), (2.3)–(2.6), and satisfying conditions (2.7), (2.4), (3.1), and (3.3). By using the reduction procedure given by some other operators F' and some different differentials $(D^n(x_{n+1,g}, g))'$, we reduce (4.2) to

$$G(\psi(\mathbf{x}'), \phi(\mathbf{x})) = \sum_{m=g+n} P_m\left(F'; g, n, \mathbf{x}_{n,g}\right)\ \mathcal{F}_{\mathcal{W},0}^{(g)}, \tag{4.5}$$

where $P_m\left(F'; g, n, \mathbf{x}_{n,g}\right)$ are functions depending on operators F', and explicitly containing genus $g \geq 0$ generalized elliptic functions. Thus, $\mathcal{F}_{\mathcal{W},n}^{(g)}$ is explicitly known and it is represented as a series of auxiliary functions $P_m\left(F'; g, n, \mathbf{x}_{n,g}\right)$ depending on F', g, n, and $\mathbf{x}_{n,g}$.

Now let us consequently apply the initial differential $D^n(x_{n+1,g}, g)$ (2.3) to reconstruct back some functions $\mathcal{F}''_{\mathcal{W}}$ starting from each of $\mathcal{F}_{\mathcal{W}}^{(g)}$ for $m = g+n$ in (4.5). Finally, we obtain (4.2) for $\mathcal{G} = \mathcal{F}''_{\mathcal{W},m}$ with

$$\mathcal{F}''_{\mathcal{W},m} = \sum_{m=g+n} D^{n-1}(x_{n-1,g}) \circ \cdots \circ D^1(x_{1,g}) \circ P_m(F'; g, n, \mathbf{x}_{n,g}).\mathcal{F}_{\mathcal{W},0}^{(g)}.$$

Since the differentials D^g and $D^n(x_{n+1,g}, g)$ are defined in that way they act on the functions $\mathcal{F}_{\mathcal{W}}^{(g)}$ only, it is clear that the action of $D^{n-1}(x_{n-1,g}) \circ \ldots \circ D^1(x_1, g)$ and multiplication by $P_m(F'; g, n, \mathbf{x}_{n,g})$ commute.

Thus, we infer that the m-th reduction cohomology H^m of (d^m, C^m) is equivalent to the factor space Com^m/G^m with the coefficients given by the coset

$$\left\{ P_m(F'; g, n, \mathbf{x}_{n,g})|_{G(x',x)=0} / P_{m-1}(F'; g, n, \mathbf{x}_{n,g})|_{G(x',x)\neq 0} \right\},$$

of genus g counterparts of elliptic functions, and relative to the subspace of covariance preserving operators F' for the the functional $G\left(\psi(\mathbf{x}'), \phi(\mathbf{x})\right)$. The H^m-th relative reduction cohomology of a vertex operator algebra V-module W is then given by the ratio of series of generalized elliptic functions recursively generated by the reduction formulas (2.2)–(2.5). $\square$

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5. Appendix: Vertex Operator Algebras

In this Subsection we recall the notion of a vertex operator algebra $(V, Y, \mathbf{1}_V, \omega)$ [9,14,19,20,26,30]. Here V is a linear space endowed with a $\mathbb{Z}$-grading $V = \bigoplus_{r\in\mathbb{Z}} V_r$, $\dim V_r < \infty$. The state $0 \neq \mathbf{1}_V \in V_0$, is called vacuum vector, $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator is a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$, with formal variable z. For any vector $v \in V$, $x = (v, z)$, we have a vertex operator $Y(x) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$. The linear operators (which are called modes) $u(n) : V \to V$ satisfy creativity $Y(v, z)\mathbf{1}_V = v + O(z)$, and lower truncation $v(n)u = 0$, conditions for each u, $v \in V$ and $n \gg 0$. The vertex operators satisfy an analogue of Jacobi identity

$$\begin{aligned}\delta(z_1, z_2, z_0)Y(x_1)Y(x_2) - \delta(z_2, z_1, -z_0)Y(x_2)Y(x_1)\\ = z_0 z_2^{-1}\delta(z_1, z_0, z_2)Y(Y(u, z_0)v, z_2),\end{aligned}$$

for $x_1 = (u, z_1)$, $x_2 = (v, z_2)$, and $\delta(z, z', z'') = \delta((z' - z'')(z''')^{-1})$. These axioms imply locality, skew-symmetry, associativity and commutativity conditions:

$$\begin{aligned}(z_1 - z_2)^N Y(x_1)Y(x_2) &= (z_1 - z_2)^N Y(x_2)Y(x_1),\\ Y(u, z)v &= e^{zL(-1)}Y(v, -z)u,\\ (z_0 + z_2)^N Y(u, z_0 + z_2)Y(v, z_2)w &= (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w,\\ u(k)Y(v, z) - Y(v, z)u(k) &= \sum_{j\geq 0}\binom{k}{j} Y(u(j)v, z)z^{k-j},\end{aligned}$$

for u, v, $w \in V$ and integers $N \gg 0$. For the conformal vector ω one has $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, where $L(n)$ satisfies the Virasoro algebra with central charge c

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}\mathrm{Id}_V, \tag{5.1}$$

where Id_V is identity operator on V. Each vertex operator satisfies the translation property $\partial_z Y(u,z) = Y\left(L(-1)u, z\right)$. The Virasoro operator $L(0)$ defined a $\mathbb{Z}$-grading with $L(0)u = ru$, for $u \in V_r$, $r \in \mathbb{Z}$, $\mathrm{wt}(u) = r$. For $v = \mathbf{1}_V$ one has $Y(\mathbf{1}_V, z) = \mathrm{Id}_V$. Note also that modes of homogeneous states are graded operators on V, i.e., for $v \in V_k$, $v(n) : V_m \to V_{m+k-n-1}$. In particular, let us define the zero mode $o(v)$ as $o(v) = v(wt(v) - 1)$ additively extending to V.

Let us recall also the square-bracket formalism [52] for a vertex operator algebra V, i.e., the quadruple $(V, Y[.,.], \mathbf{1}_V, \tilde{\omega})$ The square bracket vertex operators are given by

$$Y[v,z] = \sum_{n\in\mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1),$$

with $q_z = e^z$. Corresponding conformal vector is $\tilde{\omega} = \omega - \frac{c_V}{24}\mathbf{1}_V$. For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and $m \geq 0$,

$$v[m] = m! \sum_{i\geq m} c(wt(v), i, m) v(i), \qquad \sum_{m=0}^{i} c(wt(v), i, m) x^m = \binom{wt(v) - 1 + x}{i}.$$

Given a vertex operator algebra V, one defines the adjoint vertex operator with respect to $\rho \in \mathbb{C}$, $Y^\dagger_\rho[v,z] = Y\left[\exp\left(z\rho^{-1}L[1]\right)\left(-\rho z^{-2}\right)^{L[0]} v, \rho z^{-1}\right]$. associated with the formal Möbius map [19] $z \mapsto \frac{\rho}{z}$. An element $u \in V$ is called quasiprimary if $L(1)u = 0$. For quasiprimary u of weight $\mathrm{wt}(u)$ one has $u^\dagger(n) = (-1)^{\mathrm{wt}(u)}\ \alpha^{n+1-\mathrm{wt}(u)}\ u(2\mathrm{wt}(u) - n - 2)$.

We call a bilinear form $\langle .,. \rangle : V \times V \to \mathbb{C}$, invariant if [19,31] $\langle Y(u,z)a, b\rangle = \langle a, Y^\dagger(u,z)b\rangle$, for all a, b, $u \in V$. Note that the adjoint vertex operator $Y^\dagger(.,.)$ as well as the bilinear form $\langle .,. \rangle$, depend on α. Rewriting in terms of modes, we obtain $\langle u(n)a, b\rangle = \langle a, u^\dagger(n)b\rangle$. Choosing $u = \omega$, and for $n = 1$ implies that $\langle L(0)a, b\rangle = \langle a, L(0)b\rangle$. Thus, $\langle a, b\rangle = 0$, when $\mathrm{wt}(a) \neq \mathrm{wt}(b)$. A vertex operator algebra V is called of strong-type if $V_0 = \mathbb{C}\mathbf{1}_V$, and it is simple and self-dual, i.e., isomorphic to the dual module V' as a V-module. It is proven in [31] that a strong-type vertex operator algebra V has a unique invariant non-degenerate bilinear form up to normalization. The form $\langle .,. \rangle$ defined on a strong-type vertex operator algebra V is the unique invariant bilinear form $\langle .,. \rangle$ normalized by $\langle \mathbf{1}_V, \mathbf{1}_V\rangle = 1$. A vertex operator algebra V-module W has similar properties as V [9,19,20,26,30].

A vertex algebra V is called quasi-conformal [6] if it admits an action of the local Lie algebra of Aut $\mathcal{O}$ for which

$$[\boldsymbol{v}, Y(u,w)] = -\sum_{m\geq -1} \left((m+1)!\right)^{-1} \left(\partial_w^{m+1} v(w)\right) Y(L_m u, w),$$

with $\boldsymbol{v} = -\sum_{r\geq -1} v_r L_r$, $v(z)\partial_z = \sum_{r\geq -1} v_r z^{r+1}\partial_z$, is true for any $v \in V$, the element $L_W(-1) = -\partial_z$, as the translation operator T, $L_W(0) = -z\partial_z$. In addition it acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of Aut $\mathcal{O}^{(n)}$ acts locally nilpotently.

5.1. Vertex Operator Algebra V-Module W Bundle $\mathcal{W}$

The notion of a vertex operator algebra V bundle was introduced in [6]. In this Appendix we recall that definition of a V-module W bundle $\mathcal{W}$. The idea is to associate canonically (i.e., coordinate independently) End $\mathcal{W}$-valued sections $\mathcal{Y}_{\mathcal{W}}$ of $\mathcal{W}^*$ (the bundle dual to $\mathcal{W}$) to matrix elements of V-module W vertex operators.

Denote by Aut $\mathcal{O}$ the group of continuous automorphisms of $\mathcal{O} = \mathbb{C}[[z]]$ on an arbitrary smooth curve S and its Lie algebra $Aut\ \mathcal{O}$. Let V be a quasi-conformal vertex algebra (see the previous subsection). Its module W is graded by finite dimensional Aut $\mathcal{O}$-submodules. One defines a vertex operator algebra bundle $\mathcal{W}_S$ and its dual $\mathcal{W}^*_S$ as inductive and projective limits of vector bundles of finite rank over S, in particular, for the disc $D = \mathrm{Spec}\mathbb{C}[[z]]$, or $D_z = \mathrm{Spec}\mathcal{O}_z$ with $Aut_{D_z} = Aut\ X|_{D_z}$ and $\mathcal{W}_{D_z} = \mathcal{W}_{S|_{D_z}}$. Let Aut_z be the Aut $\mathcal{O}$-torsor of coordinates at $z \in S$. Recall that $\mathcal{W}_z = Aut_z \underset{\mathrm{Aut}\mathcal{O}}{\times} W$ is the fiber of $\mathcal{W}_{|D_z}$ at $z \in S$. Let us define End $\mathcal{W}_z$-valued meromorphic section $\mathcal{Y}_{\mathcal{W}}$ of the bundle $\mathcal{W}^*$ on the punctured disc $D_z^\times$. This section is given by the map (linear in $\overline{u}$, u, and $\mathcal{O}_z$-linear in s) $(\overline{u}, s, u) \mapsto \langle \overline{u}, \mathcal{Y}_{\mathcal{W}}(s) \cdot u\rangle$, assigning a function on $D_z^\times$ denoted by $\langle \overline{u}, \mathcal{Y}_{\mathcal{W}}(s) \cdot v\rangle$, for $\overline{u} \in \mathcal{W}^*_z$, $u \in \mathcal{W}_z$, and a regular section s of $\mathcal{W}|_{D_z}$. For a coordinate z on the disc D_z, we then obtain a z-trivialization of $\mathcal{W}$ $i_z : W[[z]] \simeq \Gamma(D_z, \mathcal{W})$, and trivializations $\mathcal{W}^* \simeq \mathcal{W}^*_z$, $\mathcal{W} \simeq \mathcal{W}_z$ of the fibers which we denote by $(\overline{u}, z)$, (u, z). Define an End $\mathcal{W}_z$-valued section $\mathcal{Y}_{\mathcal{W}}$ of $\mathcal{W}^*$ on $D_z^\times$ by

$$\begin{aligned}\mathcal{F}^{(0)}_{\mathcal{W}}(x) &= \sum_{w_k \in V_{(k)}} \langle (\overline{u}_k, z), \mathcal{Y}_{\mathcal{W}}(i_z(v)) \cdot (u_k, z)\rangle \\ &\sim \sum_{w_k \in W_{(k)}} \langle \overline{u}_k, Y(v,z) u_k\rangle = \mathcal{F}^{(0)}_{W}(x),\end{aligned} \tag{5.2}$$

where z is a coordinate on D_z. Then the section $\mathcal{Y}_{\mathcal{W}}$ is canonical, i.e., independent of the choice of coordinate z on D_z.

Let S be a smooth complex variety and $\mathcal{E} \to S$ a holomorphic vector bundle over S. We use the same notation $\mathcal{E}$ for the sheaf of holomorphic sections of $\mathcal{E}$. Let Ω be the sheaf of differentials on S. A holomorphic connection ∇ on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega$ satisfying Leibniz rule $\nabla(f\phi) = f\nabla(\phi) + \phi \otimes df$, for any holomorphic function f.

6. Appendix: The ρ-Formalism of Raising the Genus of a Riemann Surface

Here we recall so called ρ-formalism of raising the genus, i.e., a specific way of attaching a handle to a Riemann surface $\Sigma^{(g)}$ of genus g to form a genus $g+1$ Riemann surface $\Sigma^{(g+1)}$ was introduced in [51]. Let z_1, z_2 be local coordinates in the neighborhood of two separated points p_1 and p_2 on $\Sigma^{(g)}$. Consider two disks $|z_a| \leq r_a$, for $r_a > 0$ and $a = 1, 2$. r_1, r_2 required to be small enough so that the disks do no intersect. Introduce a complex parameter ρ with $|\rho| \leq r_1 r_2$ and excise the disks $\{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(g)}$, to form a twice-punctured surface $\widehat{\Sigma}^{(g)} = \Sigma^{(g)} \backslash \bigcup_{a=1,2} \{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\}$. We notate $\bar{1} = 2$, $\bar{2} = 1$. Next define annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(g)}$ with $\mathcal{A}_a = \{z_a : |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$ and identify them as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \rho, \tag{6.1}$$

to form a compact Riemann surface $\Sigma^{(g+1)} = \widehat{\Sigma}^{(g)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g+1$. The relation (6.1) parametrizes a cylinder connecting the punctured Riemann surface to itself. On $\Sigma^{(g+1)}$ we define the standard basis of cycles $\{a_1, b_1, \ldots, a_{g+1}, b_{g+1}\}$ where the set $\{a_1, b_1, \ldots, a_g, b_g\}$ is the original basis on $\Sigma^{(g)}$. Introduce a closed anti-clockwise contour $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ parametrized by z_a around the puncture at $z_a = 0$. Due to the sewing relation (6.1) $\mathcal{C}_2(z_2) \sim -\mathcal{C}_1(z_1)$ We then introduce the cycle $a_{g+1} \sim \mathcal{C}_2(z_2)$ and b_{g+1} as a path chosen in $\widehat{\Sigma}^{(g)}$ between identified points $z_1 = z_0$ and $z_2 = \rho/z_0$ on the sewn surface.

7. Appendix: Genus g Generalizations of Elliptic Functions

In this Appendix we recall [44] genus g generalizations of classical elliptic functions.

7.1. Classical Elliptic Functions

Here we recall the classical elliptic functions [29,41]. For an integer $k \geq 2$, the Eisenstein series is given by

$$E_k(\tau) = E_k(q) = \delta_{n,even} \left(-(k!)^{-1} B_k + 2((k-1)!)^{-1} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \right),$$

where $\tau \in \mathbb{H}$, $q = e^{2\pi i \tau}$, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and B_k is the $k-$th Bernoulli number. For integer $k \geq 1$, define elliptic functions $z \in \mathbb{C}$

$$P_1(z, \tau) = \frac{1}{z} - \sum_{k \geq 2} E_k(\tau) z^{k-1}, \quad P_k(z, \tau) = \frac{(-1)^{k-1}}{(k-1)!} \partial_z^{k-1} P_1(z, \tau),$$

In particular $P_2(z, \tau) = \wp(z, \tau) + E_2(\tau)$, for Weierstrass function $\wp(z, \tau)$ with periods $2\pi i$ and $2\pi i \tau$. $P_1(z, \tau)$ is related to the quasi–periodic Weierstrass $\sigma-$ function with $P_1(z + 2\pi i \tau, \tau) = P_1(z, \tau) - 1$.

7.2. Genus g Generalizations of Elliptic Functions

The generalizations of elliptic functions at genus g were proposed in [49]. Introduce a column vector $X = (X_a(m))$, indexed by $m \geq 0$ and $a \in \mathcal{I}$

$$X_a(m) = \rho_a^{-\frac{m}{2}} \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots; u(m) b_a, w_a; \ldots),$$

and a row vector $p(x) = (p_a(x, m))$, for $m \geq 0, a \in \mathcal{I}$

$$p_a(x, m) = \rho_a^{\frac{m}{2}} \partial^{(0,m)} \psi_p^{(0)}(x, w_a).$$

Let us also define column vector $G^{(g)} = \left(G_a^{(g)}(m)\right)$, for $m \geq 0, a \in \mathcal{I}$, given by

$$G^{(g)} = \sum_{k=1}^{n} \sum_{j \geq 0} \partial_k^{(j)} \, q(y_{k,g}) \, \mathcal{F}_W^{(g)}((u(j))_k \mathbf{x}_{n,g}),$$

where $q(y) = (q_a(y;m))$, for $m \geq 0$, $a \in \mathcal{I}$, is a column vector

$$q_a(y;m) = (-1)^p \rho_a^{\frac{m+1}{2}} \partial^{(m,0)} \psi_p^{(0)}(w_{-a}, y),$$

$R = (R_{ab}(m,n))$, for m, $n \geq 0$ and a, $b \in \mathcal{I}$

$$R_{ab}(m,n) = \begin{cases} (-1)^p \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}} \partial^{(m,n)} \psi_p^{(0)}(w_{-a}, w_b), & a \neq -b, \\ (-1)^p \rho_a^{\frac{m+n+1}{2}} \mathcal{E}_m^n(w_{-a}), & a = -b, \end{cases}$$

$$\mathcal{E}_m^n(y) = \sum_{\ell=0}^{2p-2} \partial^{(m)} f_\ell(y) \, \partial^{(n)} y^\ell, \quad \psi_p^{(0)}(x,y) = \frac{1}{x-y} + \sum_{\ell=0}^{2p-2} f_\ell(x) y^\ell,$$

for any Laurent series $f_\ell(x)$ for $\ell = 0, \ldots, 2p-2$. Define the matrices $\Delta_{ab}(m,n) = \delta_{m,n+2p-1}\delta_{ab}$, $\widetilde{R} = R\Delta$, and $\left(I - \widetilde{R}\right)^{-1} = \sum_{k \geq 0} \widetilde{R}^k$. Introduce $\chi(x) = (\chi_a(x;\ell))$ and $o(u; \boldsymbol{v}, \boldsymbol{y}) = (o_a(u; \boldsymbol{v}, \boldsymbol{y}; \ell))$, which are are finite row and column vectors for $a \in \mathcal{I}$, $0 \leq \ell \leq 2p-2$ with

$$\chi_a(x;\ell) = \rho_a^{-\frac{\ell}{2}} (p(x) + \widetilde{p}(x)(I - \widetilde{R})^{-1} R)_a(\ell), \quad o_a(\ell) = o_a(u; \boldsymbol{v}, \boldsymbol{y}; \ell) = \rho_a^{\frac{\ell}{2}} X_a(\ell),$$

$\widetilde{p}(x) = p(x)\Delta$. Note that $\psi_p(x,y)$ is defined by

$$\psi_p(x,y) = \psi_p^{(0)}(x,y) + \widetilde{p}(x)(I - \widetilde{R})^{-1} q(y).$$

For each $a \in \mathcal{I}_+$ introduce a vector $\theta_a(x) = (\theta_a(x;\ell))$, $0 \leq \ell \leq 2p-2$,

$$\theta_a(x;\ell) = \chi_a(x;\ell) + (-1)^p \rho_a^{p-1-\ell} \chi_{-a}(x; 2p-2-\ell).$$

We then have the following vectors of differential forms

$$P(x) = p(x) \, dx^p, \quad Q(y) = q(y) \, dy^{1-p}, \quad \widetilde{P}(x) = P(x)\Delta, \tag{7.1}$$

$$\Psi_p(x,y) = \psi_p(x,y) \, dx^p \, dy^{1-p} = \Psi_p^{(0)}(x,y) + \widetilde{P}(x)(I - \widetilde{R})^{-1} Q(y). \tag{7.2}$$

Finally, one introduces

$$\Theta_a(x;\ell) = \theta_a(x;\ell) \, dx^p, \quad O_a(u; \boldsymbol{v}, \boldsymbol{y}; \ell) = o_a(u; \boldsymbol{v}, \boldsymbol{y}; \ell) \, \boldsymbol{dy}^{\mathrm{wt}(\boldsymbol{v})}. \tag{7.3}$$

8. Appendix: Examples of Vertex Operator Algebra n-Point Functions

8.1. Vertex Operator Algebra n-Point Functions on Riemann Sphere

For $\mathbf{v}_n \in V$, and a homogeneous $u \in V$, the n-point function on the sphere is given by [19,20]

$$\mathcal{F}_W^{(0)}(\mathbf{x}_{n,0}) = \langle u', Y(x_1) \ldots Y(x_n) u \rangle,$$

while the partition function is $\mathcal{F}^{(0)}_{W,0} = \langle u'_{(a)}, u_{(b)} \rangle = \delta_{a,b}$. The reductive differentials of (2.3) are

$$D^n_1(x_{n+1,0}, 0).\mathcal{F}^{(0)}_{W}(\mathbf{x}_{n,0}) = T_1(o(v)).\mathcal{F}^{(0)}_{W}(\mathbf{x}_{n,0}), \tag{8.1}$$

$$D^n_2(x_{n+1,0}, 0).\mathcal{F}^{(0)}_{W}(\mathbf{x}_{n,0}) = z_{n+1}^{-\mathrm{wt}(v)} \sum_{k=1}^{n} \sum_{m\geq 0} f_{wt(v_{n+1,0}),m}(z_{n+1}, z_r) T_k(v(m)). \mathcal{F}^{(0)}_{W}(\mathbf{x}_{n,0}),$$

where we define $f^{(0)}_{wt(v,m}(z,w)$ is a rational function defined by

$$f^{(0)}_{n,m}(z,w) = \frac{z^{-n}}{m!}\left(\frac{d}{dw}\right)^m \frac{w^n}{z-w}, \quad \iota_{z,w} f^{(0)}_{n,m}(z,w)$$
$$= \sum_{j\in\mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1},$$

where $\iota_{z,w} : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[[z_1, z_1^{-1} \ldots, z_n z_n^{-1}]]$ are maps [19].

8.2. Vertex Operator Algebra n-Point Functions on the Torus

For $\mathbf{v}_n \in V^{\otimes n}$ the genus one n-point function is defined by

$$\mathcal{F}^{(1)}_{W}(\mathbf{x}_{n,1}) = Tr_W \left(Y\left(q_1^{L(0)} v_1, q_1\right) \ldots Y\left(q_n^{L(0)} v_n, q_n\right)\, q^{L(0)-c/24}\right),$$

for $q = e^{2\pi i\tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parametr, and c is the central charge of the Virasoro algebra of V. For any $v_{n+1,g} \in V$, $\mathbf{v}_n \in V^{\otimes n}$, the torus reduciton formula is given by [52]

$$D^{n+1}_1(x_{n+1,1}, 1).\mathcal{F}^{(1)}_{W}(\mathbf{x}_{n,1}) = \mathcal{F}^{(1)}_{W}(o(v_{n+1,g})\, \mathbf{x}_{n,1}), \tag{8.2}$$
$$D^{n+1}_2(x_{n+1,1}, 1).\mathcal{F}^{(1)}_{W}(\mathbf{x}_{n,1}) = \sum_{k=1}^{n} \sum_{m\geq 0} P_{m+1}(z_{n+1} - z_k, \tau)\, \mathcal{F}^{(1)}_{W}((v[m])_k.\, \mathbf{x}_{n,1}).$$

Here $P_m(z,\tau)$ denote Weierstrass functions defined by

$$P_m(z,\tau) = \frac{(-1)^m}{(m-1)!} \sum_{n\in\mathbb{Z}_{\neq 0}} \frac{n^{m-1} q_z^n}{1-q^n}.$$

8.3. Vertex Operator Algebra Reduction Formulas in Genus g Schottky Uniformization

In this Section we recall reduction relations for vertex operator algebra n-point functions defined on a genus g Riemann surface constructed in the Schottky uniformization procedure [7,44,45,49]. In this case, the coefficients in reduction formulas are meromorphic functions on Riemann surfaces and represent genus g generalizations of the elliptic functions [29,41]. For $2g$ vertex operator algebra V states and corresponding local coordinates $\boldsymbol{b} = (b_{-1}, b_1; \ldots; b_{-g}; b_g)$,

$\boldsymbol{w} = (w_{-1}, w_1; \ldots; w_{-g}, w_g)$, of $2g$ points $(p_{-1}, p_1; \ldots; p_{-g}, p_g)$ on the Riemann sphere consider the genus zero $2g$-point correlation function

$$\begin{aligned}\mathcal{F}_V^{(0)}(\boldsymbol{b}, \boldsymbol{w}) =& \mathcal{F}_V^{(0)}(b_{-1}, w_{-1}; b_1, w_1; \ldots; b_{-g}, w_{-g}; b_g, w_g) \\ =& \prod_{a \in \mathcal{I}_+} \rho_a^{\mathrm{wt}(b_a)} \mathcal{F}_V^{(0)}(\overline{b}_1, w_{-1}; b_1, w_1; \ldots; \overline{b}_g, w_{-g}; b_g, w_g).\end{aligned}$$

where $\mathcal{I}_+ = \{1, 2, \ldots, g\}$. Let us denote $\mathbf{b}_{+,g} = (b_1, \ldots, b_g)$, and an element of a V-tensor product $V^{\otimes g}$-basis with the dual basis $\mathbf{b}_{-,g} = (b_{-1}, \ldots, b_{-g})$, with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\rho_a}$ (cf. Appendix 5). Let w_a for $a \in \mathcal{I}$ be $2g$ formal variables and $\boldsymbol{\rho}_g = (\rho_1, \ldots, \rho_g)$ g complex parametrs. We may identify $\boldsymbol{\rho}_g$ with the canonical Schottky parametrs. One introduces the genus g partition function (zero-point function) as

$$\mathcal{F}_V^{(g)} = \mathcal{F}_V^{(g)}(\boldsymbol{w}, \boldsymbol{\rho}_g) = \sum_{\mathbf{b}_{+,g}} \mathcal{F}_V^{(0)}(\boldsymbol{b}, \boldsymbol{w}), \quad (\boldsymbol{w}, \boldsymbol{\rho}_g) = (w_{\pm 1}, \rho_1; \ldots; w_{\pm g}, \rho_g). \tag{8.3}$$

For $\mathbf{x}_{n,g} = (\mathbf{v}_{n,g}, \mathbf{y}_{n,g})$, one defines the genus g formal n-point function for $\mathbf{v}_{n,g} \in V^{\otimes n}$ and formal parametrs $\mathbf{y}_{n,g}$ by

$$\begin{aligned}\mathcal{F}_V^{(g)}(\mathbf{x}_{n,g}) = \mathcal{F}_V^{(g)}(\mathbf{x}_{n,g}; \boldsymbol{w}, \boldsymbol{\rho}) = \sum_{\mathbf{b}_{+,g}} \mathcal{F}_V^{(0)}(\mathbf{x}_{n,g}; \boldsymbol{b}, \boldsymbol{w}), \\ \mathcal{F}_V^{(0)}(\mathbf{x}_{n,g}; \boldsymbol{b}, \boldsymbol{w}) = \mathcal{F}_V^{(0)}(\mathbf{x}_n; b_{-1}, w_{-1}; \ldots; b_g, w_g).\end{aligned} \tag{8.4}$$

Let $U \subset V$ be a vertex operator subalgebra such that V admits a U-module W_α decomposition $V = \bigoplus_{\alpha \in A} W_\alpha$, over an indexing set A. For a tensor product of g modules $W_{\boldsymbol{\alpha}} = \bigotimes_{a=1}^{g} W_{\alpha_a}$, consider

$$\mathcal{F}_{W_\alpha}^{(g)}(\mathbf{x}_{n,g}) = \sum_{\mathbf{b_{+,g}} \in W_\alpha} \mathcal{F}_W^{(0)}(\mathbf{x}_{n,g}; \boldsymbol{b}, \boldsymbol{w}), \tag{8.5}$$

where here the sum is over a basis $\{\mathbf{b}_{+,g}\}$ for $W_{\boldsymbol{\alpha}}$. It follows that

$$\mathcal{F}_W^{(g)}(\mathbf{x}_{n,g}) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}} \mathcal{F}_{W_\alpha}^{(g)}(\mathbf{x}_{n,g}), \tag{8.6}$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in \boldsymbol{A}$, for $\boldsymbol{A} = A^{\otimes g}$. Finally, one defines corresponding formal n-point correlation differential forms

$$\widetilde{\mathcal{F}}_{W_\alpha}^{(g)}(\mathbf{x}_{n,g}) = \mathcal{F}_{W_\alpha}^{(g)}(\mathbf{x}_{n,g}) \prod_{k=1}^{n} dy_{k,g}^{\mathrm{wt}(\mathbf{v}_{k,g})}.$$

Corresponding differential D^g acts as

$$\begin{aligned}\widetilde{\mathcal{F}}_V^{(g+1)}(\mathbf{x}_{n,g}) = D^g . \mathcal{F}_V^{(g+1)}(\mathbf{x}_{n,g}) = D^g . \sum_{\mathbf{b}_{+,g}} \widetilde{\mathcal{F}}^{(0)}(\mathbf{x}_{n,g}; \mathbf{b}_{2g}, \mathbf{w}_{2g}) \\ = \sum_{\mathbf{b}_{+,g+1}} \widetilde{\mathcal{F}}^{(0)}(\mathbf{x}_{n,g}; \mathbf{b}_{2g+1}, \mathbf{w}_{2g+1}).\end{aligned} \tag{8.7}$$

In [49] they prove that the genus g $(n+1)$-point formal differential $\widetilde{\mathcal{F}}^{(g)}_{W_\alpha}(x_{n+1,g};\mathbf{x}_{n,g})$, for $x_{n+1,g}=(v_{n+1,g},y_{n+1,g})$, for quasiprimary vectors $v_{n+1,g}\in U$ of weight $\mathrm{wt}(v_{n+1,g})=p$ with formal parametrs $\mathbf{y}_{n+1,g}$, and general vectors $\mathbf{v}_n$ with parametrs $\mathbf{y}_n$ satisfies the reduction formulas

$$\widetilde{\mathcal{F}}^{(g)}_{W_\alpha,n+1}(\mathbf{x}_{n+1,g}) = \left(D_1^{(n+1)} + D_2^{(n+2)}\right)\cdot \widetilde{\mathcal{F}}^{(g)}_{W_\alpha,n}(\mathbf{x}_{n,g}), \tag{8.8}$$

$$D_1^{n+1}(x_{n,g},g)\cdot \widetilde{\mathcal{F}}^{(g)}_{W_\alpha,n}(\mathbf{x}_{n,g}) = \sum_{a=1}^{g}\Theta_a(y_{n+1,g})\, O_a^{W_\alpha}\ (v_{n+1,g};\mathbf{x}_{n,g}),$$

$$D_2^{n+1}(x_{n+1,g},g)\cdot \widetilde{\mathcal{F}}^{(g)}_{W_\alpha,n+1}(\mathbf{x}_{n+1,g})$$
$$= \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\Psi_p(y_{n+1,g},y_{k,g})\,\widetilde{\mathcal{F}}^{(g)}_{W_\alpha,n}((u(j))_k.\mathbf{x}_{n,g})\,dy^{j}_{k,g},$$

Here $\partial^{(0,j)}$ is given by $\partial^{(i,j)}f(x,y)=\partial_x^{(i)}\partial_y^{(j)}f(x,y)$, for a function $f(x,y)$, and $\partial^{(0,j)}$ denotes partial derivatives with respect to x and $y_{j,g}$. The forms $\Psi_p(y_{n+1,g},y_{k,g})\,dy^j_{k,g}$ are given by (7.2), $\Theta_a(x)$ is of (7.3), and $O_a^{W_\alpha}(v_{n+1,g};\mathbf{x}_{n,g})$ is (7.3).

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