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On Left T-Nilpotent Rings

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Abstract. It is shown that any ring being a sum of two left T-nilpotent subrings is left T-nilpotent. The paper contains the description of all the semigroups S such that an S-graded ring $R = \bigoplus_{s\in S} A_s$ has the property that the left T-nilpotency of all subrings among the subgroups A_s of the additive group of R implies the left T-nilpotency of R. Furthermore, this result is extended to rings R being S-sums.

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1. Introduction

We say that an associative ring R is a sum of its subrings R_1 and R_2 and write $R = R_1 + R_2$ if each element r of R can be written in the form $r = r_1 + r_2$ with $r_1 \in R_1$ and $r_2 \in R_2$. The direct inspiration for this paper was the following general problem: given a class $\mathcal{M}$ of rings, does the condition R_1, $R_2 \in \mathcal{M}$ imply $R_1 + R_2 \in \mathcal{M}$? In [5], Kegel showed that if a ring R is a sum of its nilpotent subrings R_1 and R_2, then R is nilpotent. By [16], an analogous result occurs for the case when R_1 and R_2 are nil rings of bounded index. The generalization of this result given in [15] lead to the positive answer to a long-open question related to PI rings, i.e. rings satisfying certain polynomial identities. Namely, it was shown there that if R_1 and R_2 are PI rings, then so is R. On the other hand, for many classes of rings the mentioned problem has a negative solution. For example, in [8,11] Kelarev constructed examples of non-radical rings R (in the sense of the prime as well as Levitzki radical) which are sums of their radical subrings R_1 and R_2. Moreover, in [17] Salwa gave an example of a ring R containing a regular element despite the fact that R is a sum of subrings R_1 and R_2 which are sums of its nilpotent ideals.

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In [1], Bokut' proved that any algebra over a field can be embedded in an algebra that is a sum of its three nilpotent subalgebras. This shows that the mentioned above general problem for sums of two subrings and its analogues for sums of three or more subrings are definitely different cases. However, under some additional assumptions, it is possible to obtain satisfactory positive results for sums of any number of subrings. The most general results related to the topic were obtained for rings graded by a semigroup S (see [12,13]) and rings being S-sums (see [9,14]). In particular, these results are also related to the class of nilpotent rings, PI-rings, the prime radical, as well as Levitzki and Jacobson radicals.

This article focuses mainly on the class of left T-nilpotent rings studied in [3,7,16]. It should be mentioned that in [3] it was shown that the famous Koethe problem is equivalent to the question whether a ring which is a sum of a left T-nilpotent subring and a nil subring is nil.

In [16], it was proved that if the ring R is a sum of subrings R_1 and R_2 such that R_1 is left T-nilpotent and $R_2 \in S$, where S is a supernilpotent radical, then $R \in S$. Examples of supernilpotent radicals are the prime, locally nilpotent and Jacobson radicals.

Many interesting results concerning the subject of this paper were obtained by Kelarev. In [7], he provided an example of a ring R that is a sum of two subrings R_1 and R_2 such that the Levitzki radical of R does not contain any of the hyperanihilators of R_1 and R_2. Some results directly related to the class of left T-nilpotent rings were presented in [10]. They were obtained for rings which are arbitrary finite sums of their additive subgroups. In particular, Kelarev showed there that a ring which is finite union of left T-nilpotent subrings is left T-nilpotent, too. These results were a strong inspiration for us.

In Sect. 2, we show that any ring R being a sum of two left T-nilpotent subrings, is left T-nilpotent. Furthermore, we prove that if $R = R_1 + R_2$, where R_1 is a T-nilpotent subring of R and R_2 is a subgroup of the additive group of R satisfying $R_2^d \subseteq R_1$ for some positive integer d, then R is left T-nilpotent. Notice that in [14] a similar result was obtained for the Jacobson radical with $d = 2$.

Section 3 is devoted to the description of all the semigroups S for which S-graded rings $R = \bigoplus_{s \in S} A_s$ have the property that the left T-nilpotency of all subrings among the subgroups A_s of the additive group of R implies the left T-nilpotency of R. The generalization of this result to rings being S-sums is presented in Sect. 4.

The notation used in this article is consistent with generally accepted standards. In particular, if A is a left (right) ideal of a ring R, then we write $A <_l R$ ($A <_r R$). If A is either a left or a right ideal of R, then A is said to be a one-sided ideal of R. If it is not necessary to distinguish the side of a one-sided ideal A of R, then we write briefly $A < R$. For two-sided ideals of rings and semigroups the symbol $\lhd$ is used instead. The left annihilator of a subring A in R is denoted by $l_R(A)$. The symbol $\mathbb{N}$ stands for the set of all non-negative integers.

2. Left T-Nilpotent Rings

We remind the reader that a ring R is said to be *left T-nilpotent* if for each sequence (a_n) of elements of R, there exists $n \in \mathbb{N}$ such that $a_1 \cdot \ldots \cdot a_n = 0$. There is a well-know criterion for a ring R to be T-nilpotent related to its left hyperannihilator $l(R)$ defined as a sum $\bigcup_{\alpha \geq 0} l_\alpha(R)$, where $l_0(R) = \{0\}$ and $l_\alpha(R) = \{x \in R \,|\, xR \subseteq \bigcup_{\beta<\alpha} l_\beta(R)\}$ for any ordinal number $\alpha > 0$. Namely, a ring R is left T-nilpotent if and only if $l(R) = R$.

We start with some technical lemma which will turn out to be very useful in further considerations.

Lemma 2.1. *If $\{X_n : n \in \mathbb{N}\}$ is a family of non-empty finite subsets of a ring R such that $\sum_{x\in X_1} x \cdot \ldots \cdot \sum_{x\in X_n} x \neq 0$ for every $n \in \mathbb{N}$, then there exists an infinite sequence (x_n) satisfying $x_n \in X_n$ and $x_1 \cdot \ldots \cdot x_n \neq 0$ for each $n \in \mathbb{N}$.*

Proof. Let $G = \bigcup_{n\in\mathbb{N}} G_n$ with

$$G_n = \big\{(y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \ldots \times X_n : y_1 \cdot y_2 \cdot \ldots \cdot y_n \neq 0\big\}.$$

Since $\sum_{x\in X_1} x \cdot \ldots \cdot \sum_{x\in X_n} x \neq 0$, we have

$$\sum_{x\in X_1} x \cdot \ldots \cdot \sum_{x\in X_n} x = \sum_{(y_1,\ldots,y_n)\in G_n} y_1 \cdot y_2 \cdot \ldots \cdot y_n$$

and consequently $0 < |G_n| < \infty$, for every $n \in \mathbb{N}$. Combining this with $|G| = \infty$, we infer that there exists $x_1 \in X_1$ such that $x_1 \neq 0$ and infinitely many sequences belonging to G start with x_1. We will construct the next terms of the desired sequence (x_n) inductively. Suppose that there exist $(x_1, x_2, \ldots, x_k) \in G_k$ such that infinitely many sequences belonging to G start with $(x_1, x_2, \ldots, x_k)$. Since the set X_{k+1} is finite and non-empty and $|G| = \infty$, there exists $x_{k+1} \in X_{k+1}$ such that infinitely many sequences from G start with $(x_1, \ldots, x_k, x_{k+1})$. It follows that $x_1 \cdot \ldots \cdot x_k \cdot x_{k+1} \neq 0$. By induction, we have constructed an infinite sequence (x_n) such that $x_n \in X_n$ and $x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0$ for every $n \in \mathbb{N}$. □

The following corollary is a direct consequence of Lemma 2.1.

Corollary 2.2. *Let A and B be subgroups of the additive group of a ring R such that $R = A + B$. Then the following conditions are equivalent:*

- *(i) the ring R is left T-nilpotent*
- *(ii) there is no sequence (c_n) of elements of R such that $c_n \in A \cup B$ and $c_1 \cdot \ldots \cdot c_n \neq 0$ for each $n \in \mathbb{N}$.*

The next corollary follows at once from Corollary 2.2.

Corollary 2.3. *If A and B are left T-nilpotent subrings of a ring R which is not left T-nilpotent and $R = A + B$, then there exists a sequence (c_n) of elements of $A \cup B$ such that $c_n \in A$ for infinitely many n, $c_n \in B$ for infinitely many n and $c_1 \cdot c_2 \cdot \ldots \cdot c_n \neq 0$ for each $n \in \mathbb{N}$.*

In [4, Theorem 1.5] it was shown that any left T-nilpotent left ideal of a ring R generates a left T-nilpotent ideal in R. Here we give a new simpler proof of this fact in a slightly more general form. We begin with the following

Lemma 2.4. *If A and B are left T-nilpotent subrings of a ring R such that $R = A + B$ and $A < R$, then the ring R is left T-nilpotent.*

Proof. Suppose, contrary to our claim, that a ring R satisfies all the assumptions of the lemma but it is not left T-nilpotent. There is no loss of generality in assuming that $A <_r R$. It follows from Corollary 2.3 that there exist a sequence (c_n) of elements of $A \cup B$ and its subsequence (c_{k_n}) such that $c_1 \cdot c_2 \cdot \ldots \cdot c_n \neq 0$ and $c_{k_n} \in A$ for each $n \in \mathbb{N}$. Define $d_n = c_{k_n} \cdot c_{k_n+1} \cdot \ldots \cdot c_{k_{n+1}-1} \in A$ for each $n \in \mathbb{N}$. As $A <_r R$, we get $d_n \in A$ for each $n \in \mathbb{N}$. Furthermore, for each $n \in \mathbb{N}$, $d_1 \cdot d_2 \cdot \ldots \cdot d_n = c_{k_1} \cdot \ldots \cdot c_{k_1+1} \cdot \ldots \cdot c_{k_{n+1}-1} \neq 0$ because of $c_1 \cdot c_2 \cdot \ldots \cdot c_{k_{n+1}-1} \neq 0$. Therefore, the ring A is not left T-nilpotent, a contradiction. ☐

Theorem 2.5. *If A is a one-sided left T-nilpotent ideal of a ring R, then the ideal I generated by A is left T-nilpotent.*

Proof. Without loss of generality, we may assume that $A <_l R$. Then $I = A + AR$. Assume, by way of contradiction, that the ring AR is not left T-nilpotent. Then there is a sequence (a_n) of elements of AR such that $a_1 \cdot \ldots \cdot a_n \neq 0$ for each $n \in \mathbb{N}$. In particular, each term of this sequence is a sum of elements of the form $a \cdot r$ with $a \in A$ and $r \in R$, so Lemma 2.1 implies the existence of sequences (b_n) and (r_n) such that $b_n \in A$, $r_n \in R$ and $(b_1 r_1) \cdot \ldots \cdot (b_n r_n) \neq 0$ for each $n \in \mathbb{N}$. Combining this with $(b_1 r_1) \cdot \ldots \cdot (b_n r_n) = b_1 \cdot (r_1 b_2) \cdot \ldots \cdot (r_{n-1} b_n) \cdot r_n$ and $r_1 b_2, \ldots, r_{n-1} b_n \in A$ for each $n \in \mathbb{N}$ leads to the conclusion that $(b_1, r_1 b_2, r_2 b_3, r_3 b_4, \ldots)$ is an infinite sequence of elements of A satisfying $b_1 \cdot (r_1 b_2) \cdot (r_2 b_3) \cdot \ldots \cdot (r_{n-1} b_n) \neq 0$ for each integer $n > 1$, contrary to the left T-nilpotency of A. Thus AR is a left T-nilpotent ring. Therefore, applying Lemma 2.4 to the ring $I = A + AR$, we infer that the ring I is left T-nilpotent. ☐

Let B and A be a subring of a ring R and a subgroup of the additive group of R, respectively. Suppose that $R = A + B$ and $A^2 \subseteq B$. It follows from [9] that if B is a locally nilpotent ring, then so is R. In view of [14], this statements remains true for the case when B is a Jacobson radical ring. Here we present the analogous results for left T-nilpotent rings.

Theorem 2.6. *Let B and A be a left T-nilpotent subring of a ring R and a subgroup of the additive group of R, respectively. If $R = A + B$ and $A^d \subseteq B$ for some positive integer d, then the ring R is left T-nilpotent.*

Proof. The assertion is obvious for $d = 1$. Consider any integer $d \geq 2$. Suppose, contrary to our claim, that the ring R is not left T-nilpotent. Then $l(R) \neq R$ (see the introductory remarks before Lemma 2.1). First we show that $B \nsubseteq l(R)$. If $B \subseteq l(R)$, then $A^d \subseteq l(R)$ because of $A^d \subseteq B$. Thus $A^{d-1}R = A^d + A^{d-1}B \subseteq l(R)$, whence $A^{d-1} \subseteq l(R)$. We continue in a similar fashion to obtain $A \subseteq l(R)$ after a finite number of steps. Consequently, $R = A + B \subseteq l(R)$, contrary to $l(R) \neq R$. Thus $B \nsubseteq l(R)$. Therefore, there is no loss of generality in assuming that $B \neq \{0\}$ and $l_R(R) = \{0\}$. Define $L = l_R(B)$. Then $L \neq 0$, since $L \cap B = l_B(B) \neq 0$. If $0 \neq x \in L$, then $xB = 0$, whence $xA = xR \neq 0$. Moreover, $LA^d \subseteq LB = 0$, so there exists the largest positive integer $k < d$ such that $xA^k \neq 0$ for each $x \in L \setminus \{0\}$. Hence $xA^{k+1} = 0$ for some $x \in L \setminus \{0\}$. Combining this with $xA^k \neq 0$ gives $xA^kB \neq 0$. Therefore, $xA^kb_1 \neq 0$ for some $b_1 \in B$. Suppose that for some positive integer n, elements $b_1, \ldots, b_n \in B$ such that $xA^kb_1 \cdot \ldots \cdot b_n \neq 0$ have been constructed. Then $(xA^kb_1 \cdot \ldots \cdot b_n)A^k \subseteq x(A + B)A^k = xA^{k+1} = 0$, whence $xA^kb_1 \cdot \ldots \cdot b_n \nsubseteq L = l_R(B)$, i.e. $xA^kb_1 \cdot \ldots \cdot b_nb_{n+1} \neq 0$ for some $b_{n+1} \in B$. In this way, we can construct a sequence (b_n) of elements of B such that $xA^kb_1 \cdot \ldots \cdot b_n \neq 0$ for each $n \in \mathbb{N}$. In particular, $b_1 \cdot \ldots \cdot b_n \neq 0$ for every $n \in \mathbb{N}$, contrary to the left T-nilpotency of B. This completes the proof. $\square$

Remark 2.7. Let A be any subring of a ring R. Define $A_0(R) = \{0\}$, $A_{\alpha+1}(R) = \{x \in R : xA \subseteq A_\alpha(R)\}$ for any ordinal number $\alpha > 0$, and $A_\beta(R) = \bigcup_{\alpha<\beta} A_\alpha(R)$ for a limit ordinal β. A straightforward verification shows that if $\alpha \leq \beta$, then $A_\alpha(R) \subseteq A_\beta(R)$, $A_\alpha(R)A \subseteq A_\alpha(R)$ and $A_\alpha(R) <_l R$ for each ordinal number α. Hence for $L_R(A) = \bigcup_\alpha A_\alpha(R)$ we have $L_R(A) <_l R$. Moreover, if S is a subring of R, then $A_\alpha(S) \subseteq A_\alpha(R)$, so $L_S(A) \subseteq L_R(A)$. In particular, $l_\alpha(A) \subseteq A_\alpha(R)$ for each ordinal number α, and $l(A) \subseteq L_R(A)$. Notice also that

$$A_\alpha(L_R(A)) = A_\alpha(R) \text{ for every ordinal number } \alpha. \tag{2.1}$$

Indeed, if this is not true, then there is the smallest ordinal number α such that $A_\alpha(L_R(A)) \neq A_\alpha(R)$. Since $A_0(L_R(A)) = A_0(R) = \{0\}$, α is not a limit number, i.e. $\alpha = \beta + 1$ for some ordinal number β. Furthermore, $A_\alpha(L_R(A)) \subseteq A_\alpha(R)$, so there exists $x \in A_{\beta+1}(R)$ such that $x \notin A_{\beta+1}(L_R(A))$. But $x \in L_R(A)$ and $xA \subseteq A_\beta(R) = A_\beta(L_R(A))$, whence $x \in A_{\beta+1}(L_R(A))$, a contradiction.

Lemma 2.8. *Let R be a ring with $l_R(R) = \{0\}$. If $R = A + B$ for some non-zero left T-nilpotent subrings A and B of R, then $L_R(A) \cap l_R(B) = \{0\}$.*

Proof. Suppose that $L_R(A) \cap l_R(B) \neq \{0\}$. Consequently, there exists the smallest ordinal number α such that $A_\alpha(R) \cap l_R(B) \neq \{0\}$. Since $l_R(A) \cap l_R(B) \subseteq l_R(R) = \{0\}$ and $A_1(R) = l_R(A)$, we have $A_1(R) \cap l_R(B) = \{0\}$, which shows that $\alpha > 1$. Of course, α is not a limit number, so $\alpha = \beta + 1$ for some ordinal number β, and there is a non-zero $x \in A_{\beta+1}(R)$ such that $xB = \{0\}$. Therefore, $\{0\} \neq xA \subseteq A_\beta(R)$, whence $0 \neq xa \in A_\beta(R)$ for some $a \in A$. Combining this with $A_\beta(R) \cap l_R(B) = \{0\}$ leads to $xaB \neq \{0\}$. Hence there exists $b_1 \in B$ such that $xab_1 \neq 0$. We now proceed by induction. Suppose that for some $n \in \mathbb{N}$ there exist elements $b_1, \ldots, b_n \in B$ such that $xab_1 \cdot \ldots \cdot b_n \neq 0$. As $R = A + B$, we get $ab_1 \cdot \ldots \cdot b_n = a' + b'$ for some $a' \in A$ and $b' \in B$. Combining this with $xB = \{0\}$ gives $0 \neq xab_1 \cdot \ldots \cdot b_n = x(a' + b') = xa'$. Thus, $0 \neq xa' \in A_\beta(R)$, and consequently, there exists $b_{n+1} \in B$ such that $xa'b_{n+1} \neq 0$. Therefore, $xab_1 \cdot \ldots \cdot b_n b_{n+1} \neq 0$, from here $b_1 \cdot \ldots \cdot b_n \cdot b_{n+1} \neq 0$. By induction, there exists a sequence (b_n) of elements of B such that $b_1 \cdot b_2 \cdot \ldots \cdot b_n \neq 0$ for each $n \in \mathbb{N}$, contrary to the left T-nilpotency of B. ☐

We now are able to present the main result in this section.

Theorem 2.9. *If A and B are left T-nilpotent subrings of a ring R and $R = A + B$, then the ring R is left T-nilpotent.*

Proof. Suppose, contrary our claim, that R is not left T-nilpotent. There is no loss of generality in assuming that $l_R(R) = \{0\}$, $A \neq \{0\}$ and $B \neq \{0\}$. Then $l_A(A) \neq \{0\}$, and consequently, $l_R(A) \neq \{0\}$. Moreover, $L_R(A) <_l R$ and the left T-nilpotency of A implies $A \subseteq L_R(A)$. Let $I = l_R(L_R(A))$. Since $L_R(A) <_l R$, we obtain $I \lhd R$. Suppose that $I \neq \{0\}$. As $A \subseteq L_R(A)$, we get $IA = \{0\}$. Combining this with $R = A + B$ and $l_R(R) = \{0\}$ gives $IB \neq \{0\}$. Furthermore, $IB \subseteq I$, so $(IB)A = \{0\}$. Therefore, $ib_1 \neq 0$ for some $i \in I$ and $b_1 \in B$. Assume that we have already found, for some $k \in \mathbb{N}$, elements $b_1, \ldots, b_k \in B$ such that $j = ib_1 \cdot \ldots \cdot b_k \neq 0$. Then $j \in I$, i.e. $jA = \{0\}$. Hence $jB \neq \{0\}$ and $jb_{k+1} \neq 0$ for some $b_{k+1} \in B$, and consequently, $ib_1 \cdot \ldots \cdot b_k b_{k+1} \neq 0$. Therefore, $b_1 \cdot \ldots \cdot b_n \neq 0$ for every $n \in \mathbb{N}$, contrary to the left T-nilpotency of B. Thus $I = \{0\}$. In particular, $l_{L_R(A)}(L_R(A)) = \{0\}$.

Next, $A \subseteq L_R(A)$ and $R = A + B$, so the modularity of the lattice of subgroups of the group R^+ implies $L_R(A) = A + B_1$, where $B_1 = L_R(A) \cap B$ is a left T-nilpotent ring. Moreover, $B_1 \neq \{0\}$, whence $bB_1 = \{0\}$ for some non-zero $b \in B_1$. As $b \in L_R(A)$, we get $b \in A_\alpha(R)$ for some ordinal number α. In view of (2.1), $b \in A_\alpha(L_R(A))$, i.e. $b \in L_{L_R(A)}(A)$, contrary to Lemma 2.8. ☐

In [17, Example 2], Sands gave an example of a ring R such that $R = A + B$, A is a M-nilpotent subring of R, $B \lhd R$ and $B^2 = \{0\}$, but R is not M-nilpotent. We now show that an analogous example does not exist under the assumption $A < R$.

Lemma 2.10. *If A and B are a M-nilpotent one-sided ideal and a nilpotent subring of a ring R, respectively, and $R = A + B$, then the ring R is M-nilpotent.*

Proof. There is no loss of generality in assuming that $A <_r R$. Suppose, contrary to our claim, that the ring R is not M-nilpotent. Then it contains a double sequence (x_n) such that $x_{-n} \cdot \ldots \cdot x_0 \cdot \ldots \cdot x_n \neq 0$ for every $n \in \mathbb{N}$. Since $R = A + B$, for every $k \in \mathbb{N} \cup \{0\}$ there exist $a_{\pm k} \in A$

and $b_{\pm k} \in B$ such that $x_{\pm k} = a_{\pm k} + b_{\pm k}$. Hence for every $n \in \mathbb{N}$ we have $(a_{-n} + b_{-n}) \cdot \ldots \cdot (a_0 + b_0) \cdot \ldots \cdot (a_n + b_n) \neq 0$. Therefore, for all $i = 0, 1, \ldots, n$ there exist $z_{(\pm i)} \in \{a_{(\pm k)}, b_{(\pm k)}\}$ such that $z_{-n} \cdot \ldots \cdot z_0 \cdot \ldots \cdot z_n \neq 0$. For each $n \in \mathbb{N}$, let us denote by G_n the set of all elements $(y_{-n}, \ldots, y_0, \ldots, y_n) \in \{a_{-n}, b_{-n}\} \times \ldots \times \{a_0, b_0\} \times \ldots \times \{a_n, b_n\}$ satisfying $y_{-n} \cdot \ldots \cdot y_0 \cdot \ldots \cdot y_n \neq 0$. Since

$$0 \neq (a_{-n}+b_{-n}) \cdot \ldots \cdot (a_1+b_1) \cdot \ldots \cdot (a_n+b_n) = \sum_{(y_{-n},\ldots,y_0,\ldots,y_n) \in G_n} y_{-n} \cdot \ldots \cdot y_0 \cdot \ldots \cdot y_n,$$

we have $0 < |G_n| < \infty$. Clearly the set $G = G_1 \cup G_2 \cup \ldots$ is infinite, and consequently, there exists $0 \neq c_0 \in \{a_0, b_0\}$ such that infinitely many sequences belonging to G have the zero term equal to c_0. Assume that we have already found an element $(c_{-k}, \ldots, c_0, \ldots, c_k) \in G_k$ for some $k \in \mathbb{N}$ and infinitely many sequences belonging to G start with the subsequence $(c_{-k}, \ldots, c_0, \ldots, c_k)$. As previously we infer that there exists $c_{\pm(k+1)} \in \{a_{\pm(k+1)}, b_{\pm(k+1)}\}$ such that infinitely many sequences from G begin with $(c_{-(k+1)}, c_{-k}, \ldots, c_0, \ldots, c_k, c_{k+1})$, so $c_{-(k+1)} \cdot c_{-k} \cdot \ldots \cdot c_0 \cdot \ldots \cdot c_k \cdot c_{k+1} \neq 0$. By induction, there is a double sequence (c_n) such that $c_{\pm n} \in A \cup B$ and $c_{-n} \cdot \ldots \cdot c_0 \cdot \ldots \cdot c_n \neq 0$ for every $n \in \mathbb{N}$. Since B is a nilpotent ring, there exists a subsequence (c_{k_n}) of the sequence (c_n) such that $\{c_{k_n}\} \subseteq A$. Let $d_0 = c_{k_{(-1)}} \cdot c_{k_{(-1)}+1} \cdot \ldots \cdot c_{k_1 - 1}$, $d_n = c_{k_n} \cdot c_{k_n+1} \cdot \ldots \cdot c_{k_{n+1}-1}$ and $d_{-n} = c_{k_{-n}} \cdot c_{k_{-n}+1} \cdot \ldots \cdot c_{k_{(-(n-1))}+1}$ for $n \in \mathbb{N}$. Then $\{d_n\} \subseteq A$ because of $A <_r R$. Moreover, for any fixed $n \in \mathbb{N}$ and $t = max(k_{-n}, k_n)$ we have $c_{k_{-t}} \cdot \ldots \cdot c_0 \cdot \ldots \cdot c_t \neq 0$, so $d_{-n} \cdot \ldots \cdot d_0 \cdot \ldots \cdot d_n = c_{k_{-n}} \cdot c_{k_{-n}+1} \cdot \ldots \cdot c_{k_n} \cdot c_{k_n - 1} \neq 0$. Therefore, the ring A is not M-nilpotent, a contradiction. □

3. Rings with S-Gradation

Definition 3.1. Let S be a semigroup. A ring R is said to be S-graded if there exist a family of subgroups $\{A_s \colon s \in S\}$ of the additive group of R such that $R = \oplus_{s \in S} A_s$ and $A_s A_t \subseteq A_{st}$ for all $s, t \in S$.

Remark 3.2. Notice that A_t is a subring of ring R if and only if $t = t^2$ or $t \neq t^2$ and $A_t^2 = 0$. Indeed, if A_t is a subring in R and $t \neq t^2$, then $A_t^2 \subseteq A_t \cap A_{t^2} = \{0\}$. The reverse implication is obvious.

Definition 3.3. Let S be a semigroup, and let $\mathcal{M}$ be a class of rings. The class $\mathcal{M}$ is said to be S-closed if $\mathcal{M}$ contains all the S-graded rings $R = \oplus_{s \in S} A_s$ such that all subrings among the groups A_s of the additive group of R belong to $\mathcal{M}$.

Definition 3.4. Let $\mathcal{N}$ and $\mathcal{M}$ be classes of rings satisfying $\mathcal{N} \subseteq \mathcal{M}$, and let S be a semigroup. The pair $(\mathcal{N}, \mathcal{M})$ is called S-closed if the class $\mathcal{M}$ contains all the S-graded rings $R = \oplus_{s \in S} A_s$ such that all subrings among the groups A_s of the additive group of R belong to $\mathcal{N}$.

Remark 3.5. Let $\mathcal{S}$ be the class of all semigroups S such that the class of all left T-nilpotent rings is S-closed. Then:

(i) if $S \in \mathcal{S}$, then every subsemigroup A of S belongs to $\mathcal{S}$;
(ii) if $S \in \mathcal{S}$ and $I \lhd S$, then $S/I \in \mathcal{S}$;
(iii) if $S \in \mathcal{S}$, $I \lhd S$ and $S/I \in \mathcal{S}$, then $S \in \mathcal{S}$;
(iv) if S belongs to $\mathcal{S}$, then the semigroup $S^0 = S \cup \{0\}$ with zero adjoined belongs to $\mathcal{S}$

In view of the above remark, it is sufficient to describe semigroups with zero from the class $\mathcal{S}$.

Definition 3.6. We say that a sequence (s_n) of elements of a semigroup S *satisfies the condition (q)* if for any positive integers $k < l < m$ the condition $s = s_k \cdot \ldots \cdot s_l = s_{l+1} \cdot \ldots \cdot s_m$ implies $s \neq s^2$.

Remark 3.7. Notice that the existence of a sequence that satisfying the condition (q) in a semigroup S is equivalent to the existence of such a sequence in the semigroup S^0.

From now on, each semigroup is assumed to be a semigroup with zero.

Example 3.8. Let (s_n) be a sequence in a semigroup S. If the set $M = \{s_k \cdot \ldots \cdot s_l : k, l \in \mathbb{N}, k \leq l\}$ does not contain an idempotent, then (s_n) satisfies the condition (q). In particular, if G is a subset of S and $s_n \in G$ for every $n \in \mathbb{N}$ and $e \notin M$, then (s_n) satisfies condition (q).

Example 3.9. If (s_n) be a sequence in a nil semigroup S and $s_1 \cdot \ldots \cdot s_n \neq 0$ for every $n \in \mathbb{N}$, then (s_n) satisfies condition (q).

Example 3.10. If an element x of a semigroup S is not periodic, then it follows from Example 3.8 that the sequence $(x, x, x, \ldots)$ satisfies condition (q).

Let $\mathcal{Z}$ and $\mathcal{T}$ be the classes of all zero-rings and left T-nilpotent rings, respectively. It is easily seen that if S is a semigroup and the pair $(\mathcal{Z}, \mathcal{T})$ is not S-closed, then the class $\mathcal{T}$ is not S-closed. Our main goal in this section is to show that the converse implication is also true.

Let us start with the following useful lemma:

Lemma 3.11. *For any semigroup S the following conditions are equivalent:*

(i) there exists a sequence (s_n) of elements of the semigroup S satisfying the condition (q);
(ii) the pair $(\mathcal{Z}, \mathcal{T})$ is not S-closed.

Proof. (*i*) $\Rightarrow$ (*ii*). Let A be an algebra over the field K generated by the elements $a_1, a_2, \ldots$ satisfying the following relations $a_i \cdot a_j = 0$ for any $i, j \in \mathbb{N}$ such that $j \neq i + 1$. Then $a_1 \cdot a_2 \cdot \ldots \cdot a_n \neq 0$ for every $n \in \mathbb{N}$, so A is not a left T-nilpotent ring. Let $\overline{S} = \{s_k \cdot \ldots \cdot s_l : k, l \in \mathbb{N}, k \leq l\}$. For any $s \in \overline{S}$, let A_s denote the K-subspace of the linear space A generated by all elements of the form $a_k \cdot \ldots \cdot a_l$, where $k \leq l$ are positive integers such that $s_k \cdot \ldots \cdot s_l = s$. Then $A = \bigoplus_{s \in \overline{S}} A_s$. Moreover, define $A_s = \{0\}$ for $s \in S \setminus \overline{S}$. Obviously, $A = \bigoplus_{s \in S} A_s$. An easy computation shows that $A_s A_t \subseteq A_{st}$ for all $s, t \in S$. Take any $t \in S$ such that A_t is a subring of the ring A. Then it follows from Remark 3.2 that $t = t^2$ or $A_t^2 = 0$. If $A_t^2 = 0$, then A_t is a left T-nilpotent ring. Now suppose $t = t^2$. By way of contradiction assume that $A_t^2 \neq \{0\}$. Then there exist two non-zero generators $a = a_k \cdot \ldots \cdot a_l$ and $b = a_n \cdot \ldots \cdot a_m$ of the subspace A_t such that $k \leq l$, $n \leq m$, $s_k \cdot \ldots \cdot s_l = t = s_n \cdot \ldots \cdot s_m$ and $a \cdot b \neq 0$. From the definition of the algebra A, we infer that $n = l + 1$. Furthermore, the sequence (s_n) satisfies the condition (q), so $t \neq t^2$, a contradiction. Therefore, $A_t^2 = 0$ for every $t \in S$ such that A_t is a subring of the ring A.

(ii) $\Rightarrow$ *(i)*. Suppose the pair $(\mathcal{Z}, \mathcal{T})$ is not S-closed. Then there exist a ring R which is not left T-nilpotent and a non-empty family of subgroups $\{A_t\}_{t \in T}$ of the additive group of the ring R such that $R = \bigoplus_{t \in T} A_t$, $A_s A_t \subseteq A_{st}$ for all $s, t \in T$ and $A_t^2 = 0$ for every $t \in T$ for which A_t is a subring of R. In particular, if $t \in T$ and $t = t^2$, then $A_t^2 = 0$. Hence, by Lemma 2.1, there exist sequences (s_n) of elements of S and (a_n) of elements of R such that $a_n \in A_{s_n}$ and $a_1 \cdot \ldots \cdot a_m \neq 0$ for any $n, m \in \mathbb{N}$. Now it is easy to see that the sequence (s_n) satisfies condition (q). ☐

Combining Lemma 3.11, Example 3.10 and Remark 3.5 leads to the conclusion that every semigroup S belonging to $\mathcal{S}$ is periodic.

The proof of the next lemma is partially related to that of [12, Lemma 9].

Lemma 3.12. *Any infinite simple semigroup S includes a sequence (s_n) satisfying the condition (q).*

Proof. Suppose G is an infinite subgroup of a semigroup S. First we show that there exists an infinite sequence (g_n) of elements of G such that $g_m g_{m+1} \cdots g_n \neq e$ for all positive integers $m \leq n$. We construct it inductively. Since $|G| = \infty$, there exists $g \in G$ such that $g \neq e$. Define $g_1 = g$. Assume that we have already found, for some $n \in \mathbb{N}$, the desired elements $g_1, g_2, \ldots, g_n$. Since the set $C = \{(g_m g_{m+1} \cdots g_n)^{-1} \mid m \leq n\} \cup \{e\}$ is finite, it is sufficient to take any member g' of set $G \setminus C$ and define $g_{n+1} = g'$. Of course, the sequence (g_n) satisfies condition (q).

If S is not a periodic group, the assertion follows at once from Example 3.10. Now suppose that S is a periodic semigroup. If S is 0-simple, it is completely 0-simple. In view of Rees theorem (see [2, Theorem 3.5]), we can assume that $S = \mathcal{M}^0(G^0; X, Y; P)$ for some finite group G and non-degenerated $Y \times X$-matrix P. Without loss of generality, we may assume that the set Y is infinite. Then it contains pairwise different elements $y_1, y_2, \ldots$. Take any $g_1 \in G$. Since the matrix P is non-degenerated, there exists $x_0 \in X$ such that $p_{y_1 x_0} \neq 0$, i.e. $p_{y_1 x_0} \in G$. Hence $s_1 = (g_1, x_0, y_1) \in S$. Let $s_2 = (p_{y_1 x_0}^{-1}, x_0, y_2)$. Then $s_2 \in S$ and $s_1 s_2 = (g_1, x_0, y_2)$. Suppose that we have already found, for some $n \in \mathbb{N}$, the elements $s_1, \ldots, s_n \in S$ such that $s_1 \cdot \ldots \cdot s_n = (g_1, x_0, y_n)$. Then $p_{y_n x} \in G$ for some $x \in X$, so $s_{n+1} = (p^{-1}_{y_n x}, x, y_{n+1}) \in S$ and $s_1 s_2 \cdot \ldots \cdot s_n s_{n+1} = (g_1, x_0, y_{n+1})$. Thus for any positive integers $k < l < m$ we get $s_k \cdot \ldots \cdot s_l = (h, x, y_l)$ and $s_{l+1} \cdot \ldots \cdot s_m = (h', x', y_m)$ for some $h, h' \in G$ and $x, x' \in X$. Therefore, $s_k \cdot \ldots \cdot s_l \neq s_{l+1} \cdot \ldots \cdot s_m$. Consequently, the sequence (s_n) satisfies condition (q). □

Definition 3.13. We say that a nil semigroup S is *left T-nilpotent* if for every sequence (s_n) of elements of S, there exists positive integer n such that $s_1 \cdot s_2 \cdot \ldots \cdot s_n = 0$.

Proposition 3.14. *Let S be a nil semigroup. Then the class of left T-nilpotent rings is S-closed if and only if the semigroup S is left T-nilpotent.*

Proof. (i) ⇒ (ii). Suppose, contrary our claim, that S is not left T-nilpotent. Then there exists a sequence (s_n) of elements of S satisfying $s_1 \cdot s_2 \cdot \ldots \cdot s_n \neq 0$ for every $n \in \mathbb{N}$. By virtue of Example 3.9, the sequence (s_n) satisfies the condition (q). Combining this with Lemma 3.11 leads to a contradiction.

(ii) ⇒ (i). Assume, by way of contradiction, that $R = \oplus_{s \in S} R_s$ is an S-graded ring such that $R \notin \mathcal{T}$ and $R_s \in \mathcal{T}$ for every $s \in S$ for which R_s is a subring of R. It follows from Lemma 2.1 that there is a sequence (s_n) of elements of S such that $r_n \in R_{s_n}$ and $r_1 \cdot r_2 \cdot \ldots \cdot r_n \neq 0$ for each $n \in \mathbb{N}$. Since the semigroup S is left T-nilpotent, there exists a sequence $c_1 \leq c_2 \leq \ldots$ of positive integers such that $y_t = r_{c_t} \cdot r_{c_t+1} \cdot \ldots \cdot r_{c_{t+1}} \in R_0$ for each $t \in \mathbb{N}$. Since R_0 is a ring belonging to $\mathcal{T}$, there exists $k \in \mathbb{N}$ such that $y_1 \cdot \ldots \cdot y_k = 0$, contrary to the fact that $r_1 \cdot \ldots \cdot r_n \neq 0$ for every $n \in \mathbb{N}$. □

Remark 3.15. Let S and T be any semigroup and set, respectively. Suppose that $S = \bigcup_{i \in T} S_i$, where $S_i \lhd S$ and $S_i \cap S_j = \{0\}$ for $i \neq j$ (i.e. S is a direct sum of ideals S_i). If the class $\mathcal{T}$ is S_i-closed for each $i \in T$, then it is also S-closed. Indeed, consider any S-graded ring $R = \oplus_{s \in S} R_s$ such that $R_s \in \mathcal{T}$ for every $s \in S$ for which R_s is a subring of R. Notice that $R = \oplus_{i \in T} R_{S_i}$ is a direct sum of ideals $R_{S_i} = \oplus_{s \in S_i} R_s$. Since R_{S_i} is a ring belonging to $\mathcal{T}$ for $i \in T$, we get $R \in \mathcal{T}$. Notice also that if $|T| < \infty$, then the assumption $S_i \cap S_j = \{0\}$ for $i \neq j$ can be omitted.

Definition 3.16. We say that class $\mathcal{M}$ of rings is *closed under sums of left ideals* if for any ring R and its left ideals L_1, L_2 the condition that $L_1, L_2 \in \mathcal{M}$ implies $L_1 + L_2 \in \mathcal{M}$.

Lemma 3.17. *Any class $\mathcal{M}$ of rings, which is closed under sums of one-sided ideals, subrings, and contains a class of all zero rings, is G-closed for any finite group G.*

Proof. Let $G = \{g_1, g_2, \ldots, g_n\}$ with $g_1 = e$. Consider any ring $A = \oplus_{g \in G} A_s$ with G-gradation with respect to the group G, and suppose $A_e \in \mathcal{M}$. Define the following matrix ring:

$$P = \begin{pmatrix} A_e & A_{g_1 g_2^{-1}} & \cdots & A_{g_1 g_n^{-1}} \\ A_{g_2 g_1^{-1}} & A_e & \cdots & A_{g_2 g_n^{-1}} \\ \vdots & \vdots & & \vdots \\ A_{g_n g_1^{-1}} & A_{g_n g_2^{-1}} & \cdots & A_e \end{pmatrix}.$$

Since P is a sum of left ideals of the form

$$L_i = \begin{pmatrix} 0 & \cdots & A_{g_1 g_i^{-1}} & \cdots & 0 \\ 0 & \cdots & A_{g_2 g_i^{-1}} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & A_e & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & A_{g_n g_i^{-1}} & \cdots & 0 \end{pmatrix} \in \mathcal{M},$$

we get $P \in \mathcal{M}$. It is easy to check that

$$A \ni \sum_{g \in G} a_g \overset{\phi}{\longmapsto} \begin{pmatrix} a_e & a_{g_1 g_2^{-1}} & \cdots & a_{g_1 g_n^{-1}} \\ a_{g_2 g_1^{-1}} & a_e & \cdots & a_{g_2 g_n^{-1}} \\ \vdots & \vdots & & \vdots \\ a_{g_n g_1^{-1}} & a_{g_n g_2^{-1}} & \cdots & a_e \end{pmatrix} \in P$$

is an isomorphism of the ring A onto some subring M of P. Thus $M \in \mathcal{M}$, and consequently, $A \in \mathcal{M}$. □

Corollary 3.18. *The class of left T-nilpotent rings is G-closed for any finite group G.*

The proof of the next lemma is similar to that of [18, Lemma 4.1]. We include it for completeness of this paper.

Lemma 3.19. *The class of left T-nilpotent rings is S-closed for any finite semigroup S.*

Proof. Let S be a finite semigroup and $R = \oplus_{s \in S} R_s$ be an S-graded ring such that $R_s \in \mathcal{T}$ for every $s \in S$ for which R_s is a subring of R. The proof is by induction on $|S|$. In view of the fact that class $\mathcal{T}$ is closed under extensions, it suffices to prove the lemma for semigroups S which are nilpotent or 0-simple. If S is a nilpotent semigroup, then R is a nilpotent ring. Therefore, it is sufficient to assume that S is a 0-simple semigroup. Then $S = \mathcal{M}^0(G^0; m, n; A)$ for some group G, positive integers m, n and $m \times n$ matrix A. If $S = G^0$, the assertion follows from Corollary 3.18 and (iv) of Remark 3.5. Now suppose $S \neq G^0$. Then, for $1 \leq i \leq n$, the columns B_i of the matrix semigroup $\mathcal{M}^0(G^0; m, n; a)$ are left ideals of S. This implies that R is a sum of left ideals $R_{B_i} = \oplus_{s \in B_i} R_s$

which are B_i-graded rings such that $|R_{B_i}| < |S|$. By induction hypothesis, R is a sum of left ideals belonging to $\mathcal{T}$. Finally, $R \in \mathcal{T}$. $\square$

Our next aim is to describe all the semigroups S for which the class of left T-nilpotent rings is S-closed. We begin with the following definition of some particular chain of ideals in S

Definition 3.20. Let S be a semigroup, and let $U_0(S) = \{0\}$. Consider any ordinal number $\alpha > 0$. If α is a limit number, then we define $U_\alpha(S) = \bigcup_{\gamma<\alpha} U_\gamma(S)$. Otherwise, $U_\alpha(S)$ is defined to an ideal of S such that $U_\alpha(S)/U_{\alpha-1}(S)$ decomposes into a direct sum of minimal finite ideals in $S/U_{\alpha-1}(S)$ or $U_\alpha(S)/U_{\alpha-1}(S)$ is a left T-nilpotent. Moreover, define $U(S) = \bigcup_{\alpha\geq 0} U_\alpha(S)$.

Lemma 3.21. *If S is a semigroup in which there is no sequence satisfying the condition (q), then $S = U(S)$.*

Proof. In view of Example 3.10, S is a periodic semigroup. Suppose $S/U(S) \neq \{0\}$. Obviously, in $Q = S/U(S)$ also there is no sequence satisfying the condition (q). Moreover, every nil ideal Q is left T-nilpotent by Example 3.9, so Q does not have any non-zero nil ideals.

We will show that Q has a non-zero minimal ideal. Note that if Q_1 is a right ideal of Q, then there exists a non-zero idempotent $e \in Q_1$. Indeed, otherwise, since Q_1 is a periodic semigroup and S does not have sequences that satisfy condition (q), Q_1 is left T-nilpotent. It is easy to see that an ideal H generated by Q_1 in Q is left T-nilpotent, which gives a contradiction. Assume that for some sequence (e_n) of idempotents of Q there exists an infinite chain of right ideals $e_1Q \supsetneq e_2Q \supsetneq \ldots$ of Q. Clearly, $e_ne_{n+1} = e_{n+1}$ for every $n \in \mathbb{N}$, so (e_n) satisfies condition (q), contrary to our assumption. Therefore $J = e_mQ$ is a minimal non-zero right ideal of Q for some $m \in \mathbb{N}$. Let us note that every ideal M of e_mQ such that $M^2 \neq \{0\}$ contains a non-zero right ideal MJ of Q. Since J is a minimal right ideal of Q, $MJ \subseteq M \subseteq J \subseteq MJ$. It follows that $M = J$. Hence J/N is 0-simple for some nilpotent ideal N of J, because J is not nil. Since J/N is a periodic semigroup, J/N is completely 0-simple. Therefore J/N, and consequently also J, contains a primitive idempotent. Without loss of generality we can assume that e_m is a primitive idempotent of J. Note that, if $Qf \subsetneq Qe_m$ for some non-zero idempotent $f \in Q$, we have $f = fe_m$. Clearly $Qe_mf \subsetneq Qe_m$. But $t = e_mf \in J$ is a non-zero idempotent and $te_m = e_mt = t$, so $t = e_m$. The obtained contradiction implies that Qe_mQ is a non-zero minimal ideal of Q. Using Lemma 3.12, we get that Qe_mQ is a finite ideal in Q, which is a contradiction with the definition of $U(S)$, so the proof is complete. $\square$

Theorem 3.22. *Let S be a semigroup. The class of left T-nilpotent rings is S-closed if and only if there is no sequence of elements of S satisfying the condition (q).*

Proof. First suppose $\mathcal{T}$ is an S-closed class. Then the pair $(\mathcal{Z}, \mathcal{T})$ is S-closed. It follows from Lemma 3.11 that there exists no sequence of elements of S satisfying the condition (q).

Conversely, suppose that there is no sequence of elements of S satisfying the condition (q). First we show that the class $\mathcal{T}$ is $U_\alpha(S)$-closed for any ordinal number α. Of course, this is true for $U_0(S)$. Consider any ordinal number $\alpha > 0$, and assume the class $\mathcal{T}$ is $U_\beta(S)$-closed for every ordinal $\beta < \alpha$. By Proposition 3.14, Lemma 3.19 and Remark 3.15, the class $\mathcal{T}$ is Q-closed if Q is a T-nilpotent semigroup or Q is a direct sum of minimal ideals. Moreover, the class $\mathcal{T}$ is closed under extensions, so it is enough to assume that α is a limit ordinal number. In this case, suppose that class $\mathcal{T}$ is not $U_\alpha(S)$-closed. Let $R = \oplus_{h \in U_\alpha(S)} A_h$ be a $U_\alpha(S)$-graded ring such that $R \notin \mathcal{T}$ and $A_h \in \mathcal{T}$ for every $h \in U_\alpha(S)$ for which A_h is a subring in R. In view of $R \notin \mathcal{T}$ and Lemma 2.1, we can assume that there exists a sequence (h_n) of elements of $U_\alpha(S)$ such that for every $n \in \mathbb{N}$ there exists $a_n \in A_{h_n}$ and $a_1 \cdot \ldots \cdot a_m \neq 0$ for every $m \in \mathbb{N}$. Let β be the smallest ordinal number such that $U_\beta(S) \cap H \neq \emptyset$ for $H = \{h_n \colon n \in \mathbb{N}\}$. Then there exists a set of natural numbers $C = \{c_1, c_2, \ldots, c_s\}$ such that $1 = c_1 < c_2 < \ldots < c_s$ and $h_{c_j} \cdot h_{c_j+1} \cdot \ldots \cdot h_{c_{j+1}-1} \in U_\beta(S)$ for each $j \in \{1, 2, \ldots, s-1\}$. Furthermore, $\{h_{c_s} \cdot \ldots \cdot h_l : l \in \mathbb{N}, c_s \leq l\} \cap U_\beta(S) = \emptyset$, so the set C cannot be extended infinitely many times. Indeed, if C could be extended infinitely many times, then we would get a contradiction to the fact that $a_1 \cdot \ldots \cdot a_m \neq 0$ for each $m \in \mathbb{N}$, because the class $\mathcal{T}$ is $U_\beta(S)$-closed. Let $\overline{h}_1 = h_1 \cdot h_2 \cdot \ldots \cdot h_{c_s}$, and let $\gamma_1 = \beta$. Similarly, we define $\overline{h}_2$ and the ordinal number $\gamma_2 > \gamma_1$ by using the sequence $(h_{c_s+1}, h_{c_s+2}, \ldots)$. This construction can be repeated infinitely many times. Consequently, we obtain the sequence $(\overline{h}_n)$ and the growing sequence of ordinal numbers (γ_n) such that $\overline{h}_k \cdot \ldots \cdot \overline{h}_l \in U_{\gamma_k}(S) \setminus U_{\gamma_{k-1}}(S)$ where $k, l \in \mathbb{N}, k \leq l$. It is easily seen that $(\overline{h}_n)$ satisfies the condition (q), a contradiction. Thus the class $\mathcal{T}$ is $U_\alpha(S)$-closed. Moreover, $S = U(S)$ by Lemma 3.21. □

In view of the foregoing result we have the following

Corollary 3.23. *For any semigroup S the following conditions are equivalent:*

(i) the class $\mathcal{T}$ is S-closed;
(ii) the pair $(\mathcal{Z}, \mathcal{T})$ is S-closed;
(iii) there is no sequence of elements of S satisfying the condition (q).

4. Rings Which are S-Sums

Definition 4.1. Let S be a semigroup. A ring R is called an S-sum if there exists a family of subgroups $\{R_s \colon s \in S\}$ of the additive group of R such that $R = \sum_{s \in S} R_s$ and $R_s R_t \subseteq \sum_{q \in \langle st \rangle} R_q$ for all $s, t \in S$, where $\langle st \rangle$ means the subsemigroup of S generated by st.

Remark 4.2. Let S be a semigroup. Of course, every S-graded ring is an S-sum. Much relevant information on S-sums can be found in [6].

Our goal is to extend Theorem 3.22 to rings being S-sums. As for S-graded rings, the description of semigroups S will turn to crucial. The following additional facts will be also needed.

Lemma 4.3. *Let G be a finite 2-group, and let $R = \sum_{s\in G} R_s$ be a G-sum. If R_e ieft T-nilpotent ring, then so is R.*

Proof. Each finite 2-group has a central sequence with quotients of order two. Moreover, if N is a normal divisor of the group G, then $R = \sum_{gN\in G/N} R_{gN}$ is a G/N-sum, whose initial component is R_N. Therefore, if R_N is a left T-nilpotent ring and G/N is a group of order two, then it follows from Theorem 2.6 that the ring R is left T-nilpotent. Thus the assertion follow by a straightforward induction on the order of G. ☐

Lemma 4.4. *Let S be a finite semigroup, and let $R = \sum_{s\in S} R_s$ be an S-sum such that the only subgroups of S are 2-groups. If $R_s \in \mathcal{T}$ for every $s \in S$ for which R_s is a subring of R, then $R \in \mathcal{T}$.*

Proof. If $S = G^0$ for some 2-group G, the claim follows from Lemma 4.3. In the case when $S \neq G^0$ it is enough to use inductive reasoning related to cardinality of semigroup S, as in the proof of Lemma 3.19. ☐

Lemma 4.5. *For any periodic group G which is not a 2-group, there exists a G-sum $R = \sum_{s\in G} R_s$ which is not left T-nilpotent and R_s is a zero-ring for every $s \in S$ for which R_s is a subring in R.*

Proof. Let $X = \{x, y, z\}$ be any three-element set, and let B be a ring which is not left T-nilpotent. Consider the ring $A = XB[X]$ of all commutative polynomials over B in variables x, y, z with zero constant term. Let I be the ideal of A generated by x^3, y^3, xy, xz and yz. Consider the ring $R = A/I$. Define $R_g = (zB[z] + xB + y^2B + I)/I$, $R_{g^2} = (x^2B + yB + I)/I$ and $R_s = 0$ for any $s \in G \setminus \{g, g^2\}$. Notice that $R_gR_{g^2} = R_{g^2}R_g = 0$. Since $g, g^2 \in \langle g\rangle$, we get also $R_g^2 = (zB[z] + x^2B + I)/I \subseteq \sum_{s\in\langle g\rangle} R_s$ and $R_{g^2}^2 = (y^2B + I)/I \subseteq \sum_{s\in\langle g\rangle} R_s$. Thus R is a G-sum which is not left T-nilpotent and R_s is a ring with zero multiplication for every $s \in S$ for which R_s is a subring in R. ☐

Theorem 4.6. *The following conditions are equivalent for a semigroup S:*

- *(i) every S-sum $R = \sum_{s\in S} R_s$ has the property that the left T-nilpotency of all subrings among the subgroups R_s implies the left T-nilpotency of R;*
- *(ii) each subgroup of S is 2-group and there does not exists a sequence of elements of S satisfying the condition (q).*

Proof. (i) ⇒ (ii). The first statement of *(ii)* follows at once from Lemma 4.5, while the second-one is a direct consequence of Lemma 3.11.

(ii) ⇒ (i) It follows from Lemma 3.21 that $S = U(S)$. The rest of the proof is based on Lemma 4.4 and the proof of (ii) ⇒ (i) of Proposition 3.14, and runs analogously to the proof of Theorem 3.22. □

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