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Results in Mathematics

Paired Kernels and Their Applications

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Abstract. This paper considers paired operators in the context of the Lebesgue Hilbert space on the unit circle and its subspace, the Hardy space H^2. The kernels of such operators, together with their analytic projections, which are generalizations of Toeplitz kernels, are studied. Results on near-invariance properties, representations, and inclusion relations for these kernels are obtained. The existence of a minimal Toeplitz kernel containing any projected paired kernel and, more generally, any nearly S^*-invariant subspace of H^2, is derived. The results are applied to describing the kernels of finite-rank asymmetric truncated Toeplitz operators.

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1. Introduction

Let X be a Banach space, $P \in \mathcal{L}(X)$ a projection, and $Q = I - P$ the complementary projection. An operator of the form $AP + BQ$ or $PA + QB$ with $A, B \in \mathcal{L}(X)$ is called a *paired operator* [16, 23, 24]. In this paper we consider the case when $A = M_a$ and $B = M_b$ are multiplication operators on $L^2 := L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle, with $a, b \in L^\infty := L^\infty(\mathbb{T})$, and we denote by $S_{a,b}$ and $\Sigma_{a,b}$ the operators defined on L^2 by

$$S_{a,b}f = aP^+f + bP^-f\,, \qquad \Sigma_{a,b}f = P^+(af) + P^-(bf). \tag{1.1}$$

Paired operators first appeared in the context of the theory of singular integral equations [25, 26]. Consider the canonical example of a singular integral

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operator on L^2,

$$(Lf)(t) = A(t)f(t) + B(t)(S_{\mathbb{T}}f)(t), \tag{1.2}$$

with $A, B \in L^\infty$ and

$$(S_{\mathbb{T}}f)(t) = \frac{1}{\pi i}\mathrm{PV}\int_{\mathbb{T}} \frac{f(y)}{y-t}\,dy, \qquad t \in \mathbb{T}. \tag{1.3}$$

It is well known that, denoting by $P^\pm$ the orthogonal projections from L^2 onto the Hardy spaces $H^2_+ := H^2(\mathbb{D})$ and $H^2_- = \overline{H^2_0} = (H^2_+)^\perp$, respectively, identified with closed subspaces of L^2, we have $I = P^+ + P^-$ and $S_{\mathbb{T}} = P^+ - P^-$, so that the operator L can be expressed as

$$L = (A+B)P^+ + (A-B)P^-. \tag{1.4}$$

We may write this as $S_{a,b}$ with $a = A+B$, $b = A-B$, while its adjoint is a paired operator of the second type in (1.1),

$$S^*_{a,b}f = \Sigma_{\bar{a},\bar{b}}. \tag{1.5}$$

Paired operators are also closely related to Toeplitz operators of the form

$$T_a = P^+aP^+_{|H^2}, \tag{1.6}$$

where $a \in L^\infty$ is called the *symbol* of the operator. If we represent $S_{a,b}$ in the form

$$\begin{pmatrix} P^+aP^+ & P^+bP^- \\ P^-aP^+ & P^-bP^- \end{pmatrix} : H^2_+ \oplus H^2_- \to H^2_+ \oplus H^2_-, \tag{1.7}$$

we see that paired operators are dilations of Toeplitz operators. If $b \in \mathcal{G}L^\infty$ (that is, invertible in L^∞) we can write

$$S_{a,b} = aP^+ + bP^- = b\left(\frac{a}{b}P^+ + P^-\right), \tag{1.8}$$

which is equivalent after extension [2] to the Toeplitz operator $T_{a/b}$ [4].

However, paired operators and spaces that turn out to be kernels of paired operators, called *paired kernels*, also appear in different guises, for instance in the study of dual truncated Toeplitz operators [7], in the description of scalar-type block Toeplitz kernels [14], in the characterization of the ranges of finite-rank truncated Toeplitz operators [5], and in the study of nearly invariant subspaces for shift semigroups [22].

We shall consider mainly paired operators of the form $S_{a,b}$, where the pair (a, b) is called a *symbol pair*. These operators have been considered mostly under particular conditions, such as invertibility in L^∞, for a, b or a/b [25,26]; see also [23] and references therein. The operator $S_{a,b}$ is said to be of *normal type* if $a, b \in \mathcal{G}L^\infty$. This is by far the most studied case, but some particular types of non-normal paired operators have also been considered [9,15,23,24]. We shall assume throughout the paper, more generally, that $a, b \in L^\infty$ and the pair (a, b) is *nondegenerate*; that is, a, b and $a - b$ are nonzero a.e. on $\mathbb{T}$. We also use the notation $P^\pm\phi = \phi_\pm$ for $\phi \in L^2$.

Here we shall study in particular the properties of kernels of paired operators, called *paired kernels*, and their projections into H^2_+ and H^2_-, called *projected paired kernels*. Denoting

$$\ker_{a,b} = \ker S_{a,b}, \qquad \ker^{\pm}_{a,b} = P^{\pm}\ker_{a,b}, \tag{1.9}$$

we have that $\ker T_{a/b} = P^+ \ker S_{a,b} =: \ker^+_{a,b}$ for the possibly unbounded Toeplitz operator $T_{a/b}$. If $a/b \in L^\infty$ then $\ker^+_{a,b}$ is a *Toeplitz kernel*, i.e., the kernel of a bounded Toeplitz operator; otherwise it may not be a closed subspace of $H^2_+ := P^+L^2$. However, we can define a one-to-one correspondence (see Sect. 2) between $\ker^+_{a,b}$ and $\ker_{a,b}$, where the latter is closed because it is the kernel of a bounded operator. So one can also see paired kernels as being the natural closed space generalizations of Toeplitz kernels, allowing us to study the kernels of unbounded Toeplitz operators of the form $T_{a/b}$ with $a/b \notin L^\infty$ in terms of the bounded operators $S_{a,b}$ and P^+ on L^2. It is thus natural to ask whether some known properties of Toeplitz kernels with bounded symbols can be extended or related with corresponding properties of paired kernels or their projections on H^2_+. Alternatively one can look at this as studying the question of how certain properties of a Toeplitz operator extend to a dilation of the form (1.7).

In the following sections we study several properties of paired kernels which extend or are in contrast with various known properties of Toeplitz kernels. To compare the case where $a/b \in L^\infty$ (and $\ker^+_{a,b}$ is a Toeplitz kernel) with that where $a/b \notin L^\infty$ and to illustrate some natural questions arising in the latter case, we start by considering in Sect. 3 an example which appears in the study of nearly invariant subspaces for shift semigroups [22]. This leads to the study of near invariance properties of paired kernels (Sect. 4); to the question of existence of a minimal Toeplitz kernel containing any given $\ker^+_{a,b}$ and, more generally, any space of the form $b\ker T_a$, including the closed nearly S^*-invariant subspaces of H^2_+ (Sect. 5); to investigating the relations between paired kernels for operators with connected symbol pairs (Sect. 6). In Sect. 7 the results are applied to study and describe the kernels of finite rank asymmetric truncated Toeplitz operators [10], showing in particular that, surprisingly, they do not depend on the range space if the latter is "large" enough.

2. Projected Paired Kernels

Recall that we denote the kernel of a paired operator, which we call a *paired kernel*, by

$$\ker_{a,b} = \ker S_{a,b} \tag{2.1}$$

and we call

$$\ker^{\pm}_{a,b} = P^{\pm}\ker_{a,b} = P^{\pm}\ker S_{a,b} \tag{2.2}$$

a *projected paired kernel*.

Each function $f \in L^2$ belongs to one and only one paired kernel [6, Thm. 4.6]. On the other hand, for any $\phi_+ \in H^2_+$,

$$\phi_+ \in \ker^+_{a,b} \iff \phi_+ \in \ker T_{a/b},$$

where $T_{a/b}$ is the possibly unbounded Toeplitz operator with symbol a/b defined on the domain

$$D_{a/b} = \left\{\phi_+ \in H^2_+ : \frac{a}{b}\phi_+ \in L^2\right\}.$$

Analogously, for the dual Toeplitz operator [7]

$$\check{T}_{b/a} : \check{D}_{b/a} = \left\{\phi_- \in H^2_- : \frac{b}{a}\phi_- \in L^2\right\} \to H^2_-,$$

defined by

$$\check{T}_{b/a}\phi_- = P^- \frac{b}{a}\phi_-,$$

we have that $\ker^-_{a,b} = \ker \check{T}_{b/a}$.

If $a/b \in L^\infty$ then $D_{a/b} = H^2_+$ and we say that $\ker^+_{a,b}$ is a *Toeplitz kernel*, i.e., the kernel of a bounded Toeplitz operator.

Paired kernels and their projections into $H^2_\pm$ can be related as follows.

Proposition 2.1. *The operators*

$$\mathcal{P}^+ : \ker_{a,b} \to \ker^+_{a,b}, \qquad \mathcal{P}^+\phi = \frac{b}{b-a}\phi \tag{2.3}$$

and

$$\mathcal{M}_{a/b} : \ker^+_{a,b} \to \ker^-_{a,b}, \qquad \mathcal{M}_{a/b}\phi_+ = \frac{a}{b}\phi_+ \tag{2.4}$$

are well-defined and bijective with inverses

$$(\mathcal{P}^+)^{-1} : \ker^+_{a,b} \to \ker_{a,b}\,, \qquad (\mathcal{P}^+)^{-1}\phi_+ = \left(1 - \frac{a}{b}\right)\phi_+ \tag{2.5}$$

and

$$\mathcal{M}^{-1}_{a/b} : \ker^-_{a,b} \to \ker^+_{a,b}\,, \qquad \mathcal{M}^{-1}_{a/b}\phi_- = \frac{b}{a}\phi_-, \tag{2.6}$$

and we have $\mathcal{P}^+\phi = P^+\phi$, *and* $\mathcal{M}_{a/b}(\mathcal{P}^+\phi) = P^-\phi$ *for* $\phi \in \ker_{a,b}$.

Corollary 2.2. $\dim\ker_{a,b} < \infty \iff \dim\ker^+_{a,b} < \infty \iff \dim\ker^-_{a,b} < \infty$ *and, if these dimensions are finite, then they are equal.*

The following is also an immediate consequence of Proposition 2.1, noting that if $\phi_+ = 0$ (or similarly for ϕ_-) on a set of positive measure then, by the Luzin–Privalov theorem, it is 0 a.e. on $\mathbb{T}$.

Corollary 2.3. *If* $\phi \in \ker_{a,b}$ *then* $\phi = 0 \iff \phi_+ = 0 \iff \phi_- = 0$.

Note that, although $\mathcal{P}^+$ is bounded and bijective, its inverse is not necessarily bounded when $b \notin \mathcal{G}L^\infty$, since $\ker^+_{a,b}$ may not be closed in H^2_+, as shown in an example in the next section.

Just as we can relate the kernels of Toeplitz operators with those of dual Toeplitz operators, we can also reduce the study of $\ker^-_{a,b}$ to that of $\ker^+_{\bar{b},\bar{a}}$, as the next proposition shows.

Proposition 2.4. $\ker^-_{a,b} = \bar{z}\overline{\ker^+_{\bar{b},\bar{a}}}$.

Proof. For $\phi_- \in H^2_-$ we have

$$\begin{aligned}\phi_- \in \ker^-_{a,b} &\iff a\phi_+ + b\phi_- = 0 \text{ for some } \phi_+ \in H^2\\ &\iff \bar{a}(\bar{z}\overline{\phi_+}) + \bar{b}(\bar{z}\overline{\phi_-}) = 0 \text{ for some } \phi_+ \in H^2\\ &\iff \bar{b}(\bar{z}\overline{\phi_-}) + \bar{a}\psi_- = 0 \text{ for some } \psi_- \in \overline{H^2_0}\\ &\iff \bar{z}\overline{\phi_-} \in \ker^+_{\bar{b},\bar{a}} \iff \phi_- \in \bar{z}\overline{\ker^+_{\bar{b},\bar{a}}}.\end{aligned}$$

□

From now on we shall mainly focus on the properties of the projected paired kernels $\ker^+_{a,b}$. To compare the case where $b \in \mathcal{G}L^\infty$ with that where $b \notin \mathcal{G}L^\infty$, we start by considering a particular example of the latter, which is used in [22].

3. An Example and Questions it Raises

Let θ be the singular inner function $\theta(z) = \exp \dfrac{z-1}{z+1}$ for $z \in \mathbb{T}$ and let $a = \bar{\theta}$, $b(z) = z+1$. The kernel of $S_{\bar{\theta},z+1}$ is described by

$$\bar{\theta}\phi_+ + (z+1)\phi_- = 0, \qquad \text{i.e.,} \qquad \frac{\phi_+}{z+1} = -\theta\phi_-. \tag{3.1}$$

Since the left-hand side of this last equation represents a function in the Smirnov class $\mathcal{N}_+$ and the right-hand side represents a function in L^2, we have that both belong to H^2_+ and thus, from

$$\bar{\theta}\frac{\phi_+}{z+1} = -\phi_-, \tag{3.2}$$

we conclude that $\phi_+/(z+1) \in K_\theta$, where K_θ denotes the model space $H^2_+ \ominus \theta H^2_+ = \ker T_{\bar{\theta}}$. It follows that $\ker^+_{\bar{\theta},z+1} \subseteq (z+1)K_\theta$ and the converse inclusion is easily seen to be true, so

$$\ker^+_{\bar{\theta},z+1} = (z+1)K_\theta. \tag{3.3}$$

Clearly $\ker_{\bar{\theta},z+1} = \left(1 - \frac{\bar{\theta}}{z+1}\right)\ker^+_{\bar{\theta},z+1}$ is a closed subspace of L^2, since $S_{\bar{\theta},z+1}$ is bounded, but

$$\mathcal{P}^+\ker_{\bar{\theta},z+1} = \ker^+_{\bar{\theta},z+1} = (z+1)K_\theta \tag{3.4}$$

is not closed [22, Prop 3.2], so $\mathcal{P}^+$ does not have a bounded inverse.

Note that, while $\ker^+_{\bar\theta,z+1}$ is not closed, it is nevertheless contained in a (closed) minimal Toeplitz kernel, by which we mean a Toeplitz kernel that contains $\ker^+_{\bar\theta,z+1}$ and is itself contained in any other Toeplitz kernel containing $\ker^+_{\bar\theta,z+1}$. Indeed, on the one hand,

$$(z+1)K_\theta = \ker^+_{\bar\theta,z+1} \subsetneq K_{z\theta} = \ker T_{\bar z\bar\theta}. \tag{3.5}$$

On the other hand it was shown in [11] that for any nonzero $\phi_+ \in H^2_+$ there exists a minimal Toeplitz kernel to which ϕ_+ belongs. Now, $\ker T_{\bar z\bar\theta}$ is the minimal kernel containing the function

$$f = (z+1)\frac{\theta - \theta(0)}{z} \in \ker^+_{\bar\theta,z+1},$$

since we have

$$\bar z\bar\theta f = \bar z\frac{z+1}{z}(1-\theta(0)\bar\theta) = \bar z\overline{(z+1)(1-\overline{\theta(0)}\theta)},$$

where $(z+1)(1-\overline{\theta(0)}\theta)$ is an outer function in H^2_+ (see [12, Thm, 2.2]). Thus $K_{z\theta}$ is the minimal Toeplitz kernel containing $\ker^+_{\bar\theta,z+1}$.

These results naturally raise several questions, especially when compared with some known results for Toeplitz kernels.

Question 3.1. Having shown that $\ker^+_{\bar\theta,z+1}$ is not a Toeplitz kernel, one may ask whether there is any Toeplitz kernel contained in that space. The answer is negative, due to the near invariance properties of Toeplitz kernels [11]; these imply in particular that no Toeplitz kernel can be contained in $(z+1)H^2_+$, since Toeplitz kernels are nearly $\dfrac{1}{z+1}$ invariant [11] and $\ker^+_{\bar\theta,z+1} = (z+1)K_\theta$. This equality also shows that, in contrast with Toeplitz kernels, projected paired kernels may not be nearly $\dfrac{1}{z+1}$ invariant. But do other near invariance properties of Toeplitz kernels extend to projected paired kernels? This is studied in Sect. 4, comparing the two cases where $a/b \in L^\infty$ and $a/b \notin L^\infty$.

Question 3.2. On the other hand, there exists a minimal Toeplitz kernel containing $\ker^+_{\bar\theta,z+1}$, which is $K_{\theta z}$. Is there a minimal Toeplitz kernel containing any given nontrivial $\ker^+_{a,b}$? We answer this question in the affirmative, and in a more general setting, in Sect. 5, by showing that the closure of any projected paired kernel admits a representation of the form $f \ker T_g$ with $f \in H^2_+$ and $g \in L^\infty$, and we discuss the existence of such a representation for projected paired kernels. Note that not only can every Toeplitz kernel be written as a product of the form $f \ker T_g$ [20], but we also have the same property for other projected paired kernels: for instance, $\ker^+_{\bar\theta,z+1} = (z+1)K_\theta$.

Question 3.3. The relation (3.5) can be rewritten as $\ker^+_{\bar\theta,z+1} \subsetneq \ker^+_{\bar\theta\bar z,1}$, raising the question of what may be the inclusion relations between the two projected paired kernels when we multiply the two elements of the symbol pair (a, b) by certain functions. For Toeplitz operators we have, for instance, that

$$h_- \in \overline{H^\infty} \implies \ker T_g \subseteq \ker T_{h_- g},$$

where the inclusion is strict if the inner factor of $\overline{h_-}$ is non-constant [8] and an equality if $\overline{h_-}$ is outer in H^∞; also,

$$h_+ \in H^\infty \implies h_+ \ker T_{h_+ g} \subseteq \ker T_g,$$

where the inclusion is strict if h_+ has a non-constant inner factor [8] and an equality if h_+ is invertible in H^∞. We study how analogous inclusion relations can be established for general projected paired kernels, and whether the inclusion is strict or an equality, in Sect. 6.

The results are applied to study kernels of (asymmetric) truncated Toeplitz operators in Sect. 7.

4. Near Invariance Properties

A subspace $\mathcal{S} \subseteq H^2_+$ is said to be *nearly S^*-invariant* if and only if

$$f_+ \in \mathcal{S}, f_+(0) = 0 \implies S^* f_+ \in \mathcal{S}. \tag{4.1}$$

Here S^* denotes the backward shift on H^2_+, i.e., $S^* = T_{\bar{z}}$.

Noting that $f_+(0) = 0$ means that $\bar{z} f_+ \in H^2_+$ and, in that case, $S^* f = \bar{z} f$, it is clear that (4.1) is equivalent to

$$f_+ \in \mathcal{S}, \bar{z} f_+ \in H^2_+ \implies \bar{z} f_+ \in \mathcal{S} \tag{4.2}$$

and we say, equivalently, that $\mathcal{S}$ is nearly $\bar{z}$-invariant [11]. More generally, if η is a complex-valued function defined a.e. on $\mathbb{T}$ we say that $\mathcal{S} \subseteq H^2_+$ is *nearly η-invariant* if and only if

$$f_+ \in \mathcal{S}, \eta f_+ \in H^2_+ \implies \eta f_+ \in \mathcal{S}. \tag{4.3}$$

In this case if $\eta \in L^\infty$ we can also say that $\mathcal{S}$ is nearly T_η-invariant.

Toeplitz kernels are closed nearly S^*-invariant spaces. Furthermore, in [11] a large class of functions η was described for which all Toeplitz kernels are nearly η-invariant, and which includes all functions in $\overline{H^\infty}$ and all rational functions without poles in $\mathbb{D}^e \cup \{\infty\}$, where $\mathbb{D}^e = \{z \in \mathbb{C} : |z| > 1\}$.

In particular, Toeplitz kernels are nearly $\bar{\theta}$-invariant and nearly $\frac{1}{z-z_0}$-invariant, where θ is any inner function and $z_0 \in \mathbb{T} \cup \mathbb{D}$. As a consequence of this, we conclude that no Toeplitz kernel can be contained in θH^2_+ or in $(z - z_0) H^2_+$ for $z_0 \in \mathbb{T} \cup \mathbb{D}$.

It is clear from the examples of Sect. 3 that the latter property cannot be extended to projected paired kernels in general. Other near invaiance properties of Toeplitz kernels, however, are shared with projected paired kernels.

The following results are simple consequences of the definitions at the beginning of this section. We assume that $\ker^+_{a,b} \neq \{0\}$.

Proposition 4.1. $\ker^+_{a,b}$ *is nearly η-invariant for every $\eta \in \overline{H^\infty}$.*

Corollary 4.2. $\ker^+_{a,b}$ *is nearly $\bar{\theta}$-invariant for every inner function θ, and therefore there exists a function $\phi_+ \in \ker^+_{a,b}$ such that $\phi_+ \notin \theta H^2_+$.*

Corollary 4.3. $\ker^+_{a,b}$ *is nearly R-invariant for every rational function R bounded at ∞ whose poles lie in $\mathbb{D}$.*

Naturally, if $a/b \in L^\infty$, this property can be extended to every rational R without poles in $\mathbb{D}^e \cup \{\infty\}$, since in that case $\ker^+_{a,b}$ is a Toeplitz kernel.

It follows from Proposition 4.1, in particular, that projected paired kernels are nearly S^*-invariant subspaces of H^2_+, and so are their closures, by the following result.

Proposition 4.4. *If $\mathcal{S} \subset H^2_+$ is nearly S^*-invariant, then its closure is also nearly S^*-invariant.*

Proof. Clearly we may assume without loss of generality that $\mathcal{S} \neq \{0\}$. In that case, since $\mathcal{S}$ is nearly S^*-invariant, there must exist $h \in \mathcal{S}$ with $h(0) = 1$. Now if $f \in \bar{\mathcal{S}}$ with $f(0) = 0$, then there is a sequence (f_n) in $\mathcal{S}$ with $f_n \to f$ in norm and hence $f_n(0) \to f(0) = 0$. Thus for each n we have that $f_n - f_n(0)h$ is a function in $\mathcal{S}$ vanishing at 0, so $f_n - f_n(0)h = zg_n$ for some $g_n \in \mathcal{S}$, and $\lim f_n = f = \lim zg_n$. Hence (g_n) converges to a function $g \in \bar{\mathcal{S}}$ such that $f = zg$. □

The closed nearly S^*-invariant subspaces of H^2_+ admit a representation as a product of the form uK_θ, where $u \in H^2_+$ and K_θ is a model space, which will be considered in the next section. In the case of the closure of $\ker^+_{a,b}$, u must be outer by Corollary 4.2.

The notion of near η-invariance can naturally be extended to $\ker^-_{a,b}$, replacing H^2_+ by H^2_-. The following proposition shows that it is enough to consider the problem of near invariance for $\ker^+_{a,b}$.

Proposition 4.5. $\ker^-_{a,b}$ *is nearly $\bar{\eta}$-invariant in H^2_- if and only if $\ker^+_{\bar{b},\bar{a}}$ is nearly η-invariant in H^2_+.*

Proof. This is a consequence of Proposition 2.4. Suppose that $\ker^-_{a,b}$ is nearly $\bar{\eta}$-invariant in H^2_-, i.e.,

$$\phi_- \in \ker^-_{a,b}, \quad \bar{\eta}\phi_- \in H^2_- \implies \bar{\eta}\phi_- \in \ker^-_{a,b},$$

and let

$$\phi_+ \in \ker^+_{\bar{b},\bar{a}}, \qquad \eta\phi_+ \in H^2_+.$$

Then $\bar{z}\overline{\phi_+} \in \ker^-_{a,b}$, $\bar{\eta}\bar{z}\overline{\phi_+} \in H^2_-$, so $\bar{\eta}\bar{z}\overline{\phi_+} \in \ker^-_{a,b}$ by near invariance which, by Proposition 2.4, implies that $\eta\phi_+ \in \ker^+_{\bar{b},\bar{a}}$. So $\ker^+_{\bar{b},\bar{a}}$ is nearly η-invariant in H^2_+. The converse is proved analogously. □

Since $\ker^+_{a,b}$ is nearly $\bar{z}$-invariant in H^2_+ (equivalently, nearly S^*-invariant), it follows that $\ker^-_{a,b}$ is nearly z-invariant in H^2_-. Therefore, if $\ker^\pm_{a,b} \neq \{0\}$, there exists $\phi_+ \in \ker^+_{a,b}$ with $\phi_+(0) \neq 0$ and there exists $\phi_- \in \ker^-_{a,b}$ such that $(z\phi_-)(\infty) \neq 0$ (i.e., $\overline{\psi_+(0)} \neq 0$, where $\psi_+ = \bar{z}\overline{\phi_-}$). Analogously, since $\ker^+_{a,b}$ is nearly $\frac{1}{z-z_0}$-invariant for any $z_0 \in \mathbb{D}$, there exists, for any $z_0 \in \mathbb{D}$, $\phi_+ \in \ker^+_{a,b}$ with $\phi_+(z_0) \neq 0$ and $\psi_- \in \ker^-_{a,b}$ such that $\psi_-(1/z_0) \neq 0$.

5. Minimal Toeplitz Kernels and Representations of Projected Paired Kernels

If $a/b \in L^\infty$, then $\ker^+_{a,b}$ is a Toeplitz kernel; but, in general, $\ker^+_{a,b}$ may not even be a closed subspace of H^2_+, as in the case studied in Sect. 3. There it was also shown that, although $\ker^+_{\bar{\theta},z+1}$ is not a Toeplitz kernel, one can nevertheless determine a minimal Toeplitz kernel containing it. It is thus natural to ask if such a property holds for every projected paired kernel. The answer is in the affirmative, as one of the consequences of the following theorem.

Recall that a *maximal function* ϕ_m for a Toeplitz kernel $\ker T_g$ is one such that $\ker T_a$ is the minimal kernel to which ϕ_m belongs (see [11]). Every Toeplitz kernel possesses a maximal function.

Theorem 5.1. *Let* $a \in L^\infty\backslash\{0\}$ *and* $b \in H^2_+$ *such that* $\ker T_a \neq \{0\}$ *and* $b\ker T_a \subset H^2_+$. *Then there exists a minimal Toeplitz kernel containing* $b\ker T_a$, *which is* $\ker T_{a\bar{b}/b_o}$, *where* b_o *is the outer factor of* b. *Moreover, if* ϕ_m *is a maximal function for* $\ker T_a$, *then* $b\phi_m$ *is a maximal function for* $\ker T_{a\bar{b}/b_o}$.

Proof. If $\ker T_a \neq \{0\}$, let ϕ_m be a maximal function of $\ker T_a$, and write $\phi_m = I_+O_+$ with I_+ inner and O_+ outer in H^2_+. Then

$$\ker T_a = \ker T_{\bar{z}\overline{I_+O_+}/O_+} \tag{5.1}$$

[11, Thm. 5.1]. On the other hand, there exists a minimal kernel for $b\phi_m \in b\ker T_a \subset H^2_+$ which, if $b = b_ib_o$ is an inner–outer factorization of b with b_i inner and b_o outer, is given by

$$K_{\min}(b\phi_m) = \ker T_{\frac{\bar{z}\overline{I_+b_ib_oO_+}}{b_oO_+}} = \ker T_{\frac{\bar{z}\overline{b\phi_m}}{b_oO_+}}. \tag{5.2}$$

Let us show that

$$K_{\min}(b\phi_m) \supset b\ker T_a. \tag{5.3}$$

Let ψ_+ be any non-zero element of $\ker T_a$; then (see (5.1)) $\dfrac{\bar{z}\overline{I_+O_+}}{O_+}\psi_+ \in H^2_-$ and

$$\underbrace{\frac{\bar{z}\overline{I_+O_+b_ib_o}}{O_+b_o}}_{\in L^\infty}\underbrace{(b\psi_+)}_{\in H^2_+} = \underbrace{\overline{b_o}}_{\in \overline{H^2_+}}\underbrace{\frac{\bar{z}\overline{I_+O_+}}{O_+}\psi_+}_{\in H^2_-=\bar{z}\overline{H^2_+}} \in \bar{z}\overline{H^1} \cap L^2 \subset \bar{z}\overline{H^2_+} = H^2_-. \tag{5.4}$$

Thus $b\psi_+ \in K_{\min}(b\phi_m)$ as defined in (5.2), and (5.3) holds. On the other hand, since $b\phi_m \in b\ker T_a$, any Toeplitz kernel containing $b\ker T_a$ must also contain $K_{\min}(b\phi_m)$, so the latter is the minimal kernel containing $b\ker T_a$. Finally, from $\ker T_a = \ker T_{\bar{z}\overline{I_+O_+}/O_+}$ we conclude that

$$a = \frac{\bar{z}\overline{I_+O_+}}{O_+}h_- = \frac{\bar{z}\overline{\phi_m}}{O_+}h_-$$

for some $h_- \in \mathcal{G}\overline{H^\infty}$ (by [13, Cor. 7.8]. Therefore,

$$a\frac{\bar{b}}{b_o} = \frac{\bar{z}\overline{b\phi_m}}{b_oO_+}h_-$$

and it follows from (5.2) that $K_{\min}(b\phi_+) = \ker T_{a\bar{b}/b_o}$. □

Corollary 5.2. *Every subspace of H^2_+ of the form uK_θ, where $u \in H^2_+$ and θ is an inner function, is contained in a minimal Toeplitz kernel.*

As an example, take the example of Sect. 3, namely $(z+1)K_\theta$. By Theorem 5.1 we have that the minimal kernel containing $(z+1)K_\theta$ is

$$\ker T_{\bar{\theta}\frac{\overline{z+1}}{z+1}} = \ker T_{\bar{\theta}\bar{z}} = K_{\theta z},$$

as shown before.

A well-known theorem by Hitt [21] describes the closed nearly S^*-invariant subspaces of H^2_+ as having the form $M = uK$, where $u \in H^2_+$ has unit norm, $u(0) > 0$, u is orthogonal to all elements of M vanishing at the origin, K is an S^*-invariant subspace, and the operator of multiplication by u is isometric from K into M. Naturally, one can have $K = \{0\}$ or $K = H^2_+$, but the most interesting cases are those in which K is a model space $K_\theta = \ker T_{\bar{\theta}}$.

Corollary 5.3. *For every nondegenerate (a, b) there exists a minimal kernel containing $\ker^+_{a,b}$, which coincides with $\ker^+_{a,b}$ if $a/b \in L^\infty$.*

Proof. By Proposition 4.4, the closure of $\ker^+_{a,b}$ is nearly S^*-invariant, so we have that the closure of $\ker^+_{a,b}$ is uK, where $u \in H^2_+$ is outer, by Corollary 4.2, and $K = H^2_+$ or $K = K_\theta$ with θ inner. In the latter case, the result follows from Proposition 4.1, Proposition 4.4, Hitt's theorem, and Corollary 5.2. If $K = H^2_+$, then, since $uH^2_+ \subseteq H^2_+$ is closed and u is outer, we must have $u \in \mathcal{G}H^\infty$ and $uK = H^2_+$. □

Hayashi showed in [20] that the kernel of every Toeplitz operator T_g can be written as uK_θ where u is outer, u^2 is rigid (an exposed point of the unit ball of H^1), θ is inner with $\theta(0) = 0$, and u multiplies K_θ isometrically onto the Toepliz kernel. It may happen that $u \in \mathcal{G}H^\infty$ and, in that case, we can write

$$uK_\theta = u \ker T_{\bar{\theta}} = \ker T_{\bar{\theta}u^{-1}} = \ker^+_{\bar{\theta},u}. \tag{5.5}$$

Other representations of a similar form can be found for Toeplitz kernels. For instance, if $g \in L^\infty$ admits a Wiener–Hopf factorization of the form $g = g_-\bar{\theta}g_+$ with $g_- \in \mathcal{G}\overline{H^\infty}$, $g_+ \in \mathcal{G}H^\infty$ and θ inner, then

$$\ker T_g = g_+^{-1} \ker T_{\bar{\theta}} = g_+^{-1} K_\theta = \ker^+_{\bar{\theta},g_+^{-1}}. \tag{5.6}$$

Another example where

$$\ker^+_{a,b} = b \ker T_a \tag{5.7}$$

is the case studied in Sect. 3, which is not a Toeplitz kernel. It is thus natural to ask for conditions under which (5.7) holds and, in general, what is the relation between $\ker^+_{a,b}$ and $b \ker T_a$ for $a, b \in L^\infty$ nondegenerate. We have the following.

Proposition 5.4. *For $a, b \in L^\infty$ we have that $b \ker T_a \cap H^2_+ \subseteq \ker^+_{a,b}$ and*

$$\ker^+_{a,b} = b \ker T_a \cap H^2_+ \ \text{ if and only if } \ \ker^+_{a,b} \subseteq bH^2_+. \tag{5.8}$$

Proof. We have $a \ker T_a \subseteq H^2_-$, so, for any $\phi_+ \in \ker T_a$ such that $b\phi_+ \in H^2_+$,

$$a \underbrace{(b\phi_+)}_{\in H^2_+} + b \underbrace{(-a\phi_+)}_{\in H^2_-} = 0,$$

and it follows that $b\phi_+ \in \ker^+_{a,b}$. So $b \ker T_a \cap H^2_+ \subseteq \ker^+_{a,b}$.

Now, it is clear that $b \ker T_a \cap H^2_+ = \ker^+_{a,b}$ implies that $\ker^+_{a,b} \subseteq bH^2_+$. Conversely, suppose that $\ker^+_{a,b} \subseteq bH^2_+$. Then, for any $\phi_+ \in \ker^+_{a,b}$ we have $\phi_+ = b\psi_+$ with $\psi_+ \in H^2_+$, and therefore, for some $\phi_- \in H^2_-$,

$$\begin{aligned} a\phi_+ + b\phi_- = 0 &\iff ab\psi_+ + b\phi_- = 0 \\ &\iff b(a\psi_+ + \phi_-) = 0. \end{aligned}$$

We have the standing assumption that $b \neq 0$ a.e. on $\mathbb{T}$, so $a\psi_+ = -\phi_-$. We conclude that $\psi_+ \in \ker T_a$, so $\phi_+ \in b \ker T_a \cap H^2_+$, and it follows that $\ker^+_{a,b} \subseteq b \ker T_a \cap H^2_+$. □

An immediate consequence of Proposition 5.4 is the following.

Corollary 5.5. *If $b \in H^\infty$ then $b \ker T_a \subseteq \ker^+_{a,b}$.*

It is clear from the invariance results of Sect. 4 that we can have $\ker^+_{a,b} \subseteq bH^2_+$, with $b \in H^\infty$, only if b is outer. Moreover, we have the following.

Corollary 5.6. *If $b \in H^\infty$ is outer and either (i) $a \in \mathcal{G}L^\infty$ or (ii) $b \in \mathcal{G}H^\infty$, then* $\ker^+_{a,b} = b \ker T_a$.

Proof. If $b \in \mathcal{G}H^\infty$ then $bH^2_+ = H^2_+$ and the equality follows from (5.8). If $a \in \mathcal{G}H^\infty$ then

$$a\phi_+ + b\phi_- = 0 \iff \underbrace{\frac{\phi_+}{b}}_{\in\mathcal{N}^+} = \underbrace{-\frac{\phi_-}{a}}_{\in L^2} \in \mathcal{N}^+ \cap L^2 = H^2_+,$$

so $\phi_+ \in bH^2_+$ and again the equality in Corollary 5.6 follows from (5.8). □

6. Inclusion Relations

As in the case of Toeplitz kernels [11], the near invariance properties of projected paired kernels imply certain lower bounds for the dimension of a paired kernel containing a given function. For instance, if $\phi \in \ker_{a,b}$ and $\phi_+(0) = 0$, then $\ker_{a,b}$ must also contain the function $\psi \in L^2$ with $\psi_+ = \bar{z}\phi_+$ (note that this defines ψ by Proposition 2.1). As another example, using a similar reasoning, if there exists $\phi \in \ker_{a,b}$ such that $\phi_+ \in \theta H^2_+$ or $\phi_- \in \bar{\theta} H^2_-$ where θ is inner but not a finite Blaschke product, then $\ker_{a,b}$ is infinite-dimensional.

On the other hand, it is easy to see that, if θ_1 and θ_2 are inner functions, then

$$\ker^+_{a\theta_1, b\overline{\theta_2}} \subseteq \ker^+_{a,b}, \tag{6.1}$$

We may then ask if the inclusion is strict and, in that case, how much "smaller" $\ker^+_{a\theta_1, b\overline{\theta_2}}$ is with respect to $\ker^+_{a,b}$ and, in particular, when it is $\{0\}$. More generally, one may ask what are the relations between two paired kernels, or two projected paired kernels, whose symbol pairs are related by multiplication operators.

Note that, since a nontrivial paired kernel cannot be contained in a different one, and indeed their intersection is $\{0\}$ (see [6]), obtaining, for example, a paired operator analogue of the property $\ker T_{\theta g} \subsetneq \ker T_g$ (valid for $g \in L^\infty$ and θ inner, nonconstant) is possible only by saying that $\ker_{\theta a,b}$ is isomorphic to a proper subspace of $\ker_{a,b}$. Alternatively, taking Proposition 2.1 into account, we can say it in an equivalent and simpler way as $\ker^+_{\theta a,b} \subsetneq \ker^+_{a,b}$ (cf. Proposition 6.1).

In the following propositions we present several inclusion relations between projected paired kernels which generalise similar properties valid for Toeplitz kernels, using in particular the near invariance properties of Sect. 4 to establish strict inclusions.

We recall that, for any $\eta \in L^\infty$, $\ker_{a,b} = \ker_{a\eta,b\eta}$, so $\ker^+_{a,b} = \ker^+_{a\eta,b\eta}$.

Proposition 6.1. (i) *If $h_- \in \overline{H^\infty}$ then* $\ker^+_{a,bh_-} \subseteq \ker^+_{a,b} \subseteq \ker^+_{ah_-,b}$.

(ii) *If* $\overline{h_-}$ *is outer, then*

$$(a) \quad \ker^+_{a,bh_-} = \ker^+_{a,b} \quad \textit{if } \frac{a}{bh_-} \in L^\infty,$$

$$(b) \quad \ker^+_{ah_-,b} = \ker^+_{a,b} \quad \textit{if } \frac{a}{b} \in L^\infty.$$

(iii) *If* $\overline{h_-}$ *has a non-constant inner factor, then*

$$\ker^+_{a,bh_-} \subsetneq \ker^+_{a,b} \subsetneq \ker^+_{ah_-,b}.$$

Proof. (i) $a\phi_+ + bh_-\phi_- = 0 \implies a\phi_+ + b(h_-\phi_-) = 0$ and $a\phi_+ + b\phi_- = 0 \implies (ah_-)\phi_+ + b(h_-\phi_-) = 0$.

(ii) Let now $\overline{h_-}$ be outer. We start by proving the second equality (ii)(b). We have

$$ah_-\phi_+ + b\phi_- = 0 \iff \frac{\phi_-}{h_-} = -\frac{a}{b}\phi_+ \iff \frac{\overline{\phi_-}}{\overline{h_-}} = -\overline{\frac{a}{b}\phi_+}.$$

Since the left-hand side of the last equation is in the Smirnov class $\mathcal{N}_+$ when $\overline{h_-}$ is outer and the right-hand side is in L^2 if $a/b \in L^\infty$, we have under these assumptions that $\phi_-/h_- \in H^2_-$, so $\phi_+ \in \ker^+_{a,b}$ since $a\phi_+ + b\dfrac{\phi_-}{h_-} = 0$, and thus $\ker^+_{ah_-,b} \subseteq \ker^+_{a,b}$. The equality follows from (i).

Next, we have that

$$a\phi_+ + b\phi_- = 0 \implies a\phi_+ + bh_-\frac{\phi_-}{h_-} = 0 \implies \frac{\phi_-}{h_-} = -\frac{a}{bh_-}\phi_+,$$

and we conclude analogously that $\dfrac{\phi_-}{h_-} \in H^2_-$ so $\ker^+_{a,b} \subseteq \ker^+_{a,bh_-}$. Again the equality (ii)(a) follows from (i).

(iii) Suppose that $\overline{h_-}$ has a non-constant inner factor. If $\ker^+_{a,b} \subseteq \ker^+_{a,bh_-}$ then, for any $\phi \in L^2$ such that $a\phi_+ + b\phi_- = 0$ we must also have $\psi_- \in H^2_-$ such that $a\phi_+ + bh_-\psi_- = 0$. Thus $\phi_- = h_-\psi_-$. This implies that $\ker^-_{a,b} \subseteq h_-H^2_-$, which is impossible by Corollary 4.2 because $\overline{h_-}$ has a nonconstant inner factor. So $\ker^+_{a,bh_-} \subsetneq \ker^+_{a,b}$.

A similar argument shows that $\ker^+_{a,b} \subsetneq \ker^+_{ah_-,b}$. □

Proposition 6.2. *Suppose that* $a/b \in L^\infty$ *and* $z_0 \in \mathbb{T}$. *Then* $\ker^+_{a(z-z_0),b} = \ker^+_{az,b}$.

Proof. $\ker^+_{a(z-z_0),b} = \ker^+_{az(z-z_0)/z,b} = \ker^+_{az,b}$ by Proposition 6.1(ii), since $(z - z_0)/z \in \overline{H^\infty}$ and its conjugate is $1 - \overline{z_0}z$, an outer function in H^∞. □

As an immediate corollary we have:

Corollary 6.3. *Let $g \in L^\infty$ and $z_0 \in \mathbb{T}$. Then* $\ker T_{g(z-z_0)^n} = \ker T_{gz^n}$.

Remark 6.4. This corollary allows us to generalize several results from [8, Sec. 6], in particular Theorems 6.2 and 6.7 in [8], which deal with the relations between $\ker T_g$ and $\ker T_{\theta g}$, for a finite Blaschke product θ, to the case where the symbol has zeros of integer order on $\mathbb{T}$.

Proposition 6.5. (i) *If $h_+ \in H^\infty$ then*

$$h_+ \ker^+_{ah_+,b} \subseteq \ker^+_{a,b} \quad \textit{and} \quad h_+ \ker^+_{a,b} \subseteq \ker^+_{a,bh_+}.$$

(ii) *If $h_+ \in \mathcal{G}H^\infty$, then*

$$h_+ \ker^+_{ah_+,b} = \ker^+_{a,b} = h_+^{-1} \ker^+_{a,bh_+}.$$

(iii) *If h_+ has a non-constant inner factor, then*

$$h_+ \ker^+_{ah_+,b} \subsetneq \ker^+_{a,b} \quad \textit{and} \quad h_+ \ker^+_{a,b} \subsetneq \ker^+_{a,bh_+}.$$

Proof. (i) We can write $(ah_+)\phi_+ + b\phi_- = 0$ as $a(h_+\phi_+) + b\phi_- = 0$, from which $h_+\ker^+_{ah_+,b} \subseteq \ker^+_{a,b}$; then if $\phi_+ \in \ker^+_{a,b}$ we have $a\phi_+ + b\phi_- = 0$ for some $\phi_- \in H^2_-$, so, since $a(h_+\phi_+) + (bh_+)\phi_- = 0$, we have $h_+\phi_+ \in \ker^+_{a,bh_+}$.

(ii) If $h_+ \in \mathcal{G}H^\infty$, then from $a\phi_+ + b\phi_- = 0$ we obtain $ah_+(\phi_+ h_+^{-1}) + b\phi_- = 0$, so $\ker^+_{a,b} \subseteq h_+\ker^+_{ah_+,b}$ and we have equality from (i). The second equality follows from $h_+^{-1}\ker^+_{ah_+^{-1},b} = h_+^{-1}\ker^+_{a,bh_+}$.

(iii) If h_+ has a non-constant inner factor, we cannot have $\ker^+_{a,b} \subseteq h_+H^2_+$ nor $\ker^+_{a,bh_+} \subseteq h_+H^2_+$ by the near-invariance result of Corollary 4.2. $\square$

As a consequence of Proposition 6.1 (ii) and Proposition 6.5 (ii) we have the following.

Proposition 6.6. *Let $B = B_-gB_+ \in L^\infty$, where $B_-^{\pm 1} \in \overline{H^\infty}$ and $B_+^{\pm 1} \in H^\infty$. Then*

$$\ker^+_{aB,b} = B_+^{-1}\ker^+_{ag,b} \quad \textit{and} \quad \ker^+_{a,Bb} = B_+\ker^+_{a,gb}.$$

Proof.

$$\phi_+ \in \ker^+_{aB,b} \iff \exists \phi_- : aB\phi_+ + b\phi_- = 0,$$

that is, $aB_-gB_+\phi_+ + b\phi_- = 0$.

We write this as $\exists \phi_- : ag(B_+\phi_+) + b(\phi_- B_-^{-1}) = 0$. Equivalently, $\exists \psi_- : ag(B_+\phi_+) + b\psi_- = 0$.

Finally, this holds if and only if $B_+\phi_+ \in \ker^+_{ag,b}$, i.e., $\phi_+ \in B_+^{-1}\ker^+_{ag,b}$.

The other identity is proved similarly. $\square$

Corollary 6.7. *If $h_- \in \mathcal{G}\overline{H^\infty}$ then* $\ker^+_{a,b} = \ker^+_{ah_-,b} = \ker^+_{a,h_-^{-1}b} = \ker^+_{a,h_-,b}$.

Proof. We have $\ker^+_{a,b} \subseteq \ker^+_{ah_-,b}$; conversely,

$$\begin{aligned} ah_-\phi_+ + b\phi_- = 0 &\implies a\phi_+ + b(h_-^{-1}\phi_-) = 0 \\ &\implies \phi_+ \in \ker^+_{a,b}, \end{aligned}$$

so $\ker^+_{a,b} = \ker^+_{ah_-,b} = \ker^+_{a,bh_-^{-1}}$ and, replacing h_- by h_-^{-1} we conclude that $\ker^+_{a,b} = \ker^+_{a,bh_-}$. □

Since the inclusion in (6.1) is strict when θ_1 or θ_2 is not constant or, equivalently, $\ker^+_{a\theta,b} \subsetneq \ker^+_{a,b}$ when θ is a non-constant inner function, one may ask whether the dimensions of those two spaces can be compared, as in the case of Toeplitz kernels. The following theorem generalises analogous results obtained in [3, Sec. 2] and [8, Sec. 6] for Toeplitz kernels, and can be proved in a similar way.

Theorem 6.8. *Let θ be a non-constant finite Blaschke product. Then*

(i) $\ker^+_{a\theta,b}$ *is finite-dimensional if and only if* $\ker^+_{a,b}$ *is finite-dimensional. Similarly for* $\ker^+_{a,\theta b}$.

(ii) *If* $\dim \ker^+_{a,b} = d < \infty$ *and* $\dim K_\theta \geq d$ *then* $\ker^+_{a\theta,b} = \{0\}$.

(iii) *If* $\dim \ker^+_{a,b} = d < \infty$ *and* $\dim K_\theta = k < d$, *then* $\dim \ker^+_{a\theta,b} = d - k$.

Note that by Proposition 2.1 and Corollary 2.2 the same result holds if we replace $\ker^+_{a,b}$ by $\ker_{a,b}$ etc. When $\ker^+_{a,b}$ is not finite-dimensional, one can still compare it with $\ker^+_{a\theta,b}$ by means of the following decomposition.

Theorem 6.9. *Let θ be a non-constant finite Blaschke product. If θ has k zeros in $\mathbb{D}$, counting multiplicities, in which case we can write $\theta = B_- z^k B_+$ with $B_-^{\pm 1} \in \overline{H_\infty}$ and $B_+^{\pm 1} \in H^\infty$, then there exist $\psi_{j+} \in H^2_+$ $(j = 0, \ldots, k-1)$ with $\psi_{j+}(0) = 1$, for each j such that the following direct sum decomposition holds:*

$$\begin{aligned} \ker^+_{a,b} &= z^k B_+ \ker^+_{a\theta,b} + \operatorname{span}\{\psi_{0+}, z\psi_{1+}, \ldots, z^{k-1}\psi_{(k-1)+}\} \\ &= z^k \ker^+_{az^k,b} + \operatorname{span}\{\psi_{0+}, z\psi_{1+}, \ldots, z^{k-1}\psi_{(k-1)+}\}. \end{aligned}$$

Proof. By Proposition 6.6 it is enough to consider $\ker^+_{az^k,b}$. Since $\ker^+_{a,b}$ is nearly S^*-invariant, there exists $\psi_{0+} \in \ker^+_{a,b}$ with $\psi_{0+}(0) = 1$; let ψ_{0-} be given by $a\psi_{0+} + b\psi_{0-} = 0$. Then for any $\phi_+ \in \ker^+_{a,b}$,

$$\begin{aligned} a\phi_+ + b\phi_- = 0 &\iff az\frac{\phi_+ - \phi_+(0)\psi_{0+}}{z} + a\phi_+(0)\psi_{0+} + b\phi_- = 0 \\ &\iff az\frac{\phi_+ - \phi_+(0)\psi_{0+}}{z} + b(\phi_- - \phi_+(0)\psi_{0-}) = 0, \end{aligned}$$

so

$$\tilde{\phi}_+ := \frac{\phi_+ - \phi_+(0)\psi_{0+}}{z} \in \ker^+_{az,b}$$

and $\phi_+ = z\tilde{\phi}_+ + \phi_+(0)\psi_{0+}$.

Therefore $\ker^+_{a,b} = z\ker^+_{az,b} \oplus \operatorname{span}\{\psi_0\}$.

Proceeding analogously with $\ker^+_{az,b}, \ker^+_{az^2,b}, \ldots, \ker^+_{az^{k-1},b}$, we get

$$\ker^+_{a,b} = z^k \ker^+_{az^k,b} \oplus \operatorname{span}\{\psi_{0+}, z\psi_{1+}, \ldots, z^{k-1}\psi_{(k-1)+}\},$$

where $\psi_{j+} \in H^2$ and $\psi_{j+}(0) = 1$ for all $j = 0, 1, \ldots, k-1$. $\square$

We now obtain a description of some related projected paired kernels that will be used in the next section to study truncated Toeplitz operators: $\ker^+_{p_1,p_2}$, $\ker^+_{\alpha p_1,p_2}$ and $\ker^+_{p_1,\alpha p_2} = \ker^+_{\bar{\alpha} p_1,p_2}$, where p_1 and p_2 are polynomials of degrees n_1 and n_2 respectively without common zeros, and α is an inner function. Note that $\ker^+_{p_1,p_2}$ can be related to a new class of Toeplitz-like operators introduced in [17–19]: see [6].

Proposition 6.10. *Let p_1 and p_2 satisfy the assumptions above, and let moreover $p_i = p_{i\mathbb{D}} p_{i\mathbb{T}} p_{i\mathbb{E}}$, for $i = 1, 2$, where the zeros of $p_{i\mathbb{D}}$, $p_{i\mathbb{T}}$ and $p_{i\mathbb{E}}$ are in $\mathbb{D}, \mathbb{T}$ and $\mathbb{E} = \mathbb{D}^e$, respectively, and we denote by $n_{i\mathbb{D}}, n_{i\mathbb{T}}$ and $n_{i\mathbb{E}}$ the corresponding degrees. Then*

(i) $\ker^+_{p_1,p_2} = \{0\}$ *if $n_2 \le m := n_{2\mathbb{T}} + n_{1\mathbb{T}} + n_{2\mathbb{E}} + n_{1\mathbb{D}}$;*

(ii) $\ker^+_{p_1,p_2} = p_{1\mathbb{T}} p_{2\mathbb{T}} p_{1\mathbb{D}} p_{2\mathbb{E}} \mathcal{P}_{n_2-m-1}$*, where $\mathcal{P}_\ell$, for $l \in \mathbb{N} \cup \{0\}$, denotes the space of all polynomials of degree less than or equal to ℓ, and*

$$\dim \ker^+_{p_1,p_2} = n_2 - m = n_{2\mathbb{D}} - n_{1\mathbb{D}} - n_{1\mathbb{T}}$$

if $n_2 > m$.

Proof.

$$p_1\phi_+ + p_2\phi_- = 0 \iff p_1\phi_+ = -p_2\phi_- = -\underbrace{p_{2\mathbb{T}} p_{2\mathbb{E}} p_{2\mathbb{D}}}_{\text{degree } n_2} \phi_- = q_{n_2-1},$$

where q_{n_2-1} is a polynomial of degree less than or equal to n_2-1. Since $\frac{q_{n_2-1}}{p_2} \in H^2_-$ and $\frac{q_{n_2-1}}{p_1} \in H^2_+$, q_{n_2-1} must be of the form $q_{n_2-1} = p_{2\mathbb{T}} p_{2\mathbb{E}} p_{1\mathbb{T}} p_{1\mathbb{D}} q$, where q is a polynomial. Thus if $n_2 - 1 < m$ then $q_{n_2-1} = 0$ and (i) holds. Also, if $n_2 > m$ then (ii) holds. $\square$

Proposition 6.11. *With the same assumptions and notation as in* Proposition 6.10*, let α be a nonconstant inner function. In the case that α is a finite Blaschke product, let $\deg \alpha$ denote the number of zeros of α, counting multiplicity. Let $d = n_{2\mathbb{D}} - n_{1\mathbb{D}} - n_{1\mathbb{T}} > 0$.*

(i) $\ker^+_{\alpha p_1,p_2} = \{0\}$ *if α is not a finite Blaschke product or* $\deg \alpha \ge d > 0$.

(ii) $\dim \ker^+_{\alpha p_1,p_2} = d - \deg \alpha$ *if α is a finite Blaschke product with* $\deg \alpha < d$. *In this case, factorizing $\alpha = B_- z^{\deg \alpha} B_+$ with $B_-^{\pm 1} \in \overline{H^\infty}$ and $B_+^{\pm 1} \in H^\infty$, we have that $\ker^+_{\alpha p_1,p_2} = B_+^{-1} \ker^+_{z^{\deg \alpha} p_1,p_2}$, where the projected paired kernel on the right-hand side is described in* Proposition 6.10

Proof. (i) If α is not a finite Blaschke product or $\deg\alpha \geq d$, then (i) holds by Theorem 6.8.

(ii) If $\deg\alpha < d$ then

$$\ker^+_{\alpha p_1,p_2} = \ker^+_{B_- z^{\deg\alpha} B_+ p_1,p_2} = B_+^{-1}\ker^+_{z^{\deg\alpha} p_1,p_2}$$

by Proposition 6.6.

□

Analogously, we have:

Proposition 6.12. *With the same assumptions and notation as in* Proposition 6.11*, and assuming that* $\dim\ker^+_{p_1,p_2} = d = n_{2\mathbb{D}} - n_{1\mathbb{D}} - n_{1\mathbb{T}} > 0$*, we have*

$$\ker^+_{\bar{\alpha} p_1,p_2} = \overline{B_-^{-1}}\ker^+_{\bar{z}^{\deg\alpha} p_1,p_2} = \overline{B_-^{-1}}\ker^+_{p_1,z^{\deg\alpha} p_2},$$

where $\alpha = B_- z^{\deg\alpha} B_+$ *as in Proposition 6.11 and* $\ker^+_{p_1,z^{\deg\alpha} p_2}$ *is described in* Proposition 6.10.

Proof. The first equality follows from Proposition 6.6 and the second from the definition of $\ker^+$. □

7. Kernels of Finite-Rank Asymmetric Truncated Toeplitz Operators

Paired operators can also be defined in the matricial setting. They appear in the literature [7,16,23] as operators on $(L^2)^n$ with $n \times n$ matricial coefficients $A, B \in \mathcal{L}((L^2)^n)$.

In most cases A and B are multiplication operators and in that case the paired operator takes the form

$$T_{A,B} = AP^+ + BP^- \qquad \text{with} \quad A, B \in (L^\infty)^{n\times n}. \tag{7.1}$$

One can, however, consider other possible generalizations of scalar paired operators, for instance

$$T_{A,B} : (L^2)^n \to L^2, \qquad T_{A,B} = AP^+ + BP^- \quad \text{with} \quad A, B \in \mathcal{L}((L^2)^n, L^2) \tag{7.2}$$

or, considering in particular multiplication operators,

$$\begin{aligned} &T_{A,B} = AP^+ + BP^- : (L^2)^n \to L^2 \qquad \text{with} \quad A, B \in (L^\infty)^{1\times n}, \\ &A = [a_i]^T_{i=1,\ldots,n}, \qquad B = [b_i]^T_{i=1,\ldots,n}. \end{aligned} \tag{7.3}$$

The kernels of operators of this form are defined by the Riemann–Hilbert problem

$$\sum_{i=1}^n a_i\phi_{i+} = -\sum_{i=1}^n b_i\phi_{i-}, \qquad \phi_{i\pm} \in H^2_\pm. \tag{7.4}$$

The latter have appeared in recent works: see, for instance, [1] and [14].

We apply here the results of the latter, together with those of previous sections, to study the kernels of finite-rank asymmetric truncated Toeplitz operators (ATTO for short) of the form

$$A_\phi^{\theta,\alpha} : K_\theta \to K_\alpha, \qquad A_\phi^{\theta,\alpha} = P_\alpha \phi P_{\theta|K_\theta}, \tag{7.5}$$

where θ, α are inner functions, P_θ and P_α denote the orthogonal projections from L^2 onto K_θ and K_α respectively, and $\phi \in L^\infty$ is called the *symbol* of the ATTO.

Indeed, if we define P_1 to be the projection $P_1(x, y) = x$, then we have

$$\ker A_\phi^{\theta,\alpha} = P_1 \ker T_G, \tag{7.6}$$

where G has the matrix symbol

$$G = \begin{bmatrix} \bar{\theta} & 0 \\ \phi & \alpha \end{bmatrix}. \tag{7.7}$$

Now it was shown in [14, Thm. 3.1 and Cor. 3.4] that the kernels of block Toeplitz operators with symbols of the form (7.7) are of so-called *scalar type*, i.e., can be described as scalar multiples of a fixed vector function, and can be obtained from a pair of functions $f, g \in \mathcal{F}^2$, where $\mathcal{F}$ consists of all complex-valued functions defined a.e. on $\mathbb{T}$, provided that f and g satisfy the relation $Gf = g$.

To present the results from [14] that will be used here, it is useful to settle some notation, as follows:

(i) if $f = (f_1, f_2) \in \mathcal{F}^2$, then we say that f is left-invertible in $\mathcal{F}$ if and only if there exists $\tilde{f} \in \mathcal{F}^2$ such that $\tilde{f}^T f = 1$, and, in that case, $\tilde{f}^T$ is called the left inverse of f in $\mathcal{F}$;

(ii) we define $\mathcal{H}_X = \{h \in \mathcal{H} : Xh \in \mathcal{H}\}$ for any subspace $\mathcal{H} \subseteq (L^2)^n$ and any $X \in \mathcal{F}^{n\times n}$;

(iii) for any matricial coefficients A, B with elements in $\mathcal{F}$ let us define $T_{A,B} f = AP^+ f + BP^- f$ for $f \in (L^2)^n$ and write $\mathfrak{s}_{A,B} = A(H_+^2)^n \cap B(H_-^2)^n$.

With this notation we can formulate Corollary 3.4 in [14] as follows.

Theorem 7.1. *Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ belong to $\mathcal{F}^2$ and such that $Gf = g$ with $G \in (L^\infty)^{2\times 2}$. If $\mathcal{S} := \mathfrak{s}_{(\det G) f^T, -g^T} = \{0\}$, then $\ker A_\phi^{\theta,\alpha} = \mathcal{K} f_1$, where*

$$\mathcal{K} = \tilde{f}^T [(H_+^2)^2]_{f\tilde{f}^T} \cap \tilde{g}^T [(H_-^2)^2]_{g\tilde{g}^T}. \tag{7.8}$$

We now apply Theorem 7.1 and the results of the previous sections to study the behaviour of an ATTO of the form (7.5) with finite rank. As explained in [10, Sec. 3] we are led to take the symbol to be

$$\phi = \bar{\theta} R_+ - \alpha R_- + \sum_{j=1}^{N} \frac{\bar{\theta} P^\alpha_{n_j-1}(t_j) - \alpha P^{\bar{\theta}}_{n_j-1}(t_j)}{(z - t_j)^{n_j}}, \tag{7.9}$$

where

(i) $R_\pm$ are rational functions vanishing at ∞, such that R_- has no poles in $\mathbb{D}^e \cup \mathbb{T}$ and R_+ has no poles in $\mathbb{D} \cup \mathbb{T}$;

(ii) $t_j \in \mathbb{T}$ $(j = 1, \ldots, N)$ are regular points for θ and α, i.e., θ and α are analytic in a neighbourhood of each t_j (in which case $\bar{\theta}$, which can be extended to a definition outside $\overline{\mathbb{D}}$ by $\bar{\theta}(z) = \overline{\theta(1/\bar{z})}$ for $|z| > 1$, is also analytic in a neighbourhood of t_j);

(iii) $P^{\alpha}_{n_j-1}$ and $P^{\bar{\theta}}_{n_j-1}$ are the Taylor polynomials of order $n_j - 1$, relative to the point t_j, for α and for $\bar{\theta}$, respectively.

Defining

$$R_2^+ = R_+ + \sum_{j=1}^{N} \frac{P^{\alpha}_{n_j-1}(t_j)}{(z - t_j)^{n_j}} = \frac{Q_2}{\mathcal{E} D_{2+}}, \tag{7.10}$$

$$R_1^- = R_- - \sum_{j=1}^{N} \frac{P^{\bar{\theta}}_{n_j-1}(t_j)}{(z - t_j)^{n_j}} = \frac{Q_1}{\mathcal{E} D_{1-}}, \tag{7.11}$$

where

$$\mathcal{E} = \prod_{j=1}^{N} (z - t_j)^{n_j}, \quad \text{with} \quad \deg \mathcal{E} = n_{\mathbb{T}} := \sum_{j=1}^{N} n_j, \tag{7.12}$$

and Q_1, Q_2 are polynomials, D_{2+} is the denominator of R_+, with n_- zeros (including multiplicities) in $\mathbb{D}^e$, and D_{1-} is the denominator of R_-, with n_+ zeros in $\mathbb{D}$, we can write, from (7.9),

$$\phi = \bar{\theta} R_{2+} - \alpha R_{1-}. \tag{7.13}$$

Then G in (7.7) takes the form

$$G = \begin{bmatrix} \bar{\theta} & 0 \\ \bar{\theta} R_{2+} - \alpha R_{1-} & \alpha \end{bmatrix} \tag{7.14}$$

and one can verify that $f, g \in \mathcal{F}^2$, defined by

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \theta R_{1-} \end{bmatrix} = \theta \begin{bmatrix} 1 \\ \frac{Q_1}{\mathcal{E} D_{1-}} \end{bmatrix},$$

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ R_{2+} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{Q_2}{\mathcal{E} D_{2+}} \end{bmatrix}, \tag{7.15}$$

satisfy $Gf = g$ and have left inverses, respectively,

$$\tilde{f}^T = [\bar{\theta} \quad 0], \qquad \tilde{g}^T = [1 \quad 0]. \tag{7.16}$$

To apply Theorem 7.1, we start by showing that the assumption $\mathcal{S} = \{0\}$ in Theorem 7.1 is satisfied when K_α is "large enough".

Proposition 7.2. *Let G be given by (7.14) with α such that $\dim K_\alpha \geq m := n_+ + n_- + n_{\mathbb{T}}$, and let f, g be defined as in (7.15). Then $\mathcal{S} = \mathfrak{s}_{(\det G) f^T, -g^T} = \{0\}$.*

Proof. We have $\det G = \alpha\bar{\theta}$ and $\mathcal{S}$ is defined by

$$\begin{aligned}
&\alpha\left(\phi_{1+} + \frac{Q_1}{\mathcal{E}D_{1-}}\phi_{2+}\right) = \phi_{1-} + \frac{Q_2}{\mathcal{E}D_{2+}}\phi_{2-} \\
&\iff \frac{\alpha}{D_{1-}}\underbrace{(\mathcal{E}D_{1-}\phi_{1+} + Q_1\phi_{2+})}_{\psi_+} = \frac{z^{n_-+n_{\mathbb{T}}}}{D_{2+}}\underbrace{\frac{\mathcal{E}D_{2+}\phi_{1-} + Q_2\phi_{2-}}{z^{n_-+n_{\mathbb{T}}}}}_{\psi_-} \\
&\iff \alpha D_{2+}\psi_+ = z^{n_-+n_{\mathbb{T}}}D_{1-}\psi_-,
\end{aligned} \tag{7.17}$$

where $\psi_\pm \in H^2_\pm$. So $\psi = \psi_+ + \psi_- \in \mathfrak{s}_{\alpha D_{2+}, -z^{n_-+n_{\mathbb{T}}}D_{1-}} = \{0\}$, by Corollary 2.3 and Proposition 6.11 (i) (with $p_1 = D_{2+}$, $p_2 = -z^{n_-+n_{\mathbb{T}}}D_{1-}$, $n_{2\mathbb{D}} = n_+ + n_- + n_{\mathbb{T}}$ and $n_{1\mathbb{T}} = n_{1\mathbb{D}} = 0$) and it follows that the left-hand side of (7.17) is

$$\alpha\left(\phi_{1+} + \frac{Q_1}{\mathcal{E}D_{1-}}\phi_{2+}\right) = \frac{\alpha}{\mathcal{E}D_{1-}}\psi_+ = 0$$

and, of course, analogously for the right-hand side of (7.17), so $\mathcal{S} = \{0\}$. □

Next we characterise the spaces $[(H^2_+)^2]_{f\tilde{f}^T}$ and $[(H^2_+)^2]_{g\tilde{g}^T}$ in (7.8). Note that

$$f\tilde{f}^T = \begin{bmatrix} 1 & 0 \\ R_{1-} & 0 \end{bmatrix} \qquad \text{and} \quad g\tilde{g}^T = \begin{bmatrix} 1 & 0 \\ R_{2+} & 0 \end{bmatrix}, \tag{7.18}$$

so

$$(\phi_{1+}, \phi_{2+}) \in [(H^2_+)^2]_{f\tilde{f}^T} \iff R_{1-}\phi_{1+} \in H^2_+ \iff \phi_{1+} \in \mathcal{E}D_{1-}H^2_+ \tag{7.19}$$

and

$$(\phi_{1-}, \phi_{2-}) \in [(H^2_-)^2]_{g\tilde{g}^T} \iff R_{2+}\phi_{1-} \in H^2_- \iff \phi_{1-} \in \frac{\mathcal{E}D_{2+}}{z^{n_-+n_{\mathbb{T}}}}H^2_-. \tag{7.20}$$

We can now formulate the main result of this section.

Theorem 7.3. *Let ϕ be given by (7.13) and (7.10)–(7.12), and let $\dim K_\alpha \geq m := n_+ + n_- + n_{\mathbb{T}}$. Then $\ker T_G$ and $\ker A^{\theta,\alpha}_\phi$ do not depend on α and we have*

$$\ker T_G = \mathcal{E}D_{1-}D_{2+}\ker T_{\bar{\theta}z^m}\begin{bmatrix} 1 \\ R_{1-} \end{bmatrix} \quad \textit{and} \quad \ker A^{\theta,\alpha}_\phi = \mathcal{E}D_{1-}D_{2+}\ker T_{\bar{\theta}z^m}.$$

This holds, in particular, for any infinite-dimensional K_α.

Proof. By (7.8), (7.19) and (7.20)

$$\begin{aligned}
\mathcal{K} &= [\bar{\theta} \quad 0] \begin{bmatrix} \mathcal{E} D_{1-} H^2_+ \\ H^2_+ \end{bmatrix} \cap [1 \quad 0] \begin{bmatrix} \frac{\mathcal{E} D_{2+}}{z^{n_-+n_{\mathbb{T}}}} H^2_- \\ H^2_- \end{bmatrix} \\
&= \bar{\theta} \mathcal{E} D_{1-} H^2_+ \cap \frac{\mathcal{E} D_{2+}}{z^{n_-+n_{\mathbb{T}}}} H^2_- \\
&= \bar{\theta} \mathcal{E} D_{1-} \ker^+_{\bar{\theta} D_{1-}, -\frac{D_{2+}}{z^{n_-+n_{\mathbb{T}}}}} \\
&= \bar{\theta} \mathcal{E} D_{1-} \ker^+_{\bar{\theta} D_{1-} z^{n_-+n_{\mathbb{T}}}, -D_{2+}} \\
&= \bar{\theta} \mathcal{E} D_{1-} D_{2+} \ker^+_{\bar{\theta} z^{n_++n_-+n_{\mathbb{T}}}, -1} \\
&= \bar{\theta} \mathcal{E} D_{1-} D_{2+} \ker T_{\bar{\theta} z^m},
\end{aligned}$$

where we used Proposition 6.12, and the result follows from Theorem 7.1. □

Note that, if z^m divides θ then $\ker T_{\bar{\theta} z^m}$ is the model space K_{θ/z^m}, so $\ker A^{\theta,\alpha}_{\phi}$ is isomorphic to K_{θ/z^m}.

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Declarations

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