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Results in Mathematics

Interpolator Symmetries and New Kalton-Peck Spaces

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Abstract. We study the six diagrams generated by the first three Schechter interpolators $\Delta_2(f) = f''(1/2)/2!, \Delta_1(f) = f'(1/2), \Delta_0(f) = f(1/2)$ acting on the Calderón space associated to the pair (ℓ_∞, ℓ_1). We will study the remarkable and somehow unexpected properties of all the spaces appearing in those diagrams: two new spaces (and their duals), two Orlicz spaces (and their duals) in addition to the third order Rochberg space, the standard Kalton-Peck space Z_2 and, of course, the Hilbert space ℓ_2. We will also deal with a nice test case: that of weighted ℓ_2 spaces, in which case all involved spaces are Hilbert spaces.

Mathematics Subject Classification. Primary 46B70; Secondary 46E30, 46M18.

1. Introduction

The aim of this paper is to present the seven natural Banach spaces generated by the first three interpolators of the complex interpolation method when applied to the couple (ℓ_∞, ℓ_1) at 1/2. They are three Rochberg spaces ℓ_2, Z_2 and Z_3, two Orlicz spaces ℓ_f, ℓ_g generated by the Orlicz functions $f(t) = t^2 \log t^2, g(t) = t^2 \log^4 t$; and two new spaces $\wedge, \bigcirc$. We present the basic diagram generated by these three interpolators and the six possible diagrams they generate, which produce the seven spaces just mentioned, their

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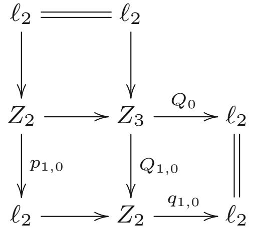
duals, and nothing more. And it is so by virtue of the symmetries of the six diagrams: some are overt (described in Sect. 2.3), but some are deeply concealed and unexpected (like those described between Proposition 3.2 and 3.7).

Following [7], we will work with a variant $\mathcal{C}$ of the Calderón space considered in [3, Section 4.1] when working with the pair (ℓ_∞, ℓ_1): if $\mathbb{S}$ is the open strip $\{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$ in the complex plane, $\mathcal{C}$ will be the space of continuous bounded functions on $\mathbb{S}$ that are also weak*-continuous as functions $f : \overline{\mathbb{S}} \longrightarrow \ell_\infty$ and that moreover are holomorphic on $\mathbb{S}$ and satisfy the boundary condition $f(k+it) \in X_k$ for each $t \in \mathbb{R}$ and $\sup_t \|f(k+it)\|_{X_k} < \infty$, valid for $k = 0, 1$. The Calderón space $\mathcal{C}$ is complete with the norm $\|f\| = \sup\{\|f(k+it)\|_k : k = 0, 1; t \in \mathbb{R}\}$. The evaluation maps $\delta_z : \mathcal{C} \longrightarrow \ell_\infty$ are continuous for all $z \in \mathbb{S}$, and given $\theta \in (0,1)$ and $p = \theta^{-1}$ one obtains $\ell_p = \{f(\theta) : f \in \mathcal{C}\}$ with the standard norm in ℓ_p equal to the quotient norm in $\|x\|_\theta = \inf\left\{\|f\| : x = f(\theta), f \in \mathcal{C}\right\}$. See [3, Lemma 4.1.1] and [5, Section 10.8] for details.

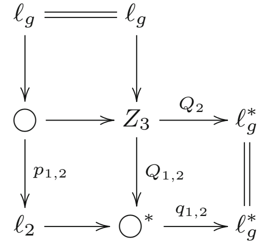
For the rest of the paper we will focus on the Hilbert space case: $\theta = 1/2$; $p = 2$. We consider the interpolators $\Delta_k : \mathcal{C} \to \ell_\infty$ defined by $\Delta_k(f) = f^{(k)}(1/2)/k!$ for $k = 0, 1, 2, \ldots$ Following Rochberg [29] (see also [6,7]), the n^{th} Rochberg space is defined as $\mathfrak{R}_n = \{(\Delta_{n-1}(f), \ldots, \Delta_0(f)) : f \in \mathcal{C}\}$ endowed with its natural quotient norm. This yields $\mathfrak{R}_1 = \ell_2$ and $\mathfrak{R}_2 = Z_2$, the Kalton-Peck space [24]. We will denote $\mathfrak{R}_3$ with the more friendly name Z_3. Among the distinguished subspaces of Z_3 we will encounter the three Orlicz spaces $\ell_2 = \{(w,0,0) \in Z_3\}$, $\ell_f = \{(0,x,0) \in Z_3\}$ and $\ell_g = \{(0,0,y) \in Z_3\}$, and the three spaces $Z_2 = \{(w,x,0) \in Z_3\}$, $\wedge = \{(w,0,y) \in Z_3\}$ and $\bigcirc = \{(0,x,y) \in Z_3\}$.

Let us now aim at diagrams: It is a fact uncovered through [4,11,24] that Z_2 admits two natural representations $0 \to \ell_2 \to Z_2 \to \ell_2 \to 0$ and $0 \to \ell_f \to Z_2 \to \ell_f^* \to 0$ as a non-trivial twisted sum that are associated to the two permutations (Δ_1, Δ_0) and (Δ_0, Δ_1). In the same way, we will show (Sect. 2) that Z_3 admits six natural representations as a twisted sum space associated with the six diagrams generated by the six permutations of the three interpolators $(\Delta_2, \Delta_1, \Delta_0)$. Indeed, if we denote $[abc]$ the diagram obtained from the permutation $(\Delta_a, \Delta_b, \Delta_c)$, the six diagrams are (we will omit the arrow $0 \to$ at the beginning and $\to 0$ at the end of the exact sequences forming the rows and columns):

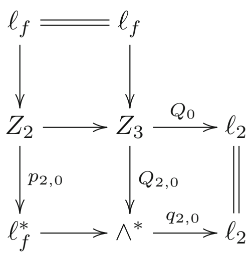
[210]



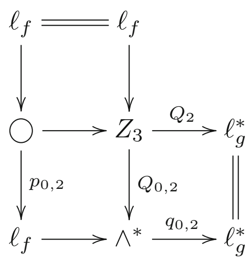
[012]



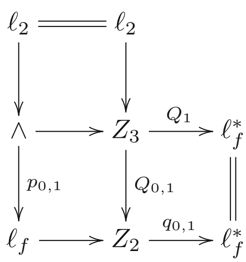
[120]



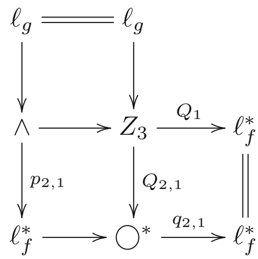
[102]



[201]



[021]



We will prove:

- **Properties shared by all spaces/sequences**
  (1) All the spaces in the diagrams are hereditarily ℓ_2 (Proposition 5.1) and have basis.
  (2) All the exact sequences are nontrivial (Corollary 5.7).
  (3) All quotient maps, except perhaps $q_{1,2}$ and $q_{2,1}$ (see below), are strictly singular (Proposition 5.14).
- **Properties similar to those of Z_2**
  (1) The spaces $\bigcirc$, $\wedge$, $\bigcirc^*$ and $\wedge^*$ admit a symmetric two-dimensional decomposition.
  (2) Z_3 admits a symmetric three-dimensional decomposition (Proposition 3.1) and it is isomorphic to its dual [6, Prop. 5.5 and Cor. 5.7].
  (3) Every infinite dimensional complemented subspace of Z_3 contains a copy of Z_3 complemented in the whole space (Proposition 5.5).
  (4) The spaces Z_3, $\wedge$ and $\wedge^*$ contain no complemented copies of ℓ_2 and admit no unconditional basis (Proposition 5.12).
  (5) Every basic sequence in Z_3 contains a subsequence equivalent to the canonical basis of one of the spaces ℓ_2, ℓ_f, ℓ_g (Theorem 5.8).
- **Properties different from those of Z_2**
  (1) None of the spaces $\bigcirc$, $\wedge$, $\bigcirc^*$ and $\wedge^*$ is isomorphic to a subspace or a quotient of Z_2 (Proposition 5.4).
  (2) $\wedge$ and $\bigcirc$ are not isomorphic to their duals (Proposition 5.10).

(3) Neither of the spaces $\wedge$ and $\wedge^*$ is isomorphic to either $\bigcirc$ or $\bigcirc^*$ (Proposition 5.11).

- **Open questions**
  (1) We have been unable to show that $\bigcirc$ (hence $\bigcirc^*$ also) contains no complemented copies of ℓ_2. From that it would follow also $q_{1,2}$ and $q_{2,1}$ are strictly singular, hence that $\bigcirc$ and $\bigcirc^*$ do not have an unconditional basis (Remark 5.15), which would complete our scheme.
  (2) We could not cover in this paper the case of interpolation at an arbitrary $\theta \neq 1/2$. In that case, the first thing one loses is duality and its associated symmetries: Z_p is no longer isomorphic to Z_p^*. The same is valid for weighted ℓ_p-spaces or weighted versions of a given space with an unconditional basis.

2. The Six Diagrams Generated by the Three Interpolators $\Delta_2, \Delta_1, \Delta_0$

A Banach space space Z is a *twisted sum of* Y *and* X if there exists an exact sequence $0 \to Y \to Z \to X \to 0$ (namely, a diagram formed by Banach spaces and continuous operators so that the kernel of each of them coincides with the image of the previous one). Twisted sums of Y and X correspond to a special type of maps $X \longrightarrow Y$, called quasi-linear maps [5,24]. We need to widen this notion as in [12,15] assuming that Y is continuously embedded in an "ambient" Hausdorff topological vector space Banach space Σ which, for us, will be a Banach or quasi-Banach space. There are indeed natural situations in which these "generalized" quasi-linear maps appear: *centralizers* between quasi-Banach function spaces [22]; *differentials* generated by two interpolators [11]; or G-actions on twisted sums [12].

Definition 2.1. A *quasi-linear map* $\Omega : X \curvearrowright Y$ *with ambient space* Σ is a homogeneous map $\Omega : X \longrightarrow \Sigma$ for which there exists a constant C such that for $x_1, x_2 \in X$,

- $\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2) \in Y$ and
- $\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\|_Y \leq C(\|x_1\|_X + \|x_2\|_X)$.

A quasi-linear map Ω as above defines a twisted sum $Y \oplus_\Omega X = \{(\beta, x) \in \Sigma \times X : \beta - \Omega(x) \in Y\}$ endowed with the quasinorm $\|(\beta, x)\|_\Omega = \|\beta - \Omega(x)\|_Y + \|x\|_X$; the embedding $\jmath : Y \longrightarrow Y \oplus_\Omega X$ given by $j(y) = (y, 0)$ is isometric and the quotient map $\pi : Y \oplus_\Omega X \longrightarrow X$ is given by $\pi(\beta, x) = x$. They define the exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus_\Omega X \longrightarrow X \longrightarrow 0$, that shall be referred to as the *exact sequence generated by* Ω. Since X and Y are complete, $(Y \oplus_\Omega X, \|(\cdot, \cdot)\|_\Omega)$ is a quasi-Banach space [14, Lemma 1.5.b]. When Y and X are B-convex Banach spaces, the quasi-norm in $Y \oplus_\Omega X$ is equivalent to a norm [20, Theorem 2.6]. This is the case for the spaces we consider in this paper.

Definition 2.2. A quasi-linear map $\Omega : X \curvearrowright Y$ with ambient space Σ is *bounded* if there exists a constant D so that $\Omega x \in Y$ and $\|\Omega x\|_Y \leq D\|x\|_X$ for each $x \in X$. It is *trivial* if there exists a linear map $L : X \longrightarrow \Sigma$ so that $\Omega - L : X \longrightarrow Y$ is bounded. Two quasilinear maps Ω_1, Ω_2 $X \curvearrowright Y$ with ambient space Σ are *boundedly equivalent* if $\Omega_1 - \Omega_2 : X \to Y$ is bounded. This implies that $\|(\cdot,\cdot)\|_{\Omega_1}$ and $\|(\cdot,\cdot)\|_{\Omega_2}$ are equivalent quasi-norms. The quasilinear maps $\Omega_1 : X_1 \curvearrowright Y_1$ and $\Omega_2 : X_2 \curvearrowright Y_2$ are *isomorphically equivalent*, denoted $\Omega_1 \simeq \Omega_2$, if there exist three isomorphisms S, T, U forming a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_1 & \longrightarrow & Y_1 \oplus_{\Omega_1} X_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\ & & {\scriptstyle S}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle U}\downarrow & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & Y_2 \oplus_{\Omega_2} X_2 & \longrightarrow & X_2 & \longrightarrow & 0. \end{array} \tag{1}$$

The following notions of domain and range generalize the classical domain and range for Ω-operators obtained from an interpolation process [8,17,18], for centralizers on function spaces [4] or for G-centralizers in suitable G-Banach spaces [12].

Definition 2.3. Let $\Omega : X \curvearrowright Y$ be a quasi-linear map with ambient space Σ. The *domain of* Ω is the set $\operatorname{Dom}\Omega = \{x \in X : \Omega x \in Y\}$, and the *range of* Ω is the set $\operatorname{Ran}\Omega = \{\beta \in \Sigma : \exists x \in X : \beta - \Omega x \in Y\}$.

Since Ω is quasi-linear, $\operatorname{Dom}\Omega$ is a linear subspace of X as well as $\operatorname{Ran}\Omega$. The space $\operatorname{Dom}\omega$ can be endowed with the quasi-norm $\|x\|_D = \|\Omega x\| + \|x\|$ so that it is isometric to the subspace $\{(0, x) \in Y \oplus_\Omega X$. The space $\operatorname{Ran}\Omega$ can be endowed with the quasi-norm $\|\beta\|_R = \inf\{\|\beta - \Omega x\| + \|x\|\}$ where the infimum is taken over all $x \in X : \beta - \Omega x \in Y$. In this way $\operatorname{Ran}\Omega$ can be identified with the quotient $(Y \oplus_\Omega X)/\operatorname{Dom}\Omega$ with quotient map $(\beta, x) \to \beta$. What is not guaranteed is that either $\operatorname{Dom}\Omega$ is a closed subspace of $Y \oplus_\Omega X$ or, equivalently, that $\operatorname{Ran}\Omega$ is Hausdorff. Now, if $\Omega : X \to \Sigma$ is continuous at 0 for some choice of the ambient space Σ then $\operatorname{Dom}\Omega$ is closed. Indeed, if $(0, x_n) \to (z, x)$ then $\|z - \Omega(x_n - x)\| + \|x_n - x\| \to 0$. Thus $x_n - x \to 0$ and, by continuity, $\Omega(x_n - x) \to 0$ in Σ; thus $\|z\|_\Sigma \leq \|z - \Omega(x_n - x)\|_\Sigma + \|\Omega(x_n - x)\|_\Sigma \to 0$, which means that $\|z\|_\Sigma = 0$ and, by Hausdorffness, $z = 0$. In fact, if $\operatorname{Ran}\Omega$ is Hausdorff then we could choose it as ambient space: the formal identity establishes a continuous inclusion $Y \to \operatorname{Ran}\Omega$ since $\|y\|_R \leq \|y\|$ (with the choice $x = 0$) and $\Omega : X \to \operatorname{Ran}\omega$.

Given an interpolation pair of Banach spaces (X_0, X_1) with ambient space Σ and associated Calderón space $\mathcal{C}$, we fix the following terminology:

- **Operator acting on the pair.** An operator $T : \Sigma \to \Sigma$ is said to *act on the pair* (X_0, X_1) if $T[X_i] \subset X_i$ for $i = 0, 1$

Interpolator. An operator $\Delta : \mathcal{C} \to \Sigma$ is an interpolator if every T acting on the pair admits an operator $T_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ such that $\Delta T_\mathcal{C} = T\Delta$.

Consistent family of interpolators. A family $\{\Delta_i : i \in I\}$ of interpolators on $\mathcal{C}$ is said to be *consistent* if for each operator T acting on the pair (X_0, X_1) there exists an operator $T_\mathcal{C}$ on $\mathcal{C}$ such that $T\Delta_i = \Delta_i T_\mathcal{C}$ for every $i \in I$.

Given a finite sequence $\{\Delta_i : i = 0, \ldots, n + k\}$ of interpolators we will consider the pair (Ψ, Φ) of interpolators $\Psi = \langle \Delta_{k+n-1}, \ldots \Delta_k\rangle : \mathcal{C} \to \Sigma^n$ and $\Phi = \langle \Delta_{k-1}, \ldots \Delta_0\rangle : \mathcal{C} \to \Sigma^k$, given by $\Psi(f) = (\Delta_{k+n-1}f, \ldots \Delta_k f)$ and $\Phi(f) = (\Delta_{k-1}f, \ldots \Delta_0 f)$. Proceeding in the standard way, see [7] and [11], we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
\ker\Psi \cap \ker\Phi & = & \ker\langle\Psi,\Phi\rangle & & \\
\downarrow & & \downarrow & & \\
\ker\Phi & \longrightarrow & \mathcal{C} & \xrightarrow{\Phi} & X_\Phi \\
{\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\langle\Psi,\Phi\rangle} & & \| \\
\Psi(\ker\Phi) & \xrightarrow{\imath} & X_{\langle\Psi,\Phi\rangle} & \xrightarrow{\rho} & X_\Phi
\end{array} \tag{2}$$

in which $X_\Phi = \Phi(\mathcal{C})$, $X_\Psi = \Psi(\mathcal{C})$, $X_{\langle\Psi,\Phi\rangle} = \langle\Psi, \Phi\rangle(\mathcal{C})$ and all the spaces are endowed with their natural quotient norms. The maps $\imath$ and ρ are defined by $\imath\Psi g = (\Psi g, 0)$ and $\rho(\Psi f, \Phi f) = \Phi f$. If $B_\Phi : X_\Phi \to \mathcal{C}$ denotes an homogeneous bounded selection for the quotient map $\Phi : \mathcal{C} \to X_\Phi$ then the *differential* associated to (Ψ, Φ) is the map $\Omega_{\Psi,\Phi} : X_\Phi \to \Sigma^n$ given by $\Omega_{\Psi,\Phi} = \Psi \circ B_\Phi$. We have that $\Omega_{\Psi,\Phi} : X_\Phi \curvearrowright \Psi(\ker\Phi)$ is a quasilinear map with ambient space Σ^n. The differential $\Omega_{\Psi,\Phi}$ is continuous at 0 and, consequently, the domain of $\Omega_{\Psi,\Phi}$ is closed, its range is Hausdorff, and one also has the *inverse* exact sequence $0 \longrightarrow \operatorname{Dom}\Omega_{\Psi,\Phi} \longrightarrow X_{\Psi,\Phi} \longrightarrow \operatorname{Ran}\Omega_{\langle\Psi,\Phi\rangle} \longrightarrow 0$. Moreover [11, Proposition 3.8]:

Proposition 2.4. *The following identities, with equivalence of norms in (1) and (2), hold:*

(1) $\operatorname{Dom}\Omega_{\Psi,\Phi} = \Phi(\ker\Psi)$.
(2) $\operatorname{Ran}\Omega_{\Psi,\Phi} = X_\Psi$.
(3) $\Omega_{\Phi,\Psi} = (\Omega_{\Psi,\Phi})^{-1}$.

From now on we will focus on the pair (ℓ_∞, ℓ_1) and the sequence of interpolators $\Delta_k : \mathcal{C} \to \ell_\infty$ given by $\Delta_k(f) = f^{(k)}(1/2)/k!$. These are interpolators because the evaluation map of the n^{th}- derivative $\delta_z^{(n)} : \mathcal{C} \to \ell_\infty$ at an interior z is continuous [6, Lemma 2.4] for each $n \in \mathbb{N}$. Moreover, each finite sequence $\{\Delta_k : n \leq k \leq m\}$ is consistent. More specifically, we will focus on diagram (2) obtained from the first three interpolators $\Delta_2, \Delta_1, \Delta_0$. There are six possible permutations of these interpolators, and therefore six different diagrams.

2.1. The Diagram $[abc]$.

Let (a, b, c) be a permutation of $(0, 1, 2)$. Observe that $\ker\langle\Delta_b, \Delta_c\rangle = \ker\Delta_b \cap \ker\Delta_c$. We denote by $[abc]$ the diagram generated by the triple $(\Delta_a, \Delta_b, \Delta_c)$:

$$
[abc] \qquad
\begin{array}{ccccc}
\Delta_a(\ker\Delta_b \cap \ker\Delta_c) & = & \Delta_a(\ker\Delta_b \cap \ker\Delta_c) & & \\
\downarrow j & & \downarrow k & & \\
\langle\Delta_a, \Delta_b\rangle(\ker\Delta_c) & \xrightarrow{\ l\ } & \langle\Delta_a, \Delta_b, \Delta_c\rangle(\mathcal{C}) & \xrightarrow{\ s\ } & \Delta_c(\mathcal{C}) \\
\downarrow q & & \downarrow r & & \| \\
\Delta_b(\ker\Delta_c) & \xrightarrow{\ i\ } & \langle\Delta_b, \Delta_c\rangle(\mathcal{C}) & \xrightarrow{\ p\ } & \Delta_c(\mathcal{C})
\end{array}
$$

where the maps are given by

- $j(\Delta_a h) = (\Delta_a h, 0), \quad k(\Delta_a h) = (\Delta_a h, 0, 0), \quad h \in \ker\Delta_b \cap \ker\Delta_c$;
- $l(\Delta_a g, \Delta_b g) = (\Delta_a g, \Delta_b g, 0), \quad q(\Delta_a g, \Delta_b g) = \Delta_b g, \quad i(\Delta_b g) = (\Delta_b g, 0)$, $g \in \ker\Delta_c$;
- $s(\Delta_a f, \Delta_b f, \Delta_c f) = \Delta_c f,\ r(\Delta_a f, \Delta_b f, \Delta_c f) = (\Delta_b f, \Delta_c f),\ p(\Delta_b f, \Delta_c f) = \Delta_c f, \quad f \in \mathcal{C}$.

2.2. The Quasi-Linear Maps

We simplify the notation for the quasi-linear maps as follows:

$$\Omega_{a,b} = \Omega_{\Delta_a,\Delta_b}; \quad \Omega_{a,\langle b,c\rangle} = \Omega_{\Delta_a,\langle\Delta_b,\Delta_c\rangle} \quad \text{and} \quad \Omega_{\langle a,b\rangle,c} = \Omega_{\langle\Delta_a,\Delta_b\rangle,\Delta_c}.$$

It follows from Proposition 2.4 that

(1) the central column of $[abc]$ is generated by $\Omega_{a,\langle b,c\rangle}$,
(2) the central row of $[abc]$ is generated by $\Omega_{\langle a,b\rangle,c}$,
(3) the lower row of $[abc]$ is generated by $q\circ\Omega_{\langle a,b\rangle,c} \simeq \Omega_{b,c}$, since $q\circ\langle\Delta_a, \Delta_b\rangle = \Delta_b$.
(4) the left column of $[abc]$ is generated by $\Omega_{a,\langle b,c\rangle} \circ i$.

2.3. Elementary Symmetries

The following equivalences are obvious, or can be derived from Proposition 2.4:

$$\Omega_{\langle b,c\rangle,a} \simeq \Omega_{\langle c,b\rangle,a}, \quad \Omega_{a,\langle b,c\rangle} \simeq \Omega_{a,\langle c,b\rangle}$$
$$(\Omega_{a,\langle b,c\rangle})^{-1} \simeq \Omega_{\langle b,c\rangle,a}, \quad (\Omega_{\langle a,b\rangle,c})^{-1} \simeq \Omega_{c,\langle a,b\rangle}, \quad (\Omega_{a,b})^{-1} \simeq \Omega_{b,a}.$$

3. Determination of the Spaces in the Diagrams

We will show that the six diagrams $[abc]$ corresponding to the permutations of $(0, 1, 2)$ can be drawn (with equivalence of norms) with the self-dual spaces $\mathfrak{R}_1 = \Delta_0(\mathcal{C}) = \ell_2$; $\mathfrak{R}_2 = \langle\Delta_1, \Delta_0\rangle(\mathcal{C}) = Z_2$, [7,24] and $\mathfrak{R}_3 = \langle\Delta_2, \Delta_1, \Delta_0\rangle(\mathcal{C})$ from now on denoted Z_3; the Orlicz spaces ℓ_f and ℓ_g and their duals, and the new spaces $\wedge$ and $\bigcirc$ and their duals. The properties of these spaces will be considered in Sect. 5. We begin showing that the spaces in the diagrams admit symmmetric Schauder decompositions and bases:

Proposition 3.1. *The unit vector basis* (e_n) *is a symmetric basis for the three Banach spaces* $\Delta_c(\mathcal{C})$, $\Delta_b(\ker\Delta_c)$ *and* $\Delta_a(\ker\Delta_b \cap \ker\Delta_c)$. *Similarly,* $\langle\Delta_a,\Delta_b\rangle(\ker\Delta_c)$ *and* $\langle\Delta_a,\Delta_b\rangle(\mathcal{C})$ *have a symmetric two-dimensional decomposition and* $\langle\Delta_a,\Delta_b,\Delta_c\rangle(\mathcal{C})$ *has a symmetric three-dimensional decomposition. Moreover, all the spaces in the diagrams admit a basis.*

Proof. Observe that since the family $\{\Delta_{n+k},\dots,\Delta_1\}$ is consistent, given an operator $T:\Sigma\to\Sigma$ acting on the pair the induced operator $T(\Delta_k f)=\Delta_k(T_{\mathcal{C}}f)$ defines an operator τ_k on $X_{\Delta_k}=\Delta_k(\mathcal{C})$ in the form $\tau_k(\Delta_k f)=\Delta_k(T_{\mathcal{C}}f)$: Indeed, if B_k is a homogeneous bounded selection for Δ_k then $\|\tau_k(\Delta_k f)\|_{X_{\Delta_k}}=\|\tau_k(\Delta_k B_k\Delta_k f)\|_{X_{\Delta_k}}=\|\Delta_k(T_{\mathcal{C}}B_k\Delta_k f)\|_{X_{\Delta_k}}\le\|\Delta_k\|\|T_{\mathcal{C}}\|\|B_k\|\|\Delta_k f\|_{X_{\Delta_k}}$. Let now X be any of the first three spaces in the statement and let P_n denote the natural projection onto the subspace generated by $\{e_1,\dots,e_n\}$. Since P_n is a norm-one operator on ℓ_∞ and ℓ_1, (P_n) is a bounded sequence of operators on X by the argument above. Clearly (e_n) is contained in X and generates a dense subspace. Since for each $x\in\operatorname{span}\{e_n:n\in\mathbb{N}\}$, $P_n x$ converges to x in X, it does the same for each $x\in X$. Thus (e_n) is a Schauder basis for X, and considering the operators associated to permutations of the basis. The argument at the beginning of the proof shows that the basis is symmetric. The remaining results on FDD's are proved in a similar way, using the operators induced by P_n in each of the spaces.

All the spaces have a basis because if (E_n) is a FDD for X with FDD-constant K and each E_n has a basis $(x_i^n)_{i=1}^{k_n}$ with basis constant $\le M$ then $\left((x_i^n)_{i=1}^{k_n}\right)_{n=1}^{\infty}$ is a basis for X with basis constant $\le KM$ [9, Proposition 6.5]. □

The next result shows that some of the spaces in the diagrams coincide. Note that algebraic equality implies isomorphism because if $\tau_1:X_1\to Y$ and $\tau_2:X_2\to Y$ are operators between Banach spaces with $\tau_1(X_1)=\tau_2(X_2)$ then the quotients $X_1/\ker\tau_1$ and $X_2/\ker\tau_2$ are isomorphic: if $T_i:X_i/\ker\tau_i\to Y$ denotes the injective operator induced by τ_i then $T_2^{-1}\circ T_1:X_1/\ker\tau_1\to X_2/\ker\tau_2$ is a closed bijective operator, which is continuous by the closed graph theorem.

Proposition 3.2. *The following equalities hold:*

(1) $\Delta_2(\ker\Delta_1\cap\ker\Delta_0)=\Delta_1(\ker\Delta_0)=\Delta_0(\mathcal{C})$,
(2) $\langle\Delta_2,\Delta_1\rangle(\ker\Delta_0)=\langle\Delta_1,\Delta_0\rangle(\mathcal{C})$,
(3) $\Delta_1(\ker\langle\Delta_0,\Delta_2\rangle)=\Delta_0(\ker\Delta_1)$.

Proof. Let $\varphi:\mathbb{S}\to\mathbb{D}$ be a conformal equivalence such that $\varphi(1/2)=0$. Since $\varphi'(1/2)\ne0$, we can define $\phi=\varphi'(1/2)^{-1}\varphi$. (1) For each $g\in\ker\Delta_0$ there is $f\in\mathcal{C}$ such that $g=\phi\cdot f$, hence $\Delta_1 g=\Delta_0 f$, and we get $\Delta_1(\ker\Delta_0)\subset\Delta_0(\mathcal{C})$.

Conversely, if $f \in \mathcal{C}$ then $g = \phi \cdot f \in \ker \Delta_0$ and $\Delta_0 f = \Delta_1 g$, so the second equality is proved. The first equality can be proved in a similar way. It was proved in [7, Theorem 4] that $j(x_1, x_0) = (x_1, x_0, 0)$ and $q(y_2, y_1, y_0) = y_0$ define an exact sequence

$$0 \longrightarrow \langle \Delta_1, \Delta_0 \rangle(\mathcal{C}) \xrightarrow{\ j\ } \langle \Delta_2, \Delta_1, \Delta_0 \rangle(\mathcal{C}) \xrightarrow{\ q\ } \Delta_0(\mathcal{C}) \longrightarrow 0,$$

and (2) follows from $\langle \Delta_2, \Delta_1, \Delta_0 \rangle(\ker \Delta_0) = \ker q$ and $\langle \Delta_1, \Delta_0, 0 \rangle(\mathcal{C}) = \operatorname{Im} j$. (3) Note that $y \in \Delta_0(\ker \Delta_1)$ if and only if $(0, y) \in \langle \Delta_1, \Delta_0 \rangle(\mathcal{C}) = \langle \Delta_2, \Delta_1 \rangle$ $(\ker \Delta_0)$; equivalently, $y \in \Delta_1(\ker \Delta_0 \cap \ker \Delta_2) = \Delta_1(\ker\langle \Delta_0, \Delta_2 \rangle)$. □

Next we identify the corner spaces as Orlicz sequence spaces. Let us consider the Orlicz functions $f(t) = t^2(\log t)^2$ and $g(t) = t^2(\log t)^4$.

Proposition 3.3. *$\Delta_0(\ker \Delta_1) = \ell_f$ and $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \ell_g$.*

Proof. The first equality was essentially proved in [24, Lemma 5.3]. With our notation,

$$\Delta_0(\ker \Delta_1) = \operatorname{Dom} \Omega_{1,0} = \{x \in \ell_2 : \Omega_{1,0} x \in \ell_2\}$$

and $\Omega_{1,0} : \ell_2 \to \ell_\infty$ is given by $\Omega_{1,0} = 2x \log(|x|/\|x\|_2)$. Thus

$$\Delta_0(\ker \Delta_1) = \{x \in \ell_2 : x \log |x| \in \ell_2\} = \ell_f.$$

Similarly, since $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \operatorname{Dom} \Omega_{\langle 2,1 \rangle,0}$ and $\Omega_{\langle 2,1 \rangle,0} : \ell_2 \to \ell_\infty \times \ell_\infty$ is given by

$$\Omega_{\langle 2,1 \rangle,0} x = \left(2x \log^2 \frac{|x|}{\|x\|_2}, 2x \log \frac{|x|}{\|x\|_2} \right)$$

(see [7]), we have $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2) = \{x \in \ell_2 : (2x \log^2 |x|, 2x \log |x|) \in Z_2\}$. Therefore $x \in \Delta_0(\ker \Delta_1 \cap \ker \Delta_2)$ if and only if $x \in \ell_2$, $2x \log |x| \in \ell_2$ and

$$2x \log^2 |x| - \Omega_{1,0}(2x \log |x|) = 2x \log^2 |x| - 4x \log |x| \log \frac{|x \log |x||}{\|2x \log |x|\|_2} \in \ell_2.$$

Since $\log |x \log |x|| = \log |x| + \log |\log |x||$, we conclude that $\Delta_0(\ker \Delta_1 \cap \ker \Delta_2)$ $= \{x \in \ell_2 : x \log^2 |x| \in \ell_2\} = \ell_g$. □

The second equality in the following result appears observed in [4, Example after Corollary 3].

Proposition 3.4. *$\Delta_2(\ker \Delta_0) = \Delta_1(\mathcal{C}) = \ell_f^*$.*

Proof. For the first equality, $\langle \Delta_1, \Delta_0 \rangle(\mathcal{C}) = \langle \Delta_2, \Delta_1 \rangle(\ker \Delta_0)$ by Proposition 3.2. Thus

$$\begin{aligned} x \in \Delta_1(\mathcal{C}) &\Leftrightarrow (x, f(1/2)) = (f'(1/2), f(1/2)) \text{ for some } f \in \mathcal{C} \\ &\Leftrightarrow (x, g'(1/2)) = (g''(1/2), g'(1/2)) \text{ for some } g \in \ker \Delta_0 \\ &\Leftrightarrow x \in \Delta_2(\ker \Delta_0). \end{aligned}$$

For the second equality, since $Z_2 = \langle \Delta_1, \Delta_0\rangle(\mathcal{C})$, we have a natural exact sequence

$$0 \longrightarrow \Delta_0(\ker \Delta_1) = \ell_f \xrightarrow{\ i\ } Z_2 \xrightarrow{\ p\ } \Delta_1(\mathcal{C}) \longrightarrow 0 \tag{3}$$

with $i(x) = (0, x)$ and $p(y, x) = y$. Moreover (see [24, Section 5]), the expression $\langle U_2(y, x), (b, a)\rangle = \langle -x, b\rangle + \langle y, a\rangle$ defines a bijective isomorphism $U_2 : Z_2 \to Z_2^*$, where $\langle \cdot, \cdot\rangle$ denotes the Riesz product. Since $i^* U_2 = p$, we get $\Delta_1(\mathcal{C}) = \ell_f^*$. □

The following three results were unexpected for us since, at first glance, the first two spaces seem to be incomparable.

Proposition 3.5. $\Delta_0(\ker \Delta_2) = \Delta_0(\ker \Delta_1) = \ell_f$.

Proof. The second equality is proved in Proposition 3.3. Moreover, the map $\Omega_{2,0} : \ell_2 \to \ell_\infty$ is given by $\Omega_{2,0} = 2x \log^2(|x|/\|x\|)$. Thus

$$\Delta_0(\ker \Delta_2) = \operatorname{Dom} \Omega_{2,0} = \{x \in \ell_2 : x \log^2 |x| \in \Delta_2(\ker \Delta_0) = \ell_f^*\}.$$

Since $\ell_f = \{x \in \ell_2 : x \log |x| \in \ell_2\}$, $\ell_f^* = \{x \in \ell_\infty : x \log^{-1} |x| \in \ell_2\}$ [27, Example 4.c.1]. Then

$$x \in \Delta_0(\ker \Delta_2) \Leftrightarrow x \in \ell_2 \ \text{ and } \ \frac{x \log^2 |x|}{\log(|x| \log^2 |x|)} = \frac{x \log^2 |x|}{\log |x| + 2 \log |\log x|} \in \ell_2.$$

Thus $x \in \Delta_0(\ker \Delta_2)$ if and only if $x \log |x| \in \ell_2$; equivalently $x \in \ell_f$. □

Like in the proof of Proposition 3.4, it was proved in [6, Proposition 5.1] that the expression

$$\langle U_3(x_2, x_1, x_0), (y_2, y_1, y_0)\rangle = \langle x_0, y_2\rangle - \langle x_1, y_1\rangle + \langle x_2, y_0\rangle$$

defines a bijective isomorphism $U_3 : Z_3 \to Z_3^*$ given by $U_3(x_2, x_1, x_0) = (x_0, -x_1, x_2)$. This fact will be a tool to prove the next result.

Proposition 3.6. $\Delta_2(\ker \Delta_1) = \Delta_2(\ker \Delta_0) = \ell_f^*$.

Proof. The second equality is proved in Proposition 3.4, and we derive the first equality from Proposition 3.5 by constructing an isomorphism from $\Delta_2(\ker \Delta_1)$ onto $\Delta_0(\ker \Delta_2)^*$ that takes e_n to e_n for every $n \in \mathbb{N}$. Recall that if M and N are closed subspaces of X with $N \subset M$ then $(M/N)^* \simeq N^\perp/M^\perp$. Thus, with the natural identifications we get

$$\Delta_0(\ker \Delta_2) \simeq \frac{\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2)}{\Delta_1(\ker \Delta_0 \cap \ker \Delta_2)} \implies \Delta_0(\ker \Delta_2)^* \simeq \frac{(\Delta_1(\ker \Delta_0 \cap \ker \Delta_2))^\perp}{(\langle \Delta_1, \Delta_0\rangle(\ker \Delta_2))^\perp}$$

and

$$\Delta_2(\ker \Delta_1) \simeq \frac{\langle \Delta_0, \Delta_2\rangle(\ker \Delta_1)}{\Delta_0(\ker \Delta_2 \cap \ker \Delta_1)},$$

and we conclude that U_3 induces an isomorphism from $\Delta_2(\ker \Delta_1)$ onto Δ_0 $(\ker \Delta_2)^*$ by showing that U_3 takes $\langle \Delta_0, \Delta_2\rangle(\ker \Delta_1)$ onto $(\Delta_1(\ker \Delta_0 \cap \ker \Delta_2))^\perp$

and $\Delta_0(\ker\Delta_2\cap\ker\Delta_1)$ onto $(\langle\Delta_1,\Delta_0\rangle(\ker\Delta_2))^\perp$. Indeed, $\Delta_1(\ker\Delta_0\cap\ker\Delta_2)$ can be identified with the subspace of the vectors $(0,y,0)$ in Z_3. Then $(\Delta_1(\ker\Delta_0\cap\ker\Delta_2))^\perp$ is the subspace of the vectors $(x,0,z)$ in Z_3^*, which coincides with $U_3\left(\langle\Delta_0,\Delta_2\rangle(\ker\Delta_1)\right)$, and similarly $\langle\Delta_1,\Delta_0\rangle(\ker\Delta_2)^\perp = U_3\left(\Delta_0(\ker\Delta_2\cap\ker\Delta_1)\right)$, and it is clear that the induced isomorphism takes e_n to e_n for every $n\in\mathbb{N}$. □

What follows is perhaps the most surprising symmetry:

Proposition 3.7. $\Delta_1(\ker\Delta_2) = \Delta_1(\ker\Delta_0) = \ell_2$.

Proof. Proposition 3.5 implies $\ker\Delta_0+\ker\Delta_1 = \ker\Delta_0+\ker\Delta_2$, from which we get

$$\Delta_1(\ker\Delta_0) = \Delta_1(\ker\Delta_0+\ker\Delta_2)\supset\Delta_1(\ker\Delta_2),$$

while Proposition 3.6 implies $\ker\Delta_2+\ker\Delta_0 = \ker\Delta_2+\ker\Delta_1$. Thus

$$\Delta_1(\ker\Delta_2) = \Delta_1(\ker\Delta_2+\ker\Delta_0)\supset\Delta_1(\ker\Delta_0).$$

□

A rich theory [1,30], see also [5, Section 10.8], contemplates Z_2 as a Fenchel-Orlicz space, with the meaning described next. A function $\varphi:\mathbb{C}^n\to[0,\infty)$ is a (quasi) Young function if it is (quasi) convex, $\varphi(0)=0$, $\lim_{t\to\infty}\varphi(tx)=\infty$ and $\varphi(zx)=\varphi(x)$ for every $z\in\mathbb{C}$ with $|z|=1$ and every $x\neq 0$. If we call two positive functions ϕ,ψ *equivalent* when ϕ/ψ is both upper and lower bounded, a quasi-convex function ϕ on $\mathbb{R}^n$ is equivalent to its convex hull $co\phi(x)=\inf\{\sum\theta_i\phi(x_i): x=\sum\theta_i x_i, \sum\theta_i=1, \theta_i\geq 0\}$. A Young function φ generates the Fenchel-Orlicz space

$$\ell_\varphi = \left\{(x^j)_{j\geq 1}\subset\mathbb{C}^n : \exists\rho>0 \text{ such that } \sum\varphi\Big(\frac{1}{\rho}x^j\Big)<\infty\right\}$$

endowed with the norm $\|(x^j)_{j\geq 1}\|_\varphi = \inf\{\rho>0 : \sum\varphi(\frac{1}{\rho}x^j)\leq 1\}$. The case $n=1$ correspond to Orlicz spaces. We will say that a quasi-Young function ϕ generates the Fenchel-Orlicz space ℓ_φ when $co\phi$ is equivalent to φ.

The Rochberg spaces associated to the scale of ℓ_p-spaces are Fenchel-Orlicz spaces in a natural way (see [16]). Indeed, given $\theta\in(0,1)$ and $n\geq 2$ there is a Young function $\varphi_n:\mathbb{C}^n\to[0,\infty)$ such that $Z_n=\ell_{\varphi_n}$. More precisely:

- ℓ_2 is ℓ_{φ_1}, the Orlicz space generated by the Orlicz function $\varphi_1(x)=|x_0|^2$.
- Z_2 is ℓ_{φ_2}, the Fenchel-Orlicz space generated by the quasi-Young function

$$\varphi_2(x_1,x_0) = |x_1 - x_0\log|x_0||^2 + |x_0|^2.$$

Keep track that $\varphi_2(x_1,0)=|x_1|^2$, so $\ell_2=\{(x,y)\in\ell_{\varphi_2} : y=0\}$; while $\varphi_2(0,x_0)=|x_0\log|x_0||^2+|x_0|^2\sim f$, so $\ell_f=\mathrm{Dom}\,\mathsf{KP}=\{(x,y)\in\ell_{\varphi_2} : x=0\}$.

- Z_3 is ℓ_{φ_3}, the Fenchel-Orlicz space generated by the quasi-Young function

$$\varphi_3(x_2, x_1, x_0) = \varphi_2(x_1, x_0) + \varphi_1(x_2 - g_{(x_1,x_0)}[2])$$

where $f[i]$ stands for $\frac{f^{(i)}(1/2)}{i!}$ and $g_x(z) = |x|^{2z-1}x$, so that $g_x[1] = 2x \log|x|$. Now, we set $g_{(x_1,x_0)} = g_{x_0} + \frac{\varphi}{k_2} g_{x_1 - g_{x_0}[1]}$, with $\varphi : \mathbb{S} \to \mathbb{D}$ a conformal map such that $\varphi(\frac{1}{2}) = 0$ and k_2 is adjusted so that $g_{(x_1,x_0)}[1] = x_1$. One therefore has $g_{(x_1,x_0)}(z) = g_{x_0}(z) + \frac{\varphi(z)}{k_2} g_{x_1 - 2x_0 \log|x_0|}(z) = |x_0|^{2z-1}x_0 + \frac{\varphi(z)}{k_2}|x_1 - 2x_0 \log|x_0||^{2z-1}(x_1 - 2x_0 \log|x_0|)$ to get, after a few tedious computations,

$$g_{(x_1,x_0)}[2] = 2x_0 \log^2|x_0| + \frac{\varphi'(1/2)}{k_2} 2(x_1 - 2x_0 \log|x_0|) \log(|x_1 - 2x_0 \log|x_0||) + \frac{\varphi''(1/2)}{2k_2}(x_1 - 2x_0 \log|x_0|).$$

- $\wedge$ is generated by $\varphi_3(x_2, 0, x_0) = \varphi_2(0, x_0) + |x_2 - g_{(0,x_0)}[2]|^2$.
- $\bigcirc$ is generated by $\varphi_3(0, x_1, x_0) = \varphi_2(x_1, x_0) + |g_{(x_1,x_0)}[2]|^2$.

4. Construction of the Diagrams

As we said before, $\langle \Delta_a, \Delta_b, \Delta_c \rangle(\mathcal{C}) \simeq Z_3$ for each permutation (a, b, c) of $(2, 1, 0)$.

Diagram [210]**:** By Proposition 3.2, $\Delta_2(\ker\Delta_1 \cap \ker\Delta_0) = \Delta_1(\ker\Delta_0) = \Delta_0(\mathcal{C}) \simeq \ell_2$ and $\langle \Delta_2, \Delta_1 \rangle(\ker\Delta_0) = \langle \Delta_1, \Delta_0 \rangle(\mathcal{C}) \simeq Z_2$. We thus get

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2
\end{array}$$

The two quasilinear maps generating the two middle sequences are $\Omega_{\langle 2,1 \rangle,0}$ and $\Omega_{2,\langle 1,0 \rangle}$; both can be found explicitly in [7] (and implicit in [29]) and also at the appropriate places in this paper.

Diagram [012]**:** By Propositions 3.3 and 3.7, $\Delta_0(\ker\Delta_1 \cap \ker\Delta_2) = \ell_g$ and $\Delta_1(\ker\Delta_2) = \ell_2$. So we have the spaces in the left column. The next result provides the spaces in the lower row.

Proposition 4.1. (a) $\Delta_2(\mathcal{C})$ *is isomorphic to* ℓ_g^*. (b) $\langle \Delta_1, \Delta_2 \rangle(\mathcal{C})$ *is isomorphic to* $\bigcirc^*$.

Proof. (a) By Proposition 3.3, $\ell_g = \Delta_0(\ker \Delta_1 \cap \ker \Delta_2)$ which is isomorphic to a closed subspace of Z_3, namely $\{(x_2, x_1, x_0) \in Z_3 : x_2 = x_1 = 0\}$. Hence $\ell_g^* \simeq Z_3^* / (\Delta_0(\ker \Delta_1 \cap \ker \Delta_2))^{\perp}$. Since $(\Delta_0(\ker \Delta_1 \cap \ker \Delta_2))^{\perp} = U_3 (\langle \Delta_0, \Delta_1 \rangle(\ker \Delta_2))$ then

$$\Delta_2(\mathcal{C}) \simeq \frac{Z_3}{\langle \Delta_0, \Delta_1 \rangle(\ker \Delta_2)} \simeq \ell_g^*.$$

(b) The space $\bigcirc = \langle \Delta_0, \Delta_1 \rangle(\ker \Delta_2)$ is isomorphic to $\{(x_2, x_1, x_0) \in Z_3 : x_2 = 0\}$, a closed subspace of Z_3. Hence $\bigcirc^* \simeq Z_3^* / (\langle \Delta_0, \Delta_1 \rangle(\ker \Delta_2))^{\perp}$. Since

$$(\langle \Delta_0, \Delta_1 \rangle(\ker \Delta_2))^{\perp} = U_3 (\Delta_0(\ker \Delta_1 \cap \ker \Delta_2))$$

we get $\langle \Delta_1, \Delta_2 \rangle(\mathcal{C}) \simeq Z_3/(\Delta_0(\ker \Delta_1 \cap \ker \Delta_2)) \simeq \bigcirc^*$. □

Thus we obtain the diagram:

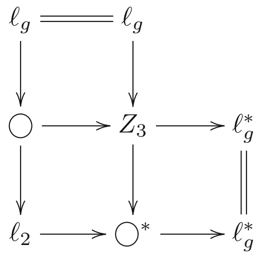
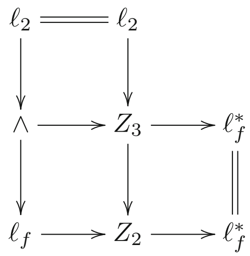


Diagram [201]**:** $\Omega_{2,\langle 0,1\rangle} \simeq \Omega_{2,\langle 1,0\rangle}$ gives the central column (coincides with that of [210]), and Propositions 3.4 and 3.5 give the lower row. Thus, we get



Arguing as in the proof of Proposition 4.1, we get (U_3 appeared before Proposition 3.6):

Proposition 4.2. $\langle \Delta_2, \Delta_0 \rangle(\mathcal{C})$ *is isomorphic to* $\wedge^* = \langle \Delta_2, \Delta_0 \rangle(\ker \Delta_1)^*$.

Proof. Since the space $\wedge = \langle \Delta_2, \Delta_0 \rangle(\ker \Delta_1)$ is isomorphic to a subspace of Z_3, we get $\wedge^* \simeq Z_3^* / (\langle \Delta_2, \Delta_0 \rangle(\ker \Delta_1))^{\perp}$. Moreover $(\langle \Delta_2, \Delta_0 \rangle(\ker \Delta_1))^{\perp} = U_3 (\Delta_1(\ker \Delta_2 \cap \ker \Delta_0))$, and therefore $\langle \Delta_2, \Delta_0 \rangle(\mathcal{C}) \simeq Z_3 / (\Delta_1(\ker \Delta_2 \cap \ker \Delta_0)) \simeq \wedge^*$. □

Diagram [120]**:** $\Omega_{\langle 1,2\rangle,0} \simeq \Omega_{\langle 2,1\rangle,0}$ gives the central row, and $\Delta_1(\ker \Delta_2 \cap \ker \Delta_0) = \ell_f$ and $\Delta_2(\ker \Delta_0) = \ell_f^*$ by Propositions 3.2, 3.3 and 3.6. Since $\wedge^* \simeq \langle \Delta_2, \Delta_0\rangle(\mathcal{C})$ by Proposition 4.2 and $\Delta_0(\mathcal{C}) = \ell_2$, we get

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \| \\
\ell_f^* & \longrightarrow & \wedge^* & \longrightarrow & \ell_2
\end{array}$$

Diagram [021]**:** $\Omega_{0,\langle 2,1\rangle} \simeq \Omega_{0,\langle 1,2\rangle}$ gives the central column and $\Omega_{\langle 0,2\rangle,1} \simeq \Omega_{\langle 2,0\rangle,1}$ gives the central row. Since $\Delta_2(\ker \Delta_1) = \ell_f^*$ by Proposition 3.6, we get

$$\begin{array}{ccccc}
\ell_g & = & \ell_g & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & Z_3 & \longrightarrow & \ell_f^* \\
\downarrow & & \downarrow & & \| \\
\ell_f^* & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_f^*
\end{array}$$

Diagram [102]**:** $\Omega_{1,\langle 0,2\rangle} \simeq \Omega_{1,\langle 2,0\rangle}$ gives the central column, and $\Omega_{\langle 1,0\rangle,2} \simeq \Omega_{\langle 0,1\rangle,2}$ gives the central row. Moreover, $\Delta_0(\ker \Delta_2) \simeq \ell_f$ by Proposition 3.5. So we get

$$\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
\bigcirc & \longrightarrow & Z_3 & \longrightarrow & \ell_g^* \\
\downarrow & & \downarrow & & \| \\
\ell_f & \longrightarrow & \wedge^* & \longrightarrow & \ell_g^*
\end{array}$$

5. Properties of the Spaces

Here we describe some isomorphic properties of the spaces in the diagrams. Recall that a Banach space X is *hereditarily* ℓ_2 if every closed infinite dimensional subspace of X contains a subspace isomorphic to ℓ_2. Being hereditarily

ℓ_2 is inherited by subspaces, but not by quotients since every separable reflexive space is a quotient of a reflexive hereditarily ℓ_2 space [2, Theorem 6.2]. To be hereditarily ℓ_2 is a three-space property [14, Theorem 3.2.d].

Proposition 5.1. *All the spaces appearing in the diagrams* [*abc*] *are hereditarily* ℓ_2.

Proof. Each infinite dimensional subspace of a reflexive Orlicz sequence space contains a copy of ℓ_p for $p \in [\alpha, \beta]$, being α (resp. β) the lower (resp. upper) Boyd index of the space [26, Proposition I.4.3, Theorem I.4.6]. Since Z_3 has type $2-\varepsilon$ and cotype $2+\varepsilon$ for each $\varepsilon > 0$, the same happens with ℓ_f and ℓ_g and their dual spaces, hence their Boyd indices are 2 and these spaces are hereditarily ℓ_2. Apply the 3-space property for all the other spaces. □

Recall from [25, Corollary 13] that if M is an Orlicz function satisfying the Δ_2-condition and $2 \leq q < \infty$ then the space ℓ_M has cotype q if and only if there exists $K > 0$ such that $M(tx) \geq Kt^q M(x)$ for all $0 \leq t, x \leq 1$. Consequently, the spaces ℓ_f and ℓ_g have cotype 2 and ℓ_f^* and ℓ_g^* have type 2. We need one more technical result:

Proposition 5.2. *Let X be a Banach space.*

(1) If X has type 2 then every subspace isomorphic to ℓ_2 is complemented.
(2) If X has an unconditional basis and cotype 2 then every subspace of X isomorphic to ℓ_2 contains an infinite dimensional subspace complemented in X.

Proof. (a) is a consequence of Maurey's extension theorem; see [19, Corollary 12.24]. (b) The following argument is similar to the proof of [28, Theorem 3.1] for subspaces of L_p, $1 < p < 2$, with an unconditional basis. Let (e_n) be an unconditional basis of X, let (x_k) be a normalized block basis of (e_n), and take a sequence (c_j) of scalars and a successive sequence (B_k) of intervals of integers so that $x_k = \sum_{i \in B_k} c_i e_i$. We consider the sequence of projections (P_k) in X defined by $P_k e_j = e_j$ if $j \in B_k$, and $P_k e_j = 0$ otherwise. Let Q_k be a norm-one projection on $\operatorname{span}\{e_j : j \in B_k\}$ onto the one-dimensional subspace generated by x_k. We claim that $Px = \sum_{k=1}^{\infty} Q_k P_k x$ defines a projection on X onto the closed subspace generated by (x_k). If $x \in X$ then $\sum_{k=1}^{\infty} P_k x$ is unconditionally converging and $\|\sum_{k=1}^{\infty} P_k x\| \leq D\|x\|$ for some $D > 0$. Moreover, since X has cotype 2, $\left(\sum_{k=1}^{\infty} \|P_k x\|^2\right)^{1/2} \leq E\|\sum_{k=1}^{\infty} P_k x\|$ for some $E > 0$. We write $Q_k P_k x = s_k x_k$ for each k. Then

$$\left(\sum_{k=1}^{\infty} |s_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^{\infty} \|P_k x\|^2\right)^{1/2} \leq E \cdot D\|x\|.$$

Hence $\sum_{k=1}^{\infty} Q_k P_k x$ converges, and it is easy to check that P is the required projection. □

Corollary 5.3. *Each infinite dimensional subspace of one of the spaces ℓ_f, ℓ_g, ℓ_f^* and ℓ_g^* contains a complemented copy of ℓ_2.*

Since $Z_2 \simeq Z_2^*$ [24], a space X is (isomorphic to) a subspace (resp. a quotient) of Z_2 if and only if X^* is a quotient (resp. a subspace) of Z_2.

Proposition 5.4. *None of the spaces $\bigcirc$, $\bigcirc^*$, $\wedge$ and $\wedge^*$ is (isomorphic to) a subspace or a quotient of Z_2.*

Proof. It was proved in [24, Theorem 5.4] that every normalized basic sequence in Z_2 has a subsequence equivalent to the basis of one of the spaces ℓ_2 or ℓ_f. Thus none of the four spaces is a subspace of Z_2 because $\bigcirc$ and $\wedge$ contain a copy of ℓ_g and $\bigcirc^*$ and $\wedge^*$ contain a copy of ℓ_f^*, as we can see in the diagrams. □

Next we extend to Z_3 some fundamental results about Z_2. The following one is in [21] for Z_2 and the proof we present is similar to that in [5, Proposition 10.9.1] for Z_2.

Proposition 5.5. *An operator $\tau : Z_3 \to X$ either is strictly singular or an isomorphism on a complemented copy of Z_3.*

Proof. Since the quotient map in the sequence $0 \to \ell_2 \to Z_3 \to Z_2 \to 0$ is strictly singular (see [7]) an operator $\tau : Z_3 \to X$ is strictly singular if and only if $\tau|_{\ell_2}$ is strictly singular. So, let τ be a non-strictly singular operator. Let us assume first that $\tau|_{\ell_2}$ is an embedding so that we can assume that $\|\tau(y, 0)\| \geq \|y\|$ for all $y \in \ell_2$. Observe the commutative diagram:

$$\begin{array}{ccccc}
\ell_2 & = & \ell_2 & & \\
\downarrow{\scriptstyle \imath} & & \downarrow{\scriptstyle (\tau,\imath)} & & \\
Z_3 & \xrightarrow{(\tau,\mathbf{id})} & X \oplus Z_3 & \longrightarrow & X \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle Q} & & \| \\
Z_2 & \longrightarrow & \mathrm{PO} & \longrightarrow & X
\end{array}$$

- The composition $Q\,(\tau, \mathbf{id})$ is strictly singular since it factors through π.
- $Q\,(\tau, \mathbf{id}) = Q(\tau, 0) + Q\,(0, \mathbf{id})$.
- $Q\,(0, \mathbf{id})$ is an embedding since

$$\begin{aligned} \|Q(0,z)\| &= \inf_{y\in\ell_2} \|(0,z) - (\tau,\imath)(y)\| = \inf_{y\in\ell_2} \|(-\tau y, z-y)\| \\ &= \inf_{y\in\ell_2} \big\{\|\tau(y,0)\| + \|z-y\|\big\} \geq \|y\| + \|z\| - \|y\| = \|z\|. \end{aligned}$$

Thus, $Q(\tau, 0)$, being the difference (or sum) between a strictly singular operator and an embedding, has to have closed range and finite dimensional kernel [27, Proposition 2.c.10] and therefore it must be an isomorphism on some finite codimensional subspace of Z_3, and the same happens to τ. All subspaces of Z_3 with codimension 3 are isomorphic to Z_3 and thus we are done.

In the general case, if τ is not strictly singular, then $\tau|_U$ is an embedding for some subspace U of ℓ_2 generated by a normalized block basis (u_n) of the canonical basis. We consider the operator $\tau_U : \ell_2 \to \ell_2$ given by $\tau_U(e_n) = u_n$, which acts on the pair. It was shown by Kalton [21] that the operator $S_U : Z_2 \longrightarrow Z_2$ defined by $S_U(e_n, 0) = (u_n, 0)$ and $S_U(0, e_n) = (\Omega_{1,0}u_n, u_n)$ is continuous and makes commutative the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle \tau_U}\big\downarrow & & \big\downarrow{\scriptstyle S_U} & & \big\downarrow{\scriptstyle \tau_U} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array} \tag{4}$$

The operator S_U can be described by the matrix $S_U = \begin{pmatrix} u & 2u\log u \\ 0 & u \end{pmatrix}$. The theory developed in [12, Proposition 7.1] explains why the upper-right entry of the matrix has to be $2u\log u$ and why there is also a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle S_U}\big\downarrow & & \big\downarrow{\scriptstyle R_U} & & \big\downarrow{\scriptstyle \tau_U} & & \\ 0 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array} \tag{5}$$

in which

$$R_U = \begin{pmatrix} u & 2u\log u & 2u\log^2 u \\ 0 & u & 2u\log u \\ 0 & 0 & u \end{pmatrix}$$

Since τ_U is an into isometry, so are S_U and R_U. Thus, $R_U(Z_3)$ is an isometric copy of Z_3. Let us show it is complemented. With that purpose, consider Z_3^U the space Z_3 constructed with each block u_n in place of e_n; namely, Z_2^U is the twisted sum space $U \oplus_{\Omega_{1,0}^U} U$ constructed with $\Omega_{1,0}^U(u) = 2\sum \lambda_n \log \frac{|u|}{\|u\|}$ for $u \in U$ and then Z_3^U is the space $Z_2^U \oplus_{\Omega_{\langle 2,1\rangle,0}^U} U$ with the corresponding definition for $\Omega_{\langle 2,1\rangle,0}^U$. We can in this way understand R_U as an operator $R'_U : Z_3^U \to Z_3$ in the obvious form: $R'_U(u_n, 0, 0) = R_U(e_n, 0, 0)$, $R'_U(0, u_n, 0) = R_U(0, e_n, 0)$ and $R'_U(0, 0, u_n) = R_U(0, 0, e_n)$. Consider the diagram

$$\begin{array}{ccc} Z_3^U & \xrightarrow{\;\;R'_U\;\;} & Z_3 \\ {\scriptstyle D_U}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ (Z_3^U)^* & \xleftarrow[\;\;(R'_U)^*\;\;]{} & Z_3 \end{array}$$

Here D_U is the obvious isomorphism between Z_3^U and $(Z_3^U)^*$ induced by D. The diagram is commutative: for normalized blocks $u_i, u_j, u_k, u_l, u_m, u_n$ one has

$$R'_U(u_i, u_j, u_k) = (u_i + 2u_j \log u_j + 2u_k \log^2 u_k, u_j + 2u_k \log u_k, u_k)$$

while the action of $D\left(u_i+2u_j\log u_j+2u_k\log^2 u_k, u_j+2u_k\log u_k, u_k\right)$ over $\left(u_l+2u_m\log u_m+2u_n\log^2 u_n, u_m+2u_n\log u_n, u_n\right)$ gives

$$(u_i+2u_j\log u_j+2u_k\log^2 u_k)u_n-(u_j+2u_k\log u_k)(u_m+2u_n\log u_n)$$
$$+u_k(u_l+2u_m\log u_m+2u_n\log^2 u_n);$$

namely

$$\delta_{in}+2\delta_{jn}\log u+2\delta_{kn}\log^2 u-\delta_{jm}-2\delta_{jn}\log u-2\delta_{km}\log u-4\delta_{kn}\log^2 u$$
$$+\delta_{kl}+2\delta_{km}\log u+2\delta_{kn}\log^2 u$$

which is $\delta_{in}-\delta_{jm}+\delta_{kl}$. Thus

$$\begin{aligned}(R'_U)^*DR'_U(u_i,u_j,u_k)(u_l,u_m,u_n) &= DR'_U(u_i,u_j,u_k)\left(R'_U(u_l,u_m,u_n)\right)\\ &= \langle R'_U(u_i,u_j,u_k), R'_U(u_l,u_m,u_n)\rangle\\ &= \delta_{in}-\delta_{jm}+\delta_{kl}\\ &= D_U(u_i,u_j,u_k)(u_l,u_m,u_n)\end{aligned}$$

Therefore, $D_U^{-1}(R'_U)^*D$ is a projection onto the range of R_U, as desired, and one can repeat the same argument as before working now with $\tau|_U$ instead of $\tau|_{\ell_2}$. □

Corollary 5.6. *Every operator from Z_3 into a twisted Hilbert space is strictly singular. In particular, Z_3 does not contain complemented copies of either Z_2 or ℓ_2.*

Proof. That Z_3 cannot be a subspace of a twisted Hilbert space was proved in [7, Prop. 12]. □

Corollary 5.7. *The six representations of Z_3 as a twisted sum in the diagrams are non-trivial.*

Proof. Since Z_3 contains no complemented copy of ℓ_2 and $Z_3 \simeq Z_3^*$ [6, Prop. 5.5 and Cor. 5.7], by Corollary 5.3 the exact sequences $Z_2 \to Z_3 \to \ell_2$, $\wedge \to Z_3 \to \ell_f^*$ and $\bigcirc \to Z_3 \to \ell_g^*$ have strictly singular quotient map, while $\ell_2 \to Z_3 \to Z_2$, $\ell_f \to Z_3 \to \wedge^*$ and $\ell_g \to Z_3 \to \bigcirc^*$ have strictly cosingular embedding. Of course, the second part is a dual result of the first one. □

In [24, Theorem 5.4] it is proved that every normalized basic sequence in Z_2 admits a subsequence equivalent to the basis of one of the spaces ℓ_2 or ℓ_f. For Z_3 we have:

Theorem 5.8. *Every normalized basic sequence in Z_3 admits a subsequence equivalent to the basis of one of the spaces ℓ_2, ℓ_f, ℓ_g.*

Proof. Let $(y_n, x_n, z_n)_n$ be a normalized basic sequence in Z_3. If $\|z_n\| \to 0$ as $n \to \infty$, we can assume that $\sum \|z_n\| < \infty$ and thus that, up to a perturbation, (y_n, x_n, z_n) is a basic sequence in Z_2; therefore it admits a subsequence equivalent to the basis of either ℓ_2 or ℓ_f [24, Theorem 5.4].

If $\|z_n\| \geq \varepsilon$ then we can assume after perturbation that there is a block basic sequence (u_n) in ℓ_2 such that $\sum \|z_n - u_n\| < \infty$. Since

$$\begin{aligned}(y_n, x_n, z_n) &= (y_n, x_n, z_n) - (\Omega_{\langle 2,1\rangle,0} u_n, u_n) + (\Omega_{\langle 2,1\rangle,0} u_n, u_n)\\ &= ((y_n, x_n) - \Omega_{\langle 2,1\rangle,0} u_n, z_n - u_n) + (\Omega_{\langle 2,1\rangle,0} u_n, u_n)\end{aligned}$$

and $z_n - u_n \to 0$ we can assume that $((y_n, x_n) - \Omega_{\langle 2,1\rangle,0} u_n, z_n - u_n)$ admits a subsequence equivalent to the basis of either ℓ_2 or ℓ_f. We conclude showing that $(\Omega_{\langle 2,1\rangle,0} u_n, u_n)$ is equivalent to the canonical basis of ℓ_g. And thus the plan is to show that $\sum (x_n \Omega_{\langle 2,1\rangle,0} u_n, \sum x_n u_n)$ converges in Z_3 if and only if $(x_n) \in \ell_g$. In order to show that, we simplify the notation: let x be a scalar sequence, let $u = (u_n)$ be the sequence of blocks and let us denote $xu = \sum x_n u_n$. Showing that $(x\Omega_{\langle 2,1\rangle,0} u, xu)$ converges in Z_3 is the same as showing that its norm is finite. Recall that for a positive normalized z one has $\Omega_{\langle 2,1\rangle,0}(z) = (2z \log^2 z, 2z \log z)$. Since

$$\begin{aligned}\|(x\Omega_{\langle 2,1\rangle,0} u, xu)\|_{Z_3} &= \|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} + \|xu\|_2\\ &= \|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} + \|xu\|_2,\end{aligned}$$

assuming $\|u_n\| = 1$ for all n and $\|xu\| = 1$, one gets

$$\begin{aligned}x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu) &= \big(x2u \log^2 u, 2x \log u\big) - \big(2xu \log^2 (xu), 2xu \log(xu)\big)\\ &= \big(2xu(\log^2 u - \log^2 xu), 2xu(\log u - \log(ux))\big)\\ &= \big(2xu(\log^2 u - (\log^2 x + \log^2 u + 2\log x \log u), -2xu \log x)\big)\\ &= \big(-2xu(\log^2 x + 2\log x \log u), -2xu \log x)\big)\end{aligned}$$

and therefore

$$\begin{aligned}&\|(x\Omega_{\langle 2,1\rangle,0} u - \Omega_{\langle 2,1\rangle,0}(xu)\|_{Z_2} = \|\Big(-2xu(\log^2 x + 2\log x \log u), -2xu \log x\Big)\|_{Z_2}\\ &\quad = \| -2xu(\log^2 x + 2\log x \log u) + 4xu \log x \log\left(2xu \log x\right)\|_2 + \|2xu \log x\|_2\\ &\quad = \|2xu \left(\log^2 x + 2\log 2 \log x + 2\log x \log\log x\right)\|_2 + \|2xu \log x\|_2.\end{aligned}$$

That means that the sequence x satisfies $x(\log^2 |x|) \in \ell_2$; namely, $x \in \ell_g$. □

This result has consequences for the structure of the spaces Z_3, $\wedge$ and $\bigcirc$.

Proposition 5.9. *Z_3 has no complemented subspace with an unconditional basis.*

Proof. If (x_n) were an unconditional basic sequence in Z_3 generating a complemented subspace, it would admit a subsequence (x_{n_k}) equivalent to the basis of one of the spaces ℓ_2, ℓ_f, ℓ_g by Theorem 5.8. Since this subsequence would generate a complemented subspace of Z_3, we would conclude that Z_3 contains a complemented copy of ℓ_2, by Corollary 5.3, which cannot happen. □

Proposition 5.10. *The spaces $\wedge$ and $\bigcirc$ are not isomorphic to their dual spaces.*

Proof. Both $\wedge$ and $\bigcirc$ are subspaces of Z_3, hence Theorem 5.8 applies. But $\wedge^*$ and $\bigcirc^*$ contain a copy of ℓ_f^*, as we can see in the diagrams, while the canonical basis of ℓ_f^* (or any of its subsequences) is not equivalent to those of ℓ_2, ℓ_f or ℓ_g. □

Proposition 5.11. *The space $\wedge$ (hence $\wedge^*$ also) is not isomorphic to either $\bigcirc$ or $\bigcirc^*$.*

Proof. The idea for the proof is to show that every weakly null sequence in $\wedge$ contains a subsequence equivalent to the canonical basis of either ℓ_2 or ℓ_g, so that $\wedge$ cannot contain either ℓ_f or ℓ_f^* and therefore it cannot be isomorphic to either $\bigcirc$ or $\bigcirc^*$. Why it is so is essentially contained in the proof of Theorem 5.8, taking into account that the elements of $\wedge$ have the form $(y, 0, z)$. Our interest lies now in showing that when (u_n) are blocks in ℓ_2 (actually in ℓ_f) and $\sum(x_n y_n, 0, u_n)$ converges in Z_3 then $x = (x_n)$ is in either ℓ_2 or ℓ_g. Using the same notation as then, since $\|(xy, 0, xu)\|_{Z_3} = \left\|(xy, 0) - \Omega_{\langle 2,1\rangle,0}(xu)\right\|_{Z_2} + \|xu\|_{\ell_2}$, and since $(xy, 0)$ and xu converge when $x \in \ell_2$, our only concern is when $\Omega_{\langle 2,1\rangle,0}(xu)$ converges in Z_2. But this means that $x \in \operatorname{Dom}\Omega_{\langle 2,1\rangle,0} = \ell_g$. □

Proposition 5.12. *The spaces $\wedge$ and $\wedge^*$ do not contain ℓ_2 complemented. Consequently, they do not have an unconditional basis.*

Proof. Consider the diagram [120]. Its lower sequence comes defined by $\triangle(x) = x\log^2 x$, obtained from the composition $\Omega_{\langle 2,1\rangle,0}x = (x\log^2 x, x\log x)$ with the projection onto the first coordinate. Let u be a sequence of disjoint blocks of the canonical basis of ℓ_2 and let $x \in \ell_2$.

$$\begin{aligned}\triangle(xu) &= xu\log^2(xu) = xu\left(\log x + \log u)^2\right) \\ &= xu\left(\log^2 x + \log^2 u + 2\log x\log u\right) \\ &= xu\log^2 x + xu\log^2 u + 2xu\log x\log u\end{aligned}$$

Observe that the second term $x \to xu\log^2 u$ is linear while the third term $x \to 2x\log xu\log u$ is $x \to \Omega_{1,0}(x)$, according to [5, Lemma 9.3.10] and up to a weight and a linear map. This map is bounded when considered with values in its range ℓ_f^*, which yields that the restriction $\triangle|_{[u]}$ is, up to a linear plus a bounded map, $\triangle$ once again. Therefore, the quotient map Q in $0 \to \ell_f^* \to \wedge^* \xrightarrow{Q} \ell_2 \to 0$ is strictly singular; hence Q^*, the embedding in its dual sequence $0 \to \ell_2 \xrightarrow{Q^*} \wedge \to \ell_f \to 0$, which is the left column in diagram [201], is strictly cosingular.

The rest is similar to [6, Prop. 15]: Assume that $\wedge^*$ contains a subspace A isomorphic to ℓ_2 complemented by some projection P. Since Q is strictly singular, there exist an infinite dimensional subspace $A' \subset \ell_2$ and a nuclear operator $K : A' \to \wedge^*$ nuclear norm $\|K\|_n < 1$ such that $I - K : A' \to A$ is a bijective isomorphism. Let N be a nuclear operator on $\wedge^*$ extending K

with $\|N\|_n < 1$. Then $I_{\wedge^*} - N$ is invertible, where $I_{\wedge^*}$ is the identity on $\wedge^*$, $(I_{\wedge^*} - N)^{-1} = \sum_{k\geq 0} N^k$, and $(I_{\wedge^*} - N) \circ P \circ (I_{\wedge^*} - N)^{-1}$ is a projection on $\wedge^*$ onto A'. This cannot be since the embedding map Q^* is strictly cosingular. Since $\wedge$ is reflexive, it cannot contain ℓ_2 complemented also. As for the second part, since $\wedge$ is a subspace of Z_3, the argument in the proof of Corollary 5.9 also proves the result. □

Corollary 5.13. *All the exact sequences appearing in the six diagrams are non-trivial.*

Proof. Corollary 5.7 showed that the sequences passing through Z_2 are non-trivial. The non-triviality for those passing through $\wedge$ and $\wedge^*$ follows from the fact that these spaces do not admit an unconditional basis (Proposition 5.12); for those passing through $\bigcirc$ follows from the fact that $\ell_f \oplus \ell_f \simeq \ell_f$ does not contain copies of ℓ_g and $\ell_g \oplus \ell_2 \simeq \ell_g$ does not contain copies of ℓ_f; and for those passing through $\bigcirc^*$ we can argue as for $\bigcirc$. □

This corollary can be improved.

Proposition 5.14. *The following maps:*

(1) $Q_0, Q_1, Q_2, Q_{1,0}, Q_{0,1}, Q_{2,0}, Q_{0,2}, Q_{1,2}, Q_{2,1}$;
(2) $p_{1,0}, p_{0,1}, p_{2,0}, p_{0,2}, p_{2,1}, p_{1,2}$; *and*
(3) $q_{1,0}, q_{0,1}, q_{2,0}, q_{0,2}$

are strictly singular.

Proof. (1) That $Q_0, Q_1, Q_2, Q_{1,0}$ and $Q_{0,1}$ are strictly singular is a consequence of Proposition 5.5, because ℓ_2, ℓ_f^*, ℓ_g^* and Z_2 do not contain Z_3. The lower part in the diagram [120]

$$\begin{array}{ccccc} Z_2 & \longrightarrow & Z_3 & \xrightarrow{Q_0} & \ell_2 \\ \downarrow{\scriptstyle p_{2,0}} & & \downarrow{\scriptstyle Q_{2,0}} & & \| \\ \ell_f^* & \longrightarrow & \wedge^* & \xrightarrow{q_{2,0}} & \ell_2 \end{array}$$

plus the technique used before shows that $Q_{2,0}$, hence $Q_{0,2}$, is strictly singular. Therefore, its restrictions $p_{2,0}$ and $p_{0,2}$ are strictly singular too. The restriction of $p_{1,2}$ to ℓ_f is the canonical inclusion of ℓ_f into ℓ_2, which is strictly singular due to the criterion [27, Theorem 4.a.10] asserting that given two Orlicz spaces ℓ_M, ℓ_N for which the canonical inclusion $\jmath : \ell_M \to \ell_N$ is continuous then $\jmath$ is strictly singular if and only if for each $B > 0$ there is a sequence $\tau_1, \ldots, \tau_n$ in $(0, 1]$ such that $\sum M(\tau_i t) \geq B \sum N(\tau_i t)$ for all $t \in [0, 1]$. Straightforward calculations yield that the canonical inclusions $\ell_g \to \ell_f$ and $\ell_f \to \ell_2$ are strictly singular. Thus, also $p_{0,2}$ is strictly singular and consequently the lower part of diagram [102]

$$\begin{array}{ccccc} \bigcirc & \longrightarrow & Z_3 & \xrightarrow{Q_2} & \ell_g^* \\ \big\downarrow{\scriptstyle p_{0,2}} & & \big\downarrow{\scriptstyle Q_{0,2}} & & \Big\| \\ \ell_f & \longrightarrow & \wedge^* & \xrightarrow{q_{0,2}} & \ell_g^* \end{array}$$

yields that $Q_{0,2}$, hence $Q_{2,0}$ too, is strictly singular. (2) the maps are restrictions of $Q_{1,0}$, $Q_{0,1}$, $Q_{2,0}$ and $Q_{0,2}$. (3) follows from Corollary 5.3 because Z_2 and $\wedge^*$ contain no complemented copy of ℓ_2. $\square$

Remark 5.15. We have been unable to prove that $q_{1,2}$ and $q_{2,1}$ are strictly singular, from where it would follow that $\bigcirc$ and $\bigcirc^*$ do not have an unconditional basis.

6. The Case of Weighted Hilbert Spaces

This is an interesting test case by its simplicity (all exact sequences are trivial and all spaces are isomorphic to Hilbert spaces), and provides some insight about what occurs in other situations. Let $w = (w_n)$ be a weight sequence (a non-increasing sequence of positive numbers such that $\lim w_n = 0$ and $\sum w_n = \infty$) and let $w^{-1} = (w_n^{-1})$. Note that $\ell_2(w)^*$ is isometric to $\ell_2(w^{-1})$.

If $\mathcal{C}$ is the Calderón spaces for the couple $(\ell_2(w^{-1}), \ell_2(w))$, an homogeneous bounded selector for the interpolator $\Delta_0 : \mathcal{C} \to \Sigma$ is $B(x)(z) = w^{2z-1}x$. Therefore $B(x)'(z) = 2w^{2z-1} \log w \cdot x$ and $\Omega_{1,0}x = \Delta_1 Bx = 2 \log w \cdot x$. The Rochberg space $\mathcal{R}_2$ will be

$$Z_2(w) = \{(y, x) : x \in \ell_2, \quad y - 2 \log w \cdot x \in \ell_2\}$$

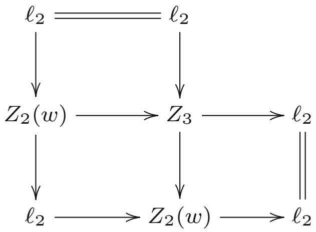
from where $\operatorname{Dom} \Omega_{1,0} = \{x \in \ell_2 : 2 \log w \cdot x \in \ell_2\} = \ell_2(\log w) = \{(0, x) \in Z_2(w)\}$ and $\operatorname{Ran} \Omega_{1,0} = \ell_2((\log w)^{-1})$ so that $(\Omega_{1,0})^{-1}x = \frac{1}{2 \log w}x$; thus Dom $(\Omega_{1,0})^{-1} = \{x \in \ell_2((\log w)^{-1}) : (\log w)^{-1} \cdot x \in \ell_2(\log w)\} = \ell_2 = \operatorname{Ran} (\Omega_{1,0})^{-1}$, as we already know.

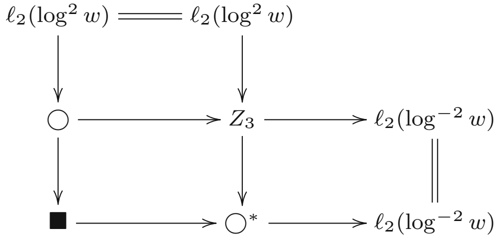
Next, $B(x)''(z) = 4w^{2z-1} \log^2 w \cdot x$, and thus $\Delta_2 B(x) = 2 \log^2 w \cdot x$. Therefore

$$\Omega_{\langle 2,1\rangle,0}(x) = (\Delta_2 B(x), \Delta_1 B(x)) = \left(2 \log^2 w \cdot x, 2 \log w \cdot x\right)$$

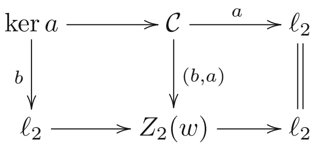
defines a linear map with domain $\operatorname{Dom} \Omega_{\langle 2,1\rangle,0} = \{x \in \ell_2 : (2 \log^2 w \cdot x, 2 \log w \cdot x) \in Z_2(w)\} = \ell_2(\log^2 w)$ since one must have $2 \log w \cdot x \in \ell_2$ and $2 \log^2 w \cdot x -$

$4\log^2 \cdot w = -2\log^2 w \cdot x \in \ell_2$. Therefore we have some parts of the first two diagrams [210] and [012]





We need to know now who are $\bigcirc = \operatorname{Dom} \Omega_{2,\langle 1,0\rangle}$ and $\blacksquare = \bigcirc/\ell_2(\log^2 w)$. To get the first of those spaces we need to know $\Omega_{2,\langle 1,0\rangle}$. Recall from the standard diagram



that if A, B are homogeneous bounded selectors for a and b then

$$W(y,x) = B(y - \Omega_{b,a}x) + Ax$$

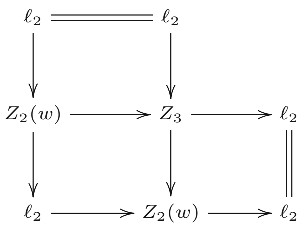
is a selector for (b,a) and therefore $\Omega_{c,(b,a)} = cW$. With this info at hand, we need a selector W for $\langle \Delta_1, \Delta_0\rangle$ to then obtain $\Omega_{2,\langle 1,0\rangle} = \Delta_2 W$. Now, the selector for Δ_0 is $Bx(z) = w^{2z-1}x$ as we already know, and the selector for $\Delta_1 : \ker \delta_0 \to \ell_2$ is $\frac{1}{\varphi'(1/2)}\varphi B$ where φ is a conformal mapping with $\varphi(1/2) = 0$. Thus, $W(y,x) = \frac{\varphi}{\varphi'(1/2)}B(y - \Omega_{1,0}x) + Bx$, and elementary calculations yield

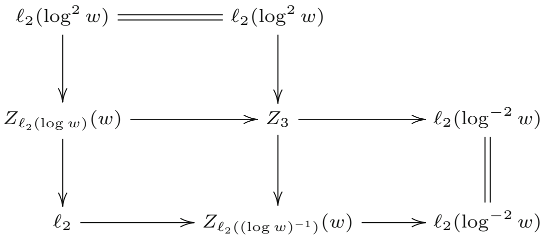
$$\begin{aligned}\Omega_{2,\langle 1,0\rangle}(y,x) = \frac{1}{2}W(y,x)''(1/2) &= \Omega_{1,0}(y - \Omega_{1,0}x)\\ &\quad + \frac{\varphi''(1/2)}{2\varphi'(1/2)}(y - \Omega_{1,0}x) + \frac{1}{2}Bx''(1/2)\\ &= 2\log w \cdot (y - 2\log w \cdot x) + \frac{\varphi''(1/2)}{2\varphi'(1/2)}(y - 2\log w \cdot x) + 2\log^2 w \cdot x.\end{aligned}$$

Setting $d = \frac{\varphi''(1/2)}{2\varphi'(1/2)}$ one gets $\Omega_{2,\langle 1,0\rangle}(y,x) = (2\log w + d)y - (2\log^2 w + 2d\log w)x$. This yields $\operatorname{Dom}\Omega_{2,\langle 1,0\rangle} = \{(y,x) \in Z_2(w) : (2\log w + d)y - (2\log^2 w + 2d\log w)x \in \ell_2\}$ and then $\operatorname{Dom}\Omega_{2,\langle 1,0\rangle}|_{\operatorname{Dom}\Omega_{1,0}} = \{(0,x) \in Z_2(w) : (2\log^2 w + 2d\log w)x \in \ell_2\} = \ell_2(\log^2 w)$. And since $dy - 2d\log wx \in \ell_2$ when $(y,x) \in Z_2(w)$ one gets

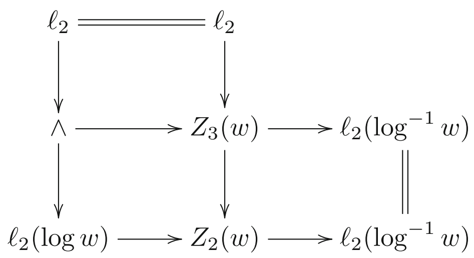
$$\begin{aligned}\bigcirc &= \{(y,x) \in Z_2(w) : (2\log w + d)y - (2\log^2 w + 2d\log w)x \in \ell_2\}\\ &= \{(y,x) \in Z_2(w) : \log wy - \log^2 wx \in \ell_2\}\\ &= \{(y,x) \in Z_2(w) : \log w(y - \log wx) \in \ell_2\}\\ &= \{(y,x) : x \in \ell_2 \quad \text{and} \quad y - \log wx \in \ell_2(\log w)\}.\end{aligned}$$

By obvious reasons we will call this space $\bigcirc = Z_{\ell_2(\log w)}(w)$. It is clear that $\bigcirc$ is a twisting $0 \longrightarrow \ell_2(\log w) \longrightarrow Z_{\ell_2(\log w)}(w) \longrightarrow \ell_2(\log w) \longrightarrow 0$ of $\ell_2(\log w)$ obtained with the *same* quasilinear map $\Omega x = 2\log wx$. This is a bonus effect of working with weighted spaces in which all maps are linear. On the other hand, $\blacksquare$ is the domain of $\Delta_2\Omega^{-1}_{2,\langle 1,0\rangle}$. We showed in Proposition 3.6 that $\Delta_0(\ker\Delta_2) = \Delta_0(\ker\Delta_1) \Longrightarrow \Delta_1(\ker\Delta_2) = \Delta_1(\ker\Delta_0)$, which in this case yields $\mathrm{Dom}\,(\Omega) = \ell_2(\log w) \Longrightarrow \blacksquare = \ell_2$. Thus, giving the analogous meaning as before to the space $Z_{\ell_2((\log w)^{-1})}(w)$, diagrams [210] and [012] are

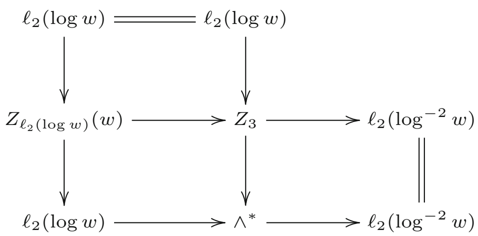


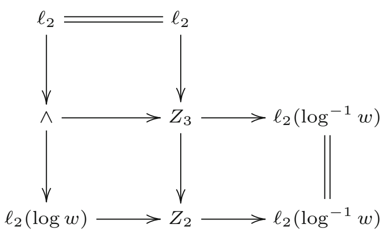


The other relevant new space appears in [201]



that we can identify as the pullback space $\wedge = \{(y, 0, x) \in Z_3\}$ generated with the map $\Omega_{2,\langle 1,0\rangle}|_{\mathrm{Dom}\,\Omega_{1,0}} x = -(2\log^2 w + 2d\log w)x$. We thus get that [102] and [201] are





The vertical sequence on the left is defined by $\Omega x = 2\log wx$ because this is the derivation associated to the interpolation couple $\big(\ell_2(w^{-1}\log w), \ell_2(w \log w)\big)_{1/2} = \ell_2(\log w)$. Since $\mathrm{Dom}\,\Omega = \{x \in \ell_2(\log w) : \log wx \in \ell_2(\log w)\} = \{x \in \ell_2(\log w) : \log^2 wx \in \ell_2\} = \ell_2(\log^2 w)$ one gets that [021] and [120] are

$$
\begin{array}{ccccc}
\ell_2(\log^2 w) & = & \ell_2(\log^2 w) & & \\
\downarrow & & \downarrow & & \\
\wedge & \longrightarrow & Z_3 & \longrightarrow & \ell_2(\log^{-1} w) \\
\downarrow & & \downarrow & & \Vert \\
\ell_2(\log^{-1} w) & \longrightarrow & \bigcirc^* & \longrightarrow & \ell_2(\log^{-1} w)
\end{array}
\qquad
\begin{array}{ccccc}
\ell_f & = & \ell_f & & \\
\downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ell_2 \\
\downarrow & & \downarrow & & \Vert \\
\ell_2(\log^{-1} w) & \longrightarrow & \wedge^* & \longrightarrow & \ell_2
\end{array}
$$

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Declarations

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

References

[1] Androulakis, G., Cazacu, C.D., Kalton, N.J.: Twisted sums, Fenchel-Orlicz spaces and property (M). Houston J. Math. **24**, 105–126 (1998)

[2] Argyros, S.A., Raikoftsalis, T.: The cofinal property of the reflexive indecomposable Banach spaces. Ann. Inst. Fourier (Grenoble) **62**, 1–45 (2012)

[3] Bergh, J., Löfström, J.: Interpolation spaces. Springer, An introduction (1976)

[4] Sánchez, F.C.: Nonlinear centralizers with values in L_0. Nonlinear Anal. **88**, 42–50 (2013)

[5] Cabello Sánchez, F., Castillo, J.M.F.: Homological methods in Banach space theory, Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, London (2023)

[6] Cabello Sánchez, F., Castillo, J.M.F., Corrêa, W.H.G.: Higher order derivatives of analytic families of Banach spaces. Studia Math. **272**, 245–297 (2023)

[7] Cabello Sánchez, F., Castillo, J.M.F., Kalton, N.J.: Complex interpolation and twisted twisted Hilbert spaces, Pacific. J. Math. **276**, 287–307 (2015)

[8] Carro, M.J., Cerdà, J., Soria, F.: Commutators and interpolation methods. Arkiv. Mat. **33**, 199–216 (1995)

[9] Casazza, P.G.: *Approximation properties,* Chapter 7 in Handbook of the Geometry of Banach spaces vol. I, W.B. Johnson and J. Lindenstrauss, eds. pp. 1131–1175. North-Holland 92001)

[10] Castillo, J.M.F., Corrêa, W.H.G., Ferenczi, V., González, M.: On the stability of the differential process generated by complex interpolation. J. Inst. Math. Jussieu **21**, 303–334 (2022)

[11] Castillo, J.M.F., Corrêa, W.H.G., Ferenczi, V., González, M.: *Differential processes generated by two interpolators*, RACSAM 114, paper 183 (2020)

[12] Castillo, J.M.F., Ferenczi, V.: Group actions on twisted sums of Banach spaces. Bull. Malaysian Math. Soc. **46**(4), 135 (2023)

[13] Castillo, J.M.F., Ferenczi, V., González, M.: Singular exact sequences generated by complex interpolation. Trans. Amer. Math. Soc. **369**, 4671–4708 (2017)

[14] Castillo, J.M.F., González, M.: *Three-space problems in Banach space theory,* Lecture Notes in Math. 1667. Springer, (1997)

[15] Castillo, J.M.F., González, M.: *Quasilinear duality and inversion in Banach spaces*, Proc. Roy Soc Edinburgh (to appear)

[16] Corrêa, W.H.G.: Complex interpolation of Orlicz sequence spaces and its higher order Rochberg spaces. Houston J. Math **48**, 111–124 (2022)

[17] Cwikel, M., Jawerth, B., Milman, M., Rochberg, R.: Differential estimates and commutators in interpolation theory. In: Berkson, E.R., Peck, N.T., Uhl, J.J. (eds.) "Analysis at Urbana II", London Mathematical Society. Lecture Notes Series, vol. 138, pp. 170–220. Cambridge Univ. Press, London (1989)

[18] Cwikel, M., Kalton, N.J., Milman, M., Rochberg, R.: A unified theory of Commutator Estimates for a class of interpolation methods. Adv. in Math. **169**, 241–312 (2002)

[19] Diestel, J., Jarchow, H., Tonge, A.: Absolutely summing operators. Cambridge Univ, Press (1995)

[20] Kalton, N.J.: The three-space problem for locally bounded F-spaces. Compositio Math. **37**, 243–276 (1978)

[21] Kalton, N.J.: *The space Z_2 viewed as a symplectic Banach space,* Proc. Research Workshop on Banach space theory (1981) Bohr-Luh Lin ed., Univ. of Iowa, 97–111 (1982)

[22] Kalton, N.J.: *Nonlinear commutators in interpolation theory,* Memoirs Amer. Math. Soc. 385. A.M.S. (1988)

[23] Kalton, N.J., Montgomery-Smith, S.: *Interpolation of Banach spaces,* Chapter 36 in Handbook of the Geometry of Banach spaces vol. II, W.B. Johnson and J. Lindenstrauss, eds. pp. 1131–1175. North-Holland (2003)

[24] Kalton, N.J., Peck, N.T.: Twisted sums of sequence spaces and the three space problem. Trans. Amer. Math. Soc. **255**, 1–30 (1979)

[25] Katirtzoglou, E.: Type and cotype of Musielak-Orlicz sequence spaces. J. Math. Anal. Appl. **226**, 431–455 (1998)

[26] Lindenstrauss, J., Tzafriri, L.: *Classical Banach spaces*, Lecture Notes in Math. 338. Springer, (1973)

[27] Lindenstrauss, J., Tzafriri, L.: *Classical Banach spaces I, sequence spaces*, Springer, (1977)

[28] Pełczyński, A., Rosenthal, H.P.: Localization techniques in L^p spaces. Studia Math. **52**, 263–289 (1975)

[29] Rochberg, R.: Higher order estimates in complex interpolation theory. Pacific J. Math. **174**, 247–267 (1996)

[30] Turett, B.: Fenchel-Orlicz spaces. Dissertationes Math. (Rozprawy Mat.) **181**, 1–55 (1980)

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