Transposed Poisson Structures

Patrícia Damas Beites, Bruno Leonardo Macedo Ferreira, and
Ivan Kaygorodov

Abstract. To present a survey on known results from the theory of transposed Poisson algebras, as well as to establish new results on this subject, are the main aims of the present paper. Furthermore, a list of open questions for future research is given.

Mathematics Subject Classification. 17A30, 17B40, 17B61, 17B63.

Keywords. Lie algebra, transposed Poisson algebra, Hom–Lie algebra, δ-derivation.

Introduction

Poisson algebras arose from the study of Poisson geometry in the 1970s and have appeared in an extremely wide range of areas in mathematics and physics, such as Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics. The study of Poisson algebras also led to other algebraic structures, such as noncommutative Poisson algebras [80], generic Poisson algebras [39,41], Poisson bialgebras [50,66], algebras of Jordan brackets and generalized Poisson algebras [8,9,27,30,61,90], Gerstenhaber algebras [46], F-manifold algebras [12], Novikov-Poisson algebras [81], quasi-Poisson algebras [7], double Poisson algebras [75], Poisson n-Lie algebras [10], etc. The study of all possible Poisson structures with a certain Lie or associative part is an important problem in the theory of Poisson algebras

[1,31,83]. Recently, a dual notion of the Poisson algebra (transposed Poisson algebra), by exchanging the roles of the two binary operations in the Leibniz rule defining the Poisson algebra, has been introduced in the paper [4] of Bai, Bai, Guo, and Wu. They have shown that the transposed Poisson algebra defined in this way not only shares common properties with the Poisson algebra, including the closure undertaking tensor products and the Koszul self-duality as an operad but also admits a rich class of identities. More significantly, a transposed Poisson algebra naturally arises from a Novikov-Poisson algebra by taking the commutator Lie algebra of the Novikov algebra. Consequently, the classic construction of a Poisson algebra from a commutative associative algebra with a pair of commuting derivations has a similar construction to a transposed Poisson algebra when there is one derivation. More broadly, the transposed Poisson algebra also captures the algebraic structures when the commutator is taken in pre-Lie Poisson algebras and to other Poisson-type algebras. The Hom- and BiHom-versions of transposed Poisson algebras are considered in [47,53]. A bialgebra theory for transposed Poisson algebras is studied in [49]. Some new examples of transposed Poisson algebras are constructed by applying the Kantor product of multiplications on the same vector space [17]. More recently, in a paper by Ferreira, Kaygorodov and Lopatkin, a relation between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras has been established [20]. These ideas were used for describing all transposed Poisson structures on the Witt algebra [20], the Virasoro algebra [20], the algebra $\mathcal{W}(a,b)$ [20], the thin Lie algebra [20], the twisted Heisenberg-Virasoro algebra [84], the Schrodinger-Virasoro algebra [84], the extended Schrodinger-Virasoro algebra [84], the 3-dimensional Heisenberg Lie algebra [84], Block Lie algebras and superalgebras [32], Witt type algebras [33], generalized Witt Lie algebras [34], Lie algebra of upper triangular matrices [35], Lie incidence algebras [36], Schrodinger algebra in $(n+1)$-dimensional space-time [82], and so on. The complete algebraic and geometric classification of complex 3-dimensional transposed Poisson algebras is given in [6].

The structure of the present article is as follows. In Sect. 1, we discuss some definitions and previous results, give some effective methods for describing transposed Poisson structures on fixed Lie algebras and apply these methods to some special types of Lie algebras. In Sect. 2, we establish many relations between transposed Poisson algebras and other well-known structures, such as generalized Poisson algebras, Gelfand-Dorfman algebras, F-manifold algebras, algebras of Jordan brackets, fields of fractions, Poisson n-Lie algebras, etc. In Sect. 3, we give a list of open questions for future research.

1. Transposed Poisson Algebras and Structures

Although all algebras and vector spaces are considered over the complex field, many results can be similarly proven for other fields. The definition of transposed Poisson algebra was given in a paper by Bai, Bai, Guo, and Wu [4].

The concept of $\frac{1}{2}$-derivation as a particular case of the notion of δ-derivation was presented in a paper due to Filippov [21] (see also [29, 89] and references therein).

Definition 1 (see [4]). Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot, \cdot]$. The triple $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ is called a transposed Poisson algebra if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [\cdot, \cdot])$ is a Lie algebra that satisfies the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

Definition 2. Let $(\mathfrak{L}, [\cdot, \cdot])$ be a Lie algebra. A transposed Poisson (algebra) structure on $(\mathfrak{L}, [\cdot, \cdot])$ is a commutative associative operation $\cdot$ on $\mathfrak{L}$ which makes $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ a transposed Poisson algebra. A transposed Poisson structure $\cdot$ on $\mathfrak{L}$ is called trivial, if $x \cdot y = 0$ for all $x, y \in \mathfrak{L}$.

Definition 3. Let $(\mathfrak{L}, [\cdot, \cdot])$ be an algebra with a multiplication $[\cdot, \cdot]$. Let φ be a linear map, and let ϕ be a bilinear map. Then φ is a $\frac{1}{2}$-derivation if it satisfies

$$\varphi[x, y] = \tfrac{1}{2}\big([\varphi(x), y] + [x, \varphi(y)]\big);$$

ϕ is a $\frac{1}{2}$-biderivation if it satisfies

$$\phi([x, y], z) = \frac{1}{2}\big([\phi(x, z), y] + [x, \phi(y, z)]\big),$$
$$\phi(x, [y, z]) = \frac{1}{2}\big([\phi(x, y), z] + [y, \phi(x, z)]\big).$$

Summarizing Definitions 1 and 3, we have the following key Lemma.

Lemma 4. *Let $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ be a transposed Poisson algebra, and let z be an arbitrary element from $\mathfrak{L}$. Then the right multiplication R_z in the commutative associative algebra $(\mathfrak{L}, \cdot)$ gives a $\frac{1}{2}$-derivation of the Lie algebra $(\mathfrak{L}, [\cdot, \cdot])$, and the bilinear map $\cdot$ gives a $\frac{1}{2}$-biderivation of $(\mathfrak{L}, [\cdot, \cdot])$.*

The main example of $\frac{1}{2}$-derivations is the multiplication by an element from the ground field. Let us call trivial $\frac{1}{2}$-derivations to such $\frac{1}{2}$-derivations. As a consequence of the following theorem, we are not interested in trivial $\frac{1}{2}$-derivations.

Theorem 5 (see [20]). *Let $\mathfrak{L}$ be a Lie algebra without non-trivial $\frac{1}{2}$-derivations. Then every transposed Poisson structure defined on $\mathfrak{L}$ is trivial.*

1.1. Unital Transposed Poisson Algebras

The present subsection gives a way to relate unital transposed Poisson algebras to generalized Poisson algebras, algebras of Jordan brackets, and quasi-Poisson algebras (see Sect. 2). Some theorems known for these types of algebras can afford interesting results and conjectures for the theory of transposed Poisson algebras.

Theorem 6 (see, [4]). *Let* $(\mathbf{A}, \cdot, [\cdot, \cdot])$ *be a complex transposed Poisson algebra. Let* $(\mathbf{A}, \cdot)$ *be a unital associative (not necessarily commutative) algebra. Then there is a derivation* $\mathfrak{D}$ *on* $(\mathbf{A}, \cdot)$ *such that*

$$[x, y] = \mathfrak{D}(x) \cdot y - x \cdot \mathfrak{D}(y). \tag{1}$$

In fact, each complex finite-dimensional semisimple commutative associative algebra is a direct sum of one-dimensional ideals. On the other hand, it is known that all derivations of this algebra are trivial and it is unital. Hence, all transposed Poisson structures on a complex finite-dimensional semisimple commutative associative algebra $(\mathbf{A}, \cdot)$ are trivial.

1.2. Transposed Poisson Structures on Nilpotent n-Lie Algebras

Let us recall the definition of transposed Poisson structures on n-Lie algebras.

Definition 7. Let $\mathfrak{L}$ be a vector space equipped with an anticommutative n-linear operation $[\cdot, \ldots, \cdot]$. Then $(\mathfrak{L}, [\cdot, \ldots, \cdot])$ is called an n-Lie algebra if

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Definition 8 (see, [4]). Let $\mathfrak{L}$ be a vector space equipped with two nonzero operations – a bilinear operation $\cdot$ and an n-linear operation $[\cdot, \ldots, \cdot]$. The triple $(\mathfrak{L}, \cdot, [\cdot, \ldots, \cdot])$ is called a transposed Poisson n-Lie algebra if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [\cdot, \ldots, \cdot])$ is an n-Lie algebra that satisfies the following compatibility condition

$$z \cdot [x_1, \ldots, x_n] = \frac{1}{n} \sum_{i=1}^{n} [x_1, \ldots, z \cdot x_i, \ldots, x_n]. \tag{2}$$

Theorem 9. *Let* $\mathfrak{L}$ *be a nilpotent k-dimensional n-Lie algebra $(n < k)$. Then $\mathfrak{L}$ admits a non-trivial transposed Poisson n-Lie structure.*

Proof. In fact, choose the basis $\{e_1, e_2, \ldots, e_k\}$ of $\mathfrak{L}$ in the following way: $e_1, e_2, \ldots, e_n \in \mathfrak{L} \backslash \mathfrak{L}^2$ and $e_k \in \mathrm{Ann}(\mathfrak{L})$. It is easy to see that the linear mapping φ given by $\varphi(e_1) = \varphi(e_2) = \ldots = \varphi(e_n) = e_k$ is a $\frac{1}{n}$-derivation of $\mathfrak{L}$. Then the commutative associative multiplication $\cdot$ given by $e_i \cdot e_j = e_k \ (1 \leq i, j \leq n)$ gives a non-trivial structure of transposed Poisson n-Lie algebra. $\qquad\square$

1.3. Transposed Poisson Structures and Quasi-Automorphisms

Let us recall the definition of quasi-automorphisms on Lie algebras.

Definition 10. Let $(\mathfrak{L}, [\cdot, \cdot])$ be a Lie algebra, and let φ be a linear map. Then φ is a quasi-automorphism on $(\mathfrak{L}, [\cdot, \cdot])$ if there exist another linear map ϕ, such that

$$[\varphi(x), \varphi(y)] = \phi[x, y].$$

Thanks to [4, Proposition 2.9], each transposed Poisson structure $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ introduces a quasi-automorphism φ_h on $(\mathfrak{L}, [\cdot, \cdot])$, such that

$\varphi_h^2[x, y] = [\varphi_h(x), \varphi_h(y)]$, where $\varphi_h(x) = h \cdot x$ for a fixed element h.

Automorphisms and scalar multiplication maps are trivial quasi-automorphisms. Hence, we have the following statement.

Proposition 11. *If a Lie algebra* $(\mathfrak{L}, [\cdot, \cdot])$ *does not admit non-trivial quasi-automorphisms, then* $(\mathfrak{L}, [\cdot, \cdot])$ *does not admit non-trivial transposed Poisson structures.*

Zhou, Feng, Kong, Wang, and Chen gave the classification of quasi-automorphisms of Borel and parabolic subalgebras of a complex finite-dimensional simple Lie algebra of rank l. Namely, they proved that in the case $l > 1$, each quasi-automorphism is trivial [77, 88]. Summarizing with Proposition 11, we have the following statement.

Lemma 12. *Let* $\mathfrak{L}$ *be a complex finite-dimensional simple Lie algebra of rank* $l > 1$, *and let* $\mathfrak{p}$ *be an arbitrary Borel or parabolic subalgebra of* $\mathfrak{L}$. *Then* $\mathfrak{p}$ *does not admit non-trivial transposed Poisson structures.*

1.4. Transposed Poisson Structures and Hom–Lie Algebra Structures

Let us recall the definition of Hom-structures on Lie algebras.

Definition 13. Let $(\mathfrak{L}, [\cdot, \cdot])$ be a Lie algebra, and let φ be a linear map. Then $(\mathfrak{L}, [\cdot, \cdot], \varphi)$ is a Hom–Lie structure on $(\mathfrak{L}, [\cdot, \cdot])$ if

$$[\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0.$$

Let us give some special cases of Hom–Lie structures.

- $(\mathfrak{L}, [\cdot, \cdot], \varphi)$ is a trivial Hom–Lie structure on $(\mathfrak{L}, [\cdot, \cdot])$ if $\varphi = \mathbb{C} \cdot \mathrm{id}$.
- $(\mathfrak{L}, [\cdot, \cdot], \varphi)$ is a central Hom–Lie structure on $(\mathfrak{L}, [\cdot, \cdot])$ if $\varphi(\mathfrak{L}) \subseteq Z(\mathfrak{L})$.
- $(\mathfrak{L}, [\cdot, \cdot], \varphi)$ is a $\mathfrak{c}$-trivial Hom–Lie structure on $(\mathfrak{L}, [\cdot, \cdot])$ if φ is a linear combination of trivial and central Hom–Lie structures.

Thanks to [4, Proposition 2.9], each transposed Poisson structure $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ introduces a Hom–Lie structure φ_h on $(\mathfrak{L}, [\cdot, \cdot])$, such that $\varphi_h(x) = h \cdot x$ for a fixed element h. Hence, we have the following statement.

Proposition 14. *If a Lie algebra* $(\mathfrak{L}, [\cdot, \cdot])$ *does not admit non-trivial* Hom–Lie *structures, then* $(\mathfrak{L}, [\cdot, \cdot])$ *does not admit non-trivial transposed Poisson structures.*

The subsequent list of algebras gives algebras without non-trivial Hom–Lie structures (known until now):

1. complex finite-dimensional simple Lie algebras (dim $\neq 3$) and superalgebras [25, 78, 85];
2. infinite-dimensional Lie superalgebras of vector fields [70, 86];
3. infinite-dimensional Cartan algebras $W(n > 1)$, $S(n)$, $H(2r)$, $K(2r + 1)$ [79].

Hence, using Proposition 14, we have the following statement.

Lemma 15. *Complex finite-dimensional simple Lie algebras (*$\dim \neq 3$*) and superalgebras, infinite-dimensional Lie superalgebras of vector fields, and infinite-dimensional Cartan algebras $W(n > 1)$, $S(n)$, $H(2r)$, $K(2r + 1)$ do not admit non-trivial transposed Poisson structures.*

Let us give one more useful trivial observation.

Proposition 16. *If a Lie algebra $(\mathfrak{L}, [\cdot, \cdot])$ with one-dimensional center $Z(\mathfrak{L})$ admits only $\mathfrak{c}$-trivial Hom–Lie structures, then $(\mathfrak{L}, [\cdot, \cdot])$ does not admit non-trivial transposed Poisson structures.*

By [59], all Hom–Lie structures on affine Kac-Moody algebras are $\mathfrak{c}$-trivial. Hence, summarizing Proposition 16 and results from [59], we have the following statement.

Lemma 17. *Complex affine Kac-Moody algebras do not admit non-trivial transposed Poisson structures.*

1.5. Transposed Poisson Structures on Simple Graded Lie Algebras of Finite Growth

Let us recall the definition of a complex simple graded Lie algebra of finite growth. A complex graded Lie algebra is a Lie algebra $\mathfrak{L}$ endowed with a decomposition $\mathfrak{L} = \bigoplus \mathfrak{L}_i$, such that $[\mathfrak{L}_i, \mathfrak{L}_j] \subseteq \mathfrak{L}_{i+j}$ for every $i, j \in \mathbb{Z}$. We always assume that every homogeneous component $\mathfrak{L}_i$ is of finite dimension. A graded Lie algebra $\mathfrak{L}$ is called simple graded if $\mathfrak{L}$ is not abelian and does not contain any non-trivial graded ideal. We say that $\mathfrak{L}$ is of finite growth if the function $n \to \dim \mathfrak{L}_n$ is bounded by some polynomial. In [62], Mathieu proved the following theorem, which was conjectured by Kac.

Theorem 18. *Let $\mathfrak{L}$ be a graded simple Lie algebra of finite growth. Then $\mathfrak{L}$ is isomorphic to one of the following Lie algebras:*

1. *a finite-dimensional simple Lie algebra;*
2. *a loop algebra;*
3. *a Cartan algebra;*
4. *the Witt algebra* W.

Xie and Liu described all Hom–Lie structures on simple graded Lie algebras of finite growth [79]. Summarizing results in [62] and [79], it is possible to prove the following theorem.

Theorem 19. *Let $\mathfrak{L}$ be a complex graded simple Lie algebra of finite growth. If $\mathfrak{L}$ admits a non-trivial Transposed Poisson structure, then it is isomorphic to one of the following Lie algebras:*

1. *the Witt algebra* W, *or*
2. *the Cartan algebra* W(1).

Proof. In fact, thanks to [79], we know that only a loop algebra, the Witt algebra W and the Cartan algebra W(1) have non-trivial Hom–Lie structures.

- It is easy to see (from the description of Hom–Lie structures), that a loop algebra does not admit non-trivial Transposed Poisson structures.
- All non-trivial transposed Poisson structures on the Witt algebra are described in [20, Theorem 20].
- The Cartan algebra W(1) admits the following multiplication table

$$[e_i, e_j] = (i - j)e_{i+j}, \quad i, j \geq -1.$$

Using the same idea that was used for the description of transposed Poisson structures on the Witt algebra [20, Theorem 20], we have that each transposed Poisson structure on W(1) is given by the following additional multiplication

$$e_i \cdot e_j = \sum_{t>0} \alpha_t e_{i+j+t},$$

for a finite set $\{\alpha_t\}$. On the other hand, each finite set $\{\alpha_t\}$ provides a structure of transposed Poisson algebra by the given way.

$\square$

Remark 20. Let us note that the Cartan algebra W(1) gives an example of a simple Lie algebra that admits only non-unital transposed Poisson structures.

2. Relations with Other Algebras

2.1. Quasi-Poisson Algebras

Recently, Billig defined quasi-Poisson algebras [7]. Namely,

Definition 21 (see [7]). Let $(\mathbf{A}, \cdot)$ be a unital commutative associative algebra and $(\mathbf{A}, \{\cdot, \cdot\})$ be a Lie algebra. $(\mathbf{A}, \cdot, \{\cdot, \cdot\})$ with a derivation $\mathfrak{D}$ of $(\mathbf{A}, \cdot)$ is called a quasi-Poisson algebra if

$$a \cdot \big(\mathfrak{D}(\{b, c\}) + \{b, c\}\big) = \{a \cdot (\mathfrak{D}(b) + b), c\} + \{b, a \cdot (\mathfrak{D}(c) + c)\} + \{a, b\} \cdot (\mathfrak{D}(c) + c) - (\mathfrak{D}(b) + b) \cdot \{a, c\}.$$

The main example of quasi-Poisson algebras is given by relation (1) [7, Lemma 6.3].

Corollary 22. *Let $(\mathbf{A}, \cdot, [\cdot, \cdot])$ be a unital transposed Poisson algebra. Then $(\mathbf{A}, \cdot, [\cdot, \cdot])$ is a quasi-Poisson algebra with the derivation $\mathfrak{D}$ obtained by the method from Theorem 6.*

2.2. Generalized Poisson Algebras and Algebras of Jordan Brackets

Let us now recall the definition of generalized Poisson bracket and Jordan brackets [8,30].

Definition 23. Let $(\mathbf{A}, \cdot)$ be a unital commutative associative algebra.

- An anticommutative product $\{\cdot, \cdot\} : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ is called a generalized Poisson bracket (it is also known as the contact bracket [90]) if $(\mathbf{A}, \{\cdot, \cdot\})$ is a Lie algebra and

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\} - \{x, 1\} \cdot y \cdot z.$$

- An anticommutative product $\{\cdot, \cdot\} : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ is called a Jordan bracket (see, [30]) if

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\} - \{x, 1\} \cdot y \cdot z,$$
$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\}$$
$$= \{x, 1\} \cdot \{y, z\} + \{y, 1\} \cdot \{z, x\} + \{z, 1\} \cdot \{x, y\}.$$

The main example of generalized Poisson brackets and Jordan brackets is given by relation (1). Hence, we have the following corollary.

Corollary 24. *Let $(\mathbf{A}, \cdot, [\cdot, \cdot])$ be a unital transposed Poisson algebra. Then $(\mathbf{A}, \cdot, [\cdot, \cdot])$ is a generalized Poisson algebra and an algebra of Jordan brackets.*

Martínez, Shestakov, and Zelmanov have shown that all Jordan brackets are embeddable into Poisson brackets [61]. In this sense, summarizing Corollary 24 and results in [61], we have the following statement.

Corollary 25. *Unital transposed Poisson brackets are embeddable into Poisson brackets.*

2.3. Algebras of Jordan Brackets

Let $(\mathbf{A}, \cdot)$ be a commutative associative algebra, and let $(\mathbf{A}, \{\cdot, \cdot\})$ be an anticommutative algebra. $(\mathbf{A}, \cdot, \{\cdot, \cdot\})$ is called an algebra of Jordan brackets if the Kantor double of $(\mathbf{A}, \cdot, \{\cdot, \cdot\})$ is a Jordan superalgebra [27]. If $(\mathbf{A}, \cdot)$ is not a unital algebra, then the conditions of Jordan brackets are equivalent to (see [28]):

$$\{\{x, y\} \cdot z, t\} + \{\{y, t\} \cdot z, x\} + \{\{t, x\} \cdot z, y\}$$
$$= \{x, y\} \cdot \{z, t\} + \{y, t\} \cdot \{z, x\} + \{t, x\} \cdot \{z, y\},$$
$$\{y \cdot t, z\} \cdot x + \{x, z\} \cdot y \cdot t = \{t \cdot x, z\} \cdot y + \{y, z\} \cdot t \cdot x,$$
$$\{t \cdot x, y \cdot z\} + \{t \cdot y, x \cdot z\} + \{x \cdot y \cdot z, t\} = \{t \cdot y, z\} \cdot x + \{t \cdot x, z\} \cdot y$$
$$+ x \cdot y \cdot \{z, t\}.$$

It is easy to see (for instance, using the identities from [4, Theorem 2.5]) that each transposed Poisson algebra is an algebra of Jordan brackets (see also

[18]). Let us recall that Poisson algebras have the same property. Hence, the variety of algebras of Jordan brackets is the first "interesting" (i.e., well-known) variety of algebras which includes all Poisson and all transposed Poisson algebras.

Corollary 26. *The Kantor double of a transposed Poisson algebra is a Jordan superalgebra.*

2.4. Gelfand–Dorfman Algebras (GD-algebras) and F-Manifold Algebras

Definition 27. A vector space $\mathbf{A}$ with a bilinear product $\cdot$ is called a Novikov algebra if

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z),$$
$$(x \cdot y) \cdot z = (x \cdot z) \cdot y.$$

Definition 28. Let $(\mathbf{A}, \cdot)$ be a Novikov algebra, and let $(\mathbf{A}, [\cdot, \cdot])$ be a Lie algebra. $(\mathbf{A}, \cdot, [\cdot, \cdot])$ is called a Gelfand-Dorfmann algebra if

$$[x, y \cdot z] - [z, y \cdot x] + [y, x] \cdot z - [y, z] \cdot x - y \cdot [x, z] = 0.$$

Definition 29. Let $(\mathbf{A}, \cdot)$ be a commutative associative algebra, and let $(\mathbf{A}, [\cdot, \cdot])$ be a Lie algebra. $(\mathbf{A}, \cdot, [\cdot, \cdot])$ is called an F-manifold algebra if

$$[x \cdot y, z \cdot t] = [x \cdot y, z] \cdot t + [x \cdot y, t] \cdot z + x \cdot [y, z \cdot t] + y \cdot [x, z \cdot t] - \\ \big(x \cdot z \cdot [y, t] + y \cdot z \cdot [x, t] + y \cdot t \cdot [x, z] + x \cdot t \cdot [y, z]\big).$$

Recently, Sartayev proved that the variety of transposed Poisson algebras coincides with the variety of commutative GD-algebras (i.e., $\cdot$ is commutative) and each commutative GD-algebra is an F-manifold algebra [68]. Let us note that each Poisson algebra is also an F-manifold algebra. Hence, the variety of F-manifold algebras gives the second "interesting" variety of algebras which includes all Poisson and all transposed Poisson algebras.

2.5. Transposed Poisson PI Algebras

The celebrated Amitsur–Levitsky theorem states that the algebra $\mathrm{Mat}_k(R)$ of $k \times k$ matrices over a commutative ring R satisfies the identity $s_{2k} = 0$, where s_k is the standard polynomial:

$$s_m(x_1, \ldots, x_m) = \sum_{\sigma \in \mathbb{S}_m} (-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(m)}.$$

Furthermore, every associative PI algebra satisfies $(s_k)^l = 0$ by Amitsur's theorem. Farkas defined customary polynomials

$$g = \sum_{\sigma \in \mathbb{S}_m} c_\sigma \{x_{\sigma(1)}, x_{\sigma(2)}\} \cdot \ldots \cdot \{x_{\sigma(2i-1)}, x_{\sigma(2i)}\}$$

and proved that every Poisson PI algebra satisfies some customary identity [16]. Farkas' theorem was established for generic Poisson algebras in [41].

Kaygorodov found an analog of customary identities for unital generalized Poisson algebras and unital algebras of Jordan brackets in [30]:

$$\mathfrak{g} = \sum_{i=0}^{[m/2]} \sum_{\sigma \in \mathbb{S}_m} c_{\sigma,i} \langle x_{\sigma(1)}, x_{\sigma(2)} \rangle \cdot \ldots \cdot \langle x_{\sigma(2i-1)}, x_{\sigma(2i)} \rangle \cdot \mathfrak{D}(x_{\sigma(2i+1)}) \cdot \ldots \cdot \mathfrak{D}(x_{\sigma(m)}), \ (3)$$

where $\langle x, y \rangle := \{x, y\} - (\mathfrak{D}(x) \cdot y - x \cdot \mathfrak{D}(y))$ and $\mathfrak{D}$ is the related derivation, i.e., $\mathfrak{D}(x) = \{x, 1\}$.

Theorem 30 (Theorem 13, [30]). *If a unital generalized Poisson algebra* **A** *satisfies a polynomial identity, then* **A** *satisfies a polynomial identity* $\mathfrak{g}$ *of type* (3).

Consequently, we obtain the following statement.

Theorem 31. [1]*If a unital transposed Poisson algebra* **A** *satisfies a polynomial identity, then* **A** *satisfies polynomial identities* $\mathfrak{g}$ *and* $\mathfrak{f}$ *of the following types*

$$\mathfrak{g}(x_1, \ldots, x_m) = \sum_{\sigma \in \mathbb{S}_m} c_\sigma \mathfrak{D}(x_{\sigma(1)}) \cdot \ldots \cdot \mathfrak{D}(x_{\sigma(m)}),$$

where $\mathfrak{D}$ *is a derivation, or*

$$\mathfrak{f}(x_1, \ldots, x_{3m}) = \sum_{\sigma \in \mathbb{S}_m} c_\sigma \prod_{k=1}^m \left([x_{\sigma(i)} \cdot x_{m+2k-1}, x_{m+2k}] \right.$$
$$\left. + [x_{\sigma(i)} \cdot x_{m+2k}, x_{m+2k-1}] \right).$$

Proof. In fact, in our case, $\langle x, y \rangle = 0$ and the identity (3) gives the identity $\mathfrak{g}$ from our statement. Now multiply the identity (3) by $\prod_{k=m+1}^{3m} x_i$ and apply the following relation, which is a consequence of the transposed Poisson identity and the generalized Poisson identity,

$$\mathfrak{D}(x) \cdot y \cdot z = -\tfrac{1}{2} \big([x \cdot y, z] + [x \cdot z, y] \big).$$

Hence, the identity $\mathfrak{g}$ gives the identity $\mathfrak{f}$.   □

---

[1]An analog of the present theorem was proven in [13]. Namely, the authors proved that if a unital transposed Poisson algebra satisfies a $[\cdot, \cdot]$-free polynomial identity (depending only on the multiplication $\cdot$), then this algebra satisfies an identity of the type $\mathfrak{D}(x_1) \cdot \ldots \cdot \mathfrak{D}(x_m) = 0$. The present results motivate the following interesting question. **Open question.** Let **A** be a commutative associative algebra with a derivation $\mathfrak{D}$ satisfying a derivation polynomial identity (i.e., a functional identity with the derivation $\mathfrak{D}$). Is there a special type of identities for **A**?

2.6. Transposed Poisson Fields

Denote by $\mathcal{Q}(\mathfrak{P})$ the field of fractions of a commutative associative algebra $\mathfrak{P}$. The Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{P}$ can be extended (see, for example, [26, 54, 58]) to a Poisson bracket on $\mathcal{Q}(\mathfrak{P})$ and

$$\left\{\frac{a}{b},\frac{c}{d}\right\} = \frac{\{a,c\}\cdot b\cdot d - \{a,d\}\cdot b\cdot c - \{b,c\}\cdot a\cdot d + \{b,d\}\cdot a\cdot c}{b^2\cdot d^2}.$$

Our aim in the present subsection is to define a similar notion to that of the transposed Poisson case.

Theorem 32. *Let* $(\mathfrak{L},\cdot,[\cdot,\cdot])$ *be a unital transposed Poisson algebra. Then* $(\mathcal{Q}(\mathfrak{L}),\bullet,[\![\cdot,\cdot]\!])$, *where*

$$\frac{a}{b}\bullet\frac{c}{d} = \frac{a\cdot c}{b\cdot d} \text{ and } \left[\!\!\left[\frac{a}{b},\frac{c}{d}\right]\!\!\right] = \frac{[a,b]\cdot c\cdot d - a\cdot b\cdot[c,d]}{b^2\cdot d^2}$$

is a transposed Poisson field.

Proof. In fact, it is easy to see that $\bullet$ is a commutative associative multiplication and $[\![\cdot,\cdot]\!]$ is anticommutative. Hence, we should check only two identities to complete the proof (for simplification of our proof the multiplication $\cdot$ will be omitted). Thanks to theorem 6, our bracket $[x,y]$ can be written as $[x,y] = \mathfrak{D}(x)y - x\mathfrak{D}(y)$ for a derivation $\mathfrak{D}$. Hence,

$$\frac{a}{b}\bullet\left[\!\!\left[\frac{c}{d},\frac{e}{f}\right]\!\!\right] - \frac{1}{2}\left(\left[\!\!\left[\frac{a}{b}\bullet\frac{c}{d},\frac{e}{f}\right]\!\!\right] + \left[\!\!\left[\frac{c}{d},\frac{a}{b}\bullet\frac{e}{f}\right]\!\!\right]\right)$$

$$= \frac{[c,d]aef - acd[e,f]}{bd^2f^2} - \frac{[ac,bd]ef - acbd[e,f] + [c,d]aebf - cd[ac,bf]}{2b^2d^2f^2}$$

$$= (2b^2d^2f^2)^{-1}\Big(2ab\mathfrak{D}(c)def - 2abc\mathfrak{D}(d)ef - 2abcd\mathfrak{D}(e)f + 2abcde\mathfrak{D}(f)$$

$$- (\mathfrak{D}(a)bcdef + ab\mathfrak{D}(c)def - a\mathfrak{D}(b)cdef - abc\mathfrak{D}(d)ef)$$

$$+ (abcd\mathfrak{D}(e)f - abcde\mathfrak{D}(f)) - (ab\mathfrak{D}(c)def - abc\mathfrak{D}(d)ef)$$

$$+ (\mathfrak{D}(a)bcdef + ab\mathfrak{D}(c)def - a\mathfrak{D}(b)cdef - abcde\mathfrak{D}(f))\Big) = 0.$$

Now we should check the Jacobi identity. For a function $f(x_1,x_2,y_1,y_2,z_1,z_2)$, we denote

$$f(x_1,x_2,y_1,y_2,z_1,z_2) + f(y_1,y_2,z_1,z_2,x_1,x_2) + f(z_1,z_2,x_1,x_2,y_1,y_2)$$

as $\circlearrowleft f(x_1,x_2,y_1,y_2,z_1,z_2)$.

Hence,

$$\circlearrowleft\left[\!\!\left[\frac{x_1}{x_2},\left[\!\!\left[\frac{y_1}{y_2},\frac{z_1}{z_2}\right]\!\!\right]\right]\!\!\right] = \circlearrowleft\left[\!\!\left[\frac{x_1}{x_2},\frac{[y_1,y_2]z_1z_2 - y_1y_2[z_1,z_2]}{y_2^2z_2^2}\right]\!\!\right]$$

$$= \frac{\circlearrowleft\left([x_1,x_2]([y_1,y_2]z_1z_2 - y_1y_2[z_1,z_2])\right)}{x_2^2y_2^2z_2^2}$$

$$- \frac{\circlearrowleft\left(x_1x_2^3[[y_1,y_2]z_1z_2 - y_1y_2[z_1,z_2],y_2^2z_2^2]\right)}{x_2^4y_2^4z_2^4}.$$

We will prove that each part of the last sum is equal to zero. For the first part of our sum, we have the following.

$$\circlearrowleft \big([x_1, x_2]([y_1, y_2]z_1 z_2 - y_1 y_2[z_1, z_2])\big)$$

$$\circlearrowleft = \big(\mathfrak{D}(x_1)x_2\mathfrak{D}(y_1)y_2 z_1 z_2 - x_1\mathfrak{D}(x_2)\mathfrak{D}(y_1)y_2 z_1 z_2 + \mathfrak{D}(x_1)x_2 y_1 \mathfrak{D}(y_2)z_1 z_2$$
$$-x_1\mathfrak{D}(x_2)y_1\mathfrak{D}(y_2)z_1 z_2 - \mathfrak{D}(x_1)x_2 y_1 y_2 \mathfrak{D}(z_1)z_2 + \mathfrak{D}(x_1)x_2 y_1 y_2 z_1 \mathfrak{D}(z_2)$$
$$-x_1\mathfrak{D}(x_2)y_1 y_2 \mathfrak{D}(z_1)z_2 + x_1\mathfrak{D}(x_2)y_1 y_2 z_1 \mathfrak{D}(z_2)\big) = 0.$$

We rewrite the second part in a sum of two other sums in the following way.

$$\circlearrowleft \big(x_1 x_2^3[[y_1, y_2]z_1 z_2 - y_1 y_2[z_1, z_2], y_2^2 z_2^2]\big) =$$
$$\circlearrowleft \big(x_1 x_2^3[\mathfrak{D}(y_1)y_2 z_1 z_2 - y_1\mathfrak{D}(y_2)z_1 z_2 - y_1 y_2 \mathfrak{D}(z_1)z_2 + y_1 y_2 z_1 \mathfrak{D}(z_2), y_2^2 z_2^2]\big)$$

$$=\circlearrowleft x_1 x_2^3 \big(\quad \mathfrak{D}(\mathfrak{D}(y_1)y_2 z_1 z_2)y_2^2 z_2^2 - \mathfrak{D}(y_1)y_2 z_1 z_2 \mathfrak{D}(y_2^2 z_2^2)$$
$$-\mathfrak{D}(y_1\mathfrak{D}(y_2)z_1 z_2)y_2^2 z_2^2 + y_1\mathfrak{D}(y_2)z_1 z_2 \mathfrak{D}(y_2^2 z_2^2)$$
$$-\mathfrak{D}(y_1 y_2 \mathfrak{D}(z_1)z_2)y_2^2 z_2^2 + y_1 y_2 \mathfrak{D}(z_1)z_2 \mathfrak{D}(y_2^2 z_2^2)$$
$$+\mathfrak{D}(y_1 y_2 z_1 \mathfrak{D}(z_2))y_2^2 z_2^2 - y_1 y_2 z_1 \mathfrak{D}(z_2)\mathfrak{D}(y_2^2 z_2^2) \big).$$

Hence, the first sub-sum is given by the following terms.

$$\circlearrowleft x_1 x_2^3 y_2^2 z_2^2 \big(\mathfrak{D}(\mathfrak{D}(y_1)y_2 z_1 z_2) - \mathfrak{D}(y_1\mathfrak{D}(y_2)z_1 z_2)$$
$$-\mathfrak{D}(y_1 y_2 \mathfrak{D}(z_1)z_2) + \mathfrak{D}(y_1 y_2 z_1 \mathfrak{D}(z_2)))$$

$$=\circlearrowleft x_1 x_2^3 y_2^2 z_2^2 \big(\mathfrak{D}^2(y_1)y_2 z_1 z_2 + \mathfrak{D}(y_1)\mathfrak{D}(y_2)z_1 z_2 + \mathfrak{D}(y_1)y_2\mathfrak{D}(z_1)z_2$$
$$+\mathfrak{D}(y_1)y_2 z_1 \mathfrak{D}(z_2)$$
$$-\mathfrak{D}(y_1)\mathfrak{D}(y_2)z_1 z_2 - y_1\mathfrak{D}^2(y_2)z_1 z_2 - y_1\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2 - y_1\mathfrak{D}(y_2)z_1\mathfrak{D}(z_2)$$
$$-\mathfrak{D}(y_1)y_2\mathfrak{D}(z_1)z_2 - y_1\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2 - y_1 y_2\mathfrak{D}^2(z_1)z_2 - y_1 y_2\mathfrak{D}(z_1)\mathfrak{D}(z_2)$$
$$+\mathfrak{D}(y_1)y_2 z_1\mathfrak{D}(z_2) + y_1\mathfrak{D}(y_2)z_1\mathfrak{D}(z_2) + y_1 y_2\mathfrak{D}(z_1)\mathfrak{D}(z_2) + y_1 y_2 z_1\mathfrak{D}^2(z_2)\big)$$

$$=\circlearrowleft \big(x_1\mathfrak{D}^2(y_1)z_1 - x_1 y_1\mathfrak{D}^2(z_1)\big)(x_2 y_2 z_2)^3$$
$$+ \circlearrowleft \big(x_2\mathfrak{D}^2(y_2)z_2 - x_2 y_2\mathfrak{D}^2(z_2)\big)x_1 y_1 z_1(x_2 y_2 z_2)^2$$
$$+2 \circlearrowleft \big(x_1 x_2\mathfrak{D}(y_1)y_2 z_1\mathfrak{D}(z_2) - x_1 x_2 y_1\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2\big)(x_2 y_2 z_2)^3$$
$$= 2 \circlearrowleft \big(x_1 x_2\mathfrak{D}(y_1)y_2 z_1\mathfrak{D}(z_2) - x_1 x_2 y_1\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2\big)(x_2 y_2 z_2)^2.$$

The second sub-sum is given by the following terms.

$$\circlearrowleft x_1 x_2^3\mathfrak{D}(y_2^2 z_2^2)\big(-\mathfrak{D}(y_1)y_2 z_1 z_2 + y_1\mathfrak{D}(y_2)z_1 z_2 + y_1 y_2\mathfrak{D}(z_1)z_2 - y_1 y_2 z_1\mathfrak{D}(z_2)\big)$$
$$= 2 \circlearrowleft x_1 x_2^3\big(-\mathfrak{D}(y_1)y_2^3\mathfrak{D}(y_2)z_1 z_2^3 - \mathfrak{D}(y_1)y_2^3 z_1 z_2^2\mathfrak{D}(z_2) + y_1 y_2(\mathfrak{D}(y_2))^2 z_1 z_2^3$$
$$+y_1 y_2^2\mathfrak{D}(y_2)z_1 z_2^2\mathfrak{D}(z_2) + y_1 y_2^2\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2^3 + y_1 y_2^3\mathfrak{D}(z_1)z_2^2\mathfrak{D}(z_2)$$
$$-y_1 y_2^2\mathfrak{D}(y_2)z_1 z_2^2\mathfrak{D}(z_2) - y_1 y_2^3 z_1 z_2^2(\mathfrak{D}(z_2))^2\big)$$
$$= 2 \circlearrowleft \big(x_2^2(\mathfrak{D}(y_2))^2 z_2^2 - x_2^2 y_2^2(\mathfrak{D}(z_2))^2\big)x_1 x_2 y_1 y_2 z_1 z_2$$
$$-2 \circlearrowleft \big(x_1 x_2\mathfrak{D}(y_1)\mathfrak{D}(y_2)z_1 z_2 - x_1 x_2 y_1 y_2\mathfrak{D}(z_1)\mathfrak{D}(z_2)\big)(x_2 y_2 z_2)^2$$
$$-2 \circlearrowleft \big(x_1 x_2\mathfrak{D}(y_1)y_2 z_1\mathfrak{D}(z_2) - x_1 x_2 y_1\mathfrak{D}(y_2)\mathfrak{D}(z_1)z_2\big)(x_2 y_2 z_2)^2$$

$$= -2 \circlearrowleft \big(x_1 x_2 \mathfrak{D}(y_1) y_2 z_1 \mathfrak{D}(z_2) - x_1 x_2 y_1 \mathfrak{D}(y_2) \mathfrak{D}(z_1) z_2\big)(x_2 y_2 z_2)^2.$$

To summarize, the sum of these two sub-sums is zero, and the Jacobi identity holds. This finishes the proof. $\qquad\square$

2.7. Transposed Poisson n-Lie Algebras and Poisson n-Lie Algebras

The following result is a direct generalization of a result from [4] for the n-ary case.

Definition 33. Let $\mathfrak{L}$ be a vector space equipped with two nonzero operations – a bilinear operation $\cdot$ and an n-linear operation $[\cdot, \ldots, \cdot]$. The triple $(\mathfrak{L}, \cdot, [\cdot, \ldots, \cdot])$ is called a Poisson n-Lie algebra if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [\cdot, \ldots, \cdot])$ is an n-Lie algebra that satisfies the following compatibility condition

$$[x \cdot y, z_2 \ldots, z_n] = x \cdot [y, z_2, \ldots, z_n] + [x, z_2, \ldots, z_n] \cdot y. \tag{4}$$

Proposition 34. *Let $(\mathfrak{L}, \cdot)$ be a commutative algebra, and let $(\mathfrak{L}, [\cdot, \ldots, \cdot])$ be an n-Lie algebra. Then $(\mathfrak{L}, \cdot, [\cdot, \ldots, \cdot])$ is both a Poisson n-Lie and a transposed Poisson n-Lie algebra if and only if*

$$x \cdot [y_1, \ldots, y_n] = [x \cdot y_1, \ldots, y_n] = 0.$$

Proof. In fact, let $u, x_1, \ldots, x_n \in \mathfrak{L}$. Substituting Eq. (2) into Eq. (4), we have

$$nx \cdot [y_1, \ldots, y_n] = \sum_{i=1}^{n} [y_1, \ldots, x \cdot y_i, \ldots, y_n]$$

$$= \sum_{i=1}^{n} x \cdot [y_1, \ldots, y_n] + \sum_{i=1}^{n} y_i \cdot [y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n].$$

Substituting Eq. (4) again into the above equality, we have

$$0 = \sum_{i=1}^{n} y_i \cdot [y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n]$$

$$= \sum_{i=1}^{n} \left(\sum_{t=1}^{i-1} [y_1, \ldots y_i \cdot y_t, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n] \right)$$

$$+ \sum_{i=1}^{n} [y_1, \ldots y_{i-1}, y_i \cdot x, y_{i+1}, \ldots, y_n]$$

$$+ \sum_{i=1}^{n} \left(\sum_{t=i+1}^{n} [y_1, \ldots y_{i-1}, x, y_{i+1}, \ldots, y_i \cdot y_t, \ldots, y_n] \right)$$

$$= \sum_{i=1}^{n} [y_1, \ldots y_{i-1}, y_i \cdot x, y_{i+1}, \ldots, y_n] = nx \cdot [y_1, \ldots, y_n].$$

Hence $x \cdot [y_1, \ldots, y_n] = 0$ and $[x \cdot y_1, \ldots, y_n] = 0$. $\qquad\square$

3. Open Questions

In what follows, we present a list where we have compiled open questions for future research related to transposed Poisson structures.

Question 1 (Pasha Zusmanovich). Formulate the notion of "transposed" for any variety of algebras/operads with two n-ary operations. Could this be done in either operadic or categorical theoretical language? This general notion should explain the appearance of "scaling by 2" in the definition of transposed Poisson algebras. What would be the transposed analogs of dendriform, post-Lie, Novikov-Poisson, etc.? What would be self-transposed algebras/operads?

Thanks to [20, Theorem 8], we know that if a Lie algebra does not admit non-trivial $\frac{1}{2}$-derivations, then it does not admit non-trivial $\frac{1}{2}$-biderivations and non-trivial transposed Poisson structures. All algebras which admit non-trivial $\frac{1}{2}$-derivations (from the present paper and [20,84]) also admit non-trivial $\frac{1}{2}$-biderivations and non-trivial transposed Poisson structures. The last results motivate the following question.

Question 2. [2]Is there a Lie algebra that admits non-trivial $\frac{1}{2}$-derivations and does not admit non-trivial $\frac{1}{2}$-biderivations? Is there a Lie algebra that admits non-trivial $\frac{1}{2}$-derivations and $\frac{1}{2}$-biderivations, but does not admit non-trivial transposed Poisson structures?

One of the first examples of non-trivial transposed Poisson algebras was given for the Witt algebra [20, Theorem 20]. The central extension of the Witt algebra is the Virasoro algebra and it has no non-trivial $\frac{1}{2}$-derivations, non-trivial $\frac{1}{2}$-biderivations and non-trivial transposed Poisson structures [20, Theorem 27]. Thanks to [79] we know that the Cartan (infinite-dimensional) algebras S_n and H_n do not have non-trivial transposed Poisson structures (and $\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations). It is easy to see that if the dimension of the annihilator of an algebra is greater than 1, then it admits non-trivial transposed Poisson structures (and $\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations). Dzhumadildaev proved that the dimensions of annihilators of central extensions of S_n and H_n are greater than 1 in [14]. It follows that there are infinite-dimensional Lie algebras without non-trivial transposed Poisson structures (and $\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations), but whose central extensions admit these non-trivial structures. The last results motivate the following question.

Question 3. Is there a Lie algebra that does not admit non-trivial transposed Poisson structures ($\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations), but whose (indecomposable) one-dimensional central extension admits them?

---

[2]From [37], there is a 6-dimensional perfect non-semisimple Lie algebra with non-trivial $\frac{1}{2}$-derivations and without non-trivial transposed Poisson algebra structures.

There are no non-trivial transposed Poisson structures defined on a complex finite-dimensional semisimple Lie algebra [20, Corollary 9]. On the other hand, the Schrödinger algebra, which is the semidirect product of a simple and a nilpotent algebra, also does not have non-trivial transposed Poisson structures (other perfect non-simple Lie algebras without non-trivial $\frac{1}{2}$-derivations, also known as Galilean algebras, can be found in [38]). Theorem 9 states that each finite-dimensional nilpotent Lie algebra has a non-trivial transposed Poisson structure ($\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations). Results from [38] state that each finite-dimensional solvable non-nilpotent Lie algebra has a non-trivial transposed Poisson structure ($\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations). The last results motivate the following question.

Question 4. [3, 4]Is there a non-perfect (non-solvable) Lie algebra that does not admit non-trivial transposed Poisson structures ($\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations)? Is there a perfect (non-semisimple) Lie algebra that admits non-trivial transposed Poisson structures ($\frac{1}{2}$-derivations, $\frac{1}{2}$-biderivations)?

The description of $\frac{1}{2}$-derivations gives a good criterion for the detection of Lie algebras that do not admit non-trivial transposed Poisson structures. Another criterion can be given by using Hom–Lie structures on Lie algebras (Proposition 14). However, the last criterion is not sufficient: the simple 3-dimensional algebra $\mathfrak{sl}_2$ has non-trivial Hom–Lie structures [78], but it does not admit non-trivial transposed Poisson structures [20, Corollary 9]. Filippov proved that each δ-derivation ($\delta \neq 0, 1$) gives a non-trivial Hom–Lie algebra structure [21, Theorem 1]. The last results motivate the following question.

Question 5. Describe Lie algebras such that each Hom–Lie algebra structure gives a $\frac{1}{2}$-derivation. Which types of Hom–Lie structures can guarantee the existence of non-trivial transposed Poisson structures on a Lie algebra?

Let us denote the associative algebra generated by $\frac{1}{2}$-derivations of $\mathfrak{L}$ as $\Delta_{\frac{1}{2}}(\mathfrak{L})$. The Witt algebra W has the following interesting property: $\mathfrak{Der}(\Delta_{\frac{1}{2}}(W)) \cong W$ [20, Theorem 19]. The last results motivate the following question.

Question 6. Is there a Lie algebra $\mathfrak{L}$ ($\not\cong W$) such that $\mathfrak{Der}(\Delta_{\frac{1}{2}}(\mathfrak{L})) \cong \mathfrak{L}$?

Thanks to [20, Corollary 9], there are no non-trivial transposed Poisson structures defined on a complex finite-dimensional semisimple Lie algebra; Theorem 6 establishes that there are no non-trivial transposed Poisson structures defined on a complex semisimple commutative associative algebra. On the other hand, there are simple Lie algebras (the Witt algebra W and the Cartan algebra W(1)) that admit non-trivial transposed Poisson structures,

---

[3]From [37], there is a 6-dimensional perfect non-semisimple Lie algebra with non-trivial $\frac{1}{2}$-derivations.

[4]From [82], there is a 9-dimensional non-perfect non-solvable Lie algebra with non-trivial $\frac{1}{2}$-derivations and non-trivial transposed Poisson structures.

but, in these cases, the associative part is not simple (Theorem 19). The last results motivate the following question.

Question 7. Is there a (not necessarily commutative) transposed Poisson algebra $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ such that its Lie and associative parts are simple (semisimple, prime, semiprime) algebras?

Combining results from Theorem 6 and [17, Proposition 28], we have that the Kantor product of a unital transposed Poisson algebra gives a new transposed Poisson algebra. Furthermore, the Kantor product of transposed Poisson algebras constructed on the Witt algebra gives new transposed Poisson algebras [17, Proposition 27]. The last results motivate the following question.

Question 8. Is there a transposed Poisson algebra such that the algebra constructed by the Kantor product of multiplications (see [17]) is not a transposed Poisson algebra?

The Kantor double gives a way for the construction of Jordan superalgebras from Poisson algebras and from algebras of Jordan brackets [27]. As we can see from Corollary 24, each unital transposed Poisson algebra, by the Kantor double process, gives a Jordan superalgebra. Moreover, the Kantor double of the commutator of a Novikov-Poisson algebra is a Jordan superalgebra [87]. Thanks to [18], each transposed Poisson superalgebra under the Kantor double process gives a Jordan superalgebra. The last results motivate the following question.

Question 9. Describe and study the class of superalgebras (= a subvariety of Jordan superalgebras) obtained by the Kantor double process from transposed Poisson algebras.

The celebrated Amitsur-Levitsky theorem states that the algebra Mat_k of $k \times k$ matrices over a commutative ring R satisfies the identity $s_{2k} = 0$. Furthermore, every associative PI algebra satisfies $(s_k)^l = 0$ by Amitsur's theorem. Farkas defined customary Poisson polynomials and proved that every Poisson PI algebra satisfies some customary identity [16]. Recently, the Farkas' theorem was established for generic Poisson algebras in [41], for unital generalized Poisson algebras in [30] and unital transposed Poisson algebras (Theorem 31). The last results motivate the following question.

Question 10. Is there a special type of polynomial identities for (not necessarily unital) transposed Poisson algebras satisfying a polynomial identity?

In 1972, Regev proved that the tensor product of two associative PI algebras is an associative PI algebra. Recently, a similar result was obtained for Poisson algebras [64]. The last results motivate the following question.

Question 11. Is a tensor product of two transposed Poisson PI algebras a transposed Poisson PI algebra?

It is known that the tensor product of two Poisson algebras gives a Poisson algebra [64], the tensor product of two transposed Poisson algebras gives a transposed Poisson algebra [4], and the tensor product of two anti-pre-Lie Poisson algebras gives an anti-pre-Lie Poisson algebra [48]. In the case of the standard multiplication of tensor Poisson algebras, it is not true for F-manifold algebras [51]. On the other hand, in the case of generalized Poisson algebras, their tensor product does not necessarily give a new generalized Poisson algebra [90]. The last results motivate the following question.

Question 12 (Pasha Zusmanovich). Find identities for algebras formed by the tensor product of a transposed Poisson algebra and a Poisson algebra.

As it was mentioned in Sects. 2.3 and 2.4, all Poisson and all transposed Poisson are algebras of Jordan brackets and F-manifold algebras. Hence, Poisson and transposed Poisson algebras are at the intersection of the varieties of F-manifold algebras and algebras of Jordan brackets. The last results motivate the following question.

Question 13. Characterize the variety of algebras obtained as the intersection of F-manifold algebras and algebras of Jordan brackets. Find a minimal variety of algebras that includes Poisson and transposed Poisson algebras.

The notion of a depolarized Poisson algebra (i.e., a binary algebra such that the commutator and the anticommutator products give a Poisson algebra) was introduced and studied in [24,60]. The last results motivate the following question.

Question 14. [5]Define and study the variety of depolarized transposed Poisson algebras.

In 1950, Hall constructed a basis for free Lie algebras. Later on: Shestakov constructed a basis for free Poisson algebras; Shestakov and Zhukavets constructed a basis for free Poisson-Malcev superalgebra with one generator; Kaygorodov, Shestakov, and Umirbaev constructed a basis for free generic Poisson algebras in [39]; and Kaygorodov constructed a basis for free unital generalized Poisson algebras in [30]. The last results motivate the following question.

Question 15. [6]Construct a basis for free transposed Poisson algebras.

In 1930, Magnus proved one of the most important theorems of the combinatorial group theory. Let $G = \langle x_1, x_2, \ldots, x_n \mid r = 1 \rangle$ be a group defined by

---

[5]Askar Dzhumadildaev proved that weak Leibniz algebras give the depolarization of transposed Poisson algebras [15].

[6]Roughly speaking, a basis of a free Poisson algebra is a basis of a polynomial ring on a free Lie algebra. Our first impression (Chengming Bai mentioned, in one of his talks, that it was also the first impression of his team) was that a basis of a free transposed Poisson algebra can be found as a free Lie algebra on a polynomial ring, but this is not true.

a single cyclically reduced relator r. If x_n appears in r, then the subgroup of G generated by $x_1, \ldots, x_{n-1}$ is a free group, freely generated by $x_1, \ldots, x_{n-1}$. He called it the Freiheitssatz and used it to give several applications, the decidability of the word problem for groups with a single defining relation among them. The Freiheitssatz was studied in various varieties of groups and algebras: Shirshov established it for Lie algebras; Romanovskii researched it for solvable and nilpotent groups; Makar-Limanov proved it for associative algebras over a field of characteristic zero. Recently it was confirmed for Novikov and right-symmetric algebras, as well as for Poisson algebras [57] and generic Poisson algebras [41]. The last results motivate the following question.

Question 16. Is the Freiheitssatz true for transposed Poisson algebras?

In 1942, Jung proved that all automorphisms of the polynomial algebra in two variables are tame. Later, in 1970, Makar-Limanov established the same result for the free associative algebra in two variables. Cohn proved that the automorphisms of a free Lie algebra with a finite set of generators are tame. Shestakov and Umirbaev proved that polynomial algebras and free associative algebras in three variables in the case of characteristic zero have wild automorphisms [69,73]. At the same time, Abdykhalykov, Mikhalev, and Umirbaev found wild automorphisms in two generated free Leibniz algebras. It was recently proved that free Poisson and generic Poisson algebras in two variables do not have wild automorphisms [39,55]. The last results motivate the following question.

Question 17. Are the automorphisms of a two-generated free transposed Poisson algebra tame?

Makar-Limanov and Umirbaev [56] proved an analog of the Bergman Centralizer Theorem: the centralizer of every non-constant element in a free Poisson algebra is a polynomial algebra in one variable. The last results motivate the following question.

Question 18 (Zerui Zhang). Describe the centralizer of a non-constant element in a free transposed Poisson algebra.

Thanks to Corollary 24, we know that each unital transposed Poisson algebra is an algebra of Jordan brackets. Martínez, Shestakov, and Zelmanov have shown that all unital Jordan brackets are embeddable into Poisson brackets [61]. Hence, all unital transposed Poisson brackets are embeddable into Poisson brackets. The last results motivate the following question.

Question 19. Are transposed Poisson brackets embeddable into Poisson brackets?

Thanks to a famous relationship between Poisson algebras and Jordan superalgebras, also known as Kantor double, the description of simple Jordan

superalgebras describes simple Poisson algebras. Also recently, simple generalized Poisson algebras and simple generalized Gerstenhaber algebras (odd generalized Poisson superalgebras) have been described [8,9]. The last results motivate the following question.

Question 20. [7]Classify simple transposed Poisson algebras.

The study of some generalizations of Poisson algebras (noncommutative Poisson algebras; Leibniz-Poisson algebras [11]; generic Poisson algebras [39], etc.) motivate the consideration of some generalizations of transposed Poisson algebras.

Question 21. Define and study the following generalizations of transposed Poisson algebras:

- (a) noncommutative transposed Poisson algebras;
- (b) transposed Poisson-Leibniz (in particular, symmetric Leibniz) algebras;
- (c) transposed generic Poisson algebras.

One of the important questions in the study of Poisson and transposed Poisson algebras is a description of (transposed) Poisson algebra structures on a certain Lie algebra or a certain associative algebra. The most useful way for a description of Poisson structures on a certain associative algebra is given by the study of derivations and biderivations of this algebra. On the other hand, a description of transposed Poisson structures on a certain Lie algebra may be given by the study of $\frac{1}{2}$-derivations and $\frac{1}{2}$-biderivations of this algebra. Recently, a constructive method for describing Poisson algebra structures on a certain Lie algebra has been given in a study of Leibniz bialgebras and symmetric Leibniz algebras [1]. The last results and Theorem 6 motivate the following question.

Question 22. Give a constructive method for describing transposed Poisson algebras on a certain non-unital associative (not necessarily commutative) algebra.

It is known that the Poisson bracket admits an extension to the field of fractions [26,39,54,58]. As we proved in Theorem 32, a unital transposed Poisson bracket admits an extension to the field of fractions. The last results motivate the following question.

Question 23. Does the bracket in Theorem 32 give an extension of a transposed Poisson algebra to the field of fractions in the non-unital case?

The universal multiplicative enveloping algebra of free Poisson and free generic Poisson fields were studied in [39,58]. In particular, in these papers, it was proved that the universal multiplicative enveloping algebra is a free ideal ring. The last results motivate the following question.

---

[7]Thanks to [18], there are no non-trivial complex finite-dimensional simple transposed Poisson algebras.

Question 24. Is the universal multiplicative enveloping algebra of a free transposed Poisson field a free ideal ring?

Multiplicative enveloping algebras are often much simpler than the algebras themselves, and in the case where the variety is not obvious, they can help a lot. A basis of the universal enveloping algebra P^e of a free Poisson algebra P was constructed and it was proved that the left dependency of a finite number of elements of P^e over P^e is algorithmically recognizable [74]. The last results motivate the following question.

Question 25 (Vladimir Dotsenko). Construct a basis of the universal enveloping algebra TP^e of a free transposed Poisson algebra TP. Is left dependency of a finite number of elements of TP^e over TP^e algorithmically recognizable?

It is known that each Novikov-Poisson algebra under commutator product on non-associative multiplication gives a transposed Poisson algebra [4]. Let us say that a transposed Poisson algebra is special if it can be embedded into a Novikov-Poisson algebra relative to the commutator bracket. Similarly, let us say that a transposed Poisson algebra is D-special (from "differentially") if it embeds into a commutative algebra with a derivation relative to the bracket $[x, y] = \mathfrak{D}(x)y - x\mathfrak{D}(y)$. Obviously, every D-special transposed Poisson algebra is a special one. It is known too that the class of special Jordan algebras (i.e. embedded into associative algebras relative to the multiplication $x \circ y = xy + yx$) is a quasivariety, but it is not a variety of algebras [72]. On the other hand, the class of all special Gelfand-Dorfman algebras (i.e., embedded into Poisson algebras with derivation relative to the multiplication $x \circ y = xd(y)$) is closed concerning homomorphisms and thus forms a variety [43]. The last results motivate the following question.

Question 26 (Pavel Kolesnikov). Does the class of special (resp., D-special) transposed Poisson algebras form a variety or a quasivariety?

Let us call special identities to all identities that hold on all special transposed Poisson algebras but do not hold on the class of all transposed Poisson algebras. The study of special and non-special identities is a popular topic in non-associative algebras (see, for example, the Jordan algebra case in [23], the Gelfand-Dorfman algebra case in [44], the case of dialgebras in [45], the Jordan trialgebra and post-Jordan algebra cases in [3], etc.). The last results motivate the following question.

Question 27 (Pavel Kolesnikov). Find the identities separating the variety generated by special (resp., D-special) transposed Poisson algebras in the variety of all transposed Poisson algebras. Find the identities separating the variety generated by D-special transposed Poisson algebras in the variety generated by special transposed Poisson algebras.

Also known is that: each two-generated Jordan algebra is special [71]; each one-generated Jordan dialgebra is special [76]; each 2-dimensional Gelfand-Dorfman algebra is special [43]. The last results motivate the following question.

Question 28. Are one-generated transposed Poisson algebras special?

Thanks to [63], all unital transposed Poisson algebras by Kantor double process give special Jordan superalgebras. On the other hand, there are examples of Poisson algebras that do not have special Kantor double [63]. The last results motivate the following question.

Question 29. Is the Kantor double of a transposed Poisson algebra a special Jordan superalgebra?

Among Gelfand-Dorfman algebras, a relevant place is occupied by special ones, i.e., those that can be embedded into differential Poisson algebras (see [44]). Namely, a Gelfand-Dorfman algebra V with operations $[\cdot, \cdot]$ and $*$ is special if there exists a Poisson algebra $(P, \cdot, \{\cdot, \cdot\})$ and derivation d such that $V \subseteq P$ and $[u, v] = \{u, v\}$, with $u * v = d(u)v$ for all $u, v \in V$. Recently, Kolesnikov and Nesterenko proved that each transposed Poisson algebra obtained from a Novikov-Poisson algebra is a special Gelfand-Dorfman algebra [42]. Sartayev proved that transposed Poisson algebras satisfy the special identities of Gelfand-Dorfman algebras [68]. The last results motivate the following question.

Question 30 (Pavel Kolesnikov, Bauyrzhan Sartayev). Are transposed Poisson algebras special Gelfand-Dorfman algebras?

From the geometric point of view, the variety of n-dimensional algebras defined by a family of polynomial identities gives a subvariety in $\mathbb{C}^{n^3}$. The geometric classification of a variety of algebras is a description of his geometric variety: defining irreducible components, their dimensions, and rigid algebras (see [2, 19, 40] and references therein). The variety of transposed Poisson algebras gives a subvariety in $\mathbb{C}^{2n^3}$. The geometric classification of complex 3-dimensional transposed Poisson algebras was given in [6]. Neretin, in his paper [65], considered n-dimensional associative, commutative, and Lie algebras, and gave the upper bound for the dimensions of their geometric varieties. The last results motivate the following question.

Question 31. Give the upper bound for dimensions of geometric varieties of n-dimensional transposed Poisson algebras.

The "Koszul" property of an (quadratic) operad is quite important. The explicit definition needs more notions on operads [22, 52]. If an operad P is Koszul, one can give the ∞-structure of such an operad, P_{∞}-algebra. For example, Lie, associative, pre-Lie, and Poisson are Koszul and hence we have L_{∞}-, A_{∞}-, PL_{∞}-, P_{∞}-algebras. These remarks motivate the following question.

Question 32 (Li Guo).[8, 9] The operad of the transposed Poisson is Koszul? If so, there might be a TP_∞-algebra structure.

Some recent works about the cohomology theory of Poisson algebras [5, 67] motivate the following question.

Question 33. Describe the cohomology theory of transposed Poisson algebras.

Poisson algebras are the semi-limit of quantization deformation of commutative associative algebras into associative algebras. F-manifold algebras are the semi-limit of quantization deformation of commutative associative algebras (or commutative pre-Lie algebras) into pre-Lie algebras. The last results motivate the following question.

Question 34 (Chengming Bai). Into which algebras are the transposed Poisson algebras the semi-limit of quantization deformation of commutative algebras?

The study of commutative and noncommutative Poisson bialgebras [50, 66] motivate the following question.

Question 35 (Chengming Bai).[10] Define and study transposed Poisson bialgebras.

The study of double Poisson algebras [75] motivates the following question.

Question 36. Define and study double transposed Poisson algebras.

Bai, Bai, Guo, and Wu proved that each transposed Poisson algebra $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ gives a transposed Poisson 3-Lie algebra by defining the following multiplication:

$$[x, y, z] = \mathfrak{D}(x) \cdot [y, z] - \mathfrak{D}(y) \cdot [x, z] + \mathfrak{D}(z) \cdot [x, y].$$

The last results motivate the following question.

---

[8]Here is a short proof of the fact that the operad TP of transposed Poisson algebras is not Koszul (communicated by Vladimir Dotsenko). Using Gröbner bases for Operads (http://irma.math.unistra.fr/~dotsenko/Operads.html), one finds that the dimensions of the components of the operad TP up to parity five are given by 1,2,6,20,74. Moreover, this operad is known to be self-dual under the Koszul duality for operads. Therefore, if it were Koszul, the exponential generating function $f(x)$ of dimensions of its components would satisfy the functional equation $f(-f(-x)) = x$. However, using the dimensions of the components listed above, we find $f(-f(-x)) = x + 7/30x^5 + O(x^6)$, and therefore the operad TP is not Koszul.
[9]Another proof was given by Askar Dzhumadildaev in [15]: he defined the weak Leibniz algebras, as algebras satisfying $(ab - ba)c = 2a(bc) - 2b(ac)$ and $a(bc - cb) = 2(ab)c - 2(ac)b$. Furthermore, he proved that weak Leibniz operad is not Koszul; operads of weak Leibniz algebras and transposed Poisson algebras are isomorphic.
[10]Recently, the present question was successfully resolved in [49].

Question 37 (Chengming Bai, Ruipu Bai, Li Guo & Yong Wu). Let $n \geq 2$ be an integer. Let $(\mathfrak{L}, \cdot, [\cdot, \ldots, \cdot])$ be a transposed Poisson n-Lie algebra, let $\mathfrak{D}$ be a derivation of $(\mathfrak{L}, \cdot)$ and $(\mathfrak{L}, [\cdot, \ldots, \cdot])$. Define an $(n+1)$-ary operation

$$[\![x_1, \ldots, x_{n+1}]\!] := \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{D}(x_i) \cdot [x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}],$$

where $\hat{x}_i$ means that the i-th entry is omitted. Is $(\mathfrak{L}, \cdot, [\![\cdot, \ldots, \cdot]\!])$ a transposed Poisson $(n+1)$-Lie algebra?

Cantarini and Kac classified all complex linearly compact (generalized) Poisson n-Lie algebras in [10], which motivates the following question.

Question 38. Give examples and classify simple transposed Poisson n-Lie algebras.

It is known that the variety of Gerstenhaber algebras coincides with the variety of odd Poisson superalgebras (see, for example, [9]). This remark motivates the following question.

Question 39. Define and study transposed Gerstenhaber algebras.

It is known [4] that the tensor product of transposed Poisson algebras is equipped with a canonical transposed Poisson algebra structure. On the level of operads, this means that the operad TPA of transposed Poisson algebras is a Hopf operad. Many natural examples of Hopf operads arise in algebraic topology when one computes the homology of a topological operad. This suggests the following question. One may define, by analogy with transposed Poisson algebras, and transposed Gerstenhaber algebras, which are given by similar axioms, but with the shifted Lie bracket of homological degree one (and appropriate signs arising from the "Koszul sign rule"). The corresponding operad seems to be a Hopf operad as well.

Question 40 (Vladimir Dotsenko). Is it the homology operad of some operad made of topological spaces? In the case of the Gerstenhaber operad, the corresponding topological operad is the operad of little 2-disks, one of the protagonists of the operad theory in its early days.

Data Availibility The manuscript has no associated data

Declarations
Conflict of interest There are no competing interests.

References

[1] Albuquerque, H., Barreiro, E., Benayadi, S., Boucetta, M., Sánchez, J.M.: Poisson algebras and symmetric Leibniz bialgebra structures on oscillator Lie algebras. J. Geom. Phys. **160**, 103939 (2021)

[2] Alvarez, M., Kaygorodov, I.: The algebraic and geometric classification of nilpotent weakly associative and symmetric Leibniz algebras. J. Algebra **588**, 278–314 (2021)

[3] Bagherzadeh, F., Bremner, M., Madariaga, S.: Jordan trialgebras and post-Jordan algebras. J. Algebra **486**, 360–395 (2017)

[4] Bai, C., Bai, R., Guo, L., Wu, Y.: Transposed Poisson algebras, Novikov–Poisson algebras, and 3-Lie algebras. J. Algebra **632**, 535–566 (2023)

[5] Bao, Y.-H., Ye, Y.: Cohomology structure for a Poisson algebra: II. Sci. China Math. **64**(5), 903–920 (2021)

[6] Beites, P., Fernández, O.A., Kaygorodov, I.: The algebraic and geometric classification of transposed Poisson algebras. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117**(2), 55 (2023)

[7] Billig, Yu.: Towards Kac-van de Leur conjecture: locality of superconformal algebras. Adv. Math. **400**, 108295 (2022)

[8] Cantarini, N., Kac, V.: Classification of linearly compact simple Jordan and generalized Poisson superalgebras. J. Algebra **313**(1), 100–124 (2007)

[9] Cantarini, N., Kac, V.: Classification of linearly compact simple rigid superalgebras. Int. Math. Res. Not. **17**, 3341–3393 (2010)

[10] Cantarini, N., Kac, V.: Classification of linearly compact simple Nambu–Poisson algebras. J. Math. Phys. **57**(5), 051701 (2016)

[11] Casas, J., Datuashvili, T.: Noncommutative Leibniz–Poisson algebras. Commun. Algebra **34**(7), 2507–2530 (2006)

[12] Dotsenko, V.: Algebraic structures of F-manifolds via pre-Lie algebras. Ann. Mat. Pura Appl. (4) **198**(2), 517–527 (2019)

[13] Dotsenko, V., Ismailov, N., Umirbaev, U.: Polynomial identities in Novikov algebras. Math. Z. **303**(3), 60 (2023)

[14] Dzhumadildaev, A.: Central extensions of infinite-dimensional Lie algebras. Funct. Anal. Appl. **26**(4), 247–253 (1992)

[15] Dzhumadildaev, A.: Weak Leibniz algebras and transposed Poisson algebras. arXiv:2308.15018

[16] Farkas, D.: Poisson polynomial identities. Commun. Algebra **26**(2), 401–416 (1998)

[17] Fehlberg, J.R., Kaygorodov, I.: On the Kantor product II. Carpathian Math. Publ. **14**(2), 543–563 (2022)

[18] Fernández Ouaridi, A.: On the simple transposed Poisson algebras and Jordan superalgebras. J. Algebra **641**, 173–198 (2024)

[19] Fernández Ouaridi, A., Kaygorodov, I., Khrypchenko, M., Volkov, Y.: Degenerations of nilpotent algebras. J. Pure Appl. Algebra **226**(3), 106850 (2022)

[20] Ferreira, B.L.M., Kaygorodov, I., Lopatkin, V.: $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115**(3), 142 (2021)

[21] Filippov, V.: δ-Derivations of Lie algebras. Sib. Math. J. **39**(6), 1218–1230 (1998)

[22] Ginzburg, V., Kapranov, M.: Koszul duality for operads. Duke Math. J. **76**(1), 203–272 (1994)

[23] Glennie, C.: Some identities valid in special Jordan algebras but not valid in all Jordan algebras. Pac. J. Math. **16**, 47–59 (1966)

[24] Goze, M., Remm, E.: Poisson algebras in terms of non-associative algebras. J. Algebra **320**(1), 294–317 (2008)

[25] Jin, Q., Li, X.: Hom-Lie algebra structures on semi-simple Lie algebras. J. Algebra **319**(4), 1398–1408 (2008)

[26] Kaledin, D.: Normalization of a Poisson algebra is Poisson. Proc. Steklov Inst. Math. **264**(1), 70–73 (2009)

[27] Kantor I.: Jordan and Lie superalgebras determined by a Poisson algebra, Algebra and analysis (Tomsk, 1989), 55–80, American Mathematical Society Translations: Series, vol. 2, p. 151. Amer. Math. Soc., Providence (1992)

[28] Kaygorodov, I.: Generalized Kantor double (Russian). Vestnik Samarskogo Gosudarstvennogo Universiteta **78**(4), 42–50 (2010)

[29] Kaygorodov, I.: δ-superderivations of semisimple finite-dimensional Jordan superalgebras. Math. Notes **91**(1–2), 187–197 (2012)

[30] Kaygorodov, I.: Algebras of Jordan brackets and generalized Poisson algebras. Linear Multilinear Algebra **65**(6), 1142–1157 (2017)

[31] Kaygorodov, I., Khrypchenko, M.: Poisson structures on finitary incidence algebras. J. Algebra **578**, 402–420 (2021)

[32] Kaygorodov, I., Khrypchenko, M.: Transposed Poisson structures on block Lie algebras and superalgebras. Linear Algebra Appl. **656**, 167–197 (2023)

[33] Kaygorodov, I., Khrypchenko, M.: Transposed Poisson structures on Witt type algebras. Linear Algebra Appl. **665**, 196–210 (2023)

[34] Kaygorodov, I., Khrypchenko, M.: Transposed Poisson structures on generalized Witt algebras and Block Lie algebras. Results Math. **78**(5), 186 (2023)

[35] Kaygorodov, I., Khrypchenko, M.: Transposed Poisson structures on the Lie algebra of upper triangular matrices. Port. Math., to appear, arXiv:2305.00727

[36] Kaygorodov, I., Khrypchenko, M.: Transposed Poisson structures on Lie incidence algebras. arXiv:2309.00332

[37] Kaygorodov, I., Khudoyberdiyev, A.: Transposed Poisson structures on solvable and perfect Lie algebras. J. Phys. A **57**(3), 035205 (2024)

[38] Kaygorodov, I., Lopatkin, V., Zhang, Z.: Transposed Poisson structures on Galilean and solvable Lie algebras. J. Geom. Phys. **187**, 104781 (2023)

[39] Kaygorodov, I., Shestakov, I., Umirbaev, U.: Free generic Poisson fields and algebras. Commun. Algebra **46**(4), 1799–1812 (2018)

[40] Kaygorodov, I., Volkov, Yu.: Degenerations of Filippov algebras. J. Math. Phys. **61**(2), 021701 (2020)

[41] Kolesnikov, P., Makar-Limanov, L., Shestakov, I.: The Freiheitssatz for generic Poisson algebras. SIGMA **10**, 115 (2014)

[42] Kolesnikov, P., Nesterenko, A.: Conformal envelopes of Novikov–Poisson algebras. Sib. Math. J. **64**, 598–610 (2023)

[43] Kolesnikov, P., Sartayev, B.: On the special identities of Gelfand–Dorfman algebras. Exp. Math. (2022). https://doi.org/10.1080/10586458.2022.2041134

[44] Kolesnikov, P., Sartayev, B., Orazgaliev, A.: Gelfand–Dorfman algebras, derived identities, and the Manin product of operads. J. Algebra **539**, 260–284 (2019)

[45] Kolesnikov, P., Voronin, V.: On special identities for dialgebras. Linear Multilinear Algebra **61**(3), 377–391 (2013)

[46] Kosmann-Schwarzbach, Y.: From Poisson algebras to Gerstenhaber algebras. Annales de l'Institut Fourier (Grenoble) **46**(5), 1243–1274 (1996)

[47] Laraiedh, I., Silvestrov, S.: Transposed Hom-Poisson and Hom-pre-Lie Poisson algebras and bialgebras. arXiv:2106.03277

[48] Liu, G., Bai, C.: Anti-pre-Lie algebras, Novikov algebras and commutative 2-cocycles on Lie algebras. J. Algebra **609**, 337–379 (2022)

[49] Liu, G., Bai, C.: A bialgebra theory for transposed Poisson algebras via anti-pre-Lie bialgebras and anti-pre-Lie-Poisson bialgebras. Commun. Contemp. Math. (2023). https://doi.org/10.1142/S0219199723500505

[50] Liu, J., Bai, C., Sheng, Y.: Noncommutative Poisson bialgebras. J. Algebra **556**, 35–66 (2020)

[51] Liu, J., Sheng, Y., Bai, C., Corrigendum to "F-manifold algebras and deformation quantization via pre-Lie algebras" [J. Algebra 559,: 467–495]. J. Algebra **574**(2021), 571–572 (2020)

[52] Loday, J.-L., Vallette, B.: Algebraic operads. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346 pp. xxiv+634. Springer, Heidelberg (2012)

[53] Ma, T., Li, B.: Transposed BiHom-Poisson algebras. Commun. Algebra **51**(2), 528–551 (2023)

[54] Makar-Limanov, L., Shestakov, I.: Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields. J. Algebra **349**, 372–379 (2012)

[55] Makar-Limanov, L., Turusbekova, U., Umirbaev, U.: Automorphisms and derivations of free Poisson algebras in two variables. J. Algebra **322**(9), 3318–3330 (2009)

[56] Makar-Limanov, L., Umirbaev, U.: Centralizers in free Poisson algebras. Proc. Am. Math. Soc. **135**(7), 1969–1975 (2007)

[57] Makar-Limanov, L., Umirbaev, U.: The Freiheitssatz for Poisson algebras. J. Algebra **328**, 495–503 (2011)

[58] Makar-Limanov, L., Umirbaev, U.: Free Poisson fields and their automorphisms. J. Algebra Appl. **15**(10), 1650196 (2016)

[59] Makhlouf, A., Zusmanovich, P.: Hom-Lie structures on Kac-Moody algebras. J. Algebra **515**, 278–297 (2018)

[60] Markl, M., Remm, E.: Algebras with one operation including Poisson and other Lie-admissible algebras. J. Algebra **299**(1), 171–189 (2006)

[61] Martínez, C., Shestakov, I., Zelmanov, E.: Jordan superalgebras defined by brackets. J. Lond. Math. Soc. **64**(2), 357–368 (2001)

[62] Mathieu, O.: Classification of simple graded Lie algebras of finite growth. Invent. Math. **108**(3), 455–519 (1992)

[63] McCrimmon, K.: Speciality and nonspeciality of two Jordan superalgebras. J. Algebra **149**(2), 326–351 (1992)

[64] Mishchenko, S., Petrogradsky, V., Regev, A.: Poisson PI algebras. Trans. Am. Math. Soc. **359**(10), 4669–4694 (2007)

[65] Neretin, Yu.: An estimate for the number of parameters defining an n-dimensional algebra. Math. USSR-Izvestiya **30**(2), 283–294 (1988)

[66] Ni, X., Bai, C.: Poisson bialgebras. J. Math. Phys. **54**(2), 023515 (2013)

[67] Penkava, M., Pichereau, A.: $\mathbb{Z}_2$-graded Poisson algebras, their deformations and cohomology in low dimensions. Transform. Groups **23**(4), 1091–1127 (2018)

[68] Sartayev, B.: Some generalizations of the variety of transposed Poisson algebras. Commun. Math. **32**(2), 55–62 (2024)

[69] Shestakov, I., Umirbaev, U.: Tame and wild automorphisms of rings of polynomials in three variables. J. Am. Math. Soc. **17**, 197–227 (2004)

[70] Sun, L., Liu, W.: Hom-Lie superalgebra structures on exceptional simple Lie superalgebras of vector fields. Open Math. **15**(1), 1332–1343 (2017)

[71] Shirshov, A.: Selected works of A. I. Shirshov. In: Contemporary Mathematicians, pp. viii+242. Birkhäuser Verlag, Basel (2009)

[72] Sverchkov, S.: A quasivariety of special Jordan algebras. Algebra Logic **22**(5), 563–573 (1983)

[73] Umirbaev, U.: The Anick automorphism of free associative algebras. J. Reine Angew. Math. **605**, 165–178 (2007)

[74] Umirbaev, U.: Universal enveloping algebras and universal derivations of Poisson algebras. J. Algebra **354**, 77–94 (2012)

[75] Van den Bergh, M.: Double Poisson algebras. Trans. Am. Math. Soc. **360**(11), 5711–5769 (2008)

[76] Voronin, V.: Special and exceptional Jordan dialgebras. J. Algebra Appl. **11**(2), 1250029 (2012)

[77] Wang, D., Chen, Z.: Quasi-automorphisms of Lie algebras. Commun. Algebra **39**(7), 2388–2395 (2011)

[78] Xie, W., Jin, Q., Liu, W.: Hom-structures on semi-simple Lie algebras. Open Math. **13**(1), 617–630 (2015)

[79] Xie, W., Liu, W.: Hom-structures on simple graded Lie algebras of finite growth. J. Algebra Appl. **16**(8), 1750154 (2017)

[80] Xu, P.: Noncommutative Poisson algebras. Am. J. Math. **116**(1), 101–125 (1994)

[81] Xu, X.: Novikov–Poisson algebras. J. Algebra **190**(2), 253–279 (1997)

[82] Yang, Y., Tang, X., Khudoyberdiyev, A.: Transposed Poisson structures on Schrodinger algebra in $(n + 1)$-dimensional space-time. arXiv:2303.08180

[83] Yao, Y., Ye, Y., Zhang, P.: Quiver Poisson algebras. J. Algebra **312**(2), 570–589 (2007)

[84] Yuan, L., Hua, Q.: $\frac{1}{2}$-(bi)derivations and transposed Poisson algebra structures on Lie algebras. Linear Multilinear Algebra **70**(22), 7672–7701 (2022)

[85] Yuan, J., Liu, W.: Hom-structures on finite-dimensional simple Lie superalgebras. J. Math. Phys. **56**(6), 061702 (2015)

[86] Yuan, J.X., Sun, L.P., Liu, W.D.: Hom-Lie superalgebra structures on infinite-dimensional simple Lie superalgebras of vector fields. J. Geom. Phys. **84**, 1–7 (2014)

[87] Zakharov, A.: Novikov–Poisson algebras and superalgebras of Jordan Brackets. Commun. Algebra **42**(5), 2285–2298 (2014)

[88] Zhou, L., Feng, C., Kong, X.: Quasi-automorphisms of Borel subalgebras (Chinese). Nanjing Daxue Xuebao Shuxue Bannian Kan **28**(1), 106–117 (2011)

[89] Zusmanovich, P.: On δ-derivations of Lie algebras and superalgebras. J. Algebra **324**(12), 3470–3486 (2010)

[90] Zusmanovich, P.: On contact brackets on the tensor product. Linear Multilinear Algebra **70**(19), 4695–4706 (2022)

Patrícia Damas Beites
Departamento de Matemática, Centro de Matemática e Aplicações (CMA-UBI)
Universidade da Beira Interior
Covilhã
Portugal
e-mail: `pbeites@ubi.pt`

Bruno Leonardo Macedo Ferreira
Federal University of Technology
Guarapuava
Brazil
e-mail: `brunolmfalg@gmail.com`

Ivan Kaygorodov
CMA-UBI
Universidade da Beira Interior
Covilhã
Portugal

and

Saint Petersburg University
St Petersburg
Russia
e-mail: `kaygorodov.ivan@gmail.com`

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.