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Results in Mathematics

Inequalities for Convex Functions and Isotonic Sublinear Functionals

Zdzisław Otachel

Abstract. In this paper, versions of the famous Jensen inequality for sublinear isotonic functionals are proved. The obtained results generalize classic Jessen's and McShane's inequalities. Applications to generalized means and to Hölder's and Minkowski's inequalities are also given.

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1. Introduction and Motivation

The notion of convex function ϕ is inextricably linked with the inequality

$$\phi\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k \phi(x_k), \tag{1}$$

where ϕ is defined on an interval I containing real numbers x_k, $(k = 1, \ldots, n)$ and w_k, $(k = 1, \ldots, n)$ are positive weights with the sum equals 1, in the sense that condition (1) can be treated as a definition of a convex function itself. Counterparts of (1) for integrals also hold true, e.g.

$$\phi\left(\int_0^1 w(t)x(t)dt\right) \leq \int_0^1 w(t)\phi(x(t))dt, \tag{2}$$

here $w : [0,1] \to [0,\infty]$ and $x : [0,1] \to I$ are integrable functions with $\int_0^1 w(t)dt = 1$.

The above facts were established in 1906 by Jensen [5], one of the founders of the convex functions theory. Presently, inequalities (1) and (2) are just called Jensen's inequalities.

Twenty five years later, Jessen [6] unified these results replacing sums and integrals in inequalities (1) and (2) by an abstract linear and positive functional A acting on a linear space L of real functions g including constant functions, such that $A(\mathbb{I}) = 1$, where $\mathbb{I}$ is the function constantly equals 1, more precisely, he showed, that

$$\phi\left(A(g)\right) \leq A(\phi(g)). \tag{3}$$

Functionals of that type are called *linear means.*

The case of multi-variable convex function ϕ in (3) was developed by McShane [7]. The inequality (3) remains true for the wider class of functionals, than linear means, namely, sublinear isotonic functionals A preserving constants, i.e. $A(\alpha\mathbb{I}) = \alpha$, $\alpha \in \mathbb{R}$. Results of that type come from Pečarić and Raşa [10] and Dragomir, Pearce and Pečarić [2] (see also [1, Th. 12.18]). Guessab and Schmeisser [3] reversed in a sense McShane's result. In fact, they proved if Jensen's type inequality (3) holds for a sublinear functional A, then A is necessary a linear mean, provided convex functions ϕ are defined on special convex subsets of $\mathbb{R}^n$.

In this article we show that Jensen-Jessen inequality (3) is met with weaker assumptions about functionals than those made in [10] or in [2]. Our results (Th. 1, Remark 2 and Th. 4) generalize mentioned Jessen's and McShane's inequalities and complement the results obtained in [10] and [2].

The layout of the paper is as follows. In the second section we collect some, useful in the sequel, basic results on convex functions and sublinear functionals. Especially, isotonic functionals of that type are discussed and many illustrative examples are included there. The next two sections consecutively contain generalizations of Jessen's and McShane's inequalities. The two subsequent sections are devoted to natural applications of our results to inequalities for generalized means and to Hölder's and Minkowski's inequalities. The final section contains conclusions and opportunities for further research.

2. Convex Functions and Isotonic and Sublinear Functionals

Let V be a linear space over real number field $\mathbb{R}$. Recall that a real-valued function ϕ defined on a convex set $K \subset V$ is a *convex function*, if the inequality $\phi(rx+sy) \leq r\phi(x)+s\phi(y)$ holds for all $x, y \in V$ and all $s, r \geq 0$ with $r+s = 1$. If the inequality is reversed we say about concave functions.

The fundamental property of convex functions is, that the function ϕ is convex on an open convex set K of a normed linear space V if and only if ϕ has support at each point x_0 of K, i.e. there exists a linear functional T on V, such that

$$\phi(x_0) + T(x - x_0) \leq \phi(x), \ x \in K,$$

see [11, Chap. IV, sec. 43, Th. B].

In case of $V = \mathbb{R}^n$, it is convenient to use the notion of the subdifferential $\partial\phi$ of ϕ instead of support,

$$\partial\phi(x_0) = \{w \in \mathbb{R}^n : \phi(x) \geq \phi(x_0) + \langle w, x - x_0 \rangle \,, \ x \in K\}, \ x_0 \in K,$$

particularly, for $n = 1$ the subdifferential $\partial\phi(x_0) = [\phi'_-(x_0), \phi'_+(x_0)]$.

A subset $C \subset V$ is a convex cone, if $rC + sC \subset C$ for all nonnegative scalars $r, s \in \mathbb{R}$. Then $x \prec y \iff y - x \in C, \ x, y \in V$ defines a preorder in V, i.e. a binary relation, which is reflexive and transitive. If, in addition, the relation is antisymmetric, we speak about partial order. It is happening if and only if $-C \cap C = \{0\}$.

A functional $A : V \to \mathbb{R}$ is said to be:

a sublinear functional (or a Banach functional), if $A(rx + sy) \leq rA(x) + sA(y)$ for all $x, y \in V$ and all $s, r \geq 0$; equivalently, the sublinear functional A is nonnegative homogeneous, $A(rx) = rA(x)$ for all real $r \geq 0$ and all $x \in V$ and subadditive, $A(x + y) \leq A(x) + A(y)$ for all $x, y \in V$;
an isotonic functional, if $x \prec y \Rightarrow A(x) \leq A(y), \ x, y \in V$;
a positive functional, if $0 \prec x \Rightarrow 0 \leq A(x), \ x \in V$;
a negative functional, if $x \prec 0 \Rightarrow A(x) \leq 0, \ x \in V$.

The epigraph of a map $f : V \supset K \to \mathbb{R}$ is the set epi $f := \{(x, t) \in K \times \mathbb{R} : f(x) \leq t\}$. It is known, f is a convex function if and only if epi f is convex; A is a sublinear functional if and only if epi A is a convex cone with $(0, -1) \notin$ epi A, (see, [14][Th. 2.1.1, Prop. 2.1.2]).

Of course, every sublinear functional is a convex function. The following properties of a sublinear functional A are easy to verify

$$A(0) = 0, \tag{4}$$

$$0 \leq A(x) + A(-x), \ x \in V, \tag{5}$$

$$A(x) - A(y) \leq A(x - y), \ x, y \in V. \tag{6}$$

Hence, it is easily seen by (6), that any negative and sublinear functional is isotonic and therefore positive, but not conversely. In case of linear functionals, isotonicity and positiveness and negativity are equivalent, one to each other. Note, properties of isotonicity or positivity are independent of sublinearity or linearity for functionals.

Sublinear functionals are e.g. upper limit operations, upper Darboux integrals, Minkowski functionals of absorbing convex sets, norms or seminorms and many others. Below, we give a few further examples illustrating such functionals.

We start from a simple

Remark 1. If A and B are (isotonic) sublinear functionals, then $\max\{A, B\}$ is like these, too.

Example 1. Let $A : V \to \mathbb{R}$ be a functional fulfilling the following sublinearity condition $A(rx + sy) \leq rA(x) + sA(y)$ for all $x, y \in V$ and all $s, r \in \mathbb{R}$.

Clearly, $A(x) = A(x+\varepsilon y-\varepsilon y) \leq A(x+\varepsilon y)-\varepsilon A(y)$. Hence $A(x)+\varepsilon A(y) \leq A(x + \varepsilon y)$ and $A(x + \varepsilon y) \leq A(x) + \varepsilon A(y)$. Thus $A(x + \varepsilon y) = A(x) + \varepsilon A(y)$. Substituting $\varepsilon = -1, +1$ we obtain $A(x - y) = A(x) - A(y)$ and $A(x + y) = A(x) + A(y)$. The first identity (for $y = x$) gives $A(0) = 0$. Linking it with the second one for $y = -x$ leads to $A(-x) = -A(x)$.

Now, according to the assumption $A(rx) \leq rA(x)$ and $A(-rx) \leq -rA(x)$ for any $r \in \mathbb{R}$. Let us assume that $A(rx) < rA(x)$ for some $r \neq 0$ and $x \in V$ with $A(x) \neq 0$. Then $A(-rx) = -A(rx) > -rA(x)$ in contrary to the assumption $A(-rx) \leq -rA(x)$.

Therefore, a functional A with $A(rx+sy) \leq rA(x)+sA(y)$ for all $x, y \in V$ and all $s, r \in \mathbb{R}$ is necessary linear.

In [12,13] one can find suggestions that the condition of sublinearity, in the above sense, is apparently weaker than linearity of functionals.

Example 2. Let V be a real linear space endowed with a partial ordering $\prec$ induced by a convex cone C. We also assume that for all $x, y \in V$ there exist $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, i.e. V is a *vector lattice*. Set $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$ and $|x| = x^+ + x^-$, where $x \in V$. It is known that $0 \prec x^+$ and $0 \prec x^-$, moreover, $x = x^+ - x^-$, $(x + y)^+ \prec x^+ + y^+$ and $(x + y)^- \prec x^- + y^-$.

For arbitrary $x, y \in V$ the following inequalities are obviously equivalent:

$$\begin{aligned} &x + y \prec x + y \\ &(x + y)^+ - (x + y)^- \prec x^+ - x^- + y^+ - y^- \\ &x^- + y^- - (x + y)^- \prec x^+ + y^+ - (x + y)^+ \end{aligned}$$

Now, let ϕ_1, ϕ_2 be linear positive functionals with $0 \leq \phi_1(x) \leq \phi_2(x),\ x \geq 0$. Then

$$\phi_1(x^- + y^- - (x + y)^-) \prec \phi_2(x^+ + y^+ - (x + y)^+)$$

or, equivalently,

$$\phi_2((x + y)^+) - \phi_1((x + y)^-) \prec \phi_2(x^+) - \phi_1(x^-) + \phi_2(y^+) - \phi_1(y^-)$$

It proves subadditivity of the functional $A(x) = \phi_2(x^+) - \phi_1(x^-)$. Negativity, positivity and nonnegative homogeneity of A is obvious. Thus A is sublinear and isotonic. In particular, $A(x) = \phi(x^+) - \alpha\phi(x^-)$ is such a functional for $0 \leq \alpha \leq 1$, when ϕ is linear and positive.

Note, that the set of sublinear functionals, the set of isotonic functionals and consequently, the set of isotonic sublinear functionals on V are convex cones.

Example 3. Let $(V, \prec)$ be a real vector lattice, as in the previous example. Since $0 \prec |x|$ and $-|x| \prec x \prec |x|$, for $A : V \to \mathbb{R}$ being a sublinear and isotonic functional,

$$-A(|x|) \leq A(-|x|) \leq A(x) \leq A(|x|).$$

The first inequality on the left is a consequence of (5). Therefore

$$|A(x)| \leq A(|x|).$$

This inequality can be treated as a version of Jensen's inequality for any sublinear and isotonic functional.

From now on, we will deal with isotonic sublinear functionals on real function spaces. Given a non-empty set E, the symbol L is reserved for a real linear space, that consists of some real-valued functions $g : E \to \mathbb{R}$, between others, the indicator function $\mathbb{I} := I_E \in V$. We assume that L is endowed with the preorder defined by the cone of all nonnegative functions from L. By analogy, any sublinear isotonic functional $A : L \to \mathbb{R}$ which is normalized in the sense that $A(\mathbb{I}) = 1$, will be called a *sublinear mean* on L. If in addition, $A(-\mathbb{I}) = -1$ we call the functional *totally normalized sublinear mean.*

Every convex combination of sublinear means is a sublinear mean. Similarly, a convex combination of totally normalized functionals gives another totally normalized functional. If ϕ is a linear mean, functionals $A(x) = \phi(x^+) - \alpha\phi(x^-)$, $0 \leq \alpha \leq 1$, considered in Ex. 2 are sublinear means, that are not totally normalized.

Example 4. Let E be a non-empty set and L be the space of all real-valued function $x : E \to \mathbb{R}$ endowed with the preorder defined by the cone of nonnegative functions.

The functional $A(x) = \sup\{x(t) : t \in E\}$ is sublinear, positive and isotonic, on the other hand, $A_1(x) = A(|x|)$ is sublinear and positive but is not isotonic. Both of them are not linear functionals.

For other examples of isotonic sublinear functionals on L, see [1], more specific examples on spaces of double sequences are to find, e.g. in [8].

3. A Generalization of Jessen Inequality for Sublinear Functionals

In the below theorem we present a generalization of Jessen [6] inequality for sublinear isotonic functionals, see [9, Th 2.4].

Theorem 1. *If ϕ is a nondecreasing nonnegative concave and continuous function defined on an interval $\mathrm{I} = [m, M]$, $m \leq 0 \leq M$, $m < M$ and A is a sublinear mean on L, then for all $g \in L$ such that $\phi \circ g \in L$ the following inequality holds*

$$A(\phi \circ g) \leq \phi(A(g)). \tag{7}$$

Proof. The assumption $\phi \circ g \in L$ forces $m \leq g(x) \leq M$ for all $x \in E$. Since A is a sublinear mean, for $-m, M \geq 0$ inequalities $0 \leq g(x) - m,\ g(x) \leq M,\ x \in E$ imply $0 \leq A(g - m\mathbb{I}) \leq A(g) - m,\ A(g) \leq M$, i.e. $m \leq A(g) \leq M$. For arbitrary $\varepsilon > 0$ there exists a linear function $f(t) = p + qt,\ t \in \mathbb{R}$, where $p, q \geq 0$, such that:

(i) $\phi(t) \leq f(t),\ 0 \leq t \leq a$, particularly, $\phi(A(g)) \leq f(A(g))$, and
(ii) $f(A(g)) \leq \phi(A(g)) + \varepsilon$.

Now, (i) ensures $\phi \circ g(t)) \leq f \circ g(t)),\ t \in E$, and consecutively, isotonicity and sublinearity of A gives

$$A(\phi \circ g) \leq A(f \circ g) = A(p\mathbb{I} + qg) \leq pA(\mathbb{I}) + qA(g) = f(A(g)) \leq \phi(A(g)) + \varepsilon.$$

The last inequality is a consequence of (ii). Finally, arbitrariness of ε leads to (7). The proof is complete. $\square$

Taking into account (5), one can express Th. 1 in terms of convex functions.

Remark 2. If ϕ is a nonincreasing nonpositive convex and continuous function defined on an interval $[m, M],\ m \leq 0 \leq M,\ m < M$ and A is a sublinear mean on L, then for all $g \in L$ such that $\phi \circ g \in L$, the following inequality holds

$$\phi(A(g)) \leq A(\phi \circ g). \tag{8}$$

In the above statements, continuity of concave or convex functions is significant.

Example 5. Let $L = \{g : [0,1] \to \mathbb{R},\ \limsup_{t \to 0} g(t) < \infty\}$, $A(g) = \limsup_{t \to 0} g(t)$ and $\phi(t) = 1,\ t \in (0,1],\ \phi(0) = 0$.

Then, for $g(t) = t,\ t \in [0,1]$ we have $A(g) = 0,\ \phi(A(g)) = 0,\ \phi \circ g(t) = \phi(t)$ and $A(\phi \circ g) = \limsup_{t \to 0} \phi(t) = 1$. Thus $A(\phi \circ g) = 1 > \phi(A(g)) = 0$ in contrary to the thesis of Th.1.

4. Generalizations of McShane [7] Results on Jensen's Inequality

For fixed $\emptyset \neq S \subset \mathbb{R}_+^{n+1}$ let us define

$$K := K_S = \{z \in \mathbb{R}^n : \langle z, a \rangle + c \geq 0,\ a \in \mathbb{R}_+^n,\ c \geq 0,\ (a, c) \in S\}, \tag{9}$$

where for $z = (z_1, \ldots, z_n), a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the symbol $\langle z, a \rangle$ stands for the standard inner product $\sum_{i=1}^n z_i a_i$ of vectors z and a. Clearly, K_S is a closed convex set such that $\mathbb{R}_+^n \subset K_S$ and $K_S + \mathbb{R}_+^n \subset K_S$.

Under settings as in the previous section, we can state as follows, cf. [9, Th 2.5].

Theorem 2. *Let $K \subset \mathbb{R}^n$ be a closed convex set of the form (9) defined for a fixed nonempty $S \subset \mathbb{R}_+^{n+1}$.*

If $f_1, \ldots, f_n \in L$ are such functions that $(f_1(t), \ldots, f_n(t)) \in K$ for any $t \in E$, then $(A(f_1), \ldots, A(f_n)) \in K$ for arbitrary sublinear mean A on L.

Proof. Let $(a_1, \ldots, a_n, c) \in S$ and a sublinear mean A on L be arbitrary chosen. If $\sum_{i=1}^n a_i f_i(t) + c \geq 0,\ t \in E$, for certain $f_1, \ldots, f_n \in L$, then by positiveness and sublinearity of A we easy get $0 \leq A(c\mathbb{I} + \sum_{i=1}^n a_i f_i) \leq c + \sum_{i=1}^n a_i A(f_i)$. Since $(a_1, \ldots, a_n, c) \in S$ is arbitrary, it is nothing but $(A(f_1), \ldots, A(f_n)) \in K$, what finishes the proof. □

Epigraphs of some continuous convex functions are closed convex sets of the form (9).

Proposition 3. *Let $K \subset \mathbb{R}^n$ be a closed convex set of the form (9) defined for a fixed nonempty $S \subset \mathbb{R}_+^{n+1}$.*

If $\phi : K \to \mathbb{R}$ is a nonpositive convex continuous function nonincreasing with respect to every variable, then epi ϕ *has the form (9), too.*

Proof. For a nonpositive convex continuous function $\phi : K \to \mathbb{R}$ being nonincreasing with respect to every variable we define the set $K' \subset \mathbb{R}^{n+1}$ as follows

$$K' = \{(z, y) \in K \times \mathbb{R} : y \geq \phi(z_0) + \langle w, z - z_0 \rangle\,,\ w \in \partial\phi(z_0),\ z_0 \in \mathrm{int}K\}. \tag{10}$$

At first we show that epi $\phi = K'$.

For any $z_0 \in \mathrm{int}K$ and $w \in \partial\phi(z_0)$ we have

$$\phi(z) \geq \phi(z_0) + \langle w, z - z_0 \rangle\,,\ z \in K. \tag{11}$$

If $(z, y) \in$ epi ϕ, then we get $y \geq \phi(z) \geq \phi(z_0) + \langle w, z - z_0 \rangle$, so $(z, y) \in K'$.

Conversely, let $(z, y) \in K'$ and assume that $\phi(z) - y > 0$. By continuity of ϕ we can find $z_0 \in \mathrm{int}K$ such that $|\phi(z) - \phi(z_0)| < \phi(z) - y$ and $z \leq z_0$. Therefore $y < \phi(z_0)$.

On the other hand, since ϕ is nonincreasing with respect to every variable, every $w \in \partial\phi(z_0)$ is nonpositive. Hence, by (10), $y - \phi(z_0) \geq \langle w, z - z_0 \rangle \geq 0$ contrary to our assumption. Thus $y \geq \phi(z)$, i.e. $(z, y) \in$ epi ϕ and finally, epi $\phi = K'$.

Now, it suffices to prove that K' is of the form (9). Let $K = K_S$, where $\emptyset \neq S \subset \mathbb{R}_+^{n+1}$. The system of inequalities, that define K' as a set of points $(z, y) \in \mathbb{R}^n \times \mathbb{R}$, one can express as follows

$$\begin{cases} \langle (z, y), (a, 0) \rangle + c \geq 0,\ (a, 0) \in \mathbb{R}_+^{n+1},\ c \geq 0,\ (a, c) \in S\}, \\ \langle (z, y), (-w, 1) \rangle + \langle w, z_0 \rangle - \phi(z_0) \geq 0,\ w \in \partial\phi(z_0),\ z_0 \in \mathrm{int}K, \end{cases} \tag{12}$$

where $\mathbb{R}^{n+1} \ni (-w, 1) \geq 0$ and $\langle w, z_0 \rangle - \phi(z_0) \geq 0$. The last estimate is a consequence of inequality (11) taken for $z = 0 \in K$ with nonnegativity of the function ϕ. The first group of inequalities of system (12) defines the set $K \times \mathbb{R}$, while the second one specifies points over and above the graph ϕ.

Now, it is easy to see, that K' has the form (9), which completes the proof. $\square$

We are ready to prove a multidimensional version of inequality (8), cf. [9, Th 2.6].

Theorem 4. *Let $K \subset \mathbb{R}^n$ be a closed convex set of the form (9) and $\phi : K \to \mathbb{R}$ be a nonpositive convex continuous function which is nonincreasing with respect to every variable.*

If $f_1, \ldots, f_n \in L$ are such functions that $(f_1(t), \ldots, f_n(t)) \in K$ for any $t \in E$ and $\phi(f_1, \ldots, f_n) \in L$, then for arbitrary sublinear mean A on L the following inequality holds

$$\phi(A(f_1), \ldots, A(f_n)) \leq A(\phi(f_1, \ldots, f_n)). \tag{13}$$

Proof. Since $(f_1(t), \ldots, f_n(t), \phi(f_1(t), \ldots, f_n(t)) \in \text{epi}\,\phi$ for every $t \in E$, based on Th. 2 and Prop. 3 we get $(A(f_1), \ldots, A(f_n), A(\phi(f_1, \ldots, f_n)) \in \text{epi}\,\phi$ for any sublinear mean A on L. It is equivalent to inequality (13) and thus the proof is finished. $\square$

5. Applications to Inequalities for Generalized Means

Let the settings from Sect. 3 still hold and let $\mathrm{I} = [0, M]$, $0 < M$ and $\psi, \chi : \mathrm{I} \to \mathrm{I}$ be continuous and strictly monotonic functions, such that $\psi(g), \chi(g) \in L$, $g \in L$, i.e. between others, $g(x) \in \mathrm{I}$, $x \in E$.

If $0 \leq \alpha \leq \psi(g) \leq \beta \leq M$, $x \in E$, then for any sublinear mean A on L we get $\alpha \leq A(\psi(g)) \leq \beta$. Thus, for the function ψ and a fixed sublinear mean A on L, the value

$$M_{A,\psi}(g) = \psi^{-1}(A(\psi(g))),\ g \in L \tag{14}$$

is well defined. We call it the *generalized mean.*

Theorem 5. *Under above assumptions, the inequality*

$$M_{A,\psi}(g) \leq M_{A,\chi}(g), \tag{15}$$

is met if $\Phi = \chi \circ \psi^{-1}$ is increasing being concave nonnegative and χ is decreasing. In case of χ is increasing the inequality is reversed.

Proof. With $g \in L$ we have also $\psi(g)$ and $\Phi(\psi(g))) = \chi(g) \in L$, by our hypotheses. Applying Th. 1 we get

$$A(\chi(g)) = A(\Phi(\psi(g))) \leq \Phi(A(\psi(g)))$$

for Φ being increasing and concave nonnegative. Hence, if χ is decreasing,

$$M_{A,\chi} = \chi^{-1}(A(\chi(g))) \geq \psi^{-1}(A(\psi(g))) = M_{A,\psi}.$$

In case of χ is increasing, the inequality is reversed. $\square$

Theorem 5 transfers to sublinear functionals the general mean value inequality originated in Hardy, Littlewood and Pólya [4, Th. 92, p. 75] and continued in [9, Th. 4.1, p. 107].

Applying the theorem to classical means $M_A^{[r]}(g) = (A(g^r))^{1/r},\ r > 0$, for isotonic sublinear functionals A, where $g(x) \geq 0,\ x \in E$ and $g^r \in L$, we easy obtain

$$s \leq r < 0 \text{ or } 0 < s \leq r \Rightarrow M_A^{[s]}(g) \leq M_A^{[r]}(g).$$

6. Applications to Hölder's and Minkowski's Inequalities

As before, let L be a linear space of real functions defined on a certain nonempty base set E, including the indicator function $\mathbb{I}$. Consider a sublinear and isotonic (but not necessary normalized) functional A on L and fix $0 \leq w \in L$ such that $A(w) > 0$. It is clear, that

$$\tilde{A}(h) = \frac{A(wh)}{A(w)}$$

is a sublinear mean defined on the space such functions $h : E \to \mathbb{R}$ including $\mathbb{I}$, that $wh \in L$.

If Φ is a nondecreasing nonnegative concave and continuous function defined on an interval I, as in Th. 1, then $\tilde{A}(\Phi(h)) \leq \Phi(\tilde{A}(h))$, i.e.

$$\frac{A(w\Phi(h))}{A(w)} \leq \Phi\left(\frac{A(wh)}{A(w)}\right), \tag{16}$$

whenever $wh,\ w\Phi(h) \in L$. This inequality leads directly to the following version of Hölder's inequality for sublinear isotonic functionals.

Theorem 6. *Under above assumptions, let* $s \in \mathbb{R} \setminus \{0, 1\}$ *and* $r = s/(s-1)$, *that is* $1/s + 1/r = 1$.

If $0 \leq p, f, g \in L$ *with* $pf^s,\ pg^r,\ pfg \in L$, *then*

$$A^{1/s}(pf^s)A^{1/r}(pg^r) \leq A(pfg), \text{ where } 0 < s < 1 \text{ or } s < 0, \tag{17}$$

$$A(pfg) \leq A^{1/s}(pf^s)A^{1/r}(pg^r), \text{ where } s > 1. \tag{18}$$

Proof. Case $0 < s < 1$. Clearly $A(pg^r) \geq 0$. If $A(pg^r) = 0$, inequality (17) holds, since $A(pfg) \geq 0$. For $A(pg^r) > 0$ apply inequality (16) to the nondecreasing nonnegative concave and continuous function $\Phi(t) = t^s,\ t \geq 0$ and $w = pg^r,\ h = fg^{-r/s}$.

The condition $s < 0$ is equivalent to $0 < r < 1$ and argumentation is analogical as above, just switch roles s and r.

Case $s > 1$. Then also $r > 1$. Since $0 \leq pfg \leq \frac{1}{s}pf^s + \frac{1}{r}pg^r$ on E, the hypothesis $A(pg^r) = 0,\ A(pf^s) = 0$ implies $A(pfg) = 0$, and inequality (18) holds. If e.g. $A(pg^r) > 0$, apply inequality (16) to the nondecreasing nonnegative concave and continuous function $\Phi(t) = t^{1/s},\ t \geq 0$ and $w = pg^r,\ h = f^s g^{-r}$. □

As a consequence of the above theorem we show a version of Minkowski's inequality for sublinear isotonic functionals.

Theorem 7. *As in Th. 6, let A be a sublinear isotonic functional on L and* $s > 1$.

If $0 \leq p, f, g \in L$ *with* $pf^s,\ pg^s,\ p(f+g)^s \in L$, *then*

$$A^{1/s}(p(f+g)^s) \leq A^{1/s}(pf^s) + A^{1/s}(pg^s). \tag{19}$$

Proof. Keeping in mind that $r = s/(s-1)$ and utilizing consecutively subadditivity of A and Hölder's type inequality from Th. 6 we obtain

$$\begin{aligned} A(p(f+g)^s) &\leq A(pf(f+g)^{s-1}) + A(pg(f+g)^{s-1}) \\ &\leq (A^{1/s}(pf^s) + A^{1/s}(pg^s))A^{1/r}(p(f+g)^s), \end{aligned}$$

what proves (19). □

7. Conclusion and Outlook

Main results of the article are included in Th.1 (together with Remark 2), Th. 2 and Th. 4. Other statements present typical consequences and applications.

Jessen's inequality (3) on convex functions was originally established for linear means, see [6] or [9, Th. 2.4]. In [10] obtained a version of this inequality, where linear means replaced by totally normalized sublinear means acting on $C(X)$, the space of all continuous real-valued functions on X being a compact Hausdorff space, endowed with supremum norm and usual ordering. A similar result can be found in [2], where the inequality is shown for totally normalized sublinear means too, but on any linear space consisting of real functions defined on any set as well as constant functions.

Our version of Jessen's inequality, presented in Th. 1 and Remark 2, is made under the same settings, but the advantage is that the strong assumption of totally normalizing for sublinear means is relaxed. However, it causes functions ϕ have to be nonpositive convex and nonincreasing or nonnegative concave and nondecreasing.

The case of multi-variable convex function ϕ in Jessen's inequality (3) for linear means was developed by McShane [7]. In Th. 4 we present an analog of McShane's result (see [9, Th. 2.6]) for sublinear means and nonpositive multi-variable convex functions which are nonincreasing with respect to each variable. Th. 2 plays an important role for the geometric formulation of Jessen's inequality included in Th. 4. It is a counterpart of McShane's result, see [9, Th. 2.5].

At the end,

a question may arise, whether Jessen's or McShane's inequalities for sublinear means are valid or have counterparts for other classes of functions related to convexity. Usually, it needs deep research, but at once the answer is yes for ϕ-convexifiable functions, cf. [15], i.e. for generally nonconvex functions, but ones which become convex by adding a convex term $c\phi$, where ϕ is a fixed convex function and $c \geq 0$.

It could also be interesting to extend the obtained results, where instead of functionals and convex functions, we consider sublinear operations and convex maps with values in a specific ordered linear space.

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Declarations

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Zdzisław Otachel
Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
Głęboka 28
20-950 Lublin
Poland
e-mail: zdzislaw.otachel@up.lublin.pl

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