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Results in Mathematics

Approximation Theorems for Complex α-Bernstein–Kantorovich Operators

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Abstract. In this paper, we introduce the complex form of α-Bernstein–Kantorovich operators. Respectively, upper quantitative estimates for the complex α-Bernstein–Kantorovich operator and its derivatives, Voronovskaya type result and the exact order of approximation of these operators are studied.

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1. Introduction

The approximation of functions with positive linear operators is one of the most important research area of applied mathematics. Especially, Bernstein polynomials play an significant role in approximation theory, thanks to their simple structure and advantages in computation. In 1912, Bernstein was introduced the classical Bernstein polynomials as follows:

$$B_\eta\left(\varphi;\tau\right)=\sum_{\rho=0}^{\eta}\varphi\left(\frac{\rho}{\eta}\right)\binom{\eta}{\rho}\tau^{\rho}(1-\tau)^{\eta-\rho},\ \eta\in\mathbb{N}\text{ and }\tau\in[0,1] \tag{1}$$

for $\varphi\in\mathbb{C}[0,1]$. For many years, many researchers focused on the discovery and modifications of Bernstein polynomials to get better convergence. Recently, Chen et al. [1] have constructed a generalization of the Bernstein operator, which depends on α as follows.

$$T_{\eta,\alpha}(\varphi;\tau)=\sum_{\rho=0}^{\eta}\varphi_\rho\ p_{\eta,\rho}^{(\alpha)}(\tau), \tag{2}$$

where $\varphi_\rho = \varphi(\frac{\rho}{\eta})$, $\alpha \in [0,1]$ and

$$p_{\eta,\rho}^{(\alpha)}(\tau) = \left[\binom{\eta-2}{\rho}(1-\alpha)\tau + \binom{\eta-2}{\rho-2}(1-\alpha)(1-\tau) + \binom{\eta}{\rho}\alpha\tau(1-\tau)\right]\tau^{\rho-1}(1-\tau)^{\eta-\rho-1}.$$

Because the α-Bernstein operators are suitable structures for continuous functions on $[0,1]$, authors constructed the following α-Bernstein–Kantorovich operators for integrable functions on $[0,1]$ and investigated their many approximation properties (See [2,3]).

$$K_{\eta,\alpha}(\varphi;\tau) = \sum_{\rho=0}^{\eta} p_{\eta,\rho}^{(\alpha)}(\tau) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \varphi(t)dt. \tag{3}$$

In addition to all these, recently, new definitions and studies have been made due to the increasing interest of researchers in complex operators in approximation problems [4–19]. One of them was introduced and studied by Nursel Çetin. In Çetin [1], introduced the α-Bernstein operator in the complex domain as follows:

$$T_{\eta,\alpha}(\varphi;\zeta) = \sum_{\rho=0}^{\eta} \varphi_\rho \; p_{\eta,\rho}^{(\alpha)}(\zeta), \tag{4}$$

where $\varphi_\rho = \varphi(\frac{\rho}{\eta})$, $\alpha \in [0,1]$, $\zeta \in \mathbb{C}$.

The above-mentioned studies motivated us to define the α-Bernstein–Kantorovich operator in the complex region as follows:

$$K_{\eta,\alpha}(\varphi;\zeta) = \sum_{\rho=0}^{\eta} p_{\eta,\rho}^{(\alpha)}(\zeta) \int_0^1 \varphi\left(\frac{\rho+t}{\eta+1}\right) dt. \tag{5}$$

where $\alpha \in [0,1]$ and $\zeta \in \mathbb{C}$. For $\alpha = 1$, these operators are reduced to the classical complex Bernstein–Kantorovich operators [6]. Throughout the paper, $\mathbb{D}_R := \{\zeta \in \mathbb{C} : |\zeta| < R\}$ be a disc in the complex plane $\mathbb{C}$ and the space of all analytic functions on $\mathbb{D}_R$ denote by $H(\mathbb{D}_R)$. Also, for $\varphi \in H(\mathbb{D}_R)$, we assume that $\varphi(\zeta) = \sum_{\mu=0}^{\infty} a_\mu \zeta^\mu$.

The paper is organized as follows. In Sect. 2, we give some auxilary results for the proofs of the main theorems. In Sect. 3, we obtain respectively, upper quantitative estimates for the $K_{\eta,\alpha}(\varphi;\zeta)$ and its derivatives on compact disks. Finally, we obtain Voronovskaja type results and the exact degrees of approximation α-Bernstein–Kantorovich operator and their derivatives.

2. Auxiliary Results

Lemma 1. *For all* $\mu \in \mathbb{N} \cup \{0\}, \eta \in \mathbb{N}, \alpha \in [0,1]$ *and* $\zeta \in \mathbb{C}$, *we have*

$$K_{\eta,\alpha}(e_\mu;\zeta) = \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\eta^j}{(\eta+1)^\mu (\mu-j+1)} T_{\eta,\alpha}(e_j;\zeta), \tag{6}$$

where $e_\mu(\zeta) = \zeta^\mu$.

Proof. From 5, we can write,

$$\begin{aligned} K_{\eta,\alpha}(e_\mu;\zeta) &= \sum_{\rho=0}^{\eta} p_{\eta,\rho}^{(\alpha)}(\zeta) \int_0^1 \left(\frac{\rho+t}{\eta+1}\right)^\mu dt \\ &= \sum_{\rho=0}^{\eta} p_{\eta,\rho}^{(\alpha)}(\zeta) \sum_{j=0}^{\mu} \int_0^1 \binom{\mu}{j} \frac{t^{\mu-j}\rho^j}{(\eta+1)^\mu} dt \\ &= \frac{1}{(\eta+1)^\mu} \sum_{\rho=0}^{\eta} p_{\eta,\rho}^{(\alpha)}(\zeta) \sum_{j=0}^{\mu} \binom{\mu}{j} \rho^j \int_0^1 t^{\mu-j} dt \\ &= \frac{1}{(\eta+1)^\mu} \sum_{j=0}^{\mu} \frac{\eta^j}{(\mu-j+1)} T_{\eta,\alpha}(e_j;\zeta). \end{aligned}$$

□

Lemma 2. *For all* $\zeta \in \mathbb{D}_r$, $\alpha \in [0,1]$ *and* $1 \le r$, *we have*

$$|K_{\eta,\alpha}(e_\mu;\zeta)| \le r^\mu, \qquad \mu,\eta \in \mathbb{N}. \tag{7}$$

Proof. We know that (see [1])

$$|T_{\eta,\alpha}(e_j;\zeta)| \le r^j.$$

From 6, we obtain

$$\begin{aligned} |K_{\eta,\alpha}(e_\mu;\zeta)| &\le \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\eta^j}{(\eta+1)^\mu (\mu-j+1)} |T_{\eta,\alpha}(e_j;\zeta)| \\ &\le \frac{1}{(\eta+1)^\mu} \sum_{j=0}^{\mu} \binom{\mu}{j} \eta^j r^\mu = r^\mu \end{aligned}$$

□

Lemma 3. *For all* $\alpha \in [0,1]$ *and* $\eta,\mu \in \mathbb{N}$, $\zeta \in \mathbb{C}$, *we have*

$$\begin{aligned} K_{\eta,\alpha}(e_\mu;\zeta) &= \frac{\zeta(1-\zeta)}{\eta} \left(K_{\eta,\alpha}(e_{\mu-1};\zeta)\right)' + \left[\zeta + \frac{(1-\zeta)}{\eta}\right] K_{\eta,\alpha}(e_{\mu-1};\zeta) \\ &\quad - \frac{\alpha(1-\zeta)}{\eta} K_\eta(e_{\mu-1};\zeta) \\ &\quad + \frac{1}{(\eta+1)^\mu} \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\eta^{j-1}}{(\mu-j+1)} \left(\frac{\eta\mu - j(\eta+1)}{\mu}\right) T_{\eta,\alpha}(e_j;\zeta) \end{aligned}$$

$$+\frac{1}{(\eta+1)^{\mu-1}}\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\mu-j)}\{B_{\eta-1}(e_{j+1};\zeta)$$
$$-B_{\eta-1}(e_j;\zeta)\} \tag{8}$$

where $K_\eta(e_\mu;\zeta)$ complex Bernstein–Kantorovich operator (see [9]).

Proof. From [1], we have

$$\begin{aligned}\frac{\zeta(1-\zeta)}{\eta}\left(T_{\eta,\alpha}(e_j;\zeta)\right)' &= T_{\eta,\alpha}(e_{j+1};\zeta)-\left[\zeta+\frac{(1-\zeta)}{\eta}\right]T_{\eta,\alpha}(e_j;\zeta)\\ &\quad-\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\{B_{\eta-1}(e_{j+1};\zeta)-B_{\eta-1}(e_j;\zeta)\}\\ &\quad+\frac{\alpha(1-\zeta)}{\eta}B_\eta(e_j;\zeta).\end{aligned} \tag{9}$$

Differentiatin g $K_{\eta,\alpha}(e_\mu;\zeta)$ with respect to $\zeta\neq 0$ and using (9), we have

$$\begin{aligned}\frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_\mu;\zeta)\right)' &= \frac{1}{(\eta+1)^{\mu}}\sum_{j=0}^{\mu}\binom{\mu}{j}\frac{\eta^{j}}{(\mu-j+1)}\frac{\zeta(1-\zeta)}{\eta}T\left({}_{\eta,\alpha}(e_j;\zeta)\right)'\\ &= \frac{1}{(\eta+1)^{\mu}}\sum_{j=0}^{\mu}\binom{\mu}{j}\frac{\eta^{j}}{(\mu-j+1)}\Bigg\{T_{\eta,\alpha}(e_{j+1};\zeta)\\ &\quad-\left[\zeta+\frac{(1-\zeta)}{\eta}\right]T_{\eta,\alpha}(e_j;\zeta)\\ &\quad-\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\{B_{\eta-1}(e_{j+1})-B_{\eta-1}(e_j)\}\\ &\quad+\frac{\alpha(1-\zeta)}{\eta}B_\eta(e_j;\zeta)\Bigg\}.\end{aligned}$$

by some calculations we get

$$\begin{aligned}K_{\eta,\alpha}(e_{\mu+1};\zeta) &= \frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_\mu;\zeta)\right)'+\left[\zeta+\frac{(1-\zeta)}{\eta}\right]K_{\eta,\alpha}(e_\mu;\zeta)\\ &\quad+\sum_{j=0}^{\mu+1}\binom{\mu+1}{j}\frac{\eta^{j}}{(\eta+1)^{\mu+1}(\mu-j+2)}T_{\eta,\alpha}(e_j;\zeta)\\ &\quad+\sum_{j=0}^{\mu}\binom{\mu}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\{B_{\eta-1}(e_{j+1})-B_{\eta-1}(e_j)\}\\ &\quad-\sum_{j=1}^{\mu+1}\binom{\mu}{j-1}\frac{\eta^{j-1}}{(\eta+1)^{\mu}(\mu-j+2)}T_{\eta,\alpha}(e_j;\zeta)-\frac{\alpha(1-\zeta)}{\eta}K_\eta(e_\mu;\zeta)\\ &= \frac{\zeta(1-\zeta)}{\eta}\left(K_\eta^{(\alpha)}(e_\mu;\zeta)\right)'+\left[\zeta+\frac{(1-\zeta)}{\eta}\right]K_{\eta,\alpha}(e_\mu;\zeta)+\frac{1}{(\mu+2)(\eta+1)^{\mu+1}}\\ &\quad+\sum_{j=1}^{\mu+1}\binom{\mu+1}{j}\frac{\eta^{j-1}}{(\eta+1)^{\mu}(\mu-j+2)}\left\{\frac{(\mu+1)\eta-j(\eta+1)}{(\mu+1)(\eta+1)}\right\}T_{\eta,\alpha}(e_j;\zeta)\end{aligned}$$

$$+\sum_{j=0}^{\mu}\binom{\mu}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\left\{B_{\eta-1}(e_{j+1})-B_{\eta-1}(e_{j})\right\}$$
$$-\frac{\alpha(1-\zeta)}{\eta}K_{\eta}(e_{\mu};\zeta)$$
$$=\frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,q}(e_{\mu};\zeta)\right)'+\left[\zeta+\frac{(1-\zeta)}{\eta}\right]K_{\eta,\alpha}(e_{\mu};\zeta)$$
$$+\sum_{j=0}^{\mu+1}\binom{\mu+1}{j}\frac{\eta^{j-1}}{(\eta+1)^{\mu+1}(\mu-j+2)}\left\{\frac{(\mu+1)\eta-j(\eta+1)}{(\mu+1)}\right\}T_{\eta,\alpha}(e_{j};\zeta)$$
$$+\sum_{j=0}^{\mu}\binom{\mu}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\left\{B_{\eta-1}(e_{j+1})-B_{\eta-1}(e_{j})\right\}$$
$$-\frac{\alpha(1-\zeta)}{\eta}K_{\eta}(e_{\mu};\zeta).$$

Here we used the identity

$$\binom{\mu}{j-1}=\binom{\mu+1}{j}\frac{j}{(\mu+1)}.$$

□

For $\eta\in\mathbb{N},\zeta\in\mathbb{C}$, $\mu\in\mathbb{N}\cup\{0\}$ and $\alpha\in[0,1]$, if we denote

$$E_{\eta,\mu,\alpha}(\zeta):=K_{\eta,\alpha}(e_{\mu};\zeta)-e_{\mu}(\zeta)-\frac{1-2\zeta}{2(\eta+1)}\mu\zeta^{\mu-1}-\frac{\zeta(1-\zeta)}{2(\eta+1)}\mu(\mu-1)\zeta^{\mu-2},$$

then it is clear that degree$(E_{\eta,\mu,\alpha}(\zeta))\leq\mu$. Using the above recurrence and by simple calculations, we get the following Lemma.

Lemma 4. *For all $\alpha\in[0,1]$ and $\eta,\mu\in\mathbb{N}$, we have*

$$E_{\eta,\mu,\alpha}(\zeta)=\frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)'+\zeta E_{\eta,\mu-1,\alpha}(\zeta)$$
$$+\frac{(\mu-1)}{\eta(\eta+1)}\zeta^{\mu-1}(1-\zeta)-\frac{1-2\zeta}{2(\eta+1)}\zeta^{\mu-1}+\frac{(1-\zeta)}{\eta}K_{\eta,\alpha}(e_{\mu-1};\zeta)$$
$$+\sum_{j=0}^{\mu}\binom{\mu}{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\left\{\frac{\mu\eta-j(\eta+1)}{\eta}\right\}T_{\eta,\alpha}(e_{j};\zeta)$$
$$+\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\left(\frac{\eta-1}{\eta}\right)^{j}\left(\frac{1-\alpha}{\eta}\right)\frac{\eta^{j}}{(\eta+1)^{\mu-1}(\mu-j)}\left\{B_{\eta-1}(e_{j+1})\right.$$
$$\left.-B_{\eta-1}(e_{j})\right\}-\frac{\alpha(1-\zeta)}{\eta}K_{\eta}(e_{\mu-1};\zeta)$$

Proof. It is immediate that $E_{\eta,0,\alpha}(\zeta)=E_{\eta,1,\alpha}(\zeta)=0$.

Using the formula (8) we get

$$E_{\eta,\mu,\alpha}(\zeta)=\frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)'+\frac{(\mu-1)}{\eta}\zeta^{\mu-1}(1-\zeta)$$

$$+\left[\zeta+\frac{(1-\zeta)}{\eta}\right]\left(E_{\eta,\mu-1,\alpha}(\zeta)+\frac{1-2\zeta}{2(\eta+1)}(\mu-1)\zeta^{\mu-2}\right.$$
$$\left.+\frac{\zeta^{\mu-2}(1-\zeta)(\mu-1)(\mu-2)}{2(\eta+1)}\right)$$
$$+\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\eta+1)^{\mu-1}(\mu-j)}\left\{B_{\eta-1}(e_{j+1})\right.$$
$$\left.-B_{\eta-1}(e_j)\right\}-\frac{\alpha(1-\zeta)}{\eta}K_{\eta}(e_{\mu-1};\zeta)$$
$$+\sum_{j=0}^{\mu}\binom{\mu}{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\left\{\frac{\mu\eta-j(\eta+1)}{\eta}\right\}T_{\eta,\alpha}(e_j;\zeta)$$
$$-\frac{1-2\zeta}{2(\eta+1)}\mu\zeta^{\mu-1}-\frac{\mu(\mu-1)\zeta^{\mu-1}(1-\zeta)}{2(\eta+1)},$$

With a simple calculation, we get the following relation.

$$E_{\eta,\mu,\alpha}(\zeta)=\frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)'+\zeta E_{\eta,\mu-1,\alpha}(\zeta)$$
$$+\frac{(\mu-1)}{\eta(\eta+1)}\zeta^{\mu-1}(1-\zeta)-\frac{1-2\zeta}{2(\eta+1)}\zeta^{\mu-1}+\frac{(1-\zeta)}{\eta}K_{\eta,\alpha}(e_{\mu-1};\zeta)$$
$$+\sum_{j=0}^{\mu}\binom{\mu}{j}\frac{\eta^{j}}{(\eta+1)^{\mu}(\mu-j+1)}\left\{\frac{\mu\eta-j(\eta+1)}{\eta}\right\}T_{\eta,\alpha}(e_j;\zeta)$$
$$+\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\frac{\eta^{j}}{(\eta+1)^{\mu-1}(\mu-j)}\left\{B_{\eta-1}(e_{j+1})\right.$$
$$\left.-B_{\eta-1}(e_j)\right\}-\frac{\alpha(1-\zeta)}{\eta}K_{\eta}(e_{\mu-1};\zeta).$$

□

3. Approximation by Complex α-Bernstein–Kantorovich-Type Operator

In this section, we start with the upper quantitative estimates for the new operators and its derivatives attached to an analytic function in a compact disk of radius $1<R$ and center 0.

Theorem 5. *Suppose that* $\alpha\in[0,1]$ *and* $\varphi\in H(\mathbb{D}_R)$.
(i) Let $r\in[1,R)$ *be arbitrary fixed. For all* $|\zeta|\le r$ *and* $\eta\in\mathbb{N}$, *we have*

$$|K_{\eta,\alpha}(\varphi;\zeta)-\varphi(\zeta)|\le\frac{M_r(\varphi)}{\eta}, \tag{10}$$

where

$$0<M_r(\varphi)=(4+\alpha)\sum_{\mu=1}^{\infty}|a_\mu|\,\mu(\mu+1)r^{\mu}$$

(ii) If $1 \le r < r_1 < R$ are arbitrary fixed and $p \in \mathbb{N}$, then for all $|\zeta| \le r$ and $\eta \in \mathbb{N}$, we have

$$\left| K^{(p)}_{\eta,\alpha}(\varphi;\zeta) - \varphi^{(p)}(\zeta) \right| \le \frac{M_{r_1}(\varphi) p! r_1}{\eta (r_1 - r)^{p+1}},$$

where $M_{r_1}(\varphi)$ is given as in (i).

Proof. (i) To estimate $|K_{\eta,\alpha}(\varphi;\zeta) - \varphi(\zeta)|$, firstly, using above recurrence 8 we get

$$\begin{aligned} K_{\eta,\alpha}(e_\mu;\zeta) - e_\mu(\zeta) &= \frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)\right)' + \zeta\left(K_{\eta,\alpha}(e_{\mu-1};\zeta) - e_{\mu-1}(\zeta)\right) \\ &\quad - \frac{\alpha(1-\zeta)}{\eta} K_\eta(e_{\mu-1};\zeta) + \frac{(1-\zeta)}{\eta} K_{\eta,\alpha}(e_{\mu-1};\zeta) \\ &\quad + \left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{\mu-1} \frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}} \left\{B_{\eta-1}(e_\mu) - B_{\eta-1}(e_{\mu-1})\right\} \\ &\quad + \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\eta^{j-1}}{(\eta+1)^\mu (\mu-j+1)} \left\{\frac{\mu\eta - j(\eta+1)}{\mu(\eta+1)}\right\} T_{\eta,\alpha}(e_j;\zeta) \\ &\quad + . \sum_{j=0}^{\mu-2} \frac{\mu-1}{\mu-1-j} \binom{\mu-2}{j} \left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^j \frac{\eta^j}{(\eta+1)^{\mu-1}(\mu-j)} \\ &\quad \left\{B_{\eta-1}(e_{j+1}) - B_{\eta-1}(e_j)\right\} \end{aligned} \tag{11}$$

and secondly, we can estimate the above two sums (11) as:

$$\begin{aligned} &\left| \frac{1}{(\eta+1)^\mu} \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\eta^j}{(\mu-j+1)} \left(1 - \frac{j}{\mu} - \frac{j}{\mu\eta}\right) T_{\eta,\alpha}(e_j;\zeta) \right| \\ &\quad \le \frac{1}{(\eta+1)^\mu} \left(\sum_{j=0}^{\mu-1} \binom{\mu-1}{j} \frac{\mu}{\mu-j} \frac{\eta^j}{\mu-j+1} \left|1 - \frac{j}{\mu} - \frac{j}{\mu\eta}\right| \right) |T_{\eta,\alpha}(e_j;\zeta)| \\ &\quad + \frac{\eta^{\mu-1}}{(\eta+1)^\mu} r^\mu \\ &\quad \le \frac{2\mu(\eta+1)^{\mu-1} + \eta^{\mu-1}}{(\eta+1)^\mu} r^\mu \le \frac{2\mu+1}{(\eta+1)} r^\mu \end{aligned}$$

and

$$\begin{aligned} &\left| \sum_{j=0}^{\mu-2} \frac{\mu-1}{\mu-1-j} \binom{\mu-2}{j} \left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^j \frac{\eta^j}{(\eta+1)^{\mu-1}(\mu-j)} \left\{B_{\eta-1}(e_{j+1}) \right.\right. \\ &\quad \left.\left. - B_{\eta-1}(e_j)\right\} \right| \\ &\le \left(\frac{1-\alpha}{\eta}\right) \frac{1}{(\eta+1)^{\mu-1}} \left(\sum_{j=0}^{\mu-2} \binom{\mu-2}{j} \frac{\mu-1}{\mu-1-j} \frac{(\eta-1)^j}{(\mu-j)} \right) \left\{|B_{\eta-1}(e_{j+1})| \right. \\ &\quad \left. + |B_{\eta-1}(e_j)|\right\} \end{aligned}$$

$$\leq \frac{(1-\alpha)(\mu-1)(\eta-1+1)^{\mu-2}}{\eta(\eta+1)^{\mu-1}}\left(r^{\mu-1}+r^{\mu-2}\right) \leq \frac{(1-\alpha)(\mu-1)}{\eta}\left(r^{\mu-1}+r^{\mu-2}\right).$$

Also, it is known that, from Theorem 1.0.8 in [9], we have

$$\left|P'_{\mu}(\zeta)\right| \leq \frac{\mu}{r}\left\|P_{\mu}\right\|_{r}, \quad \text{for all} \quad |\zeta| \leq r, \quad r \geq 1,$$

where $P_{\mu}(\zeta)$ is a complex polynomial in ζ of degree $\leq \mu$.

Therefore, from the recurrence formula 11 we get

$$\begin{aligned}
|K_{\eta,\alpha}(e_{\mu};\zeta)-e_{\mu}(\zeta)| &\leq \frac{|\zeta||1-\zeta|}{\eta}\left|(K_{\eta,\alpha}(e_{\mu};\zeta))'\right| + |\zeta|\,|K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| \\
&\quad + \frac{\alpha|1-\zeta|}{\eta}|K_{\eta}(e_{\mu-1};\zeta)| + \frac{|1-\zeta|}{\eta}|K_{\eta,\alpha}(e_{\mu-1};\zeta)| \\
&\quad + \frac{2(1-\alpha)}{\eta}\left(r^{\mu}+r^{\mu-1}\right) + \frac{2\mu+1}{(\eta+1)}r^{\mu} \\
&\quad + \frac{(1-\alpha)(\mu-1)}{\eta}\left(r^{\mu-1}+r^{\mu-2}\right) \\
&\leq \frac{r(1+r)}{\eta}\frac{\mu-1}{r}\left\|K_{\eta,\alpha}(e_{\mu-1})\right\|_{r} + r\,|K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| \\
&\quad + \frac{\alpha(1+r)}{\eta}r^{\mu-1} + \frac{(1+r)}{\eta}r^{\mu-1} + \frac{(1-\alpha)}{\eta}\left(r^{\mu}+r^{\mu-1}\right) \\
&\quad + \frac{2\mu+1}{(\eta+1)}r^{\mu} + \frac{2(1-\alpha)(\mu-1)}{\eta}\left(r^{\mu-1}+r^{\mu-2}\right) \\
&\leq \frac{2(\mu-1)}{\eta}r^{\mu} + r\,|K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| + \frac{2\alpha}{\eta}r^{\mu} + \frac{2}{\eta}r^{\mu} \\
&\quad + \frac{2(1-\alpha)}{\eta}r^{\mu} + \frac{2\mu+1}{(\eta+1)}r^{\mu} + \frac{2(1-\alpha)(\mu-1)}{\eta}r^{\mu} \\
&= r\,|K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| + \frac{4\mu+2\alpha+1+2\mu(1-\alpha)}{\eta}r^{\mu}
\end{aligned}$$

Now, by taking $\mu = 1, 2, ...,$ in the inequality

$$\begin{aligned}
&|K_{\eta,\alpha}(e_{\mu};\zeta)-e_{\mu}(\zeta)| \\
&\quad \leq \frac{4\mu+2\alpha+1+2\mu(1-\alpha)}{\eta}r^{\mu} \\
&\qquad + r\frac{4(\mu-1)+2\alpha+1+2(\mu-1)(1-\alpha)}{\eta}r^{\mu-1} \\
&\qquad + r^{2}\frac{4(\mu-2)+2\alpha+1+2(\mu-2)(1-\alpha)}{\eta}r^{\mu-2} \\
&\qquad + ... + r^{\mu-1}\frac{4+2\alpha+1+2(1-\alpha)}{\eta}r \\
&\leq \frac{4+2(1-\alpha)}{\eta}r^{\mu}(\mu+\mu-1+...+1) + \frac{\mu(2\alpha+1)}{\eta}r^{\mu} \\
&\quad = \frac{\{2+(1-\alpha)\}\mu(\mu+1)+\mu(2\alpha+1)}{\eta}r^{\mu}
\end{aligned}$$

$$\leq \frac{(4+\alpha)\,\mu\,(\mu+1)}{\eta} r^{\mu}. \tag{12}$$

Since $K_{\eta,\alpha}(\varphi;\zeta)$ is analytic in $\mathbb{D}_R$, we can write

$$K_{\eta,\alpha}(\varphi;\zeta) = \sum_{\mu=0}^{\infty} a_\mu K_{\eta,\alpha}(e_\mu;\zeta), \qquad \zeta \in \mathbb{D}_R,$$

which together with (12) immediately implies for all $|\zeta| \leq r$

$$\begin{aligned}|K_{\eta,\alpha}(\varphi;\zeta) - \varphi(\zeta)| &\leq \sum_{\mu=0}^{\infty} |a_\mu|\,|K_{\eta,\alpha}(e_\mu;\zeta) - e_\mu(\zeta)| \\ &\leq \frac{(4+\alpha)}{\eta} \sum_{\mu=1}^{\infty} |c_\mu|\,\mu\,(\mu+1)\,r^{\mu}.\end{aligned}$$

(ii) For the simultaneous approximation, denoting by γ the circle of radius $r < r_1$ and center 0 (where $r_1 > r \geq 1$), by the Cauchy's formulas it follows that for all $|\zeta| \leq r$ and $\eta \in \mathbb{N}$ we have

$$K_{\eta,\alpha}^{(p)}(\varphi;\zeta) - \varphi^{(p)}(\zeta) \leq \frac{p!}{2\pi i} \int_{\gamma} \frac{K_{\eta,\alpha}(\varphi;v) - \varphi(v)}{(v-\zeta)^{p+1}} dv,$$

which by (10) and by the inequality $|v - \zeta| \geq r_1 - r$ valid for all $|\zeta| \leq r$ and $v \in \gamma$, we have

$$\begin{aligned}\left|K_{\eta,\alpha}^{(p)}(\varphi;\zeta) - \varphi^{(p)}(\zeta)\right| &\leq \frac{p!}{2\pi} \int_{\gamma} \frac{|K_{\eta,\alpha}(\varphi;v) - \varphi(v)|}{|v-\zeta|^{p+1}} |dv| \\ &\leq \frac{p!}{2\pi} \frac{M_{r_1}(\varphi)}{\eta} \frac{2\pi r_1}{(r_1 - r)^{p+1}} = \frac{M_{r_1}(\varphi) p! r_1}{\eta (r_1 - r)^{p+1}}\end{aligned}$$

which proves (ii). □

In the next Theorem, we obtain the Voronovskaja type formula with a quantitative estimate for complex α-Bernstein–Kantorovich operators.

Theorem 6. *Let $r \in [1, R)$ and $\varphi \in H(\mathbb{D}_R)$. Then for all $\eta \in \mathbb{N}$ and $\alpha \in [0,1]$, we have*

$$\begin{aligned}&\left|K_{\eta,\alpha}(\varphi;\zeta) - \varphi(\zeta) - \frac{1-2\zeta}{2(\eta+1)}\varphi'(\zeta) - \frac{\zeta(1-\zeta)}{2(\eta+1)}\varphi''(\zeta)\right| \\ &\quad\leq \frac{1}{\eta^2} \sum_{\mu=2}^{\infty} |a_\mu| \left\{(43 - 7\alpha)\,\mu(\mu-1)^2\right\} r^{\mu} + \frac{5}{\eta^2} \sum_{\mu=1}^{\infty} \mu(\mu+1) r^{\mu}.\end{aligned}$$

Proof. Using reccurrence formula (8) and simple calculation leads us to the following relationship:

$$
\begin{aligned}
E_{\eta,\mu,\alpha}(\zeta) &= \frac{\zeta(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)' + \zeta E_{\eta,\mu-1,\alpha}(\zeta)\\
&+\frac{(\mu-1)}{\eta(\eta+1)}\zeta^{\mu-1}(1-\zeta)\\
&+\frac{1}{(\eta+1)}\left(\zeta^{\mu}-T_{\eta,\alpha}(e_{\mu};\zeta)\right)+\frac{1}{(\eta+1)}\left(1-\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\right)T_{\eta,\alpha}(e_{\mu};\zeta)\\
&-\frac{1}{2(\eta+1)}\left(1-\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\right)T_{\eta,\alpha}(e_{\mu-1};\zeta)+\frac{1}{2(\eta+1)}\left(T_{\eta,\alpha}(e_{\mu-1};\zeta)-\zeta^{\mu-1}\right)\\
&-\frac{(\mu-1)\eta^{\mu-2}}{2(\eta+1)^{\mu}}T_{\eta,\alpha}(e_{\mu-1};\zeta)+\frac{1}{(\eta+1)^{\mu}}\sum_{j=0}^{\mu-2}\binom{\mu}{j}\\
&\frac{\eta^{j}}{(\mu-j+1)}\frac{\mu\eta-j(\eta+1)}{\mu\eta}T_{\eta,\alpha}(e_{j};\zeta)\\
&-\left(K_{\eta}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)\left(\frac{\alpha(1-\zeta)}{\eta}\right)+\frac{(1-\zeta)}{\eta}\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)\\
&-\frac{1-\alpha}{\eta}\left(\frac{\eta-1}{\eta}\right)^{\mu-1}\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\left(B_{\eta-1}(e_{\mu-1})-e_{\mu-1}(\zeta)\right)\\
&+\frac{\zeta^{\mu-1}(1-\alpha)}{\eta}\left(1-\frac{(\eta-1)^{\mu-1}}{(\eta+1)^{\mu-1}}\right)+\frac{1-\alpha}{\eta}\left(\frac{\eta-1}{\eta}\right)^{\mu-1}\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\left(B_{\eta-1}(e_{\mu})-e_{\mu}(\zeta)\right)\\
&-\left(1-\frac{(\eta-1)^{\mu-1}}{(\eta+1)^{\mu-1}}\right)\frac{\zeta^{\mu}(1-\alpha)}{\eta}+\sum_{j=0}^{\mu-2}\binom{\mu-1}{j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^{j}\\
&\frac{\eta^{j}}{(\eta+1)^{\mu-1}(\mu-j)}\left\{B_{\eta-1}(e_{j+1})-B_{\eta-1}(e_{j})\right\}\\
&:=\sum_{\rho=1}^{16}I_{\rho}. \qquad (13)
\end{aligned}
$$

Firstly, the estimate of I_3 are I_8 as follows

$$
\begin{aligned}
|I_3| &\le \frac{(\mu-1)}{\eta(\eta+1)}r^{\mu-1}(1+r),\\
|I_8| &\le \frac{(\mu-1)}{2(\eta+1)^2}\left|T_{\eta,\alpha}(e_{\mu-1};\zeta)\right| \le \frac{(\mu-1)}{2(\eta+1)^2}r^{\mu-1}, \qquad (14)
\end{aligned}
$$

Secondly using the known inequality

$$
1-\prod_{\rho=1}^{\mu}\tau_{\rho}\le\sum_{\rho=1}^{\mu}(1-\tau_{\rho}),\ 0\le\tau_{\rho}\le 1,
$$

to estimate $I_5, I_6, I_9, I_{13}, I_{15}, I_{16}$.

$$
\begin{aligned}
|I_5| &\leq \frac{1}{\eta+1}\left(1-\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\right)|T_{\eta,\alpha}(e_\mu;\zeta)| \leq \frac{\mu-1}{(\eta+1)^2}r^\mu,\\
|I_6| &\leq \frac{1}{2(\eta+1)}\left(1-\frac{\eta^{\mu-1}}{(\eta+1)^{\mu-1}}\right)|T_{\eta,\alpha}(e_{\mu-1};\zeta)| \leq \frac{\mu-1}{2(\eta+1)^2}r^{\mu-1},\\
|I_9| &\leq \frac{1}{(\eta+1)^\mu}\sum_{j=0}^{\mu-2}\binom{\mu-2}{j}\frac{\mu(\mu-1)}{(\mu-j)(\mu-j-1)}\frac{\eta^j}{(\mu-j+1)}\left(1-\frac{j}{\mu}-\frac{j}{\mu\eta}\right)r^j\\
&\leq \frac{2\mu(\mu-1)(\eta+1)^{\mu-2}}{(\eta+1)^\mu}r^\mu = \frac{2\mu(\mu-1)}{(\eta+1)^2}r^\mu,\\
|I_{13}| &\leq \frac{\zeta^{\mu-1}(1-\alpha)}{\eta}\left(1-\frac{(\eta-1)^{\mu-1}}{(\eta+1)^{\mu-1}}\right) \leq \frac{2(1-\alpha)(\mu-1)}{\eta(\eta+1)}r^{\mu-1},\\
|I_{15}| &\leq \frac{\zeta^{\mu}(1-\alpha)}{\eta}\left(1-\frac{(\eta-1)^{\mu-1}}{(\eta+1)^{\mu-1}}\right) \leq \frac{2(1-\alpha)(\mu-1)}{\eta(\eta+1)}r^{\mu},\\
|I_{16}| &\leq \sum_{j=0}^{\mu-2}\binom{\mu-2}{j}\frac{\mu-1}{\mu-1-j}\left(\frac{1-\alpha}{\eta}\right)\left(\frac{\eta-1}{\eta}\right)^j\frac{\eta^j}{(\eta+1)^{\mu-1}(\mu-j)}\{B_{\eta-1}(e_{j+1})\\
&\quad -B_{\eta-1}(e_j)\}\\
&\leq \frac{(1-\alpha)(\mu-1)}{\eta(\eta+1)}\left(r^{\mu-1}+r^{\mu-2}\right). \qquad (15)
\end{aligned}
$$

Finally we estimate $I_4, I_7, I_{10}, I_{11}, I_{12}, I_{14}$. We use [9, Theorem 1.1.2], [1, Theorem 2.4] and Theorem 5(i)

$$
\begin{aligned}
&|I_4|+|I_7|+|I_{10}|+|I_{11}|+|I_{12}|+|I_{14}|\\
&\leq \frac{1}{(\eta+1)}|\zeta^\mu - T_{\eta,\alpha}(e_\mu;\zeta)| + \frac{1}{2(\eta+1)}\left|T_{\eta,\alpha}(e_{\mu-1};\zeta)-\zeta^{\mu-1}\right|\\
&\quad+\frac{\alpha(1+r)}{\eta}|K_\eta(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| + \frac{(1+r)}{\eta}|K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)|\\
&\quad+\frac{1-\alpha}{\eta}|B_{\eta-1}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)| + \frac{1-\alpha}{\eta}|B_{\eta-1}(e_\mu;\zeta)-e_\mu(\zeta)|\\
&\leq \frac{1}{(\eta+1)}\left\{\left(\frac{3\mu(\mu+1)}{2}+\mu\right)\frac{(1+r)}{\eta}r^{\mu-1} + \frac{2\mu(\mu-1)}{\eta}r^\mu\right\}\\
&\quad+\frac{1}{2(\eta+1)}\left\{\left(\frac{3(\mu-1)\mu}{2}+(\mu-1)\right)\frac{(1+r)}{\eta}r^{\mu-2} + \frac{2(\mu-1)(\mu-2)}{\eta}r^{\mu-1}\right\}\\
&\quad+\frac{\alpha(1+r)}{\eta}\left\{\frac{2(\mu-1)\mu}{\eta}r^{\mu-1}\right\} + \frac{(4+\alpha)}{\eta}\left\{\frac{2(\mu-1)\mu}{\eta}r^{\mu-1}\right\}\\
&\quad+\frac{1-\alpha}{\eta}\left\{\frac{3r(1+r)(\mu-2)(\mu-1)}{2\eta}r^{\mu-3}\right\} + \frac{1-\alpha}{\eta}\left\{\frac{3r(1+r)(\mu-1)\mu}{2\eta}r^{\mu-2}\right\}. \qquad (16)
\end{aligned}
$$

Using (12), (14), (15) and (16) in (13) finally we have

$$
|E_{\eta,\mu,\alpha}(\zeta)| \leq \frac{r(1+r)}{\eta}\left|\left(K_{\eta,\alpha}(e_{\mu-1};\zeta)-e_{\mu-1}(\zeta)\right)'\right| + r\,|E_{\eta,\mu-1,\alpha}(\zeta)|
$$

$$
\begin{aligned}
&+\frac{(\mu-1)}{\eta(\eta+1)}r^{\mu-1}(1+r)\\
&+\frac{(\mu-1)}{(\eta+1)^{2}}r^{\mu}+\frac{\mu-1}{2(\eta+1)^{2}}r^{\mu-1}+\frac{2\mu(\mu-1)}{(\eta+1)^{2}}r^{\mu}+\frac{(1-\alpha)(\mu-1)}{\eta(\eta+1)}r^{\mu-1}\\
&+\frac{(1-\alpha)(\mu-1)}{\eta(\eta+1)}r^{\mu}+\frac{(1-\alpha)(\mu-1)}{\eta(\eta+1)}\left(r^{\mu-1}+r^{\mu-2}\right)\\
&+\frac{(1+r)}{\eta(\eta+1)}\left(\frac{3\mu(\mu+1)}{2}+\mu\right)r^{\mu-1}+\frac{2\mu(\mu-1)}{\eta(\eta+1)}r^{\mu}\\
&+\frac{(1+r)}{2\eta(\eta+1)}\left(\frac{3(\mu-1)\mu}{2}+\mu-1\right)r^{\mu-2}+\frac{(\mu-1)(\mu-2)}{\eta(\eta+1)}r^{\mu-1}\\
&+\frac{2\alpha\mu(\mu-1)(1+r)}{\eta^{2}}r^{\mu-1}+\frac{(4+\alpha)(1+r)(\mu-1)\mu}{\eta^{2}}r^{\mu-1}\\
&+\frac{3(1+r)(\mu-2)(\mu-1)(1-\alpha)}{2\eta^{2}}r^{\mu-2}+\frac{3(1+r)(1-\alpha)\mu(\mu-1)}{2\eta^{2}}r^{\mu-1}\\
\leq{}&\frac{r(1+r)}{\eta}\frac{\mu-1}{r}\left\|K_{\eta,\alpha}(e_{\mu-1})-e_{\mu-1}\right\|_{r}+r\left|E_{\eta,\mu-1,\alpha}(\zeta)\right|+\frac{23\mu(\mu-1)}{\eta^{2}}r^{\mu}\\
&\frac{9\mu(\mu-1)(1-\alpha)}{\eta^{2}}r^{\mu}+\frac{3\mu(\mu-1)\alpha}{\eta^{2}}r^{\mu}+\frac{5\mu(\mu+1)}{\eta^{2}}r^{\mu}\\
\leq{}&+\frac{(\mu-1)(1+r)}{\eta}\frac{(4+\alpha)(\mu-1)\mu}{\eta}r^{\mu-1}+r\left|E_{\eta,\mu-1,\alpha}(\zeta)\right|+\frac{23\mu(\mu-1)}{\eta^{2}}r^{\mu}\\
&\frac{9\mu(\mu-1)(1-\alpha)}{\eta^{2}}r^{\mu}+\frac{3\mu(\mu-1)\alpha}{\eta^{2}}r^{\mu}+\frac{5\mu(\mu+1)}{\eta^{2}}r^{\mu}\\
\leq{}&r\left|E_{\eta,\mu-1,\alpha}(\zeta)\right|+\frac{2(4+\alpha)\mu(\mu-1)^{2}}{\eta^{2}}r^{\mu}+\frac{23\mu(\mu-1)}{\eta^{2}}r^{\mu}\\
&+\frac{9\mu(\mu-1)(1-\alpha)}{\eta^{2}}r^{\mu}+\frac{3\mu(\mu-1)\alpha}{\eta^{2}}r^{\mu}+\frac{5\mu(\mu+1)}{\eta^{2}}r^{\mu}\\
\leq{}&r\left|E_{\eta,\mu-1,\alpha}(\zeta)\right|+\frac{(43-7\alpha)\mu(\mu-1)^{2}}{\eta^{2}}r^{\mu}+\frac{5\mu(\mu+1)}{\eta^{2}}r^{\mu}
\end{aligned}
$$

As a consequence, we get

$$
\left|E_{\eta,\mu,\alpha}(\zeta)\right|\leq\frac{(43-7\alpha)\mu(\mu-1)^{2}+5\mu(\mu+1)}{\eta^{2}}r^{\mu}
$$

In conclusion,

$$
\begin{aligned}
&\left|K_{\eta,\alpha}(\varphi;\zeta)-\varphi(\zeta)-\frac{1-2\zeta}{2(\eta+1)}\varphi'(\zeta)-\frac{\zeta(1-\zeta)}{2(\eta+1)}\varphi''(\zeta)\right|\\
&\quad\leq\sum_{\mu=0}^{\infty}|a_{\mu}|\left|E_{\eta,\mu,\alpha}(\zeta)\right|
\end{aligned}
$$

$$= \frac{(43-7\alpha)}{\eta^2}\sum_{\mu=2}^{\infty}|a_\mu|\,\mu(\mu-1)^2 r^\mu + \frac{5}{\eta^2}\sum_{\mu=1}^{\infty}|a_\mu|\,\mu(\mu+1)r^\mu$$

Note that since $\varphi^{(3)}(\zeta) = \sum_{\mu=3}^{\infty} a_\mu \mu(\mu-1)(\mu-2)\zeta^{\mu-3}$ and $\varphi^{(2)}(\zeta) = \sum_{\mu=2}^{\infty} a_\mu \mu(\mu-1)\zeta^{\mu-2}$, and sumation of these two series is absolutely convergent for all $|\zeta| < R$, it easily follows that $(43-7\alpha)\sum_{\mu=2}^{\infty}|a_\mu|\,\mu(\mu-1)^2 r^\mu + 5\sum_{\mu=1}^{\infty}|a_\mu|\,\mu(\mu+1)r^\mu < \infty$. □

Remark 7. In hypothesis of in φ in Theorem 5 choosing $\alpha \in [0,1]$ as $\eta \to \infty$, it follows that

$$\lim_{\eta\to\infty}(\eta+1)\left[K_{\eta,\alpha}(\varphi;\zeta) - \varphi(\zeta)\right] = \frac{1-2\zeta}{2}\varphi'(\zeta) + \frac{\zeta(1-\zeta)}{2}\varphi''(\zeta) \quad (17)$$

Finally, in the next Theorem, we present the order of approximation in Theorem 5 is exactly $\frac{1}{\eta}$.

Theorem 8. *Suppose that* $0 \le \alpha \le 1$ *and* $\varphi \in H(\mathbb{D})_R$.

(i) If φ *is not polynomial in* ζ *of degree* $= 0$, *then for all* $r \in [1, R)$ *,we have*

$$\|K_{\eta,\alpha}(\varphi) - \varphi\|_r \sim \frac{1}{\eta}, \quad \eta \in \mathbb{N}$$

where the constants in the equivalence depend on r *and* φ.
(ii) If $1 \le r < r_1 < R$ *and* φ *is not a polynomial in* ζ *of degree* $\le \max\{1, p-1\}$ *(*$p \in \mathbb{N}$*), we have*

$$\left\|K_{\eta,\alpha}^{(p)}(\varphi) - \varphi^{(p)}\right\|_r \sim \frac{1}{\eta}, \quad \eta \in \mathbb{N}$$

in $\overline{\mathbb{D}_R}$*, where the constant in the equivalence depend on* r, r_1, φ *and* p.

Proof. (i) Taking in to account Remark 7, there exist constants $0 < D_1, D_2 < \infty$ independent of η such that

$$D_1 \le \eta\,\|K_{\eta,\alpha}(\varphi) - \varphi\|_r \le D_2$$

from which it readlily follows that

$$\frac{D_1}{\eta} \le \|K_{\eta,\alpha}(\varphi) - \varphi\|_r \le \frac{D_2}{\eta}$$

in $\overline{\mathbb{D}_r}$. Therefore, we get the desired result.

(ii) Denoting by ρ the circle of radius $r < r_1$and center 0 (where $r \in [1, r_1)$), we have the inequality $r_1 - r \le |v - \zeta|$ valid for all $|\zeta| \le r$ and $v \in \rho$.

Using the Cauchy's formula, for all $|\zeta| \le r$ and $\eta, p \in \mathbb{N}$, we obtain

$$K_{\eta,\alpha}^{(p)}(\varphi;\zeta) - \varphi^{(p)}(\zeta) = \frac{p!}{2\pi i}\int_\gamma \frac{K_{\eta,\alpha}(\varphi;v) - \varphi(v)}{(v-\zeta)^{p+1}}dv.$$

Thus, from Remark 7 we get

$$\lim_{\eta\to\infty}\eta\left[K_{\eta,\alpha}^{(p)}(\varphi;\zeta)-\varphi^{(p)}(\zeta)\right]=\left[\frac{1-2\zeta}{2}\varphi'(\zeta)+\frac{\zeta(1-\zeta)}{2}\varphi''(\zeta)\right]^{(p)}$$

uniformly in $\overline{\mathbb{D}_r}$.

It remains to show that $\varphi(\zeta)$ is not constant. Indeed, supposing that $(1-2\zeta)\varphi'(\zeta)+\zeta(1-\zeta)\varphi''(\zeta)=0$ for all $|\zeta|\le r$, that is $(\zeta(1-\zeta)\varphi'(\zeta))'=0$ for all $|\zeta|\le r$. The last equality is equivalent to $\zeta(1-\zeta)\varphi'(\zeta)=C$ for all $|\zeta|\le r$ with $\zeta\neq 0$. Therefore we get $\varphi'(\zeta)=\frac{2C}{\zeta(1-\zeta)}$, for all $|\zeta|\le r$ with $\zeta\neq 0$. But since φ is analytic in $\overline{D_r}$, we necessarily have $C=0$, which implies $\varphi'(\zeta)=0$ and $\varphi(\zeta)=const$ for all $\zeta\in\overline{D_r}$.

Therefore, there exist constant $0<M_1<M_2<\infty$ independent of η such that

$$M_1\le\eta\left\|K_{\eta,\alpha}^{(p)}(\varphi)-\varphi^{(p)}\right\|_r\le M_2,$$

which readly follows that

$$\frac{M_1}{\eta}\le\left\|K_{\eta,\alpha}^{(p)}(\varphi)-\varphi^{(p)}\right\|_r\le\frac{M_2}{\eta}$$

in $\overline{\mathbb{D}_r}$. This completes the proof. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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