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Parabolicity on Graphs

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Abstract. Large scale properties of Riemannian manifolds, in particular, those properties preserved by quasi-isometries, can be studied using discrete structures which approximate the manifolds. In a sequence of papers, M. Kanai proved that, under mild conditions, many properties are preserved by a certain (quasi-isometric) graph approximation of a manifold. One of these properties is p-parabolicity. A manifold M (respectively, a graph G) is said to be p-parabolic if all positive p-superharmonic functions on M (resp. G) are constant. This is equivalent to not having p-Green's function (i.e. a positive fundamental solution of the p-Laplace-Beltrami operator). Herein we study directly the p-parabolicity on graphs. We obtain some characterizations in terms of graph decompositions. Also, we give necessary and sufficient conditions for a uniform hyperbolic graph to be p-parabolic in terms of its boundary at infinity. Finally, we prove that if a uniform hyperbolic graph satisfies the (Cheeger) isoperimetric inequality, then it is non-p-parabolic for every $1 < p < \infty$.

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1. Introduction

Using discrete structures approximating Riemannian manifolds has proven to be a useful tool in the study of large scale properties. This is done by M. Kanai defining a graph, the ε-net of the manifold, so that the manifold and its ε-net are quasi-isometric. There are many works following Kanai's ideas or proving the relation between the large scale behavior of a manifold and some other associated graph (see, e.g., [3, 14, 23, 25, 27, 28, 36–38, 43–46, 52, 54]).

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This large-scale structure of the manifold or the corresponding graph is usually preserved by quasi-isometries, which at the same time, allow an important distortion of the local geometry. One of the main large-scale properties preserved by quasi-isometries is the Gromov hyperbolicity (see, e.g., [24,26]).

In [31–33], M. Kanai studied several geometric properties (such as isoperimetric inequalities, Poincaré-Sobolev inequalities, parabolicity, growth rate of the volume of balls, and Liouville type theorems) for a large class of Riemannian manifolds with some conditions on their local geometry. Kanai proved that these properties are preserved under quasi-isometries. Also, quasi-isometries preserve the parabolic Harnack inequality (see [17]) and several estimates on transition probabilities of random walks, such as heat kernel estimates. Moreover, Holopainen and Soardi, among other authors (see [27,28,52]), proved that the existence of non-trivial solutions of a wide class of partial differential equations is also preserved under quasi-isometries.

A manifold M (respectively, a graph G) is said to be p-parabolic if all positive p-superharmonic functions on M (resp. G) are constant. This is equivalent to not having p-Green's function (i.e. a positive fundamental solution of the p-Laplace-Beltrami operator). In [39] we studied the stability of p-parabolicity (with $1 < p < \infty$) by quasi-isometries between Riemannian manifolds weakening Kanai's assumptions. Also, we obtained some results on the p-parabolicity of graphs and trees; in particular, we characterized p-parabolicity for a large class of trees.

Our focus herein is to study p-parabolicity on (uniform) graphs. Sects. 2 and 3 provide the necessary background: in Sect. 2 we include basic definitions and main tools in the study of p-parabolicity; in Sect. 3 we recall the definition of boundary at infinity of a geodesic space, some basic results about the boundary at infinity of a hyperbolic space and a useful construction named *hyperbolic approximation* (see [13]). The hyperbolic approximation of a given metric space is a hyperbolic graph whose boundary at infinity is the given metric space.

In Sect. 4 we study the behavior of p-parabolicity in a uniform graph through certain operations as decompositions or vertex identifications.

Let X be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ let

$$(x|x')_o = \frac{1}{2}\big(d(x,o) + d(x',o) - d(x,x')\big).$$

The number $(x|x')_o$ is non-negative and it is called the *Gromov product* of x, x' with respect to o.

A metric space X is *(Gromov) hyperbolic* if it satisfies the δ-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$.

We denote by $\delta(X)$ the sharp hyperbolicity constant of X:

$$\delta(X) = \sup\Big\{ \min\{(x|z)_o, (z|y)_o\} - (x|y)_o \,\Big|\, x, y, z, o \in X \Big\}. \tag{1}$$

Hence, X is hyperbolic if and only if $\delta(X) < \infty$.

The theory of Gromov hyperbolic spaces was introduced by M. Gromov for the study of finitely generated groups (see [26]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups (see [1,24,26]). This theory has been developed from a geometric point of view to the extent of making hyperbolic spaces an important class of metric spaces to be studied on their own (see, e.g., [10,11,13,24,55]). In the last years, Gromov hyperbolicity has been intensely studied in graphs (see, e.g., [5,6,8,29,30,34,35,47,49–51,56] and the references therein).

In Sect. 5 we give necessary and sufficient conditions for a uniform hyperbolic graph to be p-parabolic in terms of the boundary at infinity of the graph.

In the last section, we prove the relation between p-parabolicity and Cheeger isoperimetric inequality for uniform hyperbolic graphs. Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [15,42]). The Cheeger isoperimetric inequality is related with many conformal invariants in graphs and Riemannian manifolds, namely the bottom of the spectrum of the Laplace-Beltrami operator, Poincaré-Sobolev inequalities, the exponent of convergence, and the Hausdorff dimensions of the sets of both escaping and bounded geodesics in negatively curved surfaces (see [2,7], [12, p.228], [16,18–22,40,41], [53, p.333]).

There is a natural connection between isoperimetric inequality and hyperbolicity. Recall that one of the definitions of Gromov hyperbolicity involves some kind of isoperimetric inequality (see [1,26]). In [36,38] we studied the relationship between the hyperbolicity and the Cheeger isoperimetric inequality in the context of graphs and Riemannian manifolds with bounded local geometry. In those works we obtained a characterization of graphs and Riemannian manifolds (with bounded local geometry) satisfying the (Cheeger) isoperimetric inequality, in terms of their Gromov boundary.

Here, in Sect. 6, we prove that if a uniform hyperbolic graph satisfies the Cheeger isoperimetric inequality, then it is non-p-parabolic for every $1 < p < \infty$.

2. Definitions and Background

A function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b, \qquad \text{for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if there exists a constant $c \geq 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [31]).

Given a graph G let us denote as $V(G)$ its vertex set and $E(G)$) its edge set. Given a function $u : V(G) \to \mathbb{R}$, define the *p-modulus of its discrete gradient* $|\nabla_G u|_p$ and its *discrete p-Dirichlet integral* $D_{p,G}(u)$, respectively, by

$$|\nabla_G u|_p(x) := \Big(\sum_{y \in N(x)} \big|u(y) - u(x)\big|^p \Big)^{1/p},$$

$$D_{p,G}(u) := \sum_{x \in V(G)} |\nabla_G u|_p^p(x) = 2 \sum_{vw \in E(G)} \big|u(v) - u(w)\big|^p,$$

where the edges are considered unoriented.

For a finite subset S of $V(G)$, the *p-capacity* of S is defined by

$$\begin{aligned} \operatorname{cap}_p S &= \operatorname{cap}_p(S, G) \\ &= \inf \Big\{ D_{p,G}(u) : u \text{ function on } V(G) \text{ with finite support, } u|_S = 1 \Big\}. \end{aligned}$$

A graph G is said to be *μ-uniform* if each vertex p of $V(G)$ has at most μ neighbors, i.e.,

$$\sup \big\{ |N(p)| \ : \ p \in V(G) \big\} \leq \mu.$$

If a graph G is μ-uniform for some constant μ we say that G is *uniform*

Theorem 1. *Given $1 < p < \infty$, a uniform graph G is p-parabolic if and only if* $\operatorname{cap}_p S = 0$ *for some (and then for every) non-empty finite subset of $S \subset V(G)$.*

For a proof of Theorem 1, see [33, Proposition 6] and [27, Final remark 5.16]. Note that the definition of discrete p-Dirichlet integral in [27] is slightly different, but both are equivalent.

Also, Corollary 7 in [32] can be trivially extended to the general case (being p-parabolic for an arbitrary $p > 1$ instead of $p = 2$) to obtain the following:

Proposition 2. *If two uniform graphs P and Q are quasi-isometric, then P is p-parabolic if so is Q.*

The *Cantor tree* (T_C, v_0) is a rooted tree such that the root, v_0, has degree two and any other vertex has degree three.

Proposition 3 [39, Corollary 5]. *The Cantor tree (T_C, v_0) is non-p-parabolic for every $1 < p < \infty$.*

Proposition 4 [39, Proposition 6]. *If a uniform graph Γ contains a non-p-parabolic subgraph Γ' for some $1 < p < \infty$, then Γ is non-p-parabolic.*

3. Boundary at Infinity and Hyperbolic Approximation

Let us recall that a *geodesic space* is a metric space such that for every couple of points there exists a geodesic joining them. A geodesic ray in a metric space X is the image of an isometric embedding $F : [0, \infty) \to X$. In this case, we say that the geodesic ray emanates from $F(0)$. A geodesic space X has a *pole* in a point v if there exists $M > 0$ such that each point of X lies in an M-neighborhood of some geodesic ray emanating from v.

Let us recall the concepts of geodesic and sequential boundary of a hyperbolic space and some basic properties. For further information and proofs we refer the reader to [10,13,24,26]. Let X be a hyperbolic space and $o \in X$ a base point.

The *relative geodesic boundary* of X with respect to the base-point o is

$$\partial_o^g X := \{[\gamma] \,|\, \gamma : [0, \infty) \to X \text{ is a geodesic ray with } \gamma(0) = o\},$$

where $\gamma_1 \sim \gamma_2$ if there exists some $K > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < K$, for every $t \geq 0$.

In fact, in the definition above the equivalence classes of the geodesic rays do not depend on the election of the base point. Therefore, the set of classes of geodesic rays is called *geodesic boundary* of X, $\partial^g X$. Herein, we do not distinguish between the geodesic ray and its image.

A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j \to \infty} (x_i | x_j)_o = \infty.$$

This property is independent of the choice of o since

$$|(x|x')_o - (x|x')_{o'}| \leq d(o, o')$$

for any $x, x', o, o' \in X$.

Two sequences $\{x_i\}, \{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i \to \infty} (x_i | x_i')_o = \infty.$$

Using the δ-inequality (1), we easily see that this defines an equivalence relation for sequences in X converging to infinity. The *sequential boundary at infinity* $\partial_\infty X$ of X is defined to be the set of equivalence classes of sequences converging to infinity.

Note that given a geodesic ray γ, the sequence $\{\gamma(n)\}$ converges to infinity and two equivalent rays induce equivalent sequences. Thus, in general, $\partial^g X \subseteq \partial_\infty X$.

We say that a metric space is *proper* if every closed ball is compact. Every uniform graph is a proper geodesic space.

Proposition 5 [10, Chapter III.H, Proposition 3.1]. *If X is a proper hyperbolic geodesic space, then the natural map from $\partial^g X$ to $\partial_\infty X$ is a bijection.*

For every $\xi, \xi' \in \partial_\infty X$, its Gromov product with respect to the base point $o \in X$ is defined as

$$(\xi|\xi')_o = \inf \liminf_{i \to \infty} (x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$.

Remark 1. [13, Lemma 2.2.2] If X is a δ-hyperbolic geodesic space, then for every pair of geodesic rays σ, σ' with $\sigma(0) = x_0 = \sigma'(0)$ and such that $\{\sigma(n)\} \in \xi$ and $\{\sigma'(n)\} \in \xi'$,

$$(\xi|\xi')_{x_0} \le \liminf_{t \to \infty} (\sigma(t)|\sigma'(t))_{x_0} \le \limsup_{t \to \infty} (\sigma(t)|\sigma'(t))_{x_0} \le (\xi|\xi')_{x_0} + 2\delta.$$

The Gromov product

$$(x|\xi)_o = \inf \liminf_{i \to \infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$.

A hyperbolic space X is said to be *visual*, if for some base point $o \in X$ there is some constant $D > 0$ such that for every $x \in X$ there is $\xi \in \partial_\infty X$ with $d(o, x) \le (x|\xi)_o + D$. Moreover, this property is independent of the choice of o.

Proposition 6 [36, Proposition 4.4]. *A proper hyperbolic geodesic space has a pole if and only if it is visual.*

A metric d on the sequential boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X$, $a > 1$ and positive constants c_1, c_2, such that

$$c_1 a^{-(\xi|\xi')_o} \le d(\xi, \xi') \le c_2 a^{-(\xi|\xi')_o}$$

for all $\xi, \xi' \in \partial_\infty X$. In this case, we say that d is a visual metric with respect to the base point o and the parameter a.

Theorem 7 [13, Theorem 2.2.7]. *Let X be a hyperbolic space. Then for any $o \in X$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a metric d on $\partial_\infty X$, which is visual with respect to o and a.*

Remark 2. Notice that for any visual metric, $\partial_\infty X$ is bounded and complete.

Proposition 8 [24, Proposition 7.9]. *If X is a proper, geodesic, hyperbolic space, then $\partial_\infty X$ is compact.*

Let us recall the following construction from [13].

A subset A in a metric space (X, d) is called *r-separated*, $r > 0$, if $d(a, a') \geq r$ for any distinct $a, a' \in A$. Note that if A is maximal with this property, then the union $\cup_{a\in A} B_r(a)$ covers X. A maximal r-separated set A in a metric space X is called an *r-approximation* of X.

A *hyperbolic approximation* of a metric space X, $\mathcal{H}(X)$, is a graph defined as follows. Fix a positive $s \leq \frac{1}{6}$ which is called the *parameter* of $\mathcal{H}(X)$. For every $k \in \mathbb{Z}$, let $A_k \in X$ be an s^k-approximation of X. For every $a \in A_k$, consider the ball $B(a, 2s^k) \subset X$. Let us define, for every k, the set $V_k^* := \{B(a, 2s^k) \mid a \in A_k\}$ and a set V_k which has a vertex corresponding to each ball in V_k^*. Then let $V = \cup_{k\in\mathbb{Z}} V_k$ be the set of vertices of the graph $\mathcal{H}(X)$. Thus, every vertex $v \in V$ corresponds to some ball $B(a, 2s^k)$ with $a \in A_k$ for some k. Let us denote the corresponding ball to $v \in V$ simply by $B(v)$.

There is a natural level function $l : V \to \mathbb{Z}$ defined by $l(v) = k$ for $v \in V_k$.

Vertices v, v' are connected by an edge if and only if they either belong to the same level, V_k, and the closed balls $\bar{B}(v), \bar{B}(v')$ intersect, $\bar{B}(v) \cap \bar{B}(v') \neq \emptyset$, or they lie on neighboring levels V_k, V_{k+1} and the ball of the upper level, V_{k+1}, is contained in the ball of the lower level, V_k.

Since A_k is an s^k-approximation of X for any $k \in \mathbb{Z}$ and $s \leq \frac{1}{6}$, every vertex in V_k has a neighbor in V_{k+1}.

An edge $vv' \subset \mathcal{H}(X)$ is called *horizontal* if its vertices belong to the same level, $v, v' \in V_k$ for some $k \in \mathbb{Z}$. Other edges are called *radial*. Consider the path metric on X for which every edge has length 1.

Note that any (finite or infinite) sequence $\{v_k\} \in V$ such that $v_k v_{k+1}$ is a radial edge for every k and such that the level function l is monotone along $\{v_k\}$, is the vertex sequence of a geodesic in $\mathcal{H}(X)$. Such a geodesic is called *radial*.

Proposition 9 [13, Proposition 6.2.10]. *For any metric space (X, d), $\mathcal{H}(X)$ is a geodesic 3-hyperbolic space.*

Assume now that X is bounded and non-trivial. Then, since $s < 1$, there is a maximal integer k with $\operatorname{diam} X < s^k$ and it is denoted by $k_0 = k_0(\operatorname{diam} X, s)$. Then, for every $k \leq k_0$ the vertex set V_k consists of one point, and therefore contains no essential information about X. Thus, the graph $\mathcal{H}(X)$ can be modified making $V_k = \emptyset$ for every $k < k_0$. This modified graph is called the *truncated hyperbolic approximation* of X, $\mathcal{H}^t(X)$. The level function l restricted to $\mathcal{H}^t(X)$ has a unique minimum, v, with $l(v) = k_0$. This point v can be considered as the natural base point of the truncated hyperbolic approximation.

Theorem 10 [13, Proposition 6.4.1]. *Let Γ be a truncated hyperbolic approximation of a complete bounded metric space (X, d). Then there is a canonical* *identification $\partial_\infty\Gamma = X$ under which the metric d of X is a visual metric on $\partial_\infty\Gamma$ with respect to the natural base point v of Γ and the parameter $a = \frac{1}{s}$.*

The following definition appeared in [36] and was inspired by the definition of a metric space having bounded geometry in [9].

Definition 1. Given a metric space (X, d), we say that X has *strongly bounded geometry* if for every $K > 0$ there exists $M > 0$ such that the following condition is satisfied: for any $\varepsilon > 0$, any ε-approximation of X, A_ε, and any $x \in X$, $|A_\varepsilon \cap B(x, K\varepsilon)| < M$.

Remark 3. Note that if $\varepsilon > \operatorname{diam} X$, then $|A_\varepsilon| = 1$ and $|A_\varepsilon \cap B(x, K\varepsilon)| \le |A_\varepsilon| = 1$. Hence, in order to check whether X has strongly bounded geometry or not, it suffices to consider $0 < \varepsilon \le \operatorname{diam} X$.

Theorem 11 [13, Corollary 7.1.6]. *Visual hyperbolic geodesic spaces X, X' with bilipschitz equivalent boundaries at infinity are roughly similar to each other. In particular, every visual hyperbolic space (i.e. every hyperbolic space with a pole) is roughly similar (and therefore, quasi-isometric) to any (truncated) hyperbolic approximation of its boundary at infinity; and any two hyperbolic approximations of a complete bounded metric space Z are roughly similar (and therefore, quasi-isometric) to each other.*

Proposition 12 [36, Proposition 4.14]. *If G is a hyperbolic uniform graph with a pole, then $\partial_\infty G$ with any visual metric has strongly bounded geometry.*

Proposition 13 [36, Proposition 4.11]. *If the metric space (X, d) has strongly bounded geometry, then $\mathcal{H}(X)$ is uniform.*

4. Parabolicity and Graph Decomposition

The following results are useful, as Proposition 4, in order to determine if a graph is p-parabolic or not.

Proposition 14. *Let G_1, G_2 be two uniform graphs such that $V_0 := \big(V(G_1) \cup V(G_2)\big) \setminus \big(V(G_1) \cap V(G_2)\big)$ is a finite set and such that the subgraphs of G_1 and G_2 induced by $V(G_1) \cap V(G_2)$ are the same. Then G_1 is p-parabolic if and only if G_2 is p-parabolic.*

Proof. Let us define for each $i = 1, 2$,

$$R := \cup_{v \in V_0} N(v), \qquad S_i := \big(V_0 \cup R\big) \cap V(G_i).$$

If $u_1 : V(G_1) \to \mathbb{R}$ has finite support and $u_1|_{S_1} = 1$, then the function $u_2 : V(G_2) \to \mathbb{R}$ defined by $u_2|_{S_2} = 1$ and $u_2|_{V(G_2) \setminus S_2} = u_1$ has finite support and $D_{p,G_2}(u_2) = D_{p,G_1}(u_1)$. Hence, $\operatorname{cap}_p(S_2, G_2) \le \operatorname{cap}_p(S_1, G_1)$. By symmetry, we have the converse inequality and Theorem 1 gives that G_1 is p-parabolic if and only if G_2 is p-parabolic. $\square$

Proposition 15. *Let G be a uniform graph and consider a finite or countable set $\{e_n\} \subseteq E(G)$ and $1 < p < \infty$. Let us consider the graph G' obtained from G by replacing each edge e_n by a path of length ℓ_n for each n.*

(1) *If G is p-parabolic, then G' is p-parabolic.*

(2) *If $\sup_n \ell_n < \infty$, then G is p-parabolic if and only if G' is p-parabolic.*

Proof. Let us fix $S \subset V(G) \subset V(G')$ and denote by v_n and w_n the incident vertices to e_n.

Consider $u : V(G) \longrightarrow \mathbb{R}$ with finite support and $u|_S = 1$. Define the function $u' : V(G') \longrightarrow \mathbb{R}$ defined as $u'|_{V(G)} := u$ and if $z \in V(G') \setminus V(G)$, then z is contained in the path of length ℓ_n for some n and

$$u'(z) := u(v_n) + d_{G'}(z, v_n)\, \frac{u(w_n) - u(v_n)}{\ell_n}.$$

It is clear that u' has finite support and $u'|_S = 1$. Since $p > 1$, we have

$$\big|u(w_n) - u(v_n)\big|^p \ge \ell_n \Big| \frac{u(w_n) - u(v_n)}{\ell_n} \Big|^p$$

and so, $D_{p,G'}(u') \le D_{p,G}(u)$. Thus, $\operatorname{cap}_p(S, G') \le \operatorname{cap}_p(S, G)$ and Theorem 1 gives that G' is p-parabolic if G is p-parabolic.

Assume now that $\sup_n \ell_n = \ell < \infty$. Consider $U : V(G') \longrightarrow \mathbb{R}$ with finite support and $U|_S = 1$. Define the function $U_0 : V(G) \longrightarrow \mathbb{R}$ defined as $U_0 := U|_{V(G)}$. It is clear that U_0 has finite support and $U_0|_S = 1$.

Fix n and denote by $\{z_j\}_{j=0}^{\ell_n}$ the vertices contained in the path of length ℓ_n with $d_{G'}(z_j, v_n) = j$ (thus, $z_0 = v_n$ and $z_{\ell_n} = w_n$). If $x_j := \big|U(z_j) - U(z_{j-1})\big|$ for $1 \le j \le \ell_n$, then

$$\big|U_0(w_n) - U_0(v_n)\big|^p \le \left(\sum_{j=1}^{\ell_n} x_j \right)^p \le \ell_n^{p-1} \sum_{j=1}^{\ell_n} x_j^p \le \ell^{p-1} \sum_{j=1}^{\ell_n} x_j^p.$$

and so, $D_{p,G}(U_0) \le \ell^{p-1} D_{p,G'}(U)$.

Thus, $\operatorname{cap}_p(S, G) \le \ell^{p-1} \operatorname{cap}_p(S, G')$ and Theorem 1 gives that G is p-parabolic if G' is p-parabolic. □

Proposition 16. *Let G be a uniform graph and $1 < p < \infty$. Consider some constant $K > 1$ and a finite or countable set $\cup_n \{v_n^j\}_{j=1}^{k_n} \subseteq V(G)$ with $\{v_n^j\}_{j=1}^{k_n} \cap \{v_m^j\}_{j=1}^{k_m} = \emptyset$ for $n \ne m$ and $1 < k_n \le K$ for every n. Let us consider the graph G' obtained from G by identifying the vertices $\{v_n^j\}_{j=1}^{k_n}$ at a single vertex v_n^* for each n, with $N(v_n^*) = \cup_{j=1}^{k_n} N(v_n^j) \setminus \{v_n^j\}_{j=1}^{k_n}$ for each n. If G' is p-parabolic, then G is p-parabolic. Furthermore, if $\cup_n \{v_n^j\}_{j=1}^{k_n}$ is a finite set, then G' is p-parabolic if and only if G is p-parabolic.*

Proof. Proposition 14 gives the second statement. Let us prove the first one.

Suppose G is μ-uniform. Therefore, G' is $(K \cdot \mu)$-uniform.

Consider $S' = \{v_1^*\}$ and a function u' on $V(G')$ with finite support and $u'|_{S'} = 1$. Let us define $S = \{v_1^j\}_{j=1}^{k_1}$ and a function u on $V(G)$ as follows: $u(v_n^j) = u'(v_n^*)$ for each n and j, and $u(v) = u'(v)$ if $v \neq v_n^j$ for every n and j. Thus, u has finite support and $u|_S = 1$.

Notice that, since G is μ-uniform, $|N(w) \cap \{v_n^j\}_{j=1}^{k_n}| \le \mu$ for any $w \in V(G)$. Therefore, $D_{p,G}(u) \le \mu \cdot D_{p,G'}(u')$. Consequently,

$$\operatorname{cap}_p(S, G) \le \mu \cdot \operatorname{cap}_p(S', G')$$

and Theorem 1 gives that G is p-parabolic, since G' is uniform and p-parabolic. □

Let us denote $\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y)$.

Proposition 17. *Let G be a uniform graph and $1 < p < \infty$. Consider some constant $D > 1$ and a finite or countable set $\cup_n \{v_n^j\}_{j=1}^{k_n} \subseteq V(G)$ with $\operatorname{diam}(\{v_n^j\}_{j=1}^{k_n}) < D$ for every n. Let us consider the graph G' obtained from G by identifying the vertices $\{v_n^j\}_{j=1}^{k_n}$ at a single vertex v_n^* for each n, with $N(v_n^*) = \cup_{j=1}^{k_n} N(v_n^j) \setminus \cup_{j=1}^{k_n} \{v_n^j\}$ for each n. Then G is p-parabolic if and only if G' is p-parabolic.*

Proof. Since G is uniform, if $\operatorname{diam}(\{v_n^j\}_{j=1}^{k_n}) < D$, then there exists some constant K such that $k_n \le K$ for every n. Thus, by Proposition 16, if G' is p-parabolic, then G is p-parabolic.

Suppose G is p-parabolic and let $v_0 \in V(G) \setminus \cup_n \{v_n^j\}_{j=1}^{k_n}$. Then, by Theorem 1, for every $i \in \mathbb{N}$ there is a function $u_i : V(G) \to \mathbb{R}$ with finite support and $u_i(v_0) = 1$ such that $D_{p,G}(u_i) < \frac{1}{i}$. Let us define $u_i^* : V(G') \to \mathbb{R}$ so that $u_i^*(w) = u_i(w)$ for every $w \in V(G) \setminus \cup_n \{v_n^j\}_{j=1}^{k_n}$ and $u_i^*(v_n^*) := u_i(v_n^1)$ for every $n \in \mathbb{N}$. Then, $u_i^*(v_0) = 1$ and u_i^* has finite support.

For each $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \big\{ vw \in E(G) : \, v \in B(v_n^1, D), \, w \in N(v) \big\},$$
$$d_n = \max \big\{ |u_i(v) - u_i(w)| : \, vw \in \mathcal{A}_n \big\}.$$

If G is μ-uniform, in $\overline{B}(v, 2D)$ there are at most

$$1 + \mu + \mu^2 + \cdots + \mu^{2D} = \frac{\mu^{2D+1} - 1}{\mu - 1}$$

vertices, for every $v \in V(G)$. Thus,

$$\begin{aligned} &\big| \big\{ k : \, \overline{B}(v_n^1, D) \cap \overline{B}(v_k^1, D) \neq \emptyset \big\} \big| \\ &\quad \le \big| \, \overline{B}(v_n^1, 2D) \setminus \{v_n^1\} \big| \\ &\quad \le \frac{\mu^{2D+1} - 1}{\mu - 1} - 1 = \frac{\mu^{2D+1} - \mu}{\mu - 1}, \end{aligned}$$

and so, an edge in $E(G)$ belongs at most to $\frac{\mu^{2D+1} - \mu}{\mu - 1}$ different sets $\mathcal{A}_n$.

Notice that, since $\operatorname{diam}(\{v_n^j\}_{j=1}^{k_n}) < D$, for every vertex $w \in V(G')$ adjacent to v_n^*,

$$|u_i^*(v_n^*) - u_i^*(w)| \le D \cdot d_n.$$

Also, since G is uniform and $\operatorname{diam}(\{v_n^j\}_{j=1}^{k_n}) < D$, there is a constant μ' such that G' is μ'-uniform. Therefore,

$$D_{p,G'}(u_i^*) \le \frac{\mu^{2D+1} - \mu}{\mu - 1}\, \mu' \cdot D^p \cdot D_{p,G}(u_i) < \frac{\mu^{2D+1} - \mu}{\mu - 1}\, \frac{\mu' D^p}{i}.$$

Hence, G' is p-parabolic by Theorem 1. □

In general, a graph obtained by attaching a countable set of p-parabolic graphs through arbitrary subsets of vertices is not necessarily a p-parabolic graph. However, the following result provides sufficient conditions for G to be p-parabolic.

Theorem 18. *Consider $1 < p < \infty$ and a finite or countable set of graphs $\{G_n\}_{n=0}^N$ $(1 \le N \le \infty)$. Let G be the graph obtained from $\{G_n\}_{n=0}^N$ as follows: given some $D \in \mathbb{N}$, a finite or countable set of vertices $\{v_{n,j}\}_{j=1}^{N_n} \subseteq V(G_n)$ with $1 \le N_n < \infty$ and $\operatorname{diam}(\{v_{n,j}\}_{j=1}^{N_n}) < D$ for each $n \ge 1$, and a set of vertices $\cup_{n=1}^N \{v_{n,j}^0\}_{j=1}^{N_n} \subseteq V(G_0)$, G is the graph obtained by identifying $v_{n,j}$ with $v_{n,j}^0$ for each $n, j \ge 1$. If G is a uniform graph, then G is p-parabolic if and only if G_n is p-parabolic for every $n \ge 0$.*

Proof. Note that G_n is uniform for every $n \ge 0$ since G is uniform.

If G is p-parabolic, then Proposition 4 gives that G_n is p-parabolic for every $n \ge 0$.

Assume now that G_n is p-parabolic for every $n \ge 0$.

By Proposition 17 it suffices to prove that the graph G' obtained from G by identifying the vertices $\{v_{n,j} = v_{n,j}^0\}_{j=1}^{N_n}$ at a single vertex v_n^* for each $n \ge 1$ is p-parabolic. Also, by Proposition 17, if G_n' is the corresponding subgraph of G' obtained by the identification of these vertices in G_n, then G_n' is p-parabolic for each $n \ge 0$. Therefore, it suffices to prove that G is p-parabolic assuming that $N_n = 1$ for every $n \ge 1$.

Since G_0 is a p-parabolic graph, given $\varepsilon > 0$ and a non-empty finite subset $S_0 \subset V(G_0)$, by Theorem 1 there exists a function u_0 on $V(G_0)$ with finite support, $u_0|_{S_0} = 1$ and $D_{p,G_0}(u_0) < \varepsilon/2$. Since u_0 has finite support and G is a uniform graph, there is K such that $u_0(v_{n,1}^0) = 0$ for every $n > K$. Since $G_1, \dots, G_K$ are p-parabolic graphs, if we choose the finite sets $S_1 = \{v_{1,1}\}, \dots, S_K = \{v_{K,1}\}$, then Theorem 1 gives that there exist functions u_n on $V(G_n)$ with finite support, $u_n(v_{n,1}) = u_0(v_{n,1}^0)$ and $D_{p,G_n}(u_n) < \varepsilon/2^{n+1}$ for each $1 \le n \le K$.

Define a function u on $V(G)$ as follows: $u|_{V(G_0)} = u_0$, $u|_{V(G_n)} = u_n$ for each $1 \le n \le K$ and $u|_{V(G_n)} = 0$ for each $n > K$. It is clear that u has finite support, $u|_{S_0} = 1$ and

$$D_{p,G}(u) = D_{p,G_0}(u_0) + \sum_{n=1}^{K} D_{p,G_n}(u_n) < \frac{\varepsilon}{2} + \sum_{n=1}^{K} \frac{\varepsilon}{2^{n+1}} < \varepsilon.$$

Since G is a uniform graph, Theorem 1 gives that G is p-parabolic. □

5. Parabolicity of Hyperbolic Graphs

Given a subset A in metric space (X, d) let us denote $\operatorname{diam}(A) := \sup_{x,y\in A} d(x, y)$.

Definition 2. A metric space (X, d) is locally perfect at the point $x \in X$ if there exist $\varepsilon_0 > 0$ and $\lambda > 1$ such that $\operatorname{diam}(B(z, \varepsilon)) > \frac{\varepsilon}{\lambda}$ for every $z \in B(x, \varepsilon_0)$ and every $0 < \varepsilon < \varepsilon_0$.

Theorem 19. *If G is a hyperbolic uniform graph with a pole such that $\partial_\infty G$ is locally perfect at some point for some visual metric, then G is non-p-parabolic for every $1 < p < \infty$.*

Proof. Suppose that $\partial_\infty G$ is locally perfect at the point x for some visual metric d_∞ with parameters ε_0 and λ. Let us assume, with no loss of generality, that $\varepsilon_0 < \operatorname{diam}(\partial_\infty G)$. Consider some parameter $s < \min\{\frac{1}{6}, \frac{1}{2\lambda}\}$ and let Γ be the truncated hyperbolic approximation of $\partial_\infty G$ with parameter s.

Now, let $m \in \mathbb{N}$ be such that $2s^m < \varepsilon_0$ and let $a \in B(x, \varepsilon_0) \cap A_m$. Let v be the vertex in Γ such that $B(v) = B(a, 2s^m)$. Since $\partial_\infty G$ is locally perfect, $\operatorname{diam}(B(a, s^m)) > \frac{s^m}{\lambda} > 2s^{m+1}$. Hence, there are two points $z_1, z_2 \in B(a, s^m)$ with $d_\infty(z_1, z_2) > 2s^{m+1}$, and since A_{m+1} is a maximal s^{m+1}-separated set, there are two points $a_1 \in B(z_1, s^{m+1}) \cap A_{m+1}$, $a_2 \in B(z_2, s^{m+1}) \cap A_{m+1}$ so that $B(a_1, 2s^{m+1}) \neq B(a_2, 2s^{m+1})$. Hence, there are two different vertices w_1, w_2 in Γ with $B(w_i) = B(a_i, 2\, s^{m+1})$ for $i = 1, 2$ and such that vw_1 and vw_2 are radial edges in Γ.

Since every vertex v as above has two adjacent vertices in the next level of Γ it follows that Γ contains a Cantor tree as a subgraph. Thus, by propositions 3 and 4, Γ is non-parabolic for every $1 < p < \infty$.

Therefore, by Theorem 11 and propositions 12 and 13, G and Γ are quasi-isometric uniform graphs and by Proposition 2, G is non-p-parabolic for every $1 < p < \infty$. □

Given two sequences of positive integers $L = \{\ell_n\}_{n=1}^\infty$ and $R = \{r_n\}_{n=1}^\infty$, with $2 \le r_n \le N$ for every $n \ge 1$ and some constant N, the *Cantor tree* $(T_{L,R}, v_0)$ is a rooted tree such that the root, v_0, has degree r_1, the vertices at distance $\ell_1 + \cdots + \ell_{n-1}$ have degree $r_n + 1$, and any other vertex has degree two. In [36] we gave the following characterization of p-parabolicity for Cantor trees.

Theorem 20. [36, Theorem 7] *Given $1 < p < \infty$ and sequences $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, the Cantor tree $(T_{L,R}, v_0)$ is p-parabolic if and only if*

$$\sum_{k=1}^{\infty} \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} = \infty.$$

Using the same ideas, we can obtain a sufficient condition for a hyperbolic uniform graph to be p-parabolic. Moreover, Theorem 22 bellow, when $p = 2$ is a consequence of Corollary 2.6 in [48]. Let us recall the following Lemma:

Lemma 21 [39, Lemma 11]. *Let $1 < p < \infty$, $a_1, \dots, a_n > 0$, $f : [0, \infty)^n \to \mathbb{R}$ given by*

$$f(y_1, \dots, y_n) = a_1 y_1^p + \cdots + a_n y_n^p,$$

and

$$D = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_k \geq 0 \text{ for } 1 \leq k \leq n \text{ and } y_1 + \cdots + y_n = 1\}.$$

Then the minimum of f on D is attained at the point y^0 with

$$y_k^0 = a_k^{-1/(p-1)} \left(\sum_{j=1}^{n} a_j^{-1/(p-1)} \right)^{-1}$$

for $1 \leq k \leq n$, and

$$\min_{y \in D} f(y) = f(y^0) = \left(\sum_{k=1}^{n} a_k^{-1/(p-1)} \right)^{-(p-1)}.$$

Theorem 22. *Let G be a hyperbolic uniform graph with a pole and some visual metric d on $\partial_\infty G$ and let $0 < s \leq \frac{1}{6}$ and $1 < p < \infty$. If there exist two sequences of positive integers $\{\ell_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$, with $r_n \geq 2$ for every $n \geq 1$,*

$$\sum_{k=1}^{\infty} \frac{\ell_k}{(r_1 \cdots r_k)^{2/(p-1)}} = \infty,$$

and for every point $x \in \partial_\infty G$, the ball $B(x, 2s^{L_{k-1}})$ contains at most r_k points which are s^{L_k}-separated (where $L_0 = 0$ and $L_k = \sum_{j=1}^{k} \ell_j$), then G is p-parabolic.

Proof. Let Γ be the truncated hyperbolic approximation of $(\partial_\infty G, d)$ with parameter s and let us denote by d_Γ the usual minimal path distance in the graph Γ. Let us assume that the maximal integer k with $\operatorname{diam} X < s^k$, k_0, satisfies $k_0 \leq 0$. Otherwise, it suffices to extend the truncated hyperbolic approximation considering V_k to be a single point for every $0 \leq k < k_0$.

Let $S = \cup_{k \leq 0} V_k$. By Proposition 8, $\partial_\infty G$ is compact. Therefore, S is finite.

For each $m \in \mathbb{N}$ let us consider a decreasing sequence $\{u_k\}_{k=0}^{m}$ with $1 = u_0$ and $u_m = 0$ and define a function $u_m : V(\Gamma) \to \mathbb{R}$ such that:

- $u_m(w) = u_0 = 1$ for every $w \in V_k$ with $k \le 0$,
- $u_m(w) = u_k$ for every $w \in V_{L_k}$ (that is, for every $w \in \Gamma_v$ such that $d_\Gamma(v,w) = L_k = \sum_{j=1}^k \ell_j$),
- $u_m(w) = u_{k-1} - \frac{u_{k-1}-u_k}{\ell_k} i$ for every w such that $d_\Gamma(v,w) = L_{k-1}+i$ for every $1 \le k \le m$ and every $1 < i < \ell_k$,
- $u_m(w) = u_m = 0$ for every w such that $d_\Gamma(v,w) \ge L_m = \sum_{j=1}^m \ell_j$.

As we saw above, V_0 is finite. For each fixed $v \in V_0$, let Γ_v be the subgraph of Γ induced by v and all the vertices in $\cup_{k>0} V_k$ which are connected to v by radial edges. Let $V_k^v := V_k \cap \Gamma_v$.

Claim: $\operatorname{cap}_p(v, \Gamma_v) = 0$.

Since each ball $B(x, 2s^{L_{k-1}})$ contains at most r_k points which are s^{L_k}-separated, then if $L_{k-1} < i \le L_k$, for each vertex w in V_{i-1} there are at most r_k vertices in V_i connected to w by radial edges. Therefore, the set V_i^v has at most $r_1 \cdots r_k$ vertices and there are at most $(r_1 \cdots r_k)^2$ radial edges between V_{i-1}^v and V_i^v (notice that the balls associated to the vertices in V_{i-1} may all intersect and a vertex from V_i can be joined by a radial edge to every vertex in V_{i-1}). Thus,

$$\frac{1}{2} D_{p,\Gamma_v}(u_m) \le \frac{|u_0 - u_1|^p}{\ell_1^p} r_1^2 \ell_1 + \cdots + \frac{|u_{m-1} - u_m|^p}{\ell_m^p} (r_1 \cdots r_m)^2 \ell_m$$
$$= \sum_{k=1}^m \frac{(r_1 \cdots r_k)^2 |u_{k-1} - u_k|^p}{\ell_k^{p-1}}.$$

By Lemma 21, with $a_k = (r_1 \cdots r_k)^2 \ell_k^{-(p-1)}$, the minimum possible value of $\frac{1}{2} D_{p,\Gamma_v}(u_m)$ is

$$\left(\sum_{k=1}^m \left((r_1 \cdots r_k)^2 \ell_k^{-(p-1)} \right)^{-1/(p-1)} \right)^{-(p-1)} = \left(\sum_{k=1}^m \frac{\ell_k}{(r_1 \cdots r_k)^{2/(p-1)}} \right)^{-(p-1)}.$$

Therefore, if $\sum_{k=1}^\infty \frac{\ell_k}{(r_1 \cdots r_k)^{2/(p-1)}} = \infty$, it follows that $\operatorname{cap}_p(v, \Gamma_v) = 0$.

Since V_0 is finite and, by the definition of u_m, the horizontal edges do not contribute to $D_{p,\Gamma}(u_m)$, it is immediate to see that $\operatorname{cap}_p(v, \Gamma) = 0$. By propositions 12 and 13, Γ is uniform. Hence, by Theorem 1, Γ is p-parabolic.

Thus, by Theorem 11, G and Γ are quasi-isometric uniform graphs, and by Proposition 2, G is p-parabolic. □

Corollary 23. *Let G be a hyperbolic uniform graph with a pole and some visual metric d on $\partial_\infty G$ and let $0 < s \le \frac{1}{6}$ and $1 < p < \infty$. If there exist a sequence of positive integers $\{\ell_n\}_{n=1}^\infty$ and a constant N such that*

$$\lim_{k \to \infty} \frac{\ell_{k+1}}{\ell_k} > N^{\frac{2}{p-1}},$$

and for every point $x \in \partial_\infty G$, the ball $B(x, 2s^{L_{k-1}})$ contains at most N points which are s^{L_k}-separated (where $L_0 = 0$ and $L_k = \sum_{j=1}^{k} \ell_j$), then G is p-parabolic.

Proof. Notice that

$$\lim_{k\to\infty} \frac{\ell_{k+1}}{\ell_k} > N^{\frac{2}{p-1}} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} \frac{\frac{\ell_{k+1}}{N^{(2k+2)/(p-1)}}}{\frac{\ell_k}{N^{2k/(p-1)}}} > 1,$$

and by the D'Alembert criterion it follows that

$$\sum_{k=1}^{\infty} \frac{\ell_k}{N^{2k/(p-1)}} = \infty. \qquad \square$$

6. Parabolicity and Isoperimetric Inequality

From Theorem 19 and Theorem 24 bellow we can immediately prove Theorem 25. This could also be obtained from Theorem 1.1 in [4]

Definition 3. The *combinatorial Cheeger isoperimetric constant* of a graph G is defined to be

$$h(G) = \inf_{U} \frac{|\partial U|}{|U|},$$

where U ranges over all non-empty finite subsets of vertices in G, $\partial U = \{v \in G \,|\, d_G(v, U) = 1\}$ and $|U|$ denotes the cardinality of the set U.

A graph G satisfies the (Cheeger) *isoperimetric inequality* if $h(G) > 0$, since this means that

$$|U| \le h(G)^{-1} |\partial U|$$

for every finite set of vertices U.

Let us recall the following from [38].

Definition 4. Given a metric space (X, d) and a constant $A > 1$, we say that (X, d) is A-*uniformly perfect* if there exists some $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and every $x \in X$ there exist a point $y \in X$ such that $\varepsilon/A < d(x, y) \le \varepsilon$. We say that (X, d) is *uniformly perfect* if there exists some A such that (X, d) is A-uniformly perfect.

Theorem 24 [38, Theorem 5]. *Given a uniform hyperbolic graph G, then $h(G) > 0$ if and only if G is an infinite graph with a pole and $\partial_\infty G$ is uniformly perfect for some visual metric.*

Theorem 25. *Given a uniform hyperbolic graph G, if $h(G) > 0$, then G is non-p-parabolic for every $1 < p < \infty$.*

Proof. Since $h(G) > 0$, by Theorem 24, G has a pole. Also, since $\partial_\infty G$ is uniformly perfect for some visual metric, there exist $A > 0$ and $\varepsilon_0 > 0$ such that $\operatorname{diam}(B(z,\varepsilon)) > \frac{\varepsilon}{A}$ for any $z \in \partial_\infty G$ and any $0 < \varepsilon < \varepsilon_0$. This means that $\partial_\infty G$ with that visual metric is locally perfect and, by Theorem 19, G is non-p-parabolic for every $1 < p < \infty$. □

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