Results Math (2024) 79:61

1422-6383/24/020001-8
published online January 12, 2024
https://doi.org/10.1007/s00025-023-02094-z

Results in Mathematics

Check for updates

Packing a Tetrahedron by Similar Tetrahedra

Janusz Januszewski and Łukasz Zielonka

Abstract. Any collection of homothetic copies of some tetrahedron T_r, whose total volume does not exceed one quarter of the volume of T_r, can be packed into T_r .

Mathematics Subject Classfication. 52C17.

Keywords. packing, tetrahedra.

1. Introduction

We say that convex bodies $C_1, C_2, \ldots$ can be *packed* into a convex body C if it is possible to apply translations and rotations to the sets C_n so that the resulting translated and rotated bodies are contained in C and have mutually disjoint interiors. It is known that if C is either a square [1], or a triangle (see [2] and [3]), or a disk [4], then any collection of homothetic copies of C, whose total area does not exceed half the area of C can be packed into C. The conjecture is that the same packing density is achievable for any planar convex body (see [5] and [6]). However, only the following upper bound was found [7]: any collection of homothetic copies of C, with total area not greater than a quarter of the area of C, can be packed into C.

Meir and Moser [8] proved that any collection of d-dimensional cubes, whose total volume does not exceed 2^{1-d} times the volume of K, can be packed into the d-dimensional cube K. In the paper [2] concerning packing of triangles, Richardson writes: "The original paper of Kranakis and Meertens [9] conjectured that Theorem 1 may be extendible to families of similar tetrahedra (in three dimensions) or even to any family of similar d-dimensional simplexes."

Both authors (Janusz Januszewski and Łukasz Zielonka) contributed equally to this work.

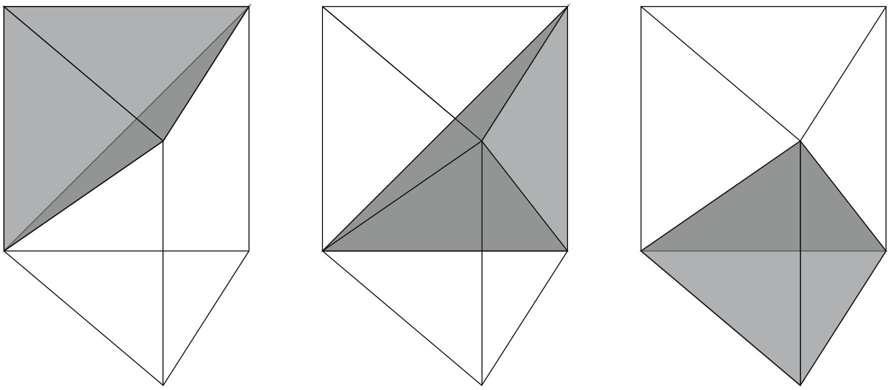

Figure 1. Partition of a prism into three congruent tetrahedra

We confirm this conjecture in three dimensions for a tetrahedron T_r of vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,0,1)$. We will show that any collection of homothetic copies of T_r, whose total volume does not exceed one quarter of the volume of T_r, can be packed into T_r.

2. Division of T_r

The unit cube can be divided into two congruent triangular prisms. Observe that such a prism can be partitioned into three tetrahedra congruent to T_r (see Fig. 1). The unit cube can be partitioned into six tetrahedra congruent to T_r.

Denote by λT_r the image of T_r in the homothety with ratio λ and center $(0,0,1)$. Moreover, let $a_1 \geq a_2 \geq \ldots$ be positive numbers, let $S_i = a_i T_r$ for $i = 1, 2, \ldots$ and let $a_1^3 + a_2^3 + \ldots \leq 1/4$. Obviously, $a_1 + a_2 \leq 1$, otherwise $a_1^3 + a_2^3 > (1/2)^3 + (1/2)^3 = 1/4$, which is a contradiction.

To describe the method of packing tetrahedra $S_1, S_2, \ldots$ into T_r, let's introduce some notations. Denote by k_1 the smallest integer such that

$$a_4 + a_{10} + \ldots + a_{6k_1-2} + a_{6k_1+1} > 1 - a_1$$

($k_1 = 3$ in Fig. 3); if $a_4 + a_7 > 1 - a_1$, then $k_1 = 1$. Moreover, let k_2 be the smallest integer such that

$$a_{6k_1+4} + a_{6k_1+10} + \ldots + a_{6k_2-2} + a_{6k_2+1} > 1 - a_1 - a_{6k_1+1}$$

($k_2 = 7$ in Fig. 3). At the end of this section we will define layers and slices. According to the packing method described in Sect. 3, tetrahedra $S_1, \ldots, S_{6k_1}$ will be packed into the first layer of the first slice; tetrahedra $S_{6k_1+1}, \ldots, S_{6k_2}$ will be packed into the second layer of the first slice. If

$$a_1 + a_{6k_1+1} + a_{6k_2+1} + \ldots + a_{6k_i+1} + a_{6k_i+2} \leq 1$$

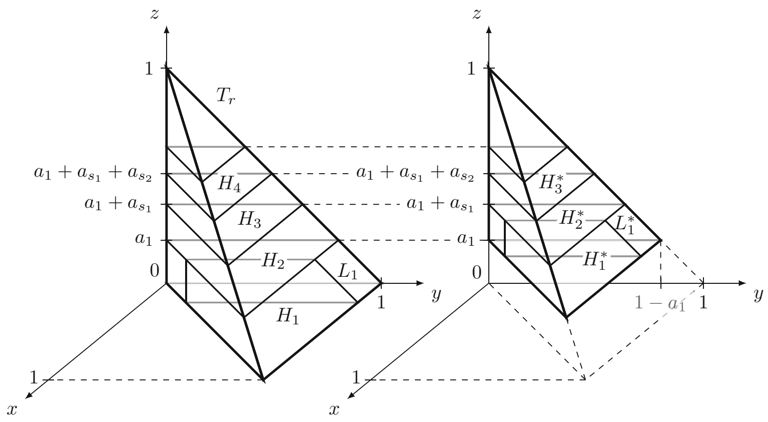

FIGURE 2. Division of T_r

for $i \geq 2$, then denote by k_{i+1} the smallest integer such that

$$a_{6k_i+4} + a_{6k_i+10} + \ldots + a_{6k_{i+1}-2} + a_{6k_{i+1}+1} > 1 - a_1 - a_{6k_1+1} - \ldots - a_{6k_i+1}.$$

If j is the smallest integer such that

$$a_1 + a_{6k_1+1} + \ldots + a_{6k_j+1} + a_{6k_j+2} > 1,$$

then we take $s_1 = a_{6k_j+2}$ ($j = 5$ and $k_j = 13$ in Fig. 3, right).

Tetrahedra will be packed into layers L_i contained in slices H_n. In Sect. 3 we will compare the total volume of tetrahedra packed into L_i with the volume of some layer L_i^*. Let

$$H_1 = T_r \setminus (1 - a_1)T_r,$$

and

$$H_1^* = (1 - a_1)T_r \setminus (1 - a_1 - a_{s_1})T_r.$$

(see Fig. 2). Moreover, let

$$L_1 = \{(x, y, z) \in H_1 : \ 0 \leq x \leq a_1\},$$
$$L_1^* = \{(x, y, z) \in H_1^* : \ 0 \leq x \leq a_{6k_1+1}\},$$
$$L_2 = \{(x, y, z) \in H_1 : \ a_1 \leq x \leq a_1 + a_{6k_1+1}\},$$
$$L_2^* = \{(x, y, z) \in H_1^* : \ a_{6k_1+1} \leq x \leq a_{6k_1+1} + a_{6k_2+1}\},$$
$$L_3 = \{(x, y, z) \in H_1 : \ a_1 + a_{6k_1+1} \leq x \leq a_1 + a_{6k_1+1} + a_{6k_2+1}\},$$
$$L_3^* = \{(x, y, z) \in H_1^* : \ a_{6k_1+1} + a_{6k_2+1} \leq x \leq a_{6k_1+1} + a_{6k_2+1} + a_{6k_3+1}\}, \text{ etc.}$$

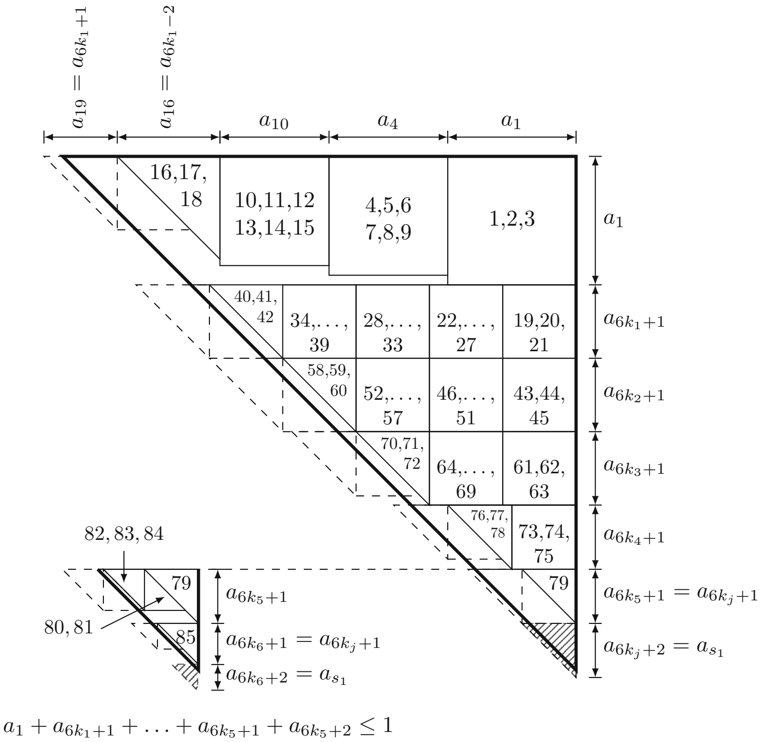

FIGURE 3. Bottom of H_1

3. Packing Method

Observe that S_1, S_2 and S_3 can be packed into the part of L_1 with $y \geq 1 - a_1$ (see Figs. 1, 3, 4). Moreover, $S_4, S_5, \ldots, S_9$ can be paced into the cube

$$\{(x, y, z) :\ 0 \leq x \leq a_4,\ 1 - a_1 - a_4 \leq y \leq 1 - a_1,\ 0 \leq z \leq a_4\}.$$

Packing method, Part I.
Tetrahedra $S_1, S_2, \ldots, S_{6k_1}$ are packed into L_1 (see Figs. 3, 4, where $k_1 = 3$). Tetrahedra $S_{6k_i+1}, S_{6k_i+2}, \ldots, S_{6k_{i+1}}$ are packed into L_{i+1}, for $i = 1, 2, \ldots, j-1$. Moreover $S_{6k_j+1} = S_{s_1-1}$ is placed in L_{j+1}. Clearly, all packed tetrahedra $S_1, \ldots, S_{s_1-1}$ are contained in H_1.

Lemma 1. *The sum of volumes of $S_2, \ldots, S_{s_1}$ is greater than the volume of H_1^*.*

Proof. It is easy to verify that $c^3/2 + d^3/2 \geq cd^2$, provided that $0 < d \leq c < 1$. Consequently,

$$\frac{1}{6}a_2^3 + \frac{1}{6}a_3^3 + \ldots + \frac{1}{6}a_7^3 \geq 3 \cdot \frac{1}{6}a_4^3 + 3 \cdot \frac{1}{6}a_7^3 \geq a_4 a_7^2 \geq a_4 a_{6k_1+1}^2.$$

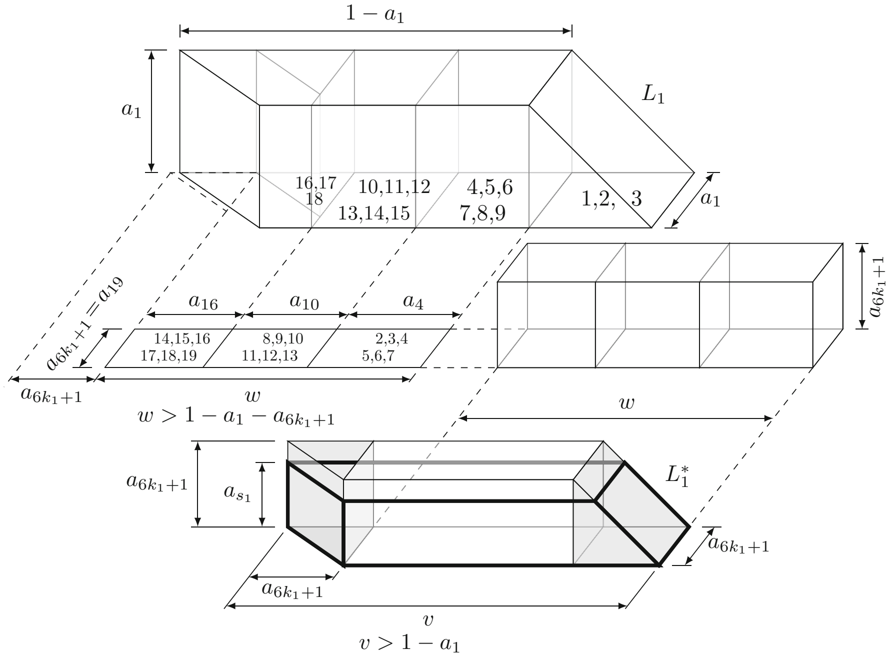

FIGURE 4. Covering L_1^* with $S_2, S_3, \ldots, S_{s_{6k_1+1}}$

Moreover, if $a_1 + a_4 + a_7 \leq 1$, then

$$\frac{1}{6}a_8^3 + \ldots + \frac{1}{6}a_{13}^3 \geq 3 \cdot \frac{1}{6}a_{10}^3 + 3 \cdot \frac{1}{6}a_{13}^3 \geq a_{10}a_{6k_1+1}^2,$$

etc. This means that that the total volume of tetrahedra $S_2, S_3, \ldots, S_{6k_1+1}$ is greater than the volume of L_1^* (see Fig. 4). For the same reason, the total volume of tetrahedra $S_{6k_i+2}, \ldots, S_{6k_{i+1}+1}$ is greater than the volume of L_{i+1}^*, for $i = 1, 2, \ldots, j-1$. This implies that the sum of volumes of $S_2, \ldots, S_{s_1}$ is greater than the volume of H_1^* (see Fig. 5).

An additional explanation is required if $a_1 + a_{6k_1+1} + \ldots + a_{6k_m+1} + a_{6k_m+2} \leq 1$, while $a_1 + a_{6k_1+1} + \ldots + a_{6k_m+1} + a_{6k_m+1} > 1$ for some integer m. We can imagine that S_{80} can be packed into L_6 in Fig. 3 (left). In this case there is enough space to pack S_{6k_m+1}, S_{6k_m+2} and S_{6k_m+3} into the part of L_{m+1} with $y \geq 1 - a_{6k_m+1}$ (see Fig. 6). Clearly, at least three next tetrahedra S_{6k_m+4}, S_{6k_m+5} and S_{6k_m+6} can be packed into this layer. Moreover, $m < j$, i.e., $S_{6k_{m+1}+1}$ is placed in the next layer in H_1. $\square$

Packing method, Part II.
Similar to Lemma 1, one can find an integer s_2 such that tetrahedra $S_{s_1}, \ldots, S_{s_2-1}$ can be packed into H_2 and that the sum of volumes of $S_{s_1+1}, \ldots S_{s_2}$ is

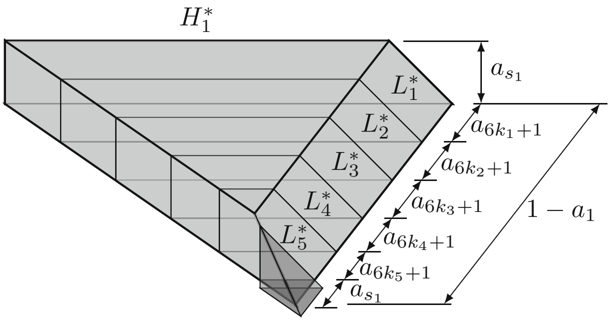

FIGURE 5. Covering H_1^* with $L_1, L_2, \ldots, L_{k_j}$ and S_{s_1}

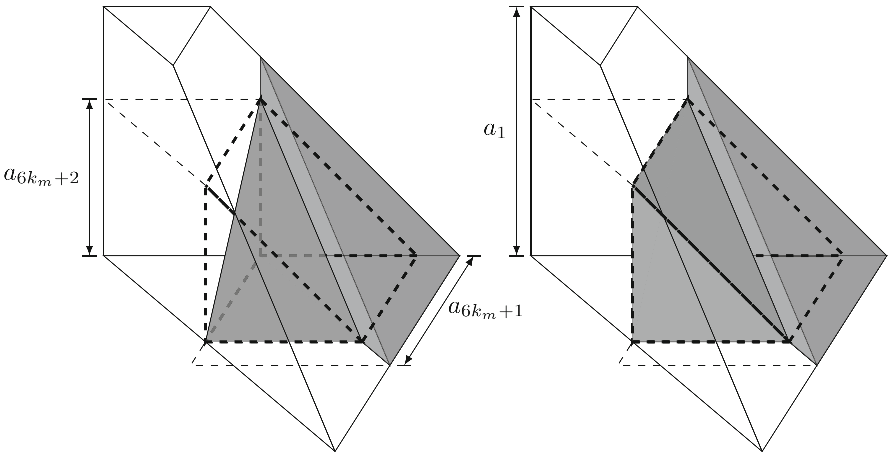

FIGURE 6. $a_1 + a_{6k_1+1} + \ldots + a_{6k_m+1} + a_{6k_m+2} \leq 1$ and $a_1 + a_{6k_1+1} + \ldots + a_{6k_m+1} + a_{6k_m+1} > 1$

greater than the volume of H_2^*, where

$$H_2 = H_1^* = (1 - a_1)T_r \setminus (1 - a_1 - a_{s_1})T_r$$

and

$$H_2^* = (1 - a_1 - a_{s_1})T_r \setminus (1 - a_1 - a_{s_1} - a_{s_2})T_r.$$

Assume that $s_2, \ldots, s_k$ are defined for $k \geq 2$ and that $a_1 + a_{s_1} + \ldots + a_{s_k} \leq 1$. Then there exists an integer s_{k+1} such that tetrahedra $S_{s_k}, \ldots, S_{s_{k+1}-1}$ can be packed into H_{k+1} and that the sum of volumes of $S_{s_k+1}, \ldots, S_{s_{k+1}}$ is greater than the volume of H_{k+1}^*, where $H_{k+1} = H_k^*$ and

$$H_{k+1}^* = (1 - a_1 - a_{s_1} - \ldots - a_{s_k})T_r \setminus (1 - a_1 - a_{s_1} - \ldots - a_{s_{k+1}})T_r,$$

provided that $a_1 + a_{s_1} + \ldots + a_{s_{k+1}} \le 1$ or

$$H^*_{k+1} = (1 - a_1 - a_{s_1} - \ldots - a_{s_k})T_r,$$

provided that $a_1 + a_{s_1} + \ldots + a_{s_{k+1}} > 1$. In the latter case we stop the packing process; there is not enough space in T_r to create a new slice of height $a_{s_{k+1}}$.

Theorem 2. *Any collection of homothetic copies of T_r, whose total volume does not exceed 1/24, can be packed into T_r.*

Proof. Let $S_1, S_2, \ldots$ be a collection of homothetic copies of T_r with total volume not greater than 1/24. Without loss of generality we can assume that $a_1 \ge a_2 \ge \ldots$, where $S_i = a_i T_r$, for $i = 1, 2, \ldots$ Assume that the tetrahedra cannot be packed into T_r by the method described above. This implies that there is an integer l such that $a_1 + a_{s_1} + \ldots + a_{s_l} > 1$. The sum of volumes of $S_2, \ldots, S_{s_l}$ is greater than the sum of volumes of $H^*_1, \ldots, H^*_{s_l}$, i.e., is greater than $\frac{1}{6}(1 - a_1)^3$. As a consequence, the total volume of all tetrahedra $S_1, S_2, \ldots$ is greater than

$$\frac{1}{6}a_1^3 + \frac{1}{6}(1 - a_1)^3 > \frac{1}{6} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{6} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{24},$$

which is a contradiction. □

Funding Not applicable.

Availability of Data and Material Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethics Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

Code Availability Not applicable.

References

[1] Moon, J.W., Moser, L.: Some packing and covering theorems. Colloq. Math. **17**, 103–110 (1967)

[2] Richardson, T.: Optimal packing of similar triangles. J. Comb. Theory Ser. A **69**, 288–300 (1995)

[3] Januszewski, J.: Optimal translative packing of homothetic triangles. Stud. Sci. Math. Hung. **46**(2), 185–203 (2009)

[4] Fekete, S.P., Keldenisch, P., Scheffer, Ch.: Packing discs into disk with optimal wors-case density. Discret. Comput. Geom. **69**, 51–90 (2023)

[5] Soifer, A.: Packing clones in convex figures. Geombinatorics **8**(1), 110–115 (1998)

[6] Novotny, P.: A note on packing clones. Geombinatorics **11**(1), 29–30 (2001)

[7] Januszewski, J.: Translative packing of a convex body by sequences of its positive homothetic copies. Acta Math. Hung. **117**(4), 349–360 (2007)

[8] Meir, A., Moser, L.: On packing of squares and cubes. J. Comb. Theory Ser. A **5**, 126–134 (1968)

[9] Kranakis, E., Meertens, L.: Optimal Packing of Triangles. Bull. Eur. Assoc. Theor. Comput. Sci. **38**, 181–185 (1989)

Janusz Januszewski and Łukasz Zielonka
Institute of Mathematics and Physics
Bydgoszcz University of Science and Technology
Al. Prof. S. Kaliskiego 7
85-789 Bydgoszcz
Poland
e-mail: januszew@pbs.edu.pl;
lukasz.zielonka@pbs.edu.pl

Received: July 20, 2023.
Accepted: December 2, 2023.