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Results in Mathematics

Convergence in Law of Iterates of Weakly Contractive in Mean Random-Valued Functions

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Abstract. We investigate the asymptotic behaviour of the sequence of forward type iterations of a given random-valued vector function on the state space being a separable and complete metric space. Assuming non-linear contraction in mean we prove that the considered sequence converges weakly to a random variable with a finite first moment and independent of the initial state. Moreover, we show that the speed of this convergence does not have to be geometric. We also present examples illustrating the result obtained.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a metric space X. Let $\mathcal{B}(X)$ denote the σ-algebra of all Borel subsets of X.

We say that $f\colon X \times \Omega \to X$ is a *random-valued* function (shortly: an *rv-function*) if it is measurable for the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n\colon X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, \mathbb{P}^\infty)$.

More exactly, one can show that for $n \in \mathbb{N}$ the n-th iterate f^n is measurable for $\mathcal{B}(X) \otimes \mathcal{A}_n$, where $\mathcal{A}_n$ denotes the σ-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$.

The iterates so defined were introduced independently in [4] and [9] with reference to functional equations (see e.g. [3,7,18]). In a broader context they form random forward iterations (see e.g. [8,13]), also known as outer iterations (see e.g. [11]). These iterates are prototypes of random dynamical systems (see [1, Section 1.1]; cf. [22]) and they have Markov property. The family $\{f(\cdot, \omega) : \omega \in \Omega\}$ forms an iterated function system (IFS for abbreviation) in which functions $f(\cdot, \omega)$ are choosing independently with probability $\mathbb{P}$. A generalization of this concept, devoted to random iteration with place-dependent probabilities, can be found, e.g., in [16, Section 3] and [25]. As in the mentioned papers we will express asymptotic behaviour of our iterates by the convergence in law. In fact, this type of convergence of iterations is closely related to the asymptotic stability of Markov operators with the kernel of the form $(x, B) \mapsto \int_\Omega \mathbb{1}_B(f(x, \omega))\mathbb{P}(d\omega)$, determined by a fixed rv-function f. For details see [14] and for a more complete point of view we refer the reader to [13,17,20] and the references therein; we only mention here that Markovian operator P is asymptotically stable, if it has an invariant measure μ^*, i.e., $P\mu^* = \mu^*$, which attracts any probability Borel measure. There are many papers in which the convergence in law of iterates of rv-functions and the stability of Markov operators with the kernel determined by an rv-function are investigated; however usually a kind of Lipschitz contraction on the rv-function considered is assumed (see e.g. [2,6,8,15]).

This paper aims to extend the results on convergence in law of iterates of random-valued functions that are mean contractive in the conventional sense to the case where only a weak (non-linear) form of the mean contractivity is assumed and examine how fast the sequence of iterates converges. The speed of convergence obtained does not have to be geometric as in the Lipschitz case. Let us mention that results [21, Theorem 9.2] and [23, Theorem 6.3.2] involve the same form of the contractivity property as that employed in the manuscript pertain to IFS with place-dependent probabilities of choosing them, but for finite many transformations and bring only asymptotic stability of IFS.

2. Preliminaries

Let (X, ρ) be a complete and separable metric space. By $\mathcal{M}_1(X)$ we denote the set of all probability measures defined on $\mathcal{B}(X)$. $\mathrm{Lip}_\alpha(X)$ denotes the set of all real functions defined on X that meet a Lipschitz condition with a constant $\alpha \in [0, \infty)$, and

$$\mathrm{Lip}_1^{\mathrm{b}}(X) = \{\varphi \in \mathrm{Lip}_1(X) : \varphi \text{ is bounded}\}.$$

It is well known (see [10, Theorem 11.3.3]) that the weak convergence of probability Borel measures on X is metrizable by the Fortet-Mourier metric $d_{FM}\colon \mathcal{M}_1(X)\times\mathcal{M}_1(X)\to[0,\infty)$ given by

$$d_{FM}(\mu,\nu)=\sup\left\{\left|\int_X \varphi d\mu-\int_X \varphi d\nu\right| : \varphi\in \mathrm{Lip}_1(X),\ |\varphi(\mathrm{x})|\le 1 \text{ for } \mathrm{x}\in \mathrm{X}\right\}.$$

Moreover we will use the Hutchinson metric (see [12,19]), also known as Wasserstein or Kantorovich-Rubinstein distance [24], defined by

$$d_H(\mu,\nu)=\sup\left\{\left|\int_X \varphi d\mu-\int_X \varphi d\nu\right| : \varphi\in \mathrm{Lip}_1^{\mathrm{b}}(X)\right\}.$$

According to [15, Lemma 3.1(i)],

$$d_H(\mu,\nu)=\sup\left\{\left|\int_X \varphi d\mu-\int_X \varphi d\nu\right| : \varphi\in \mathrm{Lip}_1(X)\right\} \quad \text{for } \mu,\nu\in\mathcal{M}_1^1(X), \tag{1}$$

where

$$\mathcal{M}_1^1(X)=\left\{\mu\in\mathcal{M}_1(X) : \int_X \rho(x,x_0)\mu(dx)<\infty\right\}$$

with an arbitrarily fixed $x_0\in X$; the definition does not depend on the choice of x_0.

Remark 2.1 [see [24, Theorem 6.18], cf. [15, Theorem 3.3 and Remark 3.2]]. The metric space $(\mathcal{M}_1^1(X), d_H|_{\mathcal{M}_1^1(X)\times\mathcal{M}_1^1(X)})$ is complete.

Remark 2.2 [see [15, Theorem 3.3]]. The set $\mathcal{M}_1^1(X)$ is dense in $(\mathcal{M}_1(X), d_{FM})$.

3. Main result

We employ the following hypothesis.

(H) (X,ρ) is a separable and complete metric space and $f\colon X\times\Omega\to X$ is an rv-function such that

$$\int_\Omega \rho(f(x,\omega),f(z,\omega))\mathbb{P}(d\omega)\le\psi(\rho(x,z)) \quad \text{for } x,z\in X$$

with a concave function $\psi\colon[0,\infty)\to[0,\infty)$.

Remark 3.1. If $\psi\colon[0,\infty)\to[0,\infty)$ is concave, then it is non-decreasing.

Proof. Suppose, towards a contradiction, that there are $t_1,t_2\in[0,\infty)$ such that $t_1<t_2$ and $\psi(t_2)<\psi(t_1)$. Fix $\alpha\in(\frac{\psi(t_2)}{\psi(t_1)},1)$ and put $t=\frac{t_2-\alpha t_1}{1-\alpha}$. Then

$$\psi(t_2)=\psi(\alpha t_1+(1-\alpha)t)\ge\alpha\psi(t_1)+(1-\alpha)\psi(t),$$

and hence $\psi(t)\le\frac{\psi(t_2)-\alpha\psi(t_1)}{1-\alpha}<0$, a contradiction. $\square$

Proposition 3.1. *Assume (H) and define* $P\colon \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ *by*

$$(P\mu)(B) = \int_X \left(\int_\Omega \mathbb{1}_B(f(x,\omega))\mathbb{P}(d\omega) \right) \mu(dx) \quad \textit{for } B \in \mathcal{B}(X). \tag{2}$$

Then

$$d_H(P\mu, P\nu) \le \psi(d_H(\mu,\nu)) \quad \textit{for } \mu,\nu \in \mathcal{M}_1^1(X).$$

Proof. Observe first that for any Borel $\varphi\colon X \to \mathbb{R}$, which is non-negative or bounded, we have

$$\int_X \varphi d(P\mu) = \int_X \left(\int_\Omega \varphi(f(x,\omega))\mathbb{P}(d\omega) \right) \mu(dx) \quad \text{for } \mu \in \mathcal{M}_1(X). \tag{3}$$

Fix $\mu,\nu \in \mathcal{M}_1^1(X)$ and denote by $\Lambda(\mu,\nu)$ the collection of all probability Borel measures λ on $X \times X$ such that

$$\lambda(B \times X) = \mu(B) \quad \text{and} \quad \lambda(X \times B) = \nu(B) \quad \text{for } B \in \mathcal{B}(X).$$

If $\lambda \in \Lambda(\mu,\nu)$, then

$$\begin{aligned} \int_{X\times X} \rho d\lambda &\le \int_{X\times X} (\rho(x,x_0) + \rho(x_0,z))\,\lambda(d(x,z)) \\ &= \int_X \rho(x,x_0)\mu(dx) + \int_X \rho(x_0,z)\nu(dz) < \infty. \end{aligned}$$

Therefore, the formula

$$T(\lambda) = \int_{X\times X} \rho d\lambda \quad \text{for } \lambda \in \Lambda(\mu,\nu)$$

defines a functional $T\colon \Lambda(\mu,\nu) \to [0,\infty)$.

If $\lambda \in \Lambda(\mu,\nu)$ and $\varphi \in \mathrm{Lip}_1^{\mathrm{b}}(X)$, then by (3), (H) and Jensen's inequality (see [10, 10.2.6]), we have

$$\begin{aligned} &\left| \int_X \varphi d(P\mu) - \int_X \varphi d(P\nu) \right| \\ &\quad = \left| \int_{X\times X} \left(\int_\Omega (\varphi(f(x,\omega)) - \varphi(f(z,\omega)))\mathbb{P}(d\omega) \right) \lambda(d(x,z)) \right| \\ &\quad \le \int_{X\times X} \left(\int_\Omega \big|\varphi(f(x,\omega)) - \varphi(f(z,\omega))\big|\mathbb{P}(d\omega) \right) \lambda(d(x,z)) \\ &\quad \le \int_{X\times X} \left(\int_\Omega \rho(f(x,\omega), f(z,\omega))\mathbb{P}(d\omega) \right) \lambda(d(x,z)) \\ &\quad \le \int_{X\times X} \psi\big(\rho(x,z)\big)\lambda(d(x,z)) \le \psi\left(\int_{X\times X} \rho(x,z)\lambda(d(x,z)) \right) \\ &\quad = \psi\big(T(\lambda)\big), \end{aligned}$$

and hence

$$d_H(P\mu, P\nu) \le \psi\big(T(\lambda)\big) \quad \text{for } \lambda \in \Lambda(\mu,\nu). \tag{4}$$

Applying (1) and the Kantorovich-Rubinstein Theorem (see [10, Theorem 11.8.2]) we conclude that there exists $\lambda_0 \in \Lambda(\mu,\nu)$ such that

$$d_H(\mu,\nu) = \inf\left\{\int_{X\times X}\rho d\lambda : \lambda \in \Lambda(\mu,\nu)\right\} = \int_{X\times X}\rho d\lambda_0.$$

This jointly with (4) implies

$$d_H(P\mu,P\nu) \le \psi(T(\lambda_0)) = \psi\left(\int_{X\times X}\rho d\lambda_0\right) = \psi(d_H(\mu,\nu)),$$

and the proof is complete. □

Corollary 3.1. *Assume (H) and let* $P\colon \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ *be the operator given by* (2)*. If there exists* $x_0 \in X$ *such that*

$$\int_\Omega \rho(f(x_0,\omega),x_0)\mathbb{P}(d\omega) < \infty, \tag{5}$$

then $P(\mathcal{M}_1^1(X)) \subset \mathcal{M}_1^1(X)$ *and for every* $n\in\mathbb{N}$ *we have*

$$d_H(P^n\mu,P^n\nu) \leqslant \psi^n(d_H(\mu,\nu)) \quad \textit{for } \mu,\nu\in\mathcal{M}_1^1(X). \tag{6}$$

Proof. If $\mu\in\mathcal{M}_1^1(X)$, then by (3) with $\varphi=\rho(\cdot,x_0)$ we obtain

$$\begin{aligned}\int_X \rho(x,x_0)(P\mu)(dx) &= \int_X\left(\int_\Omega \rho(f(x,\omega),x_0)\mathbb{P}(d\omega)\right)\mu(dx)\\ &\le \int_X\left(\int_\Omega \rho(f(x,\omega),f(x_0,\omega))\mathbb{P}(d\omega)\right)\mu(dx)\\ &\quad + \int_\Omega \rho(f(x_0,\omega),x_0)\mathbb{P}(d\omega)\\ &\le \int_X \psi(\rho(x,x_0))\mu(dx) + \int_\Omega \rho(f(x_0,\omega),x_0)\mathbb{P}(d\omega)\\ &\le \psi\left(\int_X \rho(x,x_0)\mu(dx)\right) + \int_\Omega \rho(f(x_0,\omega),x_0)\mathbb{P}(d\omega).\end{aligned}$$

Thus (5) implies $P\mu\in\mathcal{M}_1^1(X)$.

By Proposition 3.1 we see that (6) holds for $n=1$. If (6) holds for some $n\in\mathbb{N}$, then Proposition 3.1 and Remark 3.1 for $\mu,\nu\in\mathcal{M}_1^1(X)$ imply

$$\begin{aligned}d_H(P^{n+1}\mu,P^{n+1}\nu) &\le \psi(d_H(P^n\mu,P^n\nu)) \le \psi(\psi^n(d_H(\mu,\nu)))\\ &= \psi^{n+1}(d_H(\mu,\nu)),\end{aligned}$$

which completes the proof. □

Given an rv-function $f\colon X\times\Omega\to X$ we denote by $\pi_n^f(x,\cdot)$ the distribution of $f^n(x,\cdot)$, i.e.,

$$\pi_n^f(x,B) = \mathbb{P}^\infty(f^n(x,\cdot)\in B) \quad \text{for } n\in\mathbb{N}\cup\{0\},\ x\in X \text{ and } B\in\mathcal{B}(X).$$

Clearly, for every $x\in X$, $\pi_0^f(x,\cdot)=\delta_x$, the Dirac measure concentrated at x, and $\pi_1^f(x,\cdot)$ is the distribution of $f(x,\cdot)$. One can check that $\pi_{n+1}^f(x,\cdot) =$

$P\pi_f^n(x,\cdot)$ holds for $x \in X$ and $\mu \in \mathcal{M}_1(X)$ (see [14]), which implies $\pi_{n+1}^f(\mu,\cdot) = P\pi_n^f(\mu,\cdot)$ for any $\mu \in \mathcal{M}_1(X)$, with $\pi_n^f(\mu,\cdot) = \int_X \pi_n^f(x,\cdot)\mu(dx)$, and shows that operator P given by (2) is the transition operator of the sequence of iterates under consideration. It turns out that this operator is asymptotically stable. In fact, the following theorem gives what follows.

Theorem 3.1. *Assume (H) with ψ satisfying also*

$$\psi(t) < t \quad \textit{for } t \in (0,\infty). \tag{7}$$

If (5) holds with some $x_0 \in X$, then the operator $P\colon \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ given by (2) admits an invariant measure $\pi^f \in \mathcal{M}_1^1(X)$ and

$$d_H(P^n\mu, \pi^f) \le \psi^n(d_H(\mu, \pi^f)) \quad \textit{for } \mu \in \mathcal{M}_1^1(X) \textit{ and } n \in \mathbb{N}, \tag{8}$$

$$d_H(\pi_n^f(x,\cdot), \pi^f) \le \psi^n(d_H(\delta_x, \pi^f)) \le \psi^n\left(\int_X \rho(x,z)\pi^f(dz)\right) \quad \textit{for } x \in X \textit{ and } n \in \mathbb{N}. \tag{9}$$

Moreover,

$$\lim_{n\to\infty} d_{FM}(P^n\mu, \pi^f) = 0 \quad \textit{for } \mu \in \mathcal{M}_1(X).$$

Proof. From Corollary 3.1, Remark 2.1 and the Boyd-Wong Theorem (see [5, Theorem 1]) we conclude that there exists a measure $\pi^f \in \mathcal{M}_1^1(X)$ such that

$$P\pi^f = \pi^f$$

and

$$\lim_{n\to\infty} d_H(P^n\mu, \pi^f) = 0 \quad \text{for } \mu \in \mathcal{M}_1^1(X). \tag{10}$$

By Corollary 3.1 we have (8).

To get the first inequality in (9), observe that by a simple induction for all $x \in X$ and $n \in \mathbb{N} \cup \{0\}$ we have

$$P^n\delta_x = \pi_n^f(x,\cdot); \tag{11}$$

indeed, $P^0\delta_x = \delta_x = \pi_0^f(x,\cdot)$ and if (11) holds for some $n \in \mathbb{N} \cup \{0\}$, then

$$\begin{aligned}(P^{n+1}\delta_x)(B) &= (P(\pi_n^f(x,\cdot)))(B) = \int_X \left(\int_\Omega \mathbb{1}_B(f(z,\omega))\mathbb{P}(d\omega)\right)\pi_n^f(x,dz)\\ &= \int_{\Omega^\infty}\left(\int_\Omega \mathbb{1}_B(f(f^n(x,\omega),\omega'))\mathbb{P}(d\omega')\right)\mathbb{P}^\infty(d\omega)\\ &= \int_{\Omega^\infty} \mathbb{1}_B(f^{n+1}(x,\omega))\mathbb{P}^\infty(d\omega) = \pi_{n+1}^f(x,B)\end{aligned}$$

for every $B \in \mathcal{B}(\mathrm{X})$ (cf. [14, Proposition 2.1]). Since $\delta_x \in \mathcal{M}_1^1(X)$ for every $x \in X$, Corollary 3.1 and (11) imply $\pi_n^f(x,\cdot) \in \mathcal{M}_1^1(X)$ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, the first inequality in (9) follows from (8) and (11).

For the prove of the second inequality in (9), note that if $x \in X$ and $\varphi \in \mathrm{Lip}_1^{\mathrm{b}}(X)$, then

$$\begin{aligned}\left|\int_X \varphi d\delta_x - \int_X \varphi d\pi^f\right| &= \left|\varphi(x) - \int_X \varphi(z)\pi^f(dz)\right| \\ &\le \int_X |\varphi(x) - \varphi(z)|\pi^f(dz) \le \int_X \rho(x,z)\pi^f(dz).\end{aligned}$$

Hence

$$d_H(\delta_x, \pi^f) \le \int_X \rho(x,z)\pi^f(dz) \quad \text{for } x \in X,$$

which jointly with Remark 3.1 gives the second inequality in (9).

It remains to prove the moreover part. For this purpose, we fix $\mu \in \mathcal{M}_1(X)$.

If $\varphi \in \mathrm{Lip}_1(X)$ and $|\varphi(x)| \le 1$ for every $x \in X$, then due to (H) and (7) the function $\phi\colon X \to \mathbb{R}$ given by

$$\phi(x) = \int_\Omega \varphi(f(x,\omega))\mathbb{P}(d\omega)$$

belongs to $\mathrm{Lip}_1(X)$ and $|\phi(x)| \le 1$ for every $x \in X$, and moreover, by (3), for every $\nu \in \mathcal{M}_1(X)$ we have

$$\left|\int_X \varphi d(P\mu) - \int_X \varphi d(P\nu)\right| = \left|\int_X \phi d\mu - \int_X \phi d\nu\right| \le d_{FM}(\mu,\nu),$$

whence

$$d_{FM}(P\mu, P\nu) \le d_{FM}(\mu,\nu) \quad \text{for } \mu,\nu \in \mathcal{M}_1(X).$$

Fix $\varepsilon > 0$. By Remark 2.2 there exists $\nu \in \mathcal{M}_1^1(X)$ such that $d_{FM}(\mu,\nu) \le \frac{\varepsilon}{2}$ and by (10) there exists $n_0 \in \mathbb{N}$ such that $d_H(P^n\nu, \pi^f) \le \frac{\varepsilon}{2}$ for every $n \ge n_0$. Hence

$$\begin{aligned}d_{FM}(P^n\mu, \pi^f) &\le d_{FM}(P^n\mu, P^n\nu) + d_{FM}(P^n\nu, \pi^f) \\ &\le d_{FM}(\mu,\nu) + d_H(P^n\nu, \pi^f) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for every $n \ge n_0$. □

Remark 3.2. Assume $f\colon X \times \Omega \to X$ is an rv-function. If for some $x \in X$ the sequence $(\pi_n^f(x,\cdot))_{n\in\mathbb{N}}$ converges weakly to a π, then $\mathrm{supp}\pi \subset \mathrm{cl}f(X \times \Omega)$.

Proof. Suppose there exists $x_0 \in \mathrm{supp}\pi \cap (X \backslash \mathrm{cl}f(X \times \Omega))$ and let B be a closed ball in X with center at x_0 and contained in $X \backslash \mathrm{cl}f(X \times \Omega)$. Since $x_0 \in \mathrm{supp}\pi$, it follows that $\pi(B) > 0$, and by Urysohn's lemma there is a continuous $\varphi : X \to [0,1]$ such that

$$\varphi(z) = 1 \text{ for } z \in B \quad \text{ and } \quad \varphi(z) = 0 \text{ for } z \in \mathrm{cl}f(X \times \Omega).$$

Then $\varphi\big(f^n(x,\omega)\big) = 0$ for $n \in \mathbb{N}$ and $\omega \in \Omega^\infty$, and

$$\int_X \varphi(z)\pi(dz) = \lim_{n\to\infty} \int_X \varphi(z)\pi_n^f(x,dz) = \lim_{n\to\infty} \int_{\Omega^\infty} \varphi(f^n(x,\omega))\mathbb{P}^\infty(d\omega) = 0.$$

Hence

$$\pi(B) = \int_B \varphi d\pi = 0,$$

a contradiction. □

By Remark 3.1 we can set

$$\psi(\infty) = \lim_{x\to\infty} \psi(x)$$

and consider ψ as a mapping of $[0,\infty]$ into itself.

Remark 3.3. Since by the Remark 3.2 the support of the invariant measure π^f from Theorem 3.1 is included in the closure of $f(X\times\Omega)$, it follows that for all $x\in X$ and $y\in f(X\times\Omega)$, we have

$$\int_X \rho(x,z)\pi^f(dz) \le \rho(x,y) + \int_{\mathrm{supp}\pi^f} \rho(y,z)\pi^f(dz) \le \rho(x,y) + \mathrm{diam} f(X\times\Omega).$$

Therefore, (9) yields

$$d_H\left(\pi_n^f(x,\cdot),\pi^f\right) \le \psi^n\big(\mathrm{dist}(x, f(X\times\Omega)) + \mathrm{diam} f(X\times\Omega)\big)$$

for all $x\in X$ and $n\in\mathbb{N}$.

4. Examples

Fix $\xi \in L^1(\Omega,\mathcal{A},\mathbb{P})$, a non-zero $\eta\in L^1(\Omega,\mathcal{A},\mathbb{P})$, and put $\alpha = \frac{1}{\|\eta\|_1}$. Let $\psi\colon [0,\infty)\to[0,\infty)$ be a concave function such that $\eta\psi(x)+\xi$ is non-negative for every $x\in[0,\infty)$ and

$$\frac{|\psi(\alpha x)-\psi(\alpha z)|}{\alpha} \le \psi(|x-z|) < |x-z| \quad \text{for } x,z\in[0,\infty) \text{ with } x\neq z. \tag{12}$$

Note that ψ is non-expansive and $\psi(0)=0$. In particular, the formula

$$f(x,\omega) = \eta(\omega)\psi(\alpha x) + \xi(\omega) \quad \text{for } x\in[0,\infty) \text{ and } \omega\in\Omega$$

defines a random affine map $f\colon [0,\infty)\times\Omega\to[0,\infty)$. It is clear that (5) holds with any $x_0\in[0,\infty)$ and

$$\int_\Omega |f(x,\omega)-f(z,\omega)|\mathbb{P}(d\omega) \le \psi(|x-z|) \quad \text{for } x,z\in[0,\infty).$$

In consequence (H) holds, and one can apply Theorem 3.1. Note that neither [2, Theorem 3.1] nor [8, Theorem 1.1] do not apply, whenever

$$\psi \notin \bigcup_{\alpha\in(0,1)} \mathrm{Lip}_\alpha([0,\infty)). \tag{13}$$

Let π^f be the measure resulting from Theorem 3.1. Then (9) yields

$$d_H\left(\pi_n^f(x,\cdot),\pi^f\right) \le \psi^{n-1}(\psi(\infty)) \quad \text{for } x\in[0,\infty) \text{ and } n\in\mathbb{N}. \tag{14}$$

Example 4.1. Consider $\psi\colon [0,\infty) \to [0,\infty)$ given by

$$\psi(t) = \frac{t}{1+t}.$$

It is easy to see that ψ is concave, satisfies (12) with $\alpha = 1$, (13) holds, and (14) leads to

$$d_H\left(\pi_n^f(x,\cdot), \pi^f\right) \le \psi^{n-1}(1) = \frac{1}{n} \quad \text{for } x \in [0,\infty) \text{ and } n \in \mathbb{N}.$$

Example 4.2. Consider now $\psi\colon [0,\infty) \to [0,\infty)$ given by

$$\psi(t) = \arctan t.$$

It is easy to check that ψ is concave, satisfies (12) with $\alpha = 1$, and (13) holds. Now, (14) gives

$$d_H\left(\pi_n^f(x,\cdot), \pi^f\right) \le \psi^{n-1}\left(\frac{\pi}{2}\right) \quad \text{for } x \in [0,\infty) \text{ and } n \in \mathbb{N}.$$

Applying [18, Theorem 1.3.6] we conclude that for every $c \in \left(\sqrt{\frac{3}{2}}, \infty\right)$ there exists $n_0 \in \mathbb{N}$ such that

$$d_H\left(\pi_n^f(x,\cdot), \pi^f\right) \le \frac{c}{\sqrt{n}} \quad \text{for } x \in [0,\infty) \text{ and } n \ge n_0.$$

Example 4.3. Fix $\psi_0\colon [0,\infty) \to [0,\infty)$ of the form $\psi_0(t) = \frac{t}{1+t}$ or $\psi_0(t) = \arctan t$, and consider $\psi\colon [0,\infty) \to [0,\infty)$ given by

$$\psi(t) = \sin\psi_0(t).$$

Note that ψ is concave and (13) holds. Observe also that ψ satisfies (12) with $\alpha = 2$; indeed, for all $x \neq z$ we have

$$\begin{aligned} |\sin\psi_0(2x) - \sin\psi_0(2z)| &\le 2\sin\frac{|\psi_0(2x) - \psi_0(2z)|}{2} \le 2\sin\frac{\psi_0(2|x-z|)}{2} \\ &\le 2\sin\psi_0(|x-z|) \le 2|x-z|. \end{aligned}$$

Since $\psi \le \psi_0$, we conclude that (14) implies

$$d_H\left(\pi_n^f(x,\cdot), \pi^f\right) \le \psi_0^{n-1}(\psi_0(\infty)) \quad \text{for } x \in [0,\infty) \text{ and } n \in \mathbb{N}.$$

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