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Results in Mathematics

Polynomial Equations for Additive Functions I: The Inner Parameter Case

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Abstract. The aim of this sequence of work is to investigate polynomial equations satisfied by additive functions. As a result of this, new characterization theorems for homomorphisms and derivations can be given. More exactly, in this paper the following type of equation is considered

$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x^{q_i}) = 0 \qquad (x \in \mathbb{F}),$$

where n is a positive integer, $\mathbb{F} \subset \mathbb{C}$ is a field, $f_i, g_i : \mathbb{F} \to \mathbb{C}$ are additive functions and p_i, q_i are positive integers for all $i = 1, \ldots, n$.

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1. Introduction and Preliminaries

Equations satisfied by additive functions play an important role not only in the theory of commutative algebra, but also in the theory of functional equations. It is an important and challenging question how special morphisms (such as homomorphisms and derivations) can be characterized among additive mappings in general. In this paper classes of multivariable algebraic equations are introduced with appropriate solutions as field homomorphisms and derivations.

Concerning all the cases we consider here, the involved additive functions are defined on a field $\mathbb{F} \subset \mathbb{C}$ and have values in the complex field, therefore we introduce the preliminaries in this setting.

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Birkhäuser

We adopt the standard notations, that is, $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the set of complex numbers, respectively.

Henceforth we assume $\mathbb{F} \subset \mathbb{C}$ to be a field.

Definition 1. We say that a function $f: \mathbb{F} \to \mathbb{C}$ is *additive* if it fulfills the so-called *Cauchy functional equation*, that is,

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathbb{F}).$$

An additive function $d: \mathbb{F} \to \mathbb{C}$ is termed to be a *derivation* (of order 1) if it also fulfills the *Leibniz equation*, i.e.,

$$d(xy) = d(x)y + xd(y) \qquad (x, y \in \mathbb{F}).$$

An additive function $\varphi: \mathbb{F} \to \mathbb{C}$ is said to be a *h*omomorphism if it is multiplicative as well, in other words, besides additivity we also have

$$\varphi(xy) = \varphi(x)\varphi(y) \qquad (x, y \in \mathbb{F}).$$

If $\mathbb{F} = \mathbb{C}$ and φ is an isomorphism, then φ is called a *complex automorphism.*

Certain well-known equations are especially important. For instance, the additive solutions of following equation on a ring R with char$R \neq 2$

$$f(x^2) = 2xf(x) \qquad (x \in)$$

are derivations, under some assumptions, where the additive mapping f acts. As an extension of such type of results in [1,2,5] the additive solutions of the equations

$$\sum_{i=0}^{n} x^i f_{n+1-i}(x^{n+1-i}) = 0 \qquad (x \in R)$$

and

$$\sum_{i=0}^{n} f(x^{p_i})x^{q_i} = 0 \qquad (x \in \mathbb{F})$$

were described, here R denotes a ring with char$(R) \geq n$, while $\mathbb{F} \subset \mathbb{C}$ is a field. By a polynomial equation of additive functions we mean an equation of the form

$$P(f_1^{r_1}(x^{s_1}), \ldots, f_n^{r_n}(x^{s_n})) = 0,$$

where $P: \mathbb{C}^n \to \mathbb{C}$ is a n-variable polynomial, r_i, s_i are positive integers and f_i denote the unknown additive functions. Without further restrictions (e.g., on the polynomial P or on the parameters r_i, s_i), the above equation is unfortunately too general for its solutions to be fully determined. Indeed, it is not too hard to specify a polynomial P and parameters r_i, s_i so that the above equation is satisfied by all additive functions. Therefore, in this series, we will focus on the following classes of equations.

$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x^{q_i}) = 0,$$
$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x)^{q_i} = 0, \qquad (x \in \mathbb{F})$$
$$\sum_{i=1}^{n} f_i(x)^{p_i}g_i(x)^{q_i} = 0,$$

In this paper the most impressive equation, namely

$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x^{q_i}) = 0, \qquad (x \in \mathbb{F}) \tag{1}$$

will be studied from this list with a fruitful theoretical description. (This we call 'inner parameter case', as the parameters p_i, q_i are exponents of the variable x, so they act on the domain of the functions f_i, g_i, respectively.) Under some natural conditions equation (1) is satisfied by compositions of (higher order) derivations and homomorphisms. The purpose of this paper is about the converse by showing proper characterizations for the solutions of (1) in the class of additive functions.

Structure of the Paper

In Sect. 2 the most important notations, terminology and theoretical background is summarized. Concerning the notions of polynomials, generalized polynomials, exponentials and exponential polynomials, here we follow the monograph of Székelyhidi [10]. Besides these notions, decomposable functions, introduced by Shulman in [9], will play a key role in the second section. We show that all solutions of equation (1) are decomposable functions. After that a result of Laczkovich will be used, who proved in [7] that on unital commutative topological semigroups, decomposable mappings are generalized exponential polynomials. On fields these are closely related to higher order derivations that were introduced by [8, 11]. There will also be cases, when we will restrict to finitely generated subfields of $\mathbb{F}$ since on them higher order derivations are differential operators. The latter concept is significantly easier to calculate. This makes it possible to determine the exact upper bound for their orders.

The main results of the paper can be found in the third section. At first some elementary yet important lemmata serve to settle reasonable conditions for the parameters p_i, q_i such as the Homogenization Principle (see Lemma 9), which ensures that the parameters satisfy

$$p_i + q_i = N \qquad (i = 1, \ldots, n).$$

Based on the remarks and the examples of Sect. 3.1, we will provide characterization theorems for Eq. (1) under the following conditions

C(i) the positive integers $p_1, \dots, p_n$ satisfy $p_1 < \cdots < p_n$;
C(ii) for all $i = 1, \dots, n$ we have $p_i + q_i = N$;
C(iii) for all $i, j \in \{1, \dots, n\}$, $i \neq j$ we have $p_i \neq q_j$.

Further, according to Lemma 10, the solutions of the above functional equations are sufficient to determine 'up to equivalence'. This is because the functions f_i and g_i fulfill equation (1), if and only if for any automorphism $\varphi\colon \mathbb{C} \to \mathbb{C}$, the functions $\varphi \circ f_i$ and $\varphi \circ g_i$ also fulfill (1), $i = 1, \dots, n$.

Looking at Eq. (1), it yields a technical problem that there is only one independent variable in the equation. At the same time, the involved functions are assumed to be additive. Thus the *polarization formula* for multi-additive functions can be used in the *symmetrization method*, which allows us to enlarge the number of independent variables from one to N.

In Lemma 13 it is shown that the functions f_i and g_i satisfies the system of equations given by this method are decomposable functions thus generalized exponential polynomials on the group $\mathbb{F}^{\times}$. Theorem 14 says that for any $i \in \{1, \dots, n\}$, in the variety of the functions f_i and g_i there is exactly one exponential m_i. Focusing on the irreducible solutions, this means that all f_i (resp. g_i) are of the form $P_i \cdot m$ (resp. $Q_i \cdot m$), where P_i (and Q_i) are (generalized) polynomials and m is a unique exponential function.

Translating the problem to higher order derivations there can be found a natural basis of compositions of derivations of order 1 by using moment generating functions. Applying the arithmetic of derivations we get a sharp upper bound, which is $n-1$ for the order of derivation solutions under some conditions, see Theorem 20. We close this section with the study of some important special cases. In Conjecture 21 and Open Problem 1 we pose problems on the exact order of the (higher order) derivation solutions in different settings.

2. Notation, Terminology and Theoretical Background

2.1. Polynomials and Generalized Polynomials

Definition 2. Let G, S be commutative semigroups (written additively), $n \in \mathbb{N}$ and let $A\colon G^n \to S$ be a function. We say that A is *n-additive* if it is a homomorphism of G into S in each variable. If $n = 1$ or $n = 2$ then the function A is simply termed to be *additive* or *biadditive*, respectively.

The *diagonalization* or *trace* of an n-additive function $A\colon G^n \to S$ is defined as

$$A^*(x) = A\left(x, \dots, x\right) \qquad \left(x \in G\right).$$

As a direct consequence of the definition each n-additive function $A\colon G^n \to S$ satisfies

$$A(x_1, \dots, x_{i-1}, kx_i, x_{i+1}, \dots, x_n) = kA(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \qquad (x_1, \dots, x_n \in G)$$

for all $i = 1, \dots, n$, where $k \in \mathbb{N}$ is arbitrary. The same identity holds for any $k \in \mathbb{Z}$ provided that G and S are groups, and for $k \in \mathbb{Q}$, provided that G and S are linear spaces over the rationals. For the diagonalization of A we have

$$A^*(kx) = k^n A^*(x) \qquad (x \in G)\,.$$

The above notion can also be extended for the case $n = 0$ by letting $G^0 = G$ and by calling 0-additive any constant function from G to S.

One of the most important theoretical results concerning multiadditive functions is the so-called *Polarization formula*, that briefly expresses that every n-additive symmetric function is *uniquely* determined by its diagonalization under some conditions on the domain as well as on the range. Suppose that G is a commutative semigroup and S is a commutative group. The action of the *difference operator* Δ on a function $f\colon G \to S$ is defined by the formula

$$\Delta_y f(x) = f(x+y) - f(x) \qquad (x, y \in G)\,.$$

Note that the addition in the argument of the function is the operation of the semigroup G and the subtraction means the inverse of the operation of the group S.

Theorem 1 (Polarization formula)**.** *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y_1, \dots, y_m \in G$ we have*

$$\Delta_{y_1,\dots,y_m} A^*(x) = \begin{cases} 0 & \text{if } m > n \\ n!A(y_1, \dots, y_m) & \text{if } m = n. \end{cases}$$

Corollary 2. *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y \in G$*

$$\Delta_y^n A^*(x) = n!A^*(y).$$

Lemma 3. *Let $n \in \mathbb{N}$ and suppose that the multiplication by $n!$ is surjective in the commutative semigroup G or injective in the commutative group S. Then for any symmetric, n-additive function $A\colon G^n \to S$, $A^* \equiv 0$ implies that A is identically zero, as well.*

Definition 3. Let G and S be commutative semigroups, a function $p\colon G \to S$ is called a *generalized polynomial* from G to S, if it has a representation as the sum of diagonalizations of symmetric multi-additive functions from G to S. In other words, a function $p\colon G \to S$ is a generalized polynomial if and only if, it has a representation

$$p = \sum_{k=0}^{n} A_k^*,$$

where n is a nonnegative integer and $A_k\colon G^k \to S$ is a symmetric, k-additive function for each $k = 0, 1, \dots, n$. In this case we also say that p is a generalized polynomial *of degree at most n*.

Let n be a nonnegative integer, functions $p_n\colon G \to S$ of the form

$$p_n = A_n^*,$$

where $A_n\colon G^n \to S$ is a symmetric and n-additive mapping, are the so-called *generalized monomials of degree* n.

In this subsection $(G, \cdot)$ is assumed to be a commutative group (written multiplicatively).

Definition 4. *Polynomials* are elements of the algebra generated by additive functions over G. More exactly, a mapping $f\colon G \to \mathbb{C}$ is called a *polynomial* if there is a positive integer n, there exists a (classical) complex polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ in n variables and there are additive functions $a_k\colon G \to \mathbb{C}$ $(k = 1, \dots, n)$ such that

$$f(x) = P(a_1(x), \dots, a_n(x)) \qquad (x \in G)\,.$$

Remark 1. We recall that the elements of $\mathbb{N}^n$ for any positive integer n are called (n-dimensional) *multi-indices*. Addition, multiplication and inequalities between multi-indices of the same dimension are defined component-wise. Further, we define x^α for any n-dimensional multi-index α and for any $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$ by

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$$

where we always adopt the convention $0^0 = 0$. We also use the notation $|\alpha| = \alpha_1 + \dots + \alpha_n$. With these notations any polynomial of degree at most N on the commutative semigroup G has the form

$$p(x) = \sum_{|\alpha| \le N} c_\alpha a(x)^\alpha \qquad (x \in G)\,,$$

where $c_\alpha \in \mathbb{C}$ and $a = (a_1, \dots, a_n)\colon G \to \mathbb{C}^n$ is an additive function. Furthermore, the *homogeneous term of degree* k of p is

$$\sum_{|\alpha| = k} c_\alpha a(x)^\alpha.$$

Lemma 4 (Lemma 2.7 of [10]). *Let G be a commutative group, n be a positive integer and let*

$$a = (a_1, \dots, a_n)\,,$$

where $a_1, \dots, a_n$ are linearly independent complex valued additive functions defined on G. Then the monomials $\{a^\alpha\}$ for different multi-indices are linearly independent.

Definition 5. A function $m\colon G \to \mathbb{C}$ is called an *exponential* function if it satisfies

$$m(xy) = m(x)m(y) \qquad (x, y \in G)\,.$$

Furthermore, on a(n) *(generalized) exponential polynomial* we mean a linear combination of functions of the form $p{\cdot}m$, where p is a (generalized) polynomial and m is an exponential function.

The following lemma shows that generalized exponential polynomial functions are linearly independent. Although it can be stated in a more general way (see [10]), we adopt it to our situation, when the functions are complex valued.

Lemma 5 (Lemma 4.3 of [10])**.** *Let G be a commutative group, n a positive integer, $m_1, \ldots, m_n\colon G \to \mathbb{C}\,(i = 1, \ldots, n)$ be distinct nonzero exponentials and $p_1, \ldots, p_n\colon G \to \mathbb{K}\,(i = 1, \ldots, n)$ be generalized polynomials. If $\sum_{i=1}^{n} p_i \cdot m_i$ is identically zero, then for all $i = 1, \ldots, n$ the generalized polynomial p_i is identically zero.*

Additionally, we will need the analogous statement for polynomial expressions of generalized exponential polynomials which was proved in [5].

Theorem 6. *Let $\mathbb{K}$ be a field of characteristic 0 and k, l, N be positive integers such that $k, l \le N$. Let $m_1, \ldots, m_k\colon \mathbb{K}^\times \to \mathbb{C}$ be distinct exponential functions that are additive on $\mathbb{K}$, let $a_1, \ldots, a_l\colon \mathbb{K}^\times \to \mathbb{C}$ be additive functions that are linearly independent over $\mathbb{C}$ and for all $|s| \le N$ let $P_s\colon \mathbb{C}^l \to \mathbb{C}$ be classical complex polynomials of l variables. If*

$$\sum_{|s|\le N} P_s(a_1, \ldots, a_l) m_1^{s_1} \cdots m_k^{s_k} = 0$$

then for all $|s| \le N$, the polynomials P_s vanish identically.

Definition 6. Let G be a commutative group and $V \subseteq \mathbb{C}^G$ a set of functions. We say that V is *translation invariant* if for every $f \in V$ the function $\tau_g f \in V$ also holds for all $g \in G$, where

$$\tau_g f(h) = f(hg) \qquad (h \in G)\,.$$

In view of Theorem 10.1 of Székelyhidi [10], any finite dimensional translation invariant linear space of complex valued functions on a commutative group consists of exponential polynomials. This implies that if G is a commutative group, then any function $f\colon G \to \mathbb{C}$, satisfying the functional equation

$$f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y) \qquad (x, y \in G)$$

for some positive integer n and functions $g_i, h_i\colon G \to \mathbb{C}$ $(i = 1, \ldots, n)$, is an exponential polynomial of degree at most n.

This enlightens the connection between generalized polynomials and polynomials. It is easy to see that each polynomial, that is, any function of the form

$$x \longmapsto P(a_1(x), \dots, a_n(x)),$$

where n is a positive integer, $P\colon \mathbb{C}^n \to \mathbb{C}$ is a (classical) complex polynomial in n variables and $a_k\colon G \to \mathbb{C}$ $(k = 1, \dots, n)$ are additive functions, is a generalized polynomial. The converse however is in general not true. A complex-valued generalized polynomial p defined on a commutative group G is a polynomial *if and only if* its variety (the linear space spanned by its translates) is of *finite* dimension.

Henceforth, not only the notion of (exponential) polynomials, but also that of *decomposable functions* will be used. The basics of this concept are due to Shulman [9], besides this we heavily rely on the work of Laczkovich [7].

Definition 7. Let G be a group and $n \in \mathbb{N}, n \geq 2$. A function $F\colon G^n \to \mathbb{C}$ is said to be *decomposable* if it can be written as a finite sum of products $F_1 \cdots F_k$, where all F_i depend on disjoint sets of variables.

Remark 2. Without loss of generality we can suppose that $k = 2$ in the above definition, that is, decomposable functions are those mappings that can be written in the form

$$F(x_1, \dots, x_n) = \sum_E \sum_j A_j^E B_j^E$$

where E runs through all non-void proper subsets of $\{1, \dots, n\}$ and for each E and j the function A_j^E depends only on variables x_i with $i \in E$, while B_j^E depends only on the variables x_i with $i \notin E$.

Theorem 7. *Let G be a commutative topological semigroup with unit. A continuous function $f\colon G \to \mathbb{C}$ is a generalized exponential polynomial* if and only if *there is a positive integer $n \geq 2$ such that the mapping*

$$G^n \ni (x_1, \dots, x_n) \longmapsto f(x_1 + \cdots + x_n)$$

is decomposable.

The notion of derivations can be extended in several ways. We will employ the concept of higher order derivations according to Reich [8] and Unger and Reich [11]. For further results on characterization theorems on higher order derivations consult e.g. [1–3,5].

Definition 8. Let $\mathbb{F} \subset \mathbb{C}$ be a field. The identically zero map is the only *derivation of order zero*. For each $n \in \mathbb{N}$, an additive mapping $f\colon \mathbb{F} \to \mathbb{C}$ is termed to be a *derivation of order n*, if there exists $B\colon \mathbb{F} \times \mathbb{F} \to \mathbb{C}$ such that B is a bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and

$$f(xy) - xf(y) - f(x)y = B(x, y) \qquad (x, y \in \mathbb{F}).$$

The set of derivations of order n of the ring R will be denoted by $\mathscr{D}_n(\mathbb{F})$.

Remark 3. Since $\mathscr{D}_0(\mathbb{F}) = \{0\}$, the only bi-derivation of order zero is the identically zero function, thus $f \in \mathscr{D}_1(\mathbb{F})$ if and only if

$$f(xy) = xf(y) + f(x)y \qquad (x, y \in \mathbb{F}),$$

that is, the notions of first order derivations and derivations coincide. On the other hand for any $n \in \mathbb{N}$ the set $\mathscr{D}_n(\mathbb{F}) \setminus \mathscr{D}_{n-1}(\mathbb{F})$ is nonempty because $d_1 \circ \cdots \circ d_n \in \mathscr{D}_n(\mathbb{F})$, but $d_1 \circ \cdots \circ d_n \notin \mathscr{D}_{n-1}(R)$, where $d_1, \ldots, d_n \in \mathscr{D}_1(\mathbb{F})$ are non-identically zero derivations.

For our future purposes the notion of differential operators will also be important, see [6].

Definition 9. Let $\mathbb{F} \subset \mathbb{C}$ be a field. We say that the map $D\colon \mathbb{F} \to \mathbb{C}$ is a *differential operator of order at most n* if D is the linear combination, with coefficients from $\mathbb{F}$, of finitely many maps of the form $d_1 \circ \cdots \circ d_k$, where $d_1, \ldots, d_k$ are derivations on $\mathbb{F}$ and $k \leq n$. If $k = 0$ then we interpret $d_1 \circ \cdots \circ d_k$ as the identity function. We denote by $\mathscr{O}_n(\mathbb{F})$ the set of differential operators of order at most n defined on $\mathbb{F}$. We say that the order of a differential operator D is n if $D \in \mathscr{O}_n(\mathbb{F}) \setminus \mathscr{O}_{n-1}(\mathbb{F})$ (where $\mathscr{O}_{-1}(\mathbb{F}) = \emptyset$, by definition).

Remark 4. The term *differential operator* is justified by the following fact. Let $\mathbb{K} = \mathbb{Q}(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$. Then $\mathbb{K}$ is the field of all rational functions of $t_1, \ldots, t_k$ with rational coefficients. It is clear that

$$d_i = \frac{\partial}{\partial t_i}$$

is a derivation on $\mathbb{K}$ for every $i = 1, \ldots, k$. Therefore, every differential operator

$$D = \sum_{i_1+\cdots+i_k \leq n} c_{i_1,\ldots,i_k} \cdot \frac{\partial^{i_1+\cdots+i_k}}{\partial t_1^{i_1} \cdots \partial t_k^{i_k}},$$

where the coefficients $c_{i_1,\ldots,i_k}$ belong to $\mathbb{K}$, is a differential operator of order at most n, and also conversely, if D is a differential operator of order at most n on the field $\mathbb{K} = \mathbb{Q}(t_1, \ldots, t_k)$, then D is of the above form.

The main result of [6] is Theorem 1.1 that reads in our settings as follows.

Theorem 8. *Let $\mathbb{F} \subset \mathbb{C}$ be a field and let n be a positive integer. Then, for every function $D\colon \mathbb{F} \to \mathbb{C}$, the following are equivalent.*

(i) $D \in \mathscr{D}_n(\mathbb{F})$
(ii) $D \in \mathrm{cl}\,(\mathscr{O}_n(\mathbb{F}))$
(iii) *D is additive on $\mathbb{F}$, $D(1) = 0$, and D/j, as a map from the group $\mathbb{F}^{\times}$ to $\mathbb{C}$, is a generalized polynomial of degree at most n. Here j stands for the identity map defined on $\mathbb{F}$.*

3. Results

3.1. Elementary Observations: Reduction of the Problem

This part begins with some elementary, yet fundamental observations. As the following lemmata show, the original problem can be reduced to a more simpler equation.

Lemma 9 (Homogenization). *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \dots, p_n, q_1, \dots, q_n$ be fixed positive integers. Assume that the additive functions $f_1, \dots, f_n, g_1, \dots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1), that is,*

$$\sum_{i=1}^{n} f_i(x^{p_i}) g_i(x^{q_i}) = 0$$

for each $x \in \mathbb{F}$. If the set $\{p_1, \dots, p_n\}$ has a partition $\mathcal{P}_1, \dots, \mathcal{P}_k$ with the property

$$\text{if } p_\alpha, p_\beta \in \mathcal{P}_j \text{ for a certain index} j, \text{ then } p_\alpha + q_\alpha = p_\beta + q_\beta,$$

then the system of equations

$$\sum_{p_\alpha \in \mathcal{P}_j} f_\alpha(x^{p_\alpha}) g_\alpha(x^{q_\alpha}) = 0 \qquad (x \in \mathbb{F}, j = 1, \dots, k)$$

is satisfied.

Proof. Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \dots, p_n, q_1, \dots, q_n$ be fixed positive integers. Assume that the additive functions $f_1, \dots, f_n, g_1, \dots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1) for each $x \in \mathbb{F}$. Assume further that the set $\{p_1, \dots, p_n\}$ has a partition $\mathcal{P}_1, \dots, \mathcal{P}_k$ with the property

$$\text{if } p_\alpha, p_\beta \in \mathcal{P}_j \text{ for a certain index} j, \text{ then } p_\alpha + q_\alpha = p_\beta + q_\beta.$$

Observe that for all $i = 1, \dots, n$, the mapping

$$\mathbb{F} \ni x \longmapsto f_i(x^{p_i}) g_i(x^{q_i})$$

is a generalized monomial of degree $p_i + q_i$. Indeed, it is the diagonalization of the symmetric $(p_i + q_i)$-additive mapping

$$\mathbb{F}^{p_i+q_i} \ni (x_1, \dots, x_{p_i+q_i}) \longmapsto f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(p_i+q_i)}).$$

Since $\mathbb{F} \subset \mathbb{C}$, we necessarily have $\mathbb{Q} \subset \mathbb{F}$. Let now $r \in \mathbb{Q}$ be arbitrary and substitute rx in place of x in Eq. (1) to get

$$\sum_{i=1}^{n} f_i((rx)^{p_i}) g_i((rx)^{q_i}) = 0 \qquad (r \in \mathbb{Q}, x \in \mathbb{F}).$$

Using the $\mathbb{Q}$-homogeneity of the additive functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$, we deduce

$$\begin{aligned} 0 &= \sum_{i=1}^{n} f_i((rx)^{p_i}) g_i((rx)^{q_i}) = \sum_{i=1}^{n} f_i(r^{p_i} x^{p_i}) g_i(r^{q_i} x^{q_i}) \\ &= \sum_{i=1}^{n} r^{p_i+q_i} f_i(x^{p_i}) g_i(x^{q_i}) = \sum_{j=1}^{k} \sum_{p_\alpha \in \mathcal{P}_j} r^{p_\alpha+q_\alpha} f_\alpha(x^{p_\alpha}) g_\alpha(x^{q_\alpha}) \\ &\quad (r \in \mathbb{Q}, x \in \mathbb{F}). \end{aligned}$$

Note that the right hand side of this equation is a (classical) polynomial in r which is identically zero. Thus all of its coefficients should be (identically) zero, yielding that the system of equations

$$\sum_{p_\alpha \in \mathcal{P}_j} f_\alpha(x^{p_\alpha}) g_\alpha(x^{q_\alpha}) = 0 \qquad (x \in \mathbb{F}, j = 1, \ldots, k)$$

is fulfilled. □

Remark 5. The above lemma guarantees that *ab initio*

$$p_i + q_i = N \qquad (i = 1, \ldots, n)$$

can be assumed. Otherwise, after using the above homogenization, we get a system of functional equations in which this condition is already fulfilled. For instance, due to the above lemma, if the additive functions $f_1, \ldots, f_5 \colon \mathbb{F} \to \mathbb{C}$ and $g_1, \ldots, g_5 \colon \mathbb{F} \to \mathbb{C}$ satisfy equation

$$\begin{aligned} &f_1(x^{24}) g_1(x^5) + f_2(x^{20}) g_2(x^9) + f_3(x^{19}) g_3(x^{10}) \\ &\quad + f_4(x^{13}) g_4(x^7) + f_5(x^{12}) g_4(x^8) = 0 \qquad (x \in \mathbb{F}) \end{aligned}$$

then the equations

$$f_1(x^{24}) g_1(x^5) + f_2(x^{20}) g_2(x^9) + f_3(x^{19}) g_3(x^{10}) = 0 \qquad (x \in \mathbb{F})$$

and

$$f_4(x^{13}) g_4(x^7) + f_5(x^{12}) g_4(x^8) = 0 \qquad (x \in \mathbb{F})$$

are also fulfilled (separately).

Remark 6. At first glance the assumption that $p_1, \ldots, p_n$ are different seems a reasonable and sufficient supposition. Clearly, if the parameters are not necessarily different then we cannot expect anything special for the form of the involved additive functions. Indeed, let $L \subset \mathbb{C}^n$ be a linear subspace and let $f_1, \ldots, f_n \colon \mathbb{F} \to \mathbb{C}$ and $g_1, \ldots, g_n \colon \mathbb{F} \to \mathbb{C}$ be additive functions such that $\mathrm{rng}(f) \subset L$ and $\mathrm{rng}(g) \subset L^\perp$, where

$$f(x) = (f_1(x), \ldots, f_n(x)) \quad \text{and} \quad g(x) = (g_1(x), \ldots, g_n(x)) \quad (x \in \mathbb{F}).$$

In this case

$$\sum_{i=1}^{n} f_i(x)g_i(x) = \langle f(x), g(x)\rangle = 0 \qquad (x \in \mathbb{F}).$$

This shows the necessity of the above assumption. Unfortunately, the sufficiency fails to hold. To see this, let p and q be positive integers and $f\colon \mathbb{F} \to \mathbb{C}$ be an *arbitrary* additive function and define the complex-valued functions f_1, g_1, f_2, g_2 on $\mathbb{F}$ by

$$f_1(x) = f(x) \quad g_1(x) = f(x) \quad f_2(x) = if(x) \quad g_2(x) = if(x) \qquad (x \in \mathbb{F}).$$

An immediate computation shows that we have

$$f_1(x^p)g_1(x^q) + f_2(x^p)g_2(x^q) = 0 \qquad (x \in \mathbb{F}).$$

In view of the above remarks, from now on, the following assumptions are adopted.

- C(i) the positive integers $p_1, \dots, p_n$ are arranged in a strictly increasing order, i.e., $p_1 < \dots < p_n$;
- C(ii) for all $i = 1, \dots, n$ we have $p_i + q_i = N$;
- C(iii) for all $i, j \in \{1, \dots, n\}$, $i \neq j$ we have $p_i \neq q_j$.

Remark 7. Define the relation $\sim$ on $\mathbb{F}^{\mathbb{C}}$ by $f \sim g$ if and only if there exists an automorphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ f = g$. Obviously $\sim$ is an equivalence relation on $\mathbb{F}^{\mathbb{C}}$ that induces a partition on $\mathbb{F}^{\mathbb{C}}$.

Lemma 10 (Equivalence)**.** *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \dots, p_n, q_1, \dots, q_n$ be fixed positive integers fulfilling the conditions C(i)–C(iii) of Remark 6. Assume that the additive functions $f_1, \dots, f_n$, $g_1, \dots, g_n\colon \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1). Then for an arbitrary automorphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ the functions $\varphi \circ f_1, \dots, \varphi \circ f_n, \varphi \circ g_1, \dots, \varphi \circ g_n$ also fulfill equation (1).*

3.2. Structure of Solutions

We can always restrict ourselves to the case when all the involved functions are non-identically zero. Otherwise, the number of the terms appearing in Eq. (1) can be reduced.

Lemma 11 (Symmetrization)**.** *Let k and n be positive integers, $\mathbb{F} \subset \mathbb{C}$ be a field and $m_1, \dots, m_n\colon \mathbb{F} \to \mathbb{C}$ be monomials of degree k. If*

$$\sum_{i=1}^{n} m_i(x) = 0$$

holds for all $x \in \mathbb{F}$, then

$$\sum_{i=1}^{n} M_i(x_1, \dots, x_k) = 0$$

is fulfilled for all $x_1, \ldots, x_k$, *where for all* $i = 1, \ldots, n$, *the mapping* $M_i : \mathbb{F}^k \to \mathbb{C}$ *is the uniquely determined symmetric,* k*-additive function such that*

$$M_i(x, \ldots, x) = m_i(x) \qquad (x \in \mathbb{F}).$$

Proof. Let k and n be positive integers, $\mathbb{F} \subset \mathbb{C}$ be a field and $m_1, \ldots, m_n : \mathbb{F} \to \mathbb{C}$ be monomials of degree k and assume that

$$\sum_{i=1}^{n} m_i(x) = 0$$

holds for all $x \in \mathbb{F}$. Since for all $i = 1, \ldots, n$, the function m_i is a monomial of degree k, there exists a symmetric, k-additive function $M_i : \mathbb{F}^k \to \mathbb{C}$ such that we have

$$M_i(x, \ldots, x) = m_i(x) \qquad (x \in \mathbb{F}).$$

Obviously the mapping $\sum_{i=1}^{n} m_i$ is a monomial of degree k which is, by the assumptions, identically zero. To this monomial there also corresponds a symmetric and k-additive mapping, namely

$$\mathbb{F}^k \ni (x_1, \ldots, x_k) \longmapsto \sum_{i=1}^{n} M_i(x_1, \ldots x_k).$$

Observe that the trace of this symmetric and k-additive mapping is identically zero. At the same time, due to the Polarization formula (Theorem 1), every symmetric and k-additive function is *uniquely* determined by its trace. Thus

$$\sum_{i=1}^{n} M_i(x_1, \ldots, x_k) = 0$$

for all $x_1, \ldots, x_k \in \mathbb{F}$. □

Lemma 12 (Symmetrization). *Let* n *be a positive integer,* $\mathbb{F} \subset \mathbb{C}$ *be a field and* $p_1, \ldots, p_n, q_1, \ldots, q_n$ *be fixed positive integers fulfilling conditions C(ii), i.e., there is a* $N \in \mathbb{N}$ *such* $p_i + q_i = N$ *for all* $i = 1, \ldots, n$. *Assume that the additive functions* $f_1, \ldots, f_n, g_1, \ldots, g_n : \mathbb{F} \to \mathbb{C}$ *satisfy functional equation* (1) *for each* $x \in \mathbb{F}$. *Then*

$$\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} \sum_{i=1}^{n} f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) \cdot g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(N)}) = 0$$

holds for all $x_1, \ldots, x_N \in \mathbb{F}$.

Proof. Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(ii). Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n : \mathbb{F} \to \mathbb{C}$ satisfy functional equation

$$\sum_{i=1}^{n} f_i(x^{p_i}) g_i(x^{q_i}) = 0$$

for each $x \in \mathbb{F}$. Due to the additivity of the functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ for all $i = 1, \ldots, n$, the mapping

$$x \longmapsto f_i(x^{p_i})g_i(x^{q_i})$$

is a monomial of degree $p_i + q_i = N$. Further, it is the trace of the symmetric and N-additive mapping

$$\begin{aligned} &F_i(x_1, \ldots, x_N) \\ &\quad = \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) \cdot g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(N)}) \\ &\qquad (x_1, \ldots, x_N \in \mathbb{F}). \end{aligned}$$

Therefore, the statement follows from Lemma 11. □

3.3. Solutions of Eq. (1)

The main purpose of the subsection is to describe under the conditions C(i)–C(iii), the solution space of Eq. (1). We first prove that solutions of Eq. (1) are decomposable functions on the multiplicative group $\mathbb{F}^{\times}$. In view of Laczkovich [7], this immediately yields that the solutions of Eq. (1) are generalized exponential polynomials of this group.

Lemma 13. *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(i) and C(ii). Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1) for each $x \in \mathbb{F}$. Then all the functions $f_1, \ldots, f_n$ as well as $g_1, \ldots, g_n$ are decomposable functions of the group $\mathbb{F}^{\times}$.*

Proof. Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(i) and C(ii).

Let us assume first that condition C(iii) is also satisfied. Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1) for each $x \in \mathbb{F}$. Let

$$S = \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_n\}.$$

Then $\max S = \max\{p_n, q_1\}$. By condition C(iii), we have $p_n \neq q_1$. Without the loss of generality $p_n > q_1$ can be assumed, otherwise we follow a similar argument. In view of Lemma 12, we have

$$\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} \sum_{i=1}^{n} f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) \cdot g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(N)}) = 0$$

for all $x_1, \ldots, x_N \in \mathbb{F}$, or after some rearrangement,

$$\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} f_n(x_{\sigma(1)} \cdots x_{\sigma(p_n)}) \cdot g_n(x_{\sigma(p_n+1)} \cdots x_{\sigma(N)}) \\ = -\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} \sum_{i=1}^{n-1} f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) \cdot g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(N)}) \\ \left(x_1, \ldots, x_N \in \mathbb{F}^\times\right).$$

Let now

$$x_{p_n+1} = \cdots = x_N = 1,$$

then the above identity says that $g_n(1) \cdot f_n$ is decomposable. If $g_n(1)$ were zero, but g_n would not be identically zero, then there would exist $a \in \mathbb{F}^\times$ such that $g_n(a) \neq 0$. In this case the above substitutions should be modified to

$$x_{p_n+1} = a, \ x_{p_n+2} = \cdots = x_N = 1,$$

to get the same conclusion.

If $p_n = q_j$ for some $j = \{1, \ldots, n\}$ (i.e., C(iii) does not hold), then without loss of generality we may assume that $j = 1$, otherwise we may change the role of f_i and g_i, and p_i and q_i, respectively, and proceed as above. If $p_n = q_1$, then we have

$$\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_n} f_n(x_{\sigma(1)} \cdots x_{\sigma(p_n)}) \cdot g_n(x_{\sigma(p_n+1)} \cdots x_{\sigma(N)}) \\ + \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_n} g_1(x_{\sigma(1)} \cdots x_{\sigma(p_n)}) \cdot f_1(x_{\sigma(p_n+1)} \cdots x_{\sigma(N)}) \\ = -\frac{1}{N!} \sum_{\sigma \in \mathscr{S}_n} \sum_{i=2}^{n-1} f_i(x_{\sigma(1)} \cdots x_{\sigma(p_i)}) \cdot g_i(x_{\sigma(p_i+1)} \cdots x_{\sigma(N)}) \\ \left(x_1, \ldots, x_N \in \mathbb{F}^\times\right).$$

This equation with the substitutions

$$x_{p_n+1} = \cdots = x_N = 1,$$

yields that a linear combination of f_n and g_1 is decomposable. If $\{f_n, g_1\}$ is linearly dependent, then this obviously means that both f_n and g_1 are decomposable functions. If this system is not linearly dependent, then there exist $a, b \in \mathbb{F}^\times$, $a \neq b$ and different complex constants c_1 and c_2 such that

$$f_n(a) = c_1 g_1(a) \qquad f_n(b) = c_2 g_1(b).$$

With the substitutions

$$x_{p_n+1} = a, \ x_{p_n+2} = \cdots = x_N = 1,$$

and

$$x_{p_n+1} = b, \ x_{p_n+2} = \cdots = x_N = 1,$$

we get that the functions

$$(x_1, \dots, x_{p_1}) \longmapsto g_1(x_1 \cdots x_{p_1}) f_n(a) + f_n(x_1 \cdots x_{p_1}) g_1(a)$$

and

$$(x_1, \dots, x_{p_1}) \longmapsto g_1(x_1 \cdots x_{p_1}) f_n(b) + f_n(x_1 \cdots x_{p_1}) g_1(b)$$

are decomposable. Since finite linear combinations of decomposable functions are also decomposable, it follows that f_n and g_1 are decomposable, separately.

After that, let us consider the set $S \setminus \{p_n\}$ and apply the above argument for this set. With this step-by-step descending argument the statement of the lemma follows. □

Remark 8. We emphasize that condition C(iii) in Lemma 13 has not been assumed. Thus, the fact that the additive solutions of (1) are decomposable functions can be deduced only under the conditions C(i) and C(ii). On the other hand, for our purpose to describe the solutions more concretely we have to assume also condition C(iii) to avoid further difficulties. We believe however that most of our methods can work similarly only under the conditions C(i) and C(ii).

Remark 9. Note also that in general we cannot state more than that the involved functions $f_1, \dots, f_n$ and $g_1, \dots, g_n$ are decomposable functions on the commutative group $\mathbb{F}^{\times}$. In other words, we can only state that the solutions of the functional equation in question are higher order derivations (see below Corollary 15). In general it is not true that the solutions of this functional equation are differential operators (i.e., exponential polynomials of the multiplicative group $\mathbb{F}^{\times}$). To see this, let us consider the functional equation

$$x f_1(x^6) + x^2 f_2(x^5) + x^3 f_3(x^4) = 0 \qquad (x \in \mathbb{F}).$$

Indeed, using the results of [5], we deduce that $f_1, f_2, f_3 \in \mathscr{D}_2(\mathbb{F})$.

Due to a result of Laczkovich [7] and under the assumptions of Lemma 13, there exist a positive integer l, there are generalized polynomials $P_{k,i}, Q_{k,i} \colon \mathbb{F}^{\times} \to \mathbb{C}$ ($k = 1, \dots, l$ and $i = 1, \dots, n$) and there exist linearly independent exponentials $m_1, \dots, m_l \colon \mathbb{F}^{\times} \to \mathbb{C}$ such that

$$f_i(x) = \sum_{k=1}^{l} P_{k,i}(x) m_k(x) \text{ and } g_i(x) = \sum_{k=1}^{l} Q_{k,i}(x) m_k(x) \qquad \left(x \in \mathbb{F}^{\times}\right).$$

Since the generalized polynomials on any finitely generated field are polynomials and the exponentials $m_1, \dots, m_k$ are linearly independent, we can apply Theorem 6. This implies that after substituting the above form the functions $f_1, \dots, f_n$ and $g_1, \dots, g_n$ and using that for each $\kappa \in \mathbb{N}$ and $k = 1, \dots, n$, we have

$$m_k(x^{\kappa}) = m_k(x)^{\kappa} \qquad \left(x \in \mathbb{F}^{\times}\right),$$

especially

$$\sum_{i=1}^{n} P_{k,i}(x^{p_i})Q_{k,i}(x^{q_i}) = 0 \qquad \left(x \in \mathbb{F}^{\times}\right)$$

follows for all $k = 1, \ldots, l$. This tells us that it is enough to solve Eq. (1) for generalized polynomials of the group $\mathbb{F}^{\times}$. In fact, we can prove the following.

Theorem 14. *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(i)–C(iii). Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n : \mathbb{F} \to \mathbb{C}$ satisfy functional equation (1) for each $x \in \mathbb{F}$. Then there exists a positive integer l, there exist exponentials $m_i : \mathbb{F}^{\times} \to \mathbb{C}$ and there are generalized polynomials $P_i, Q_i : \mathbb{F}^{\times} \to \mathbb{C}$ of degree at most l such that*

$$f_i(x) = P_i(x)m_i(x) \qquad \text{and} \qquad g_i(x) = Q_i(x)m_i(x) \qquad \left(x \in \mathbb{F}^{\times}\right)$$

for each $i = 1, \ldots, n$.

Proof. As we saw above, under the hypothesis of the lemma, if the functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ solve Eq. (1), then there exist a positive integer l, there are generalized polynomials $P_{k,i}, Q_{k,i} : \mathbb{F}^{\times} \to \mathbb{C}$ ($k = 1, \ldots, l$ and $i = 1, \ldots, n$) and there exist linearly independent exponentials $m_1, \ldots, m_l : \mathbb{F}^{\times} \to \mathbb{C}$ such that

$$f_i(x) = \sum_{k=1}^{l} P_{k,i}(x)m_k(x) \text{ and } g_i(x) = \sum_{k=1}^{l} Q_{k,i}(x)m_k(x) \qquad \left(x \in \mathbb{F}^{\times}\right).$$

Assume to the contrary that $l \geq 2$.

Let

$$S = \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_n\}.$$

Then due to conditions C(i)–C(iii) we have $\max S = \max\{p_n, q_1\}$. Similarly as in Lemma 13, without the loss of generality $p_n > q_1$ can be assumed, otherwise we follow a similar argument. By our assumption $l \geq 2$, this yields that there exist different exponential terms in f_1 and g_1 with nonzero polynomial coefficients. For the sake of simplicity, suppose that these different exponentials are m_1 and m_2. Since $\max S = p_1$, the term $m_1^{p_1} m_2^{q_1}$ appears only in $f_1(x^{p_1})g_1(x^{q_1})$ while expanding Eq. (1). Since generalized polynomials $P_{k,i}$ and exponentials m_k satisfies the conditions of Theorem 6 for every finitely generated subfield of $\mathbb{F}$, the coefficient of the above-mentioned term which is $P_{1,1}(x^{p_1})Q_{1,1}(x^{q_1})$ has to vanish on $\mathbb{F}$. From this we can deduce that $P_{1,1}$ or $Q_{1,1}$ is identically zero, contrary to our assumption. Thus

$$f_1(x) = P_1(x)m_1(x) \qquad \text{and} \qquad g_1(x) = Q_1(x)m_1(x) \qquad (x \in \mathbb{F}),$$

with appropriate generalized polynomials $P_1, Q_1 : \mathbb{F}^{\times} \to \mathbb{C}$ and exponential $m_1 : \mathbb{F}^{\times} \to \mathbb{C}$.

Suppose now that there is a positive integer k, less than n such that for all $i=1,\dots,k$ we have

$$f_i(x)=P_i(x)m_i(x) \text{ and } g_i(x)=Q_i(x)m_i(x) \qquad \left(x\in\mathbb{F}^{\times}\right).$$

Assume that in the representation of f_{k+1} and g_{k+1} there are different exponentials with nonzero polynomial coefficients, say m_{j_1} and m_{j_2}. Observe that while expanding Eq. (1), the term $m_{j_1}^{p_{k+1}}m_{j_2}^{q_{k+1}}$ appears only at once, namely in the product $f_{k+1}(x^{p_{k+1}})g_{k+1}(x^{q_{k+1}})$. Again, due to Theorem 6, we deduce that the appropriate polynomial term, that is,

$$P_{k+1,j_1}(x^{p_{k+1}})Q_{k+1,j_2}(x^{q_{k+1}})$$

has to vanish. This proves that necessarily

$$f_{k+1}(x)=P_{k+1}(x)m_{k+1}(x) \text{ and } g_{k+1}=Q_{k+1}(x)m_{k+1}(x) \qquad \left(x\in\mathbb{F}^{\times}\right)$$

hold. This shows that there exist exponentials $m_1,\dots,m_n\colon\mathbb{F}^{\times}\to\mathbb{C}$ and generalized polynomials $P_1,\dots,P_n$ and $Q_1,\dots,Q_n$ on the group $\mathbb{F}^{\times}$ such that for all $i=1,\dots,n$

$$f_i(x)=P_i(x)m_i(x) \qquad \text{and} \qquad g_i=Q_i(x)m_i(x) \qquad \left(x\in\mathbb{F}^{\times}\right).$$

□

Remark 10. Due to conditions C(i)–C(iii), Eq. (1) has the form

$$\begin{aligned}0&=\sum_{i=1}^{n}f_i(x^{p_i})g_i(x^{q_i})\\&=\sum_{i=1}^{n}P_i(x^{p_i})m_i(x)^{p_i}Q_i(x^{q_i})m_i(x)^{q_i}=\sum_{i=1}^{n}P_i(x^{p_i})Q_i(x^{q_i})m_i(x)^{N}\\&\quad\left(x\in\mathbb{F}^{\times}\right).\end{aligned}$$

If the exponentials appearing on the right hand side would be different, then by Theorem 6, their coefficients would be zero. This implies however that there exists a proper subset $J\subset\{1,\dots,n\}$ such that

$$\sum_{j\in J}f_j(x^{p_j})g_j(x^{q_j})=0 \text{ as well as } \sum_{j\notin J}f_j(x^{p_j})g_j(x^{q_j})=0 \qquad \left(x\in\mathbb{F}^{\times}\right). \quad (2)$$

This leads to the following definition of irreducible solutions.

Definition 10. A system of solutions $\{f_1,\dots,f_n,g_1,\dots,g_n\}$ of Eq. (1) is called *irreducible* if it does not satisfy a sub-term of (1). Otherwise, we say that a system of solutions is *reducible.*

Clearly, a system of solutions $\{f_1,\dots,f_n,g_1,\dots,g_n\}$ of (1) which fulfills (2) is a reducible solution. On the other hand, the argument in Remark 10 shows that every solution of (1) can be given as a sum of irreducible solutions of disjoint sub-terms of (1). Therefore, we restrict ourselves to the irreducible case, since every reducible solution can be deduced as a sum of irreducible solutions.

Corollary 15. *Under the conditions of Theorem 14, suppose that system of functions* $\{f_1, \ldots, f_n, g_1, \ldots, g_n\}$ *is an irreducible solution of Eq.* (1). *Then there exists an exponential* $m: \mathbb{F}^\times \to \mathbb{C}$ *and there are generalized polynomials* $P_i, Q_i: \mathbb{F}^\times \to \mathbb{C}$ *such that for each* $i = 1, \ldots, n$

$$f_i(x) = P_i(x)m(x) \qquad \textit{and} \qquad g_i(x) = Q_i(x)m(x) \qquad \left(x \in \mathbb{F}^\times\right). \tag{3}$$

In other words, for each $i = 1, \ldots, n$ *there exists higher order derivations* $D_i, \widetilde{D_i}: \mathbb{F} \to \mathbb{C}$ *such that*

$$f_i(x) \sim D_i(x) \qquad \textit{and} \qquad g_i(x) \sim \widetilde{D_i}(x) \qquad \left(x \in \mathbb{F}^\times\right),$$

where $\sim$ *in the latter two equations is the equivalence relation defined in Remark 7.*

Proof. By Remark 10, all of the exponentials m_i have to be the same in the description of the solutions of Theorem 14. Therefore, Eq. (3) describes the irreducible solutions of (1).

Using Lemma 10, solutions of Eq. (1) are enough to be determined up to the equivalence relation $\sim$ defined in Remark 7. Accordingly, we can suppose that the exponential m in the above representation is the identity mapping. Hence, in view of Theorem 8 we get that the functions $f_1, \ldots, f_n$ as well as $g_1, \ldots, g_n$ are (or more precisely, are equivalent to) higher order derivations, as we stated. □

3.4. The Order of Higher Order Derivation Solutions

Every higher order derivation on $\mathbb{F}$ is a differential operator on any finitely generated subfield of $\mathbb{F}$ (see Theorem 8 and [6]). Hence on these fields the solutions are differential operators. Moreover, if the solutions on any finitely generated subfield of $\mathbb{F}$ are differential operators of order at most n, then every solution on $\mathbb{F}$ is a derivation of order n. From now on, instead of finding solutions as higher order derivations we are looking for differential operators as solutions.

For this purpose, our next aim is to understand the arithmetic of the composition of derivations of the form $d_1 \circ \cdots \circ d_r$ that are building blocks of differential operators. First we show that there is a natural form of composition of derivations that can be taken as a standard basis.

For this target, the notion of moment function sequences turn out to be useful. Here we follow [4]. A *composition* of a nonnegative integer n is a sequence of nonnegative integers $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ such that

$$n = \sum_{k=1}^{\infty} \alpha_k.$$

For a positive integer r, an r-*composition* of a nonnegative integer n is a composition $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ with $\alpha_k = 0$ for $k > r$.

Given a sequence of variables $x = (x_k)_{k\in\mathbb{N}}$ and compositions $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ and $\beta = (\beta_k)_{k\in\mathbb{N}}$ we define

$$\alpha! = \prod_{k=1}^{\infty} \alpha_k, \quad |\alpha| = \sum_{k=1}^{\infty} \alpha_k, \quad x^{\alpha} = \prod_{k=1}^{\infty} x_k^{\alpha_k}, \quad \binom{\alpha}{\beta} = \prod_{k=1}^{\infty} \binom{\alpha_k}{\beta_k}.$$

Furthermore, $\beta \leq \alpha$ means that $\beta_k \leq \alpha_k$ for all $k \in \mathbb{N}$ and $\beta < \alpha$ stands for $\beta \leq \alpha$ and $\beta \neq \alpha$.

Definition 11. Let G be a commutative group, r a positive integer, and for each multi-index α in $\mathbb{N}^r$ let $f_\alpha \colon G \to \mathbb{C}$ be a continuous function. We say that $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ is a *generalized moment sequence of rank* r, if

$$f_\alpha(x+y) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y) \tag{4}$$

holds whenever x, y are in G. The function f_0, where 0 is the zero element in $\mathbb{N}^r$, is called the *generating function* of the sequence.

Theorem 16. *Let G be a commutative group, r a positive integer, and for each α in $\mathbb{N}^r$ let $f_\alpha \colon G \to \mathbb{C}$ be a function. If the sequence of functions $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ forms a generalized moment sequence of rank r, then there exists an exponential $m \colon G \to \mathbb{C}$ and a sequence of complex-valued additive functions $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ such that for every multi-index α in $\mathbb{N}^r$ and x in G we have*

$$f_\alpha(x) = B_\alpha(a(x)) m(x),$$

where B_α denotes the multivariate Bell polynomial *corresponding to the multi-index α.*

Remark 11. It is well-known that every polynomial $P \colon \mathbb{C}^r \to \mathbb{C}$ $(r \in \mathbb{N})$ can be given as a linear combination of Bell polynomials B_α, where $\alpha \in \mathbb{N}^r$. Hence, momentum generating functions generate the exponential polynomial functions on G. By Lemma 4, polynomials of the form $B_\alpha \circ a$ are linearly independent over $\mathbb{C}$.

Lemma 17. *Let $\mathbb{F} \subset \mathbb{C}$ be a field, r be a positive integer and $d_1, \ldots, d_r \colon \mathbb{F} \to \mathbb{F}$ be linearly independent derivations. For all multi-index $\alpha \in \mathbb{N}^r$, $\alpha = (\alpha_1, \ldots, \alpha_r)$ define the function $\varphi_\alpha \colon \mathbb{F} \to \mathbb{C}$ by*

$$\begin{aligned} \varphi_\alpha(x) &= d^\alpha(x) = d_1^{\alpha_1} \circ \cdots \circ d_r^{\alpha_r}(x) \\ &= \underbrace{d_1 \circ \cdots \circ d_1}_{\alpha_1 \, times} \circ \cdots \circ \underbrace{d_r \circ \cdots \circ d_r}_{\alpha_r \ times}(x) \\ &\quad \left(x \in \mathbb{F}^\times\right). \end{aligned}$$

Then $(\varphi_\alpha)_{\alpha\in\mathbb{N}^r}$ is a generalized moment sequence of rank r on the commutative group $\mathbb{F}^\times$ and the generating function of the sequence is the identity function.

Proof. Let $\mathbb{F} \subset \mathbb{C}$ be a field, r be a positive integer and $d_1, \ldots, d_r : \mathbb{F} \to \mathbb{F}$ be linearly independent derivations. For all multi-index $\alpha \in \mathbb{N}^r$, $\alpha = (\alpha_1, \ldots, \alpha_r)$ define the function $\varphi_\alpha : \mathbb{F} \to \mathbb{C}$ as in the lemma. We prove the statement by induction of the length of the multi-index α. Assume that $\alpha \in \mathbb{N}^r$ and $|\alpha| = 1$. Then there exists $i \in \{1, \ldots r\}$ such that $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$ and

$$\begin{aligned}\varphi_\alpha(xy) &= d_i^{\alpha_i}(xy) = d_i(xy) = xd_i(y) + yd_i(x)\\ &= \varphi_0(x)\varphi_\alpha(y) + \varphi_0(y)\varphi_\alpha(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \varphi_\beta(x)\varphi_{\alpha-\beta}(y)\\ &\qquad \left(x, y \in \mathbb{F}^\times\right).\end{aligned}$$

Let now $\alpha \in \mathbb{N}^r$ and assume that the statement is true for all multi-indices $\beta \in \mathbb{N}^r$ with $\beta < \alpha$. Then

$$\begin{aligned}\varphi_\alpha(xy) &= d^\alpha(xy) = d_1^{\alpha_1} \circ \cdots \circ d_r^{\alpha_r}(xy) = \underbrace{d_1 \circ \cdots d_1}_{\alpha_1 \text{ times}} \circ \cdots \circ \underbrace{d_r \circ \cdots \circ d_r}_{\alpha_r \text{ times}}(xy)\\ &= \underbrace{d_1 \circ \cdots \circ d_1}_{\alpha_1 \text{ times}} \circ \cdots \circ \underbrace{d_{r-1} \circ \cdots \circ d_{r-1}}_{\alpha_{r-1} \text{ times}} \left(d_r^{\alpha_r}(xy)\right)\\ &= \underbrace{d_1 \circ \cdots \circ d_1}_{\alpha_1 \text{ times}} \circ \cdots \circ \underbrace{d_{r-1} \circ \cdots \circ d_{r-1}}_{\alpha_{r-1} \text{ times}} \left(\sum_{\beta_r=0}^{\alpha_r} \binom{\alpha_r}{\beta_r} d_r^{\beta_r}(x) \cdot d_r^{\alpha_r-\beta_r}(y) \right)\\ &= \sum_{\beta_r=0}^{\alpha_r} \binom{\alpha_r}{\beta_r} \underbrace{d_1 \circ \cdots \circ d_1}_{\alpha_1 \text{ times}} \circ \cdots \circ \underbrace{d_{r-1} \circ \cdots \circ d_{r-1}}_{\alpha_{r-1} \text{ times}} \left(d_r^{\beta_r}(x) \cdot d_r^{\alpha_r-\beta_r}(y)\right)\\ &= \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_{r-1}=0}^{\alpha_{r-1}} \sum_{\beta_r=0}^{\alpha_r} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_{r-1}}{\beta_{r-1}} \binom{\alpha_r}{\beta_r}\\ &\quad \times d_1^{\beta_1} \circ \cdots \circ d_r^{\beta_r}(x) \cdot d_1^{\alpha_1-\beta_1} \circ \cdots \circ d_r^{\alpha_r-\beta_r}(y)\\ &= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \varphi_\beta(x)\varphi_{\alpha-\beta}(y) \qquad \left(x, y \in \mathbb{F}^\times\right).\end{aligned}$$

□

Corollary 18. *By Lemma 17 and Remark 11 implies that the functions* $d^\alpha = d_1^{\alpha_1} \circ \cdots \circ d_r^{\alpha_r}$ *constitute a basis of the differential operators in* $\mathbb{F}^\times$. *Since every* d^α *is additive, by Lemma 4, all elements of the system* $\{d^\alpha(x)\}$, *where* $\alpha \in \cup_{r\in\mathbb{N}}\mathbb{N}^r$ *are algebraically independent.*

A consequence of the algebraic independence of the elements of d^α, *where* $\alpha \in \cup_{r\in\mathbb{N}}\mathbb{N}^r$ *is the following. Let* $P \in \mathbb{C}[x_1, \ldots, x_n]$ *be a polynomial,* $d_1, \ldots d_r$ *be derivations as in Lemma 17 and* $\alpha_1, \ldots, \alpha_n \in \cup_{r\in\mathbb{N}}\mathbb{N}^r$. *Then the following polynomial form*

$$P(d^{\alpha_1}(x), \ldots, d^{\alpha_n}(x)) = 0 \qquad (x \in G)$$

holds if and only if

$$P(\hat{d}^{\,|\alpha_1|}(x), \ldots, \hat{d}^{\,|\alpha_n|}(x)) = 0 \qquad (x \in G),$$

where $\hat{d}$ *is an arbitrary derivation (of order 1). In other words, we can substitute* $d_1, \ldots, d_r$ *by* $\hat{d}$ *in* $d^{\alpha_1}, \ldots, d^{\alpha_n}$.

By Corollary 18, it would be desirable to calculate $d^k = \underbrace{d \circ \cdots \circ d}_{k\text{times}}$, where d is a derivation (of order 1) and $k \in \mathbb{N}$. Lemma 17, together with [4, Proposition 1], implies the following statement.

Proposition 19. *Let* $\mathbb{F} \subset \mathbb{C}$ *be a field and* $d\colon \mathbb{F} \to \mathbb{C}$ *a derivation. For all positive integer* k *we define the function* d^k *on* $\mathbb{F}$ *by*

$$d^k(x) = \underbrace{d \circ \cdots \circ d}_{k\ times}(x) \qquad (x \in \mathbb{F}).$$

Then for all positive integer p *we have*

$$d^k(x_1 \cdots x_p) = \sum_{\substack{l_1, \ldots, l_p \geq 0 \\ l_1 + \cdots + l_p = k}} \binom{k}{l_1, \ldots, l_p} \prod_{t=1}^{p} d^{l_t}(x_t) \qquad (x_1, \ldots, x_p \in \mathbb{F}),$$

where the conventions $d^0 = \mathrm{id}$ *and* $\binom{k}{l_1, \ldots, l_p} = \frac{k!}{l_1! \cdots l_p!}$ *are adopted. Especially, for all positive integer* p*, we have*

$$d^k(x^p) = \sum_{\substack{l_1, \ldots, l_p \geq 0 \\ l_1 + \cdots + l_p = k}} \binom{k}{l_1, \ldots, l_p} \cdot d^{l_1}(x) \cdots d^{l_p}(x) \qquad (x_1, \ldots, x_p \in \mathbb{F}),$$

Reordering the previous expression we can get the following

$$d^k(x^p) = \sum_{\substack{j_1 + \cdots + j_s = p' < p \\ j_1 + 2j_2 + \cdots + sj_s = k}} \binom{k}{\underbrace{1, \ldots, 1}_{j_1}, \ldots, \underbrace{s, \ldots, s}_{j_s}} \prod_{t=1}^{s} \frac{1}{(j_t!)}$$
$$\times \binom{p}{\underbrace{1, \ldots, 1}_{p'}} \cdot (d(x))^{j_1} \cdots (d^s(x))^{j_s} \cdot x^{p-p'}, \qquad (x \in \mathbb{F})$$

where $j_1, \ldots, j_s$ *denotes the number of* $d(x), \ldots, d^s(x)$ *in a given composition of* $d^k(x^p)$.

With the above results, we are now ready to prove an upper bound for the order of derivations appearing in Theorem 14.

Theorem 20. *Under the hypotheses of Theorem 14, the solutions f_i and g_i are derivations D_i and $\widetilde{D}_i$ for all $i = 1, \ldots, n$. Let k and l denote the maximal orders of derivations D_i and $\widetilde{D}_i$, respectively. Suppose that there exists some i' such that the order of $D_{i'}$ and $\widetilde{D}_{i'}$ is exactly k and l, respectively. Then for all $i = 1, \ldots, n$ the order of D_i and $\widetilde{D}_i$ is less or equal to $n-1$.*

Proof. Assume contrary that the maximal order k of the above derivations $D_1, \ldots, D_n$ is greater than $n-1$. The argument for the case when the maximal order l of $\widetilde{D}_1, \ldots, \widetilde{D}_n$ is greater than $n-1$ is analogous.

By our assumption there exists an index i' such that the orders of the derivations $D_{i'}$ and $\widetilde{D}_{i'}$ is exactly k and l, respectively. It is important to note that then the sum of the orders of D_i and $\widetilde{D}_i$ is at most $k+l$ for any i, and it is exactly $k+l$ for some indices if and only if the corresponding D and $\widetilde{D}$ is of order k and of order l, respectively. Furthermore, by the algebraic independence of higher order derivations, there the number of these indices are at least two.

From now on we assume that $\mathbb{F}$ is finitely generated. Indeed, if we verify the statement for any finitely generated subfield of a field $\mathbb{F}$, then it holds for $\mathbb{F}$ itself, as well. On finitely generated fields all of these higher order derivations can be represented as differential operators, that is, on finitely generated fields we have

$$D_i(x) = \sum_{|\alpha| \leq k} \lambda_{i,\alpha} d_i^{\alpha}(x) \qquad \text{and} \qquad \widetilde{D}_i(x) = \sum_{|\beta| \leq l} \widetilde{\lambda}_{i,\beta} \widetilde{d}_i^{\beta}(x) \qquad (x \in \mathbb{F})$$

with appropriate complex constants $\lambda_{i,\alpha}$, $\widetilde{\lambda}_{i,\beta}$, $(|\alpha| \leq k,\ |\beta| \leq l,\ i = 1, \ldots, n)$ and higher order derivations $d_i^{\alpha}, \widetilde{d}_i^{\beta} \colon \mathbb{F} \to \mathbb{C}$ defined in Lemma 17. Further, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n} D_i(x^{p_i}) \widetilde{D}_i(x^{q_i}) \\ &= \sum_{i=1}^{n} \left(\sum_{|\alpha| \leq k} \lambda_{i,\alpha} d_i^{\alpha}(x^{p_i}) \right) \cdot \left(\sum_{|\beta| \leq l} \widetilde{\lambda}_{i,\beta} \widetilde{d}_i^{\beta}(x^{q_i}) \right) \qquad (x \in \mathbb{F}). \end{aligned}$$

If we expand the right hand side of the above identity with the aid of Proposition 19, we get an expression of the following polynomial form

$$\begin{aligned} P(x, d_1(x), \ldots, d_k(x), \widetilde{d}_1(x), \ldots, \widetilde{d}_l(x), \ldots, d_1^k(x), \ldots, \\ d_k^k(x), \widetilde{d}_1^l(x), \ldots, \widetilde{d}_l^l(x)) = 0 \qquad (x \in \mathbb{F}). \end{aligned}$$

If this identity can be satisfied by different functions, it can also be satisfied by a single one. By Corollary 18, this enables us to substitute the functions $d_i^{\alpha}, \widetilde{d}_i^{\beta}$ $(i = 1, \ldots, n, |\alpha| \leq k, |\beta| \leq l)$ with suitable compositions of a given derivation d of order 1. In other words, instead of the above identity, we can restrict ourselves to

$$\sum_{i=1}^{n}\left(\sum_{j=0}^{k}\lambda_{i,j}d^{j}(x^{p_i})\right)\cdot\left(\sum_{j=0}^{l}\widetilde{\lambda}_{i,j}d^{j}(x^{q_i})\right)=0 \qquad (x\in\mathbb{F})$$

with appropriate complex constants $\lambda_{i,j}\quad(i=1,\ldots,n,\ j=0,\ldots,k)$, and $\widetilde{\lambda}_{i,j}$ $(i=1,\ldots,n,\ j=0,\ldots,l)$ and derivation $d\colon\mathbb{F}\to\mathbb{C}$. By our assumptions there are some i' such that $\lambda_{i',k}\neq 0$ and $\widetilde{\lambda}_{i',l}\neq 0$.

Dividing the above sum to smaller ones, we get

$$\begin{aligned}&\sum_{i=1}^{n}\left(\sum_{j=0}^{k}\lambda_{i,j}d^{j}(x^{p_i})\right)\cdot\left(\sum_{j=0}^{l}\widetilde{\lambda}_{i,j}d^{j}(x^{q_i})\right)\\&=\sum_{i=1}^{n}\left(\sum_{j=0}^{k-1}\lambda_{i,j}d^{j}(x^{p_i})+\lambda_{i,k}d^{k}(x^{p_i})\right)\cdot\left(\sum_{j=0}^{k-1}\widetilde{\lambda}_{i,j}d^{j}(x^{q_i})+\widetilde{\lambda}_{i,l}d^{l}(x^{q_i})\right)\\&\sum_{i=1}^{n}\Big[S(p_i,k-1)S(q_i,l-1)+\widetilde{\lambda}_{i,l}d^{l}(x^{q_i})S(p_i,k-1)\\&+\lambda_{i,k}d^{k}(x^{p_i})S(q_i,l-1)+\lambda_{i,k}\widetilde{\lambda}_{i,k}d^{k}(x^{p_i})d^{l}(x^{q_i})\Big]=0\\&(x\in\mathbb{F}),\end{aligned}$$

where

$$S(p_i,k-1)=\sum_{j=0}^{k-1}\lambda_{i,j}d^{j}(x^{p_i})\text{ and }S(q_i,l-1)=\sum_{j=0}^{l-1}\widetilde{\lambda}_{i,j}d^{j}(x^{q_i}) \qquad (x\in G).$$

Note that, by the algebraic independence used in Corollary 18, this sum splits into separate terms of the form $d^{s}(x^{p_i})d^{t}(x^{q_i})$, where $s+t$ is a fixed number. By our assumption, when $s+t=k+l$, then the only way is that $s=k, t=l$. This implies that using Proposition 19 we get that

$$\begin{aligned}0&=\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,l}d^{k}(x^{p_i})d^{l}(x^{q_i})\\&=\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\\&\times\left(\sum_{\substack{j_1+\cdots+j_s=p'<p_i\\ j_1+2j_2+\cdots+sj_s=k}}\binom{k}{\underbrace{1,\ldots,1}_{j_1},\ldots,\underbrace{s,\ldots,s}_{j_s}}\binom{p_i}{\underbrace{1,\ldots,1}_{p'}}\prod_{t=1}^{s}\frac{(d^{t}(x))^{j_t}}{(j_t!)}x^{p_i-p'}\right)\\&\times\left(\sum_{\substack{j_1+\cdots+j_s=q'<q_i\\ j_1+2j_2+\cdots+sj_s=l}}\binom{l}{\underbrace{1,\ldots,1}_{j_1},\ldots,\underbrace{s,\ldots,s}_{j_s}}\binom{q_i}{\underbrace{1,\ldots,1}_{q'}}\prod_{t=1}^{s}\frac{(d^{t}(x))^{j_t}}{(j_t!)}x^{q_i-q'}\right)\end{aligned}$$

while the rest in the above sum can be computed similarly.

Case 1. If $k < l$, then we compute the coefficients of the terms

$$(d(x))^j d^{k-j}(x) d^l(x) \ (j = 0, \ldots, k-1). \tag{5}$$

For each $j = 0, \ldots, k-1$ this can be taken from the expansion of

$$\lambda_{i,k}\widetilde{\lambda}_{i,l} d^k(x^{p_i}) d^l(x^{q_i})$$

in only one way. Namely, by splitting $d^k(x^{p_i})$ into $j+1$ parts and $d^l(x^{q_i})$ into one. Then the corresponding coefficients are

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} \binom{k}{\underbrace{1, \ldots, 1}_{j}} \frac{1}{j!} \binom{p_i}{\underbrace{1, \ldots, 1}_{j+1}} \binom{q_i}{1} = 0.$$

Since each of the terms contains $\binom{k}{\underbrace{1, \ldots, 1}_{j}} \frac{1}{j!} = \binom{k}{j}$, this can be eliminated from the above equation. The corresponding equations ($j = 0, \ldots, k$) can be written in the following matrix equation

$$\begin{pmatrix} \binom{p_1}{1}\binom{q_1}{1} & \cdots & \binom{p_n}{1}\binom{q_n}{1} \\ \binom{p_1}{1,1}\binom{q_1}{1} & \cdots & \binom{p_n}{1,1}\binom{q_n}{1} \\ \vdots & \ddots & \vdots \\ \binom{p_1}{\underbrace{1, \ldots, 1}_{k}}\binom{q_1}{1} & \cdots & \binom{p_n}{\underbrace{1, \ldots, 1}_{k}}\binom{q_n}{1} \end{pmatrix} \cdot \begin{pmatrix} \lambda_{1,k}\widetilde{\lambda}_{1,l} \\ \lambda_{2,k}\widetilde{\lambda}_{2,l} \\ \vdots \\ \lambda_{n,k}\widetilde{\lambda}_{n,l} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here we note that as p_i's are all different positive integers, it follows that $p_i \geq n$ for some i and hence the first n rows of the matrix are not identically zero, as $k > n-1$. It is straightforward to verify that the first n row of above matrix equation is equivalent to

$$\begin{pmatrix} p_1 & \cdots & p_n \\ p_1^2 & \cdots & p_n^2 \\ \vdots & \ddots & \vdots \\ p_1^n & \cdots & p_n^n \end{pmatrix} \cdot \begin{pmatrix} q_1\lambda_{1,k}\widetilde{\lambda}_{1,l} \\ q_2\lambda_{2,k}\widetilde{\lambda}_{2,l} \\ \vdots \\ q_n\lambda_{n,k}\widetilde{\lambda}_{n,l} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since this is a Vandermonde type matrix with different p_i's, the only solution of this homogeneous linear system is the zero vector, i.e., $q_i\lambda_{i,k}\widetilde{\lambda}_{i,l} = 0$ for all $i = 1, \ldots, n$. This contradicts to our assumption that there is some i' for which $\lambda_{i',k} \neq 0$ and $\widetilde{\lambda}_{i',l} \neq 0$ ($q_{i'} \neq 0$ as $q_i \neq 0$ for all $i \in \{1, \ldots, k\}$).

Case 2. If $k = l$, then we compute the coefficients of the terms

$$(d(x))^{2j}(d^{k-j}(x))^2 \qquad (j = 0, \ldots, k-1). \tag{6}$$

If $j < \dfrac{k}{2}$, then this term can be taken from the expansion of

$$\lambda_{i,k}\widetilde{\lambda}_{i,l} d^k(x^{p_i}) d^k(x^{q_i})$$

in only one way. Namely, by splitting $d^k(x^{p_i})$ and $d^k(x^{q_i})$ into $j+1$ parts. Then the corresponding coefficients are

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} \left(\binom{k}{\underbrace{1,\ldots,1}_{j}} \frac{1}{j!} \right)^2 \binom{p_i}{\underbrace{1,\ldots,1}_{j+1}} \binom{q_i}{\underbrace{1,\ldots,1}_{j+1}} = 0.$$

Since each of the terms contains $\left(\binom{k}{\underbrace{1,\ldots,1}_{j}} \frac{1}{j!} \right)^2 = \binom{k}{j}^2$, this can be eliminated from the above equations. These equations for $j = 0, \ldots, \lceil k/2 \rceil - 1$ can be written in the following matrix equation

$$\begin{pmatrix} \binom{p_1}{1}\binom{q_1}{1} & \cdots & \binom{p_n}{1}\binom{q_n}{1} \\ \binom{p_1}{1,1}\binom{q_1}{1,1} & \cdots & \binom{p_n}{1,1}\binom{q_n}{1,1} \\ \vdots & \ddots & \vdots \\ \binom{p_1}{\underbrace{1,\ldots,1}_{\lceil k/2\rceil-1}}\binom{q_1}{\underbrace{1,\ldots,1}_{\lceil k/2\rceil-1}} & \cdots & \binom{p_n}{\underbrace{1,\ldots,1}_{\lceil k/2\rceil-1}}\binom{q_n}{\underbrace{1,\ldots,1}_{\lceil k/2\rceil-1}} \end{pmatrix} \cdot \begin{pmatrix} \lambda_{1,k}\widetilde{\lambda}_{1,k} \\ \lambda_{2,k}\widetilde{\lambda}_{2,k} \\ \vdots \\ \lambda_{n,k}\widetilde{\lambda}_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here we note that as p_i, q_i are all different positive integers, it follow that $\max_i \{p_i, q_i\} \geq 2n$ and hence the first $n' = \min(n, \lceil k/2 \rceil - 1)$ rows of the matrix in not identically zero, as $k > n-1$. It is straightforward to verify that the first n' row of above matrix equation is equivalent to

$$\begin{pmatrix} p_1 q_1 & \cdots & p_n q_n \\ p_1^2 q_1^2 & \cdots & p_n^2 q_n^2 \\ \vdots & \ddots & \vdots \\ p_1^{n'} q_1^{n'} & \cdots & p_n^{n'} q_n^{n'} \end{pmatrix} \cdot \begin{pmatrix} \lambda_{1,k}\widetilde{\lambda}_{1,k} \\ \lambda_{2,k}\widetilde{\lambda}_{2,l} \\ \vdots \\ \lambda_{n,k}\widetilde{\lambda}_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that if $n' = n$, then we are done with a Vandermonde matrix argument similar as it is used in Case 1. So from now on we assume that $n' = \lceil k/2 \rceil - 1$. This also means that $k < 2n$, thus the maximal order of the corresponding derivations is at most $2n-1$.

If $j \geq \dfrac{k}{2}$, then the term $(d(x))^{2j}(d^{k-j}(x))^2$ can be taken from the expansion of

$$\lambda_{i,k}\widetilde{\lambda}_{i,l} d^k(x^{p_i}) d^k(x^{q_i})$$

in three ways. One is as above, when we split both $d^k(x^{p_i})$ and $d^k(x^{q_i})$ into $j+1$ parts. Another one is when $d^k(x^{p_i})$ is split into k parts giving $d(x)^k$, and $d^k(x^{q_i})$ provides $(d^{k-j}(x))^2(d(x))^{2j-k}$. The third one is given by changing the role of p_i and q_i.

Then the corresponding coefficients are

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\Big(\binom{k}{\underbrace{1,\ldots,1}_{j}}\frac{1}{(j!)}\Big)^2\binom{p_i}{\underbrace{1,\ldots,1}_{j}}\binom{q_i}{\underbrace{1,\ldots,1}_{j}}$$
$$+\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\binom{k}{\underbrace{1,\ldots,1}_{k}}\frac{1}{(k!)}\binom{k}{\underbrace{1,\ldots,1}_{2j-k},j}\frac{1}{2}\frac{1}{(2j-k)!}$$
$$\times\Big(\binom{p_i}{\underbrace{1,\ldots,1}_{k}}\binom{q_i}{\underbrace{1,\ldots,1}_{2j-k+2}}+\binom{p_i}{\underbrace{1,\ldots,1}_{2j-k+2}}\binom{q_i}{\underbrace{1,\ldots,1}_{k}}\Big)=0.$$

First we show that the second sum has to vanish for all $j=\lceil k/2\rceil,\ldots,k-1$. In such cases, the coefficients

$$\binom{k}{\underbrace{1,\ldots,1}_{k}}\frac{1}{k!}\binom{k}{\underbrace{1,\ldots,1}_{2j-k},j}\frac{1}{2}\frac{1}{(2j-k)!}$$

are the same is each summand, it is enough to show that

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\Big(\binom{p_i}{\underbrace{1,\ldots,1}_{k}}\binom{q_i}{\underbrace{1,\ldots,1}_{2j-k+2}}+\binom{p_i}{\underbrace{1,\ldots,1}_{2j-k+2}}\binom{q_i}{\underbrace{1,\ldots,1}_{k}}\Big)=0. \tag{7}$$

This clearly holds, since the term $(d(x))^{2j+1}d^{2k-2j-1}(x)$ in the expansion of $d^k(x^{p_i})d^k(x^{q_i})$ for $j=\lceil k/2\rceil,\ldots,k-1$ can be given in exactly two ways. Either $(d(x))^k$ stems from $d^k(x^{p_i})$ and $(d(x))^{2j-k+1}\cdot d^{2k-2j-1}(x)$ stems from $d^k(x^{q_i})$, or reversely changing the role of p_i and q_i. Hence we get

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\binom{k}{\underbrace{1,\ldots,1}_{2j-k+1}}\cdot\frac{1}{(2j-k+1)!}$$
$$\times\Big(\binom{p_i}{\underbrace{1,\ldots,1}_{k}}\binom{q_i}{\underbrace{1,\ldots,1}_{2j-k+2}}+\binom{p_i}{\underbrace{1,\ldots,1}_{2j-k+2}}\binom{q_i}{\underbrace{1,\ldots,1}_{k}}\Big)=0,$$

which is equivalent to Eq. (7). Thus, for all $j=\lceil k/2\rceil,\ldots,k-1$ it follows

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\Big(\binom{k}{\underbrace{1,\ldots,1}_{j}}\frac{1}{(j!)}\Big)^2\binom{p_i}{\underbrace{1,\ldots,1}_{j}}\binom{q_i}{\underbrace{1,\ldots,1}_{j}}=0$$

This implies that

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,k}\binom{p_i}{\underbrace{1,\ldots,1}_{j}}\binom{q_i}{\underbrace{1,\ldots,1}_{j}}=0$$

hold for all $j=0,\ldots,k-1$. Thus we get

$$\begin{pmatrix} \binom{p_1}{1}\binom{q_1}{1} & \cdots & \binom{p_n}{1}\binom{q_n}{1} \\ \binom{p_1}{1,1}\binom{q_1}{1,1} & \cdots & \binom{p_n}{1,1}\binom{q_n}{1,1} \\ \vdots & \ddots & \vdots \\ \binom{p_1}{\underbrace{1,\ldots,1}_{k}}\binom{q_1}{\underbrace{1,\ldots,1}_{k}} & \cdots & \binom{p_n}{\underbrace{1,\ldots,1}_{k}}\binom{q_n}{\underbrace{1,\ldots,1}_{k}} \end{pmatrix}\cdot\begin{pmatrix}\lambda_{1,k}\widetilde{\lambda}_{1,k}\\ \lambda_{2,k}\widetilde{\lambda}_{2,k}\\ \vdots\\ \lambda_{n,k}\widetilde{\lambda}_{n,k}\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},$$

where the first n rows of the matrix is not identically zero as p_i and q_i are different, hence there is an index i' such that $p_{i'}\geq n$ and $q_{i'}\geq n$. Thus this system consisting the first n rows is equivalent to

$$\begin{pmatrix} p_1q_1 & \cdots & p_nq_n\\ p_1^2q_1^2 & \cdots & p_n^2q_n^2\\ \vdots & \ddots & \vdots\\ p_1^nq_1^n & \cdots & p_n^nq_n^n\end{pmatrix}\cdot\begin{pmatrix}\lambda_{1,k}\widetilde{\lambda}_{1,k}\\ \lambda_{2,k}\widetilde{\lambda}_{2,l}\\ \vdots\\ \lambda_{n,k}\widetilde{\lambda}_{n,k}\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}.$$

By the usual Vandermonde argument as in Case 1, the only solution of this homogeneous linear system is the zero vector, i.e., $\lambda_{i,k}\widetilde{\lambda}_{i,l}=0$ for all $i=1,\ldots,n$. This contradicts our assumption that there is some i' for which $\lambda_{i',k}\neq 0$ and $\widetilde{\lambda}_{i',k}\neq 0$.

Case 3. If $k>l$, then we prove by induction in j that

$$\sum_{i=1}^{n}\lambda_{i,k}\widetilde{\lambda}_{i,l}p_i^{j+1-s}q_i^s=0$$

holds for every $j=0,\ldots,n-1$ and every $s=0,\ldots,j$.

In each step we consider how $(d(x))^jd^{k-j}(x)d^l(x)$ can be given from the expansion $d^k(x^{p_i})d^l(x^{q_i})$. There are three possible ways, where this term can stem from.

(a) $(d(x))^jd^{k-j}(x)$ stems from $d^k(x^{p_i})$ and $d^l(x)$ stems from $d^l(x^{q_i})$. This can happen for every

$j \in \{0, \ldots, k\}$. In this case the coefficient of $(d(x))^j d^{k-j}(x) d^l(x)$ is

$$\sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{\underbrace{1, \ldots, 1}_{j}} \frac{1}{j!} \binom{p_i}{\underbrace{1, \ldots, 1}_{j+1}} \binom{q_i}{1}$$
$$= \sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{j} \binom{p_i}{\underbrace{1, \ldots, 1}_{j+1}} q_i.$$

(b) $d^{k-j}(x) d^l(x) (d(x))^{j-l}$ stems from $d^k(x^{p_i})$ and $(d(x))^l$ stems from $d^l(x^{q_i})$. This can happen if $j \geq l$. In this case the coefficient of $(d(x))^j d^{k-j}(x) d^l(x)$ is

$$\sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{k-j, l, \underbrace{1, \ldots, 1}_{j-l}} \frac{1}{(j-l)!} \binom{l}{\underbrace{1, \ldots, 1}_{l}} \frac{1}{l!} \binom{p_i}{\underbrace{1, \ldots, 1}_{j-l+2}} \binom{q_i}{\underbrace{1, \ldots, 1}_{l}}$$
$$= \sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{k-j, j-l, l} \binom{p_i}{\underbrace{1, \ldots, 1}_{j-l+2}} \binom{q_i}{\underbrace{1, \ldots, 1}_{l}}.$$

(c) $d^l(x)(d(x))^{k-l}$ stems from $d^k(x^{p_i})$ and $(d(x))^{l-(k-j)} d^{k-j}(x)$ stems from $d^l(x^{q_i})$. This can happen if $l \geq k-j$. In this case the coefficient of $(d(x))^j d^{k-j}(x) d^l(x)$ is

$$\sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{k-l, \underbrace{1, \ldots, 1}_{l}} \frac{1}{l!} \binom{l}{k-j, \underbrace{1, \ldots, 1}_{l-(k-j)}} \frac{1}{(l-(k-j))!}$$
$$\times \binom{p_i}{\underbrace{1, \ldots, 1}_{l+1}} \binom{q_i}{\underbrace{1, \ldots, 1}_{l-(k-j)+1}}$$
$$= \sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} \binom{k}{k-l, k-j, l-(k-j)} \binom{p_i}{\underbrace{1, \ldots, 1}_{l+1}} \binom{q_i}{\underbrace{1, \ldots, 1}_{l-(k-j)+1}}.$$

For $j = 0$ (and hence $s = 0$) only the first term takes into account. This means that

$$\sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} p_i q_i = 0.$$

So the inductive hypothesis holds for $j = 0$.

Now we assume that

$$\sum_{i=1}^{n} \lambda_{i,k} \widetilde{\lambda}_{i,l} p_i^{j'+1-s} q_i^s = 0$$

holds for every $j' = 0, \ldots, j-1$ and every $s = 0, \ldots, j'$. We prove that

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l}p_i^{j+1-s}q_i^s = 0$$

holds for every $s = 0, \ldots, j$, as well. Generally, some of the previous compositions are possible for a given j but the following argument works in all cases, however we just prove it when all compositions discussed below appear in the expansion. Thus we assume that the coefficient of $(d(x))^j d^{k-j}(x) d^l(x)$ is

$$0 = \sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l}\Bigg[\binom{k}{j}\binom{p_i}{\underbrace{1,\ldots,1}_{j+1}}q_i + \binom{k}{k-j, j-l, l}\binom{p_i}{\underbrace{1,\ldots,1}_{j-l+2}}\binom{q_i}{\underbrace{1,\ldots,1}_{l}} + \binom{k}{k-l, k-j, l-(k-j)}\binom{p_i}{\underbrace{1,\ldots,1}_{l+1}}\binom{q_i}{\underbrace{1,\ldots,1}_{l-(k-j)+1}}.\Bigg]$$

By the inductive hypothesis and the fact that $j < n \leq \min\{p'_i, q'_i\}$ for some i', this is equivalent to

$$0 = \sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l}\left(\binom{k}{j}p_i^{j+1}q_i + \binom{k}{k-j, j-l, l}p_i^{j-l+2}q_i^l + \binom{k}{k-l, k-j, l-(k-j)}p_i^{l+1}q_i^{l-(k-j)+1}\right).$$

Note that the expressions $p_i^{j+1-s}q_i^s$ for $s = 0, \ldots, j$ can be interchanged in the following sense. By C(ii), $p_i + q_i = N$, thus we have that $p_i^{j+1-s}q_i^s = Np_i^{j-s}q_i^s - p_i^{j-s}q_i^{s+1}$. As

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l}p_i^{j-s}q_i^s = 0$$

for every $s = 0, \ldots, j-1$ by the inductive hypothesis, the term $Np_i^{j-s}q_i^s$ can be eliminated. After several repetition of this step we get that

$$0 = \sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} \left(\binom{k}{j} p_i^{j+1} q_i \pm \binom{k}{k-j, j-l, l} p_i^{j+1} q_i \pm \binom{k}{k-l, k-j, l-(k-j)} p_i^{j+1} q_i \right).$$

We also note that using this interchange rule and the inductive hypothesis it is clear that

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} p_i^{j+1-s} q_i^s = 0$$

for any $s = 0, \ldots, j$ is equivalent to

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} p_i^{j+1} q_i = 0.$$

Therefore, to finish the proof it is enough to show that

$$\binom{k}{j} \pm \binom{k}{k-j, j-l, l} \pm \binom{k}{k-l, k-j, l-(k-j)} \neq 0.$$

This is equivalent to verify that

$$\frac{1}{j!} \pm \frac{1}{(j-l)!(l!)} \pm \frac{1}{(k-l)!(l-(k-j)!)} \neq 0.$$

Multiplying by $j!$ this lead to

$$1 \pm \binom{j}{l} \pm \binom{j}{k-l} \neq 0.$$

It is straightforward to show using the growth of $\binom{n}{k}$ in k (if $k \leq n/2$) that one term dominates the others if $l \neq k-l$ and $l \neq j-(k-l)$. Thus in these cases this (weighted) sum is nonzero. If $l = k-l$ or $l = j-(k-l)$, then either their sign is the same and hence the sum is nonzero, or their sign is different and hence they eliminate each other, hence the sum is 1 which is nonzero.

Summarizing, we get that

$$\binom{k}{j} \pm \binom{k}{k-j, j-l, l} \pm \binom{k}{k-l, k-j, l-(k-j)} \neq 0$$

and hence

$$\sum_{i=1}^{n} \lambda_{i,k}\widetilde{\lambda}_{i,l} p_i^{j+1} q_i = 0, \tag{8}$$

which is equivalent to the inductive hypothesis for j as we noted above.

Thus Eq. (8) holds for every $j = 0, \ldots, n-1$. In matrix form this means that

$$\begin{pmatrix} 1 & \ldots & 1 \\ p_1 & \ldots & p_n \\ \vdots & \ddots & \vdots \\ p_1^{n-1} & \ldots & p_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} p_1 q_1 \lambda_{1,k} \widetilde{\lambda}_{1,k} \\ p_2 q_2 \lambda_{2,k} \widetilde{\lambda}_{2,l} \\ \vdots \\ p_n q_n \lambda_{n,k} \widetilde{\lambda}_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since this matrix is a Vandermonde type matrix with different p_i's, as Case 1 and Case 2, this implies that, the only solution of this homogeneous linear system is the zero vector, i.e., $p_i q_i \lambda_{i,k} \widetilde{\lambda}_{i,l} = 0$ for all $i = 1, \ldots, n$. This contradicts to our assumption that there is some i' for which $\lambda_{i',k} \neq 0$ and $\widetilde{\lambda}_{i',l} \neq 0$ (and $p_i \neq 0$, $q_i \neq 0$ by C(i)). This also finishes the proof of the theorem, thus the order of all derivations involved in (1) is at most $n-1$.

□

Remark 12. The upper bound appearing in the above theorem is sharp. To see this, let n, N be positive integers, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and let $f \colon \mathbb{F} \to \mathbb{C}$ be an additive function for which

$$\sum_{i=1}^{n} \lambda_i f(x^i) x^{N-i} = 0$$

is fulfilled for all $x \in \mathbb{F}$. Then $f \in \mathscr{D}_{n-1}(\mathbb{R})$ if and only if $\lambda_i = (-1)^i \binom{n}{i}$ for all $i = 1, \ldots, n$, see [1,3,5].

Remark 13. The proof of Theorem 20 in all the cases is based on the fact that the matrix

$$\begin{pmatrix} \binom{p_1}{1}\binom{q_1}{1} & \cdots & \binom{p_n}{1}\binom{q_n}{1} \\ \binom{p_1}{1,1}\binom{q_1}{1} + \binom{p_1}{1}\binom{q_{1,1}}{1} & \cdots & \binom{p_n}{1}\binom{q_n}{1,1} + \binom{p_n}{1,1}\binom{q_n}{1} \\ \vdots & \ddots & \vdots \\ \binom{p_1}{\underbrace{1,\ldots,1}_{n-1}}\binom{q_1}{\underbrace{1,\ldots,1}_{n-1}} & \cdots & \binom{p_n}{\underbrace{1,\ldots,1}_{n-1}}\binom{q_n}{\underbrace{1,\ldots,1}_{n-1}} \end{pmatrix}$$

has rank n, though it is far from being trivial to find the proper sub-matrix, which verifies that.

On the other hand, the situation is much more complicated, if the maximal order k of D_i, and the maximal order l of $\widetilde{D_i}$ is not uniquely determined. Namely, if there are several different pairs (k_i, l_i) so that $k_i + l_i = K$, where K is constant and k_i is the maximum order of D_i, l_i is the maximal order of $\widetilde{D_i}$. Then the equations first can only be determined for subsets of the index set, which satisfy some nontrivial relations. In this case the first task is to show that the problem can be formalized separately for the index sets, which seems a very hard problem in full generality. In this case our method can be applied.

Theorem 20 and Remark 13 motivates our conjecture, that we verified for $n \leq 4$.

Conjecture 21. *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(i)–C(iii). Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy Eq. (1). Then every solution is a generalized exponential polynomial function of degree at most $n-1$ on $\mathbb{F}^\times$. In particular, if*

$$f_i(x) = D_i(x) \qquad \text{and} \qquad g_i(x) = \widetilde{D}_i(x) \qquad \left(x \in \mathbb{F}^\times\right) \tag{9}$$

for each $i = 1, \ldots, n$, then the order of $D_i, \widetilde{D}_i$ is at most $n-1$.

This conjecture leads to the following more general open question.

Open Question 1. *Is it true that every nonzero additive, irreducible solutions $f_1, \ldots, f_n$ of*

$$P(f_1(x^{p_1}), \ldots, f_n(x^{p_n})) = 0, \ \text{with} \ \ P(0, \ldots, 0) = 0$$

are derivations of order at most $n-1$ and the identity function up to a homomorphism, if $P : \mathbb{C} \to \mathbb{C}$ is polynomial and $p_1 \ldots, p_n$ are distinct positive integers?

Finally we highlight some important special cases when Theorem 20 gives the proper bound of the order of the derivations.

Corollary 22. *Let n be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $p_1, \ldots, p_n, q_1, \ldots, q_n$ be fixed positive integers fulfilling conditions C(i)–C(iii). Assume that the additive functions $f_1, \ldots, f_n, g_1, \ldots, g_n \colon \mathbb{F} \to \mathbb{C}$ satisfy Eq. (1) as an irreducible solution. Then $f_i \sim D_i$ and $g_i \sim \widetilde{D_i}$, where D_i and $\widetilde{D_i}$ are higher order derivations. Assume further that one of the following holds.*

- *(A) All D_i have the same order. This is the case, when $f_i(x) = c_i f(x)\,(x \in \mathbb{F})$ for some nonzero constants $c_i \in \mathbb{C}$, $i \in \{1, \ldots, n\}$.*
- *(B) All $\widetilde{D_i}$ have the same order. This is the case, when $g_i(x) = c_i g(x)\,(x \in \mathbb{F})$ for some nonzero constants $c_i \in \mathbb{C}$, $i \in \{1, \ldots, n\}$.*
- *(C) $f_i = c_i \cdot g_i$ for all $i \in \{1, \ldots, n\}$ with some nonzero constants $c_i \in \mathbb{C}$, $i = 1, \ldots, n$.*

Then the order D_i and $\widetilde{D_i}$ is at most $n-1$.

Proof. Theorem 14 implies that every solution f_i (resp. g_i) is an exponential polynomial of the form $P_i \cdot m$ (resp. $Q_i \cdot m$), which means that $f_i \sim D_i$ and $g_i \sim \widetilde{D_i}$ for some derivations D_i and $\widetilde{D_i}$.

- (A) Let k denote the order of D_i, and let l be the maximal order of $\widetilde{D_i}$. Now we are in the position to apply Theorem 20.
- (B) Similar to (A), since in case of Eq. (1), the role of the parameters $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ is symmetric.

(c) As the maximal degree of f_i is the same as the maximal degree of g_i and it is taken for the same index we can apply Theorem 20.

□

Special Cases of Eq. (1). In this subsection we consider special cases of Eq. (1). All the equations we consider here are of the form

$$f_1(x^{p_1})g_1(x^{q_1}) + f_2(x^{p_2})g_2(x^{q_2}) = 0 \qquad (x \in \mathbb{F}).$$

Here $f_1, f_2, g_1, g_2 \colon \mathbb{F} \to \mathbb{C}$ denote the unknown additive functions and the parameters p_1, p_2, q_1, q_2 fulfill conditions C(i)–C(iii). Due to the results of the previous section, we get that

$$\begin{aligned} f_i(x) &\sim \lambda_{i,0}x + \lambda_{i,1}d_i(x) \\ g_i(x) &\sim \mu_{i,0}x + \mu_{i,1}\widetilde{d}_i(x) \end{aligned} \qquad (x \in \mathbb{F}, i = 1, 2)$$

with appropriate complex constants $\lambda_{i,j}, \mu_{i,j}$ $(i = 1, 2,\ j = 0, 1)$ and derivations $d_i, \tilde{d}_i \colon \mathbb{F} \to \mathbb{C}$ $(i = 1, 2)$.

Corollary 23. *Let N be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and p, q be different positive integers (strictly) less than N and assume that $q \neq N - p$. If the additive functions $f, g \colon \mathbb{F} \to \mathbb{C}$ satisfy*

$$f(x^p)f(x^{N-p}) = g(x^q)g(x^{N-q}) \qquad (x \in \mathbb{F}),$$

then

(A) either there exists a homomorphism $\varphi \colon \mathbb{C} \to \mathbb{C}$, a derivation $d \colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = \varphi(d(x)) \qquad \text{and} \qquad g(x) = \alpha\varphi(d(x)) \qquad (x \in \mathbb{F}),$$

where $\alpha = \dfrac{p(N-p)}{q(N-q)}$,

(B) or there exists a homomorphism $\varphi \colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1) \cdot \varphi(x) \qquad g(x) = \pm f(1) \cdot \varphi(x) \qquad (x \in \mathbb{F}).$$

Corollary 24. *Let N be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and p, q be different positive integers (strictly) less than N and assume that $q \neq N - p$. If the additive functions $f, g \colon \mathbb{F} \to \mathbb{C}$ satisfy*

$$f(x^p)g(x^{N-p}) = \kappa f(x^q)g(x^{N-q}) \qquad (x \in \mathbb{F}).$$

Then

(A) if $\kappa \notin \left\{1, \dfrac{p(N-p)}{q(N-q)}\right\}$, then f is identically zero,

(B) if $\kappa = 1$, then the only possibility is that

$$f(x) = f(1) \cdot \psi(x) \qquad \text{and} \qquad g(x) = f(1) \cdot \psi(x) \qquad (x \in \mathbb{F}),$$

where $\psi \colon \mathbb{F} \to \mathbb{C}$ is a homomorphism, *(C) if* $\kappa = \dfrac{p(N-p)}{q(N-q)}$*, then there exists a homomorphism* $\varphi\colon \mathbb{C} \to \mathbb{C}$ *and derivations* $d_1, d_2\colon \mathbb{F} \to \mathbb{C}$ *such that*

$$f(x) = \varphi(d_1(x)) \qquad \text{and} \qquad g(x) = \varphi(d_2(x)) \qquad (x \in \mathbb{F}).$$

Both results implies that the nonzero additive solutions of equation

$$f(x^p)f(x^{N-p}) = \kappa f(x^q)f(x^{N-q}) \qquad (x \in \mathbb{F}).$$

are derivations of order 1 ($\kappa = \dfrac{p(N-p)}{q(N-q)}$) or the identity function ($\kappa = 1$) up to a homomorphism.

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