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Results in Mathematics

On Continuous Parameter Dependence of Roots of Analytic Functions

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Abstract. Let $\mathcal{A}_n(\Omega)$ be the set of analytic functions on a domain Ω of the complex plane which have n roots in Ω, counted with multiplicity. In this note we consider functions in $\mathcal{A}_n(\Omega)$ which depend continuously on a parameter. A simple short proof shows that the set of roots in the Hausdorff metric depends continuously on the parameter. If the parameter space is connected and all roots are known to lie in one of two disjoint open subsets Ω_1, Ω_2 of the complex plane, then the number of the roots, counted with multiplicity, in Ω_1 and Ω_2, respectively, is independent of the parameter. Each set of roots generates a unique monic polynomial. It is shown that the map which associates with each function in $\mathcal{A}_n(\Omega)$ the corresponding monic polynomial is continuous when $\mathcal{A}_n(\Omega)$ is equipped with the topology of uniform convergence on compact subsets of Ω. Possible applications are indicated.

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1. Introduction

When an analytic function depends analytically on a parameter, then it is well known that locally the roots can be arranged in such a way that they depend continuously on the parameter, and even analytically away from branching points. This is essentially done for polynomial dependence, see e. g. the exposition in Knopp [9], and particular the theorem on page 122 therein. A slightly different presentation can be found in [7, Appendix A]. Such dependence occurs for example when one considers eigenvalues of linear operators depending on a parameter. Kato [8] gives a detailed account of the results for the matrix case in Chapter II, see in particular Section 1.8. In [8, Chapter II Section 5.1] continuous dependence on a parameter is considered, whereas [8, Chapter VII] deals with various aspects of analytic dependence for operators in Hilbert spaces. These operators are mostly finite dimensional or have a compact resolvent, so that their eigenvalues can be found (locally) as the zeros of a determinant, which is analytic in the eigenvalue and depends analytically or at least continuously on the parameter. Therefore the dependence of roots of analytic function on a parameter is at the core of the investigation. Indeed, if an analytic function depends analytically on the parameter, then locally the roots can be expressed in Puiseux series with respect to the parameter, see e. g. [2, Appendix A 5.4, Theorem 3] or [10, Chapter 9].

The particular case of monic polynomials of a fixed degree has attracted more attention; see [5,6] and [3] and the references therein. The set of coefficients is identified with $\mathbb{C}^n$, and the roots, with multiplicities taken into account, are identified as equivalences classes of elements in $\mathbb{C}^n$. It is shown in [5,6] and [3] that these metric spaces of coefficients and roots are homeomorphic.

When dealing with analytic functions instead of polynomials, no such homeomorphism can exist, unless one considers equivalence classes of analytic functions on a fixed domain with fixed numbers of roots. Indeed, in applications such equivalence would often not attract the main attention. One is rather interested in properties of the roots such as their location, for which simple continuity or connectedness arguments suffice. In this context, the notion of continuity has to be precisely defined by a suitable topology in the set of roots. Indeed, the Hausdorff metric on the set of finite nonempty subsets of $\mathbb{C}$ and its multiset variant serves this purpose. The multiset variant has been used in [5] for a clear exposition of the homeomorphism between monic polynomials of degree n and the multisets of their roots.

Let $\mathcal{A}_n(\Omega)$ be the set of analytic functions on a connected open set $\Omega \subset \mathbb{C}$ which have n roots in Ω, counted with multiplicity. In Sect. 2 the roots of analytic functions $f(x, \cdot) \in \mathcal{A}_n(\Omega)$ depending on the parameter x in a topological space will be investigated. It is shown that the map from the parameter space to the set of roots, equipped with the Hausdorff metric, is continuous when f is continuous. In particular, if all roots lie in two disjoint open subsets of $\mathbb{C}$, then the number of the roots, counted with multiplicity, in each of these sets, is independent of the parameter if the parameter space is connected.

With each analytic function as described above, one can consider the monic polynomial generated by its roots. It is shown in Sect. 3 that this map from $\mathcal{A}_n(\Omega)$ to the set of the corresponding polynomials is continuous in the canonical topologies.

2. Roots of Analytic Functions Depending Continuously on a Parameter

Throughout this note let X be a nonempty topological space, Ω a nonempty connected open subset of $\mathbb{C}$, n a positive integer, and $f : X \times \Omega \to \mathbb{C}$ a continuous function such that $f(x, \cdot) \in \mathcal{A}_n(\Omega)$ for each $x \in X$.

We equip the set of roots with the Hausdorff metric. More precisely, let $\mathcal{F}$ be the set of all finite nonempty subsets of $\mathbb{C}$. We recall that the distance $d(z, B)$ between $z \in \mathbb{C}$ and $B \in \mathcal{F}$ is defined by $d(z, B) = \min\{|z-w| : w \in B\}$. Furthermore, for $A, B \in \mathcal{F}$ let $d(A, B) = \max\{d(z, B) : z \in A\}$. The Hausdorff metric d_H on $\mathcal{F}$ is then defined by

$$d_H(A, B) = \max\{d(A, B), d(B, A)\}, \quad A, B \in \mathcal{F}.$$

It is well known that d_H is indeed a metric on $\mathcal{F}$, see e. g. [1, Theorem 1.12.13], since $\mathcal{F}$ is a subset of the set of nonempty compact subsets of $\mathbb{C}$. Observe that

$$d_H(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{d_H(A_1, B_1), d_H(A_2, B_2)\}, \quad A_1, A_2, B_1, B_2 \in \mathcal{F}, \tag{1}$$

see e. g. [1, Theorem 1.12.15], which can easily be extended to finite unions.

The open and closed ball in a metric space (R, ρ) about $r \in R$ with radius $\varepsilon > 0$ are denoted by $B_\varepsilon(r) = \{s \in R : \rho(s, r) < \varepsilon\}$ and $\overline{B}_\varepsilon(r) = \{s \in R : \rho(s, r) \leq \varepsilon\}$, respectively. When applying this notation, the underlying metric space will be clear from the context.

Define the function $Z : X \to \mathcal{F}$ such that for each $x \in X$, $Z(x)$ is the set of roots of $f(x, \cdot)$. For $x \in X$ let n_x be the number of elements of $Z(x)$. For each subset $\widehat{Z}$ of $Z(x)$ let $m_x(\widehat{Z})$ be the total multiplicity of the roots of $f(x, \cdot)$ which belong to $\widehat{Z}$.

Lemma 1. *Let $y \in X$ and $\varepsilon > 0$ and enumerate $Z(y) = \{w_1, \ldots, w_{n_y}\}$. Then there is $\varepsilon_0 > 0$ such that $\varepsilon_0 \leq \varepsilon$, such that $\overline{B}_{\varepsilon_0}(w_j) \subset \Omega$ for $j = 1, \ldots, n_y$, and such that $\varepsilon_0 < \frac{1}{2}\min\{|w_j - w_k| : 1 \leq j < k \leq n_y\}$ when $n_y > 1$. Furthermore, for each $0 < \varepsilon_1 \leq \varepsilon_0$ there is a neighbourhood U of y such that each $x \in U$ satisfies $m_x(Z(x) \cap B_{\varepsilon_1}(w_j)) = m_y(\{w_j\})$ for $j = 1, \ldots, n_y$ and*

$$Z(x) = \bigcup_{j=1}^{n_y} (Z(x) \cap B_{\varepsilon_1}(w_j)). \tag{2}$$

Proof. The statement regarding the existence and choice of ε_0 is obvious since Ω is open and $Z(y)$ is finite. Note that the disks $\overline{B}_{\varepsilon_0}(w_j)$, $j = 1, \ldots . n_y$, are mutually disjoint.

Now fix $j \in \{1, \ldots, n_y\}$ and $0 < \varepsilon_1 \leq \varepsilon_0$. Then w_j is the only element of $Z(y)$, that is, the only root of $f(y, \cdot)$, belonging to $\overline{B}_{\varepsilon_1}(w_j)$. Since the boundary $\partial B_{\varepsilon_1}(w_j)$ of $B_{\varepsilon_1}(w_j)$ is compact, $\delta_j := \min\{|f(y, z)| : z \in \partial B_{\varepsilon_1}(w_j)\} > 0$. Then there is a neighbourhood U_j of y such that $|f(x, z) - f(y, z)| < \delta_j$ for all $x \in U_j$ and $z \in \partial B_{\varepsilon_1}(w_j)$; first locally on $\partial B_{\varepsilon_1}(w_j)$ due to the continuity of f and then globally by a compactness argument. Now Rouché's theorem, see e. g. [4, Theorem 3.8], proves that $m_x(Z(x) \cap B_{\varepsilon_1}(w_j)) = m_y(\{w_j\})$ for all $x \in U_j$. Putting $U := \bigcap_{j=1}^{n_y} U_j$ it follows that U is a neighbourhood of y and that

$$n = m_x(Z(x)) \geq m_x \left(\bigcup_{j=1}^{n_y} (Z(x) \cap B_{\varepsilon_1}(w_j)) \right) = m_y(Z(y)) = n, \quad x \in U.$$

Hence (2) is proved since any roots outside the union on the right hand side would increase $m_x(Z(x))$. $\square$

Theorem 2. *The function Z is continuous.*

Proof. Let $y \in X$ and $\varepsilon > 0$ and let $\varepsilon_1 = \varepsilon_0$ and U be as in Lemma 1. Let $x \in U$. Since the distance of each element $w \in Z(x) \cap B_{\varepsilon_0}(w_j)$ to w_j is less then ε_0, it follows that $d_H(Z(x) \cap B_{\varepsilon_0}(w_j), \{w_j\}) < \varepsilon_0 \leq \varepsilon$ for $1 \leq j \leq n_y$. Then (1) and (2) show that $d_H(Z(x), Z(y)) < \varepsilon$. Thus $Z(U) \subset B_\varepsilon(Z(y))$, proving that Z is continuous at y. $\square$

Recall that a topological space Y is called connected if and only if $\emptyset$ and Y are the only subsets of Y which are open as well as closed. Equivalently, Y is connected if and only if Y is not the disjoint union of two nonempty open subsets. Also recall that continuous functions map connected topological spaces onto connected topological spaces since preimages of disjoint open subsets are disjoint open subsets. Hence the following result is obvious.

Corollary 3. *Assume that X is connected. Then $Z(X)$ is connected.*

Theorem 4. *Let Ω_1 and Ω_2 be two disjoint open subsets of $\mathbb{C}$ such that $Z(x) \subset \Omega_1 \cup \Omega_2$ for all $x \in X$. Then the maps $M_{\Omega_p} : X \to \mathbb{Z}$ defined by $M_{\Omega_p}(x) = m_x(Z(x) \cap \Omega_p)$, $x \in X$, $(p = 1, 2)$ are continuous.*

Proof. Since the statement is symmetric in Ω_1 and Ω_2, it is sufficient to consider $p = 1$. Let $Z_q(x) = Z(x) \cap \Omega_q$, $x \in X$, $(q = 1, 2)$ and choose any $y \in X$. There is $0 \leq l \leq n_y$ such that we can write $Z_1(y) = \{w_1, \ldots, w_l\}$ and $Z_2(y) = \{w_{l+1}, \ldots, w_{n_y}\}$. Let ε_0 be as in Lemma 1. Since Ω_1 and Ω_2 are open, we can choose $0 < \varepsilon_1 \leq \varepsilon_0$ such that $B_{\varepsilon_1}(w_j) \subset \Omega_p$ when $j = 1, \ldots, l$ for $p = 1$ and $j = l+1, \ldots, n_y$ for $p = 2$. Then another application of Lemma 1 with the neighbourhood U of y as defined there shows that $m_x(Z_1(x)) = m_y(Z_1(y))$ for all $x \in U$. Hence $M_{\Omega_1}(U) \subset \{m_y(Z_1(y))\}$, which shows that M_{Ω_1} is continuous at y. $\square$

Corollary 5. *Assume that X is connected and that there are two disjoint open subsets Ω_1 and Ω_2 of $\mathbb{C}$ such that $Z(x) \subset \Omega_1 \cup \Omega_2$ for all $x \in X$. Then $m_x(Z(x) \cap \Omega_p)$, $x \in X$, $(p = 1, 2)$ is independent of $x \in X$.*

Proof. Since the functions M_{Ω_p} $(p = 1, 2)$ are continuous by Theorem 4 and since X is connected, $M_{\Omega_p}(X)$ must be connected. Because nonempty connected subsets of $\mathbb{Z}$ are singletons, the result follows. $\square$

A typical application of Corollary 5 would be that the space X is an interval $[a,b]$ and that the analytic functions $f(x,\cdot)$ with common domain have, say, no real roots, and that the roots of $f(a,\cdot)$ have positive imaginary parts. If now the domain has countably many disjoint subregions which contain all roots of $f(x,\cdot)$, $x\in[a,b]$, and the number of the roots, counted with multiplicity, in each of these subregions is independent of x, then also all roots of $f(b,\cdot)$ have positive imaginary parts. Meaningful examples have to be given in context and will be considered elsewhere.

One such application, still in preparation, will occur in work on the direct and inverse spectral problem for boundary value problems of a class of ordinary differential equations. Here the coefficients of the differential equation as well as the boundary conditions depend on the spectral parameter. Several entire functions occur, and to find the desired location of the roots of some of those entire functions, parameter dependence will be used to reduce this task to simpler entire functions whose location of roots is known to have the desired properties.

The assumption that the number of roots in Ω, counted with multiplicity, is constant is crucial for the continuity results in this section. Indeed, letting $X=[1,4]$, Ω the open unit disc and $f(x,z)=\sin(xz)$ for $x\in X$ and $z\in\Omega$, it is clear that $Z(x)=\{0\}$ for $x\in[1,\pi]$, whereas $Z(x)=\{0,-\pi x^{-1},\pi x^{-1}\}$ for $x\in(\pi,4]$. But $d_H(Z(x),Z(\pi))=\pi x^{-1}$ when $x\in(\pi,4]$ shows that Z is not continuous at π.

3. Canonical Polynomials

For $h\in\mathcal{A}_n(\Omega)$ with roots $(v_1,\ldots,v_n)$ written as unordered n-tuples, a unique monic polynomial

$$\pi_{\Omega,n}(h)(z):=\prod_{j=1}^{n}(z-v_j) \tag{3}$$

is defined. We will investigate the question if $\pi_{\Omega,n}$ is continuous. Here we have the natural topology on $\mathcal{A}_n(\Omega)$ induced by the standard topology of uniform convergence on compact subset of $\mathcal{A}(\Omega)$, the space of analytic functions on Ω. The space of monic polynomials of degree n will be identified with the space $\mathbb{C}^n$ of its coefficients as in [5,6] and [3], and we will use the notation $\mathcal{P}_{n,1}$ for it as in [5]. The order of the coefficients as entries of vectors in $\mathbb{C}^n$ is arbitrary but fixed.

To prove the continuity of $\pi_{\Omega,n}$ we need some preparation.

We consider the function $F:\mathcal{A}_n(\Omega)\times\Omega\to\mathbb{C}$ defined by

$$F(h,z)=h(z),\quad h\in\mathcal{A}_n(\Omega),\ z\in\Omega. \tag{4}$$

Lemma 6. *The function F is continuous.*

Proof. Let $g \in \mathcal{A}_n(\Omega)$, $w \in \Omega$ and $\varepsilon > 0$. Since g is continuous, there is $\delta > 0$ such that $\overline{B}_\delta(w) \subset \Omega$ and $|g(z) - g(w)| < \frac{\varepsilon}{2}$ for all $z \in \overline{B}_\delta(w)$. The set

$$U := \left\{ h \in \mathcal{A}_n(\Omega) : \forall z \in \overline{B}_\delta(w) \ |h(z) - g(z)| < \frac{\varepsilon}{2} \right\}$$

is a neighbourhood of g. Then

$$|F(h,z) - F(g,w)| = |h(z) - g(w)| \leq |h(z) - g(z)| + |g(z) - g(w)| < \varepsilon$$

for $h \in U$ and $z \in B_\delta(w)$ shows that F is indeed continuous at (g, w). □

We need to strengthen Theorem 2. To this end, let $k \in \mathbb{N}$ and consider the set of all non-ordered k-tuples $(u_1, \ldots, u_k)$ of complex numbers. These can either be considered as multisets as in [5] or as equivalence classes of elements in $\mathbb{C}^n$ as in [6] and [3], where $a, b \in \mathbb{C}^n$ are equivalent if there is a permutation σ of the components such that $b = \sigma(a)$. In the notation of [5] we write $\mathcal{Z}_k$ for the set of unordered k-tuples. Let S_k denote the group of permutations of the set $\{1, \ldots, k\}$. Writing $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_k)$ for two elements in $\mathcal{Z}_k$,

$$d_k(U, V) := \min_{\tau \in S_k} \max_{1 \leq j \leq k} |u_j - v_{\tau(j)}|$$

defines a metric d_k on $\mathcal{Z}_k$, see [5, Proposition 3.1]. Above, U and V are ordered representations of the unordered tuples, and the permutations $\tau \in S_k$ on V yield all representations of the class represented by V. Although the definition may look unsymmetric in U and V, one also may apply permutations to U, which can be combined with the permutations of V to the form given because S_k is a group.

We need to concatenate unordered tuples. For $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_l)$ we write $U \cup V = (u_1, \ldots, u_k, v_1, \ldots, v_l)$. It is clear that $U \cup V$ and $V \cup U$ represent the same element in $\mathcal{Z}_{k+l}$.

Lemma 7. *Let $k, l \in \mathbb{N}$ and $U_1, U_2 \in \mathcal{Z}_k$, $V_1, V_2 \in \mathcal{Z}_l$. Then*

$$d_{k+l}(U_1 \cup V_1, U_2 \cup V_2) \leq \max\{d_k(U_1, U_2), d_l(V_1, V_2)\}.$$

Proof. Let $S_{k,l}$ be the subset of all permuations in S_{k+l} which permute the first k and last l numbers only. A permutation $\omega \in S_{k,l}$ can be uniqely written as $\omega = (\sigma, \tau)$, where $\sigma \in S_k$ operates on the first k numbers and $\tau \in S_l$ operates on the last l numbers. For $p = 1, 2$ write $U_p = (u_{p,1}, \ldots, u_{p,k})$ and $V_p = (v_{p,1}, \ldots, v_{p,l})$. Then

$$
\begin{aligned}
&d_{k+l}(U_1 \cup V_1, U_2 \cup V_2) \\
&\quad \le \min_{(\sigma,\tau)\in S_{k,l}} \max\left\{ \max_{1\le j\le k} |u_{1,j} - u_{2,\sigma(j)}|, \max_{1\le j\le l} |v_{1,j} - v_{2,\tau(j)}| \right\} \\
&\quad = \max\left\{ \min_{\sigma\in S_k} \max_{1\le j\le k} |u_{1,j} - u_{2,\sigma(j)}|, \min_{\tau\in S_l} \max_{1\le j\le l} |v_{1,j} - v_{2,\tau(j)}| \right\} \\
&\quad = \max\{d_k(U_1, U_2), d_l(V_1, V_2)\}.
\end{aligned}
$$

□

Clearly, Lemma 7 can be extended to finite unions by repeated application.

Let f be as in Sect. 2, and define $\mathfrak{Z}_{n,f} : X \to \mathcal{Z}_n$ such that $\mathfrak{Z}_{n,f}(x)$ is the unordered n-tuple of roots of $f(x, \cdot)$ in Ω with multiplicity taken into account.

Theorem 8. *The function $\mathfrak{Z}_{n,f}$ is continuous.*

Proof. Let $y \in X$ and $\varepsilon > 0$ and let $\varepsilon_1 = \varepsilon_0$ and U be as in Lemma 1. Let $x \in U$ and $j \in \{1, \dots, n_y\}$. By Lemma 1 the roots of $f(y, \cdot)$ and $f(x, \cdot)$ in $B_{\varepsilon_0}(w_j)$, counted with multiplicity, are m-tuples of the form $W = (w_j, \dots, w_j)$ and $V = (v_1, \dots, v_m)$, respectively, with $m = m_y(\{w_j\})$. It follows that $|v_p - w_j| < \varepsilon_0$ for all $p = 1, \dots, m$, which gives $d_m(V, W) < \varepsilon_0 \le \varepsilon$. An application of (2) and Lemma 7 shows that $d_n(\mathfrak{Z}_{n,f}(x), \mathfrak{Z}_{n,f}(y)) < \varepsilon$. Hence $\mathfrak{Z}_{n,f}(U) \subset B_\varepsilon(\mathfrak{Z}_{n,f}(y))$ in $\mathcal{Z}_n$, which proves that $\mathfrak{Z}_{n,f}$ is continuous at y. □

Let $\zeta : \mathcal{Z}_n \to \mathcal{P}_{n,1}$ be defined by

$$
\zeta(V) = \prod_{j=1}^{n} (z - v_j), \quad V = (v_1, \dots, v_n) \in \mathcal{Z}_n.
$$

In [6, Theorem A], [5, Theorem 4.4] and [3, Theorem 2.8] the following statement has been proved.

Theorem 9. *The function ζ is a homeomorphism.*

Theorem 10. *The function $\pi_{\Omega,n} : \mathcal{A}_n(\Omega) \to \mathcal{P}_{n,1}$ is continuous.*

Proof. By Lemma 6, Theorem 8 applies in particular to $X = \mathcal{A}_n(\Omega)$ and $f = F$. Clearly, $\pi_{\Omega,n} = \zeta \circ \mathfrak{Z}_{n,F}$. Hence an application of Theorems 8 and 9 completes the proof. Of course, only the continuity of ζ is needed from Theorem 9, which is the easier part following from Vieta's theorem; see the discussion in the cited references. □

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