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A Note on Invariant Description of $SU(2)$–Structures in Dimension 5

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Abstract. We develop an invariant approach to $SU(2)$–structures on spin 5–manifolds. We characterize (via spinor approach) the subspaces in the spinor bundle which induce the same group isomorphic to $SU(2)$. Moreover, we show how to induce quaternionic structure on the contact distribution of the considered $SU(2)$–structure. We show the invariance of certain components of the covariant derivative $\nabla\varphi$, where φ is any spinor field defining $SU(2)$–structure. This shows, as expected, that (at least some of) the intrinsic torsion modules can be derived invariantly with the spinorial approach. We conclude with the explicit description of the intrinsic torsion and the characteristic connection.

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Introduction

Holonomy plays an important role in Riemannian geometry. It measures the behavior of parallel displacement with respect to the Levi-Civita connection. The celebrated theorem by Berger states that the list of possible (restricted) holonomy groups is limited to a few cases. In the list there are geometries, called *exceptional*, which appear only in certain dimensions:

– G_2 in dimension 7.
– Spin(7) in dimension 8,

There are two additional geometries, $SU(3)$ in dimension 6 and $SU(2)$ in dimension 5, which are also called exceptional (despite the fact that they fall into general category of $SU(n)$–structures in dimension n or $n+1$ as a codimension one distribution). This is due to the fact that each of these geometries in dimension k induces appropriate geometry in dimension $k+1$.

In very interesting articles [4,6,15] the authors study these exceptional geometries from the spinorial point of view. In fact, it can be shown that the unit spinor field induces an exceptional geometry. However, in the $SU(2)$ case the choice of the unit spinor is not unique.

In this article, we study $SU(2)$–geometry from the perspective of spinors, focusing on the invariant approach, i.e., independent on the choice of the defining spinor. We concentrate on the action of vectors on spinors. Let us be more precise. Consider the spinor representation $\rho : \mathrm{Spin}(5) \to \mathrm{End}(\Delta)$, where $\Delta = \mathbb{C}^4$. Let $\varphi_0 \in \Delta$ be a unit spinor (defining a group $SU(2)$). Then an $SU(2)$–structure on a spin 5–dimensional manifold M is a subbundle P in the bundle $\mathrm{Spin}(M)$ with the structure group $SU(2)$ or, equivalently, a choice of the unit spinor field φ and a subbundle P of all frames u such that $\varphi = [u, \varphi_0]$.

We show that there are different choices of spinors in Δ defining the same $SU(2)$ [3]. In fact, we show that the subspace $V \subset \Delta$ of spinors defining the same group $SU(2)$ is of real dimension four. We characterize these spaces from two perspective: complex and quaternionic. The quaternionic approach seems to be well known to the experts in the field, whereas the complex approach is most likely new.

The other case is concerned with a characterization of the intrinsic torsion modules. It is well known [9] that the intrinsic torsion is characterized by the covariant derivative $\nabla\varphi$, where φ is a defining spinor. We study the invariance of this approach. We show that for another spinor ψ defining the same $SU(2)$–structure a certain decomposition of $\nabla\psi$ induces the same components. To derive the invariance, we slightly modify the quaternionic structure on Δ and use the action of two–forms on spinors.

We begin, in the first section, with an algebraic approach to spinors defining $SU(2)$, or equivalently, its Lie algebra $\mathfrak{su}(2)$. Majority of the results in this section is well known, however it is hard to find appropriate citations. We give a characterization of the subspaces of spinors defining a given Lie algebra $\mathfrak{su}(2)$ in terms of complex and quaternionic structures. Moreover, we study a correspondence between the complex structures on the spinor space Δ and an associated four dimensional space D of vectors acting on spinors. We show how to obtain, in a canonical way, a quaternionic structure on D by the invariant spinorial approach. It is interesting, that the map which assigns a complex structure (from the quaternionic structure) to a unit spinor is, in fact, the Hopf fibration. Moreover, we show nonexistence of a complex structure on D induced from the complex structure on V.

In the second section, we show how algebraic approach developed in the first section induces a $SU(2)$–structure on a 5–dimensional spin manifold. Moreover, we show relations with the approaches from [6] and [9].

In the final—third—section, we show that with a slight modification the spinorial approach developed in [6] is invariant, i.e., independent on the choice of a defining spinor. Moreover, we derive explicit formula for the intrinsic torsion as well as for the characteristic connection.

1. Decomposition of the Space of Spinors

1.1. Spin Representation

Consider a real Clifford algebra Cl_5. Then the irreducible representation Δ of Cl_5 is complex, $\Delta = \mathbb{C}^4$. It can be given by the following action of the vectors $e_i \in \mathbb{R}^5 \subset \mathrm{Cl}_5$ [1]:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

There is a Hermitian product $(\cdot,\cdot)$ on Δ such that the spinor representation is unitary and the Clifford product by vectors is skew–symmetric. Denote by $\langle\cdot,\cdot\rangle$ the inner product, which is the real part of $(\cdot,\cdot)$,

$$\langle\varphi,\psi\rangle = \mathrm{Re}(\varphi,\psi), \quad \varphi,\psi \in \Delta.$$

Fix a (unit) spinor $\varphi \in \Delta$ and define

$$W_\varphi = \{x\cdot\varphi \mid x \in \mathbb{R}^5\}.$$

The action $\mathbb{R}^5 \ni x \mapsto x\cdot\varphi \in W_\varphi$ is an isomorphism (see, for example, the proof of Lemma 1.1 below), i.e., $\dim W_\varphi = 5$.

We begin with the first well–known easy observation (see for example Lemma 6.2 in [10]).

Lemma 1.1. *There is a unique vector $y = y_\varphi$ such that $y\cdot\varphi = i\varphi$ and a unique complex 2–dimensional subspace V_φ of W_φ. They satisfy*

$$W_\varphi = V_\varphi \oplus \langle i\varphi\rangle.$$

Moreover, there is a real 4–dimensional subspace $D_\varphi \subset \mathbb{R}^5$ such that $V_\varphi = D_\varphi\cdot\varphi$ and

$$\Delta = V_\varphi \oplus V_\varphi^\perp,$$

where for any $\psi \in V_\varphi^\perp$ we have

$$y\cdot\psi = -i\psi.$$

Proof. Writing $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \in \mathbb{C}^4$ the action $R_\varphi : \mathbb{R}^5 \to \mathbb{C}^4$, $R_\varphi(x) = x \cdot \varphi$ is represented by the matrix

$$R_\varphi = \begin{pmatrix} i\varphi^4 & \varphi^4 & -i\varphi^3 & \varphi^3 & i\varphi^1 \\ i\varphi^3 & \varphi^3 & i\varphi^4 & \varphi^4 & i\varphi^2 \\ i\varphi^2 & -\varphi^2 & -i\varphi^1 & -\varphi^1 & -i\varphi^3 \\ i\varphi^1 & \varphi^1 & i\varphi^2 & -\varphi^2 & -i\varphi^4 \end{pmatrix}.$$

It can be checked that the rank of R_φ is 5 as a real 5×8 matrix and is equal to the rank of extended block matrix $(R_\varphi \, i\varphi^\top)$. Moreover, if x is a solution to $R_\varphi(x) = i\varphi$, then

$$-\|x\|^2\varphi = x \cdot x \cdot \varphi = x \cdot (i\varphi) = -\varphi.$$

It implies $\|x\| = 1$. Thus, there is only one solution to the equation $R_\varphi(x) = i\varphi$ and this solution has unit norm. Denote it by $y = y_\varphi$ and let the action of a vector $y = \sum_i y^i e_i \in \mathbb{R}^5$ on Δ be denoted by L_y. Then L_y is represented by a matrix

$$\begin{pmatrix} iy^5 & 0 & -iy^3 + y^4 & iy^1 - y^2 \\ 0 & iy^5 & iy^1 + y^2 & iy^3 + y^4 \\ -iy^3 - y^4 & iy^1 - y^2 & -iy^5 & 0 \\ iy^1 + y^2 & iy^3 - y^4 & 0 & -iy^5 \end{pmatrix}.$$

It is not hard to check that there exists $\tilde{\varphi}$ orthogonal and $\mathbb{C}$–linearly independent with φ such that $y \cdot \tilde{\varphi} = i\tilde{\varphi}$.

We show that V_φ equals $\langle \varphi, \tilde{\varphi} \rangle_{\mathbb{C}}^\perp$. For x orthogonal to y, we have

$$(x \cdot \varphi, \psi) = (y \cdot x \cdot \varphi, y \cdot \psi) = -(x \cdot y \cdot \varphi, y \cdot \psi) = -(x \cdot \varphi, \psi), \quad \psi \in \{\varphi, \tilde{\varphi}\}.$$

This implies $(x \cdot \varphi, \tilde{\varphi}) = 0$ and $(x \cdot \varphi, \varphi) = 0$ for any x orthogonal to y. Put

$$V_\varphi = \{x \cdot \varphi \mid \langle x, y \rangle = 0\}. \tag{1}$$

By the above V_φ is orthogonal with respect to $(\cdot, \cdot)$ to the complex space spanned by φ and $\tilde{\varphi}$. Thus, V_φ is complex. By dimensional reasons V_φ is the unique complex 2–dimensional subspace in W_φ. Moreover, $D_\varphi = \{x \in \mathbb{R}^5 \mid \langle x, y \rangle = 0\}$. $\square$

We define an action of skew–forms on Δ as usual

$$\left(\sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} \right) \cdot \varphi = \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} e_1 \cdot \ldots e_k \cdot \varphi.$$

Denote by $\mathfrak{su}(2)_\varphi$ the anihilator of this action on two–forms (for a given φ). It is well known that $\mathfrak{su}(2)_\varphi$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$. Moreover let $\mathbb{R}^4_\varphi$ be the subspace of $\mathfrak{so}(5)$ of 2–forms ω such that $\omega \wedge y^\flat = 0$. In other words $\omega \in \mathbb{R}^4_\varphi$ if $\omega = \alpha \wedge y^\flat$ for some 1–form on D_φ. The action of such forms equals

$$(\alpha \wedge y^\flat) \cdot \varphi = \alpha \cdot (i\varphi) = i\alpha^\sharp \cdot \varphi \in V_\varphi.$$

Since $\dim \mathfrak{su}(2)_\varphi = 3$, it follows that the subspace $\mathfrak{so}(5) \cdot \varphi \subset \Delta$ is 7–dimensional and, clearly, orthogonal to φ. Hence, $\mathfrak{so}(5) \cdot \varphi = \langle \varphi \rangle^\perp$.

1.2. Complex Approach

Definition. We say that a complex 2–dimensional subspace V in Δ is *admissible* if for any $\varphi \in V^{\perp}$ we have $V \subset W_{\varphi}$.

Lemma 1.2. *Fix a (unit) spinor $\varphi \in \Delta$. Then V_{φ} is admissible.*

Proof. By Lemma 1.1 $V_{\varphi} = V_{\tilde{\varphi}}$, where $\tilde{\varphi}$ is as in the proof of Lemma 1.1. Take any linear combination ψ of φ and $\tilde{\varphi}$. Then, $y_{\psi} = y_{\varphi}$, hence $D_{\psi} = D_{\varphi}$. Moreover, for x orthogonal to y_{ψ} we have

$$x \cdot \psi = ax \cdot \varphi + bx \cdot \tilde{\varphi} \in V_{\varphi}$$

for some $a, b \in \mathbb{C}$. Thus $V_{\psi} = V_{\varphi}$. □

We are now going to show that the construction of y_{φ}, as well as D_{φ}, is independent of a unit spinor φ in the orthogonal complement $V^{\perp}$ of an admissible space V.

Lemma 1.3. *Let V be an admissible subspace. Then*

1. *y_{φ} coincide for all $\varphi \in V^{\perp}$,*
2. *D_{φ} coincide for all $\varphi \in V^{\perp}$,*
3. *$V = V_{\varphi}$ for any $\varphi \in V^{\perp}$.*

Proof. Fix $\varphi \in V^{\perp}$. By Lemma 1.1 there is a unique y such that $y \cdot \varphi = i\varphi$ and $W_{\varphi} = V_{\varphi} \oplus \langle i\varphi \rangle$, where $V_{\varphi} = \{x \cdot \varphi \mid \langle x, y \rangle = 0\}$. Thus $D_{\varphi} = \langle y \rangle^{\perp}$. Moreover, there is $\tilde{\varphi}$ which is $\mathbb{C}$–linearly independent with φ and such that $y \cdot \tilde{\varphi} = i\tilde{\varphi}$ and $\Delta = V_{\varphi} \oplus \langle \varphi, \tilde{\varphi} \rangle_{\mathbb{C}}$. Since V is maximal complex in W_{φ}, by admissibility we have $V_{\varphi} = V$. This proves the third condition.

Now, take any $\psi \in V^{\perp}$. Then, $\psi = a\varphi + b\tilde{\varphi}$ for some $a, b \in \mathbb{C}$. Thus $y \cdot \psi = i\psi$, which implies $D_{\psi} = D_{\varphi}$, what proves the first and the second condition. □

Theorem 1.4. *Assume V is admissible. Then $\mathfrak{su}(2)_{\varphi}$ coincide for all $\varphi \in V^{\perp}$. Conversely, for a maximal space U such that all $\mathfrak{su}(2)_{\varphi}$ coincide for $\varphi \in U$ the orthogonal complement $U^{\perp}$ is admissible.*

Proof. Assume V is admissible. Take an orthonormal $\mathbb{C}$–basis $\{\varphi, \tilde{\varphi}\}$ of $V^{\perp}$. By Lemma 1.3 $D_{\varphi} = D_{\tilde{\varphi}}$. We will write just D. Choose an orthonormal basis (e_j) of D and let (u_j) be a basis of D such that $e_j \cdot \varphi = u_j \cdot \tilde{\varphi}$. Then

$$\langle u_j, u_k \rangle = \langle u_j \cdot \tilde{\varphi}, u_k \cdot \tilde{\varphi} \rangle = \langle e_j \cdot \varphi, e_k \cdot \varphi \rangle = \langle e_j, e_k \rangle.$$

Hence (u_j) is also orthonormal. Moreover,

$$0 = \langle \varphi, \tilde{\varphi} \rangle = \langle e_j \cdot \varphi, e_j \cdot \tilde{\varphi} \rangle = \langle u_j \cdot \tilde{\varphi}, e_j \cdot \tilde{\varphi} \rangle = \langle e_j, u_j \rangle.$$

Fixing e_1 and the corresponding u_1, we may take $e_2 = u_1$. Hence, $u_1 \cdot \varphi = u_2 \cdot \tilde{\varphi}$. We have

$$\begin{aligned} \langle e_1, u_2 \rangle &= \langle e_1 \cdot \varphi, u_2 \cdot \varphi \rangle = \langle u_1 \cdot \tilde{\varphi}, u_2 \cdot \varphi \rangle \\ &= -\langle u_2 \cdot \tilde{\varphi}, u_1 \cdot \varphi \rangle = -\langle u_1 \cdot \varphi, u_1 \cdot \varphi \rangle = -1. \end{aligned}$$

Since both e_1 and u_2 are unit, it follows that $u_2 = -e_1$. In particular, $\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{u_1, u_2\}$. Analogously, we set $e_4 = u_3$ and we obtain $u_4 = -e_3$. Consider the following 2–forms

$$\omega_0 = e_1 \wedge u_1 - e_3 \wedge u_3, \quad \omega_1 = e_1 \wedge e_3 + u_1 \wedge u_3, \quad \omega_2 = e_1 \wedge u_3 - u_1 \wedge e_3. \tag{2}$$

It is not hard to check that $\omega_j \cdot \varphi = \omega_j \cdot \tilde{\varphi} = 0$ and ω_j are linearly independent. In particular $\omega_j \cdot \psi = 0$ for any $\mathbb{C}$–linear combination of φ and $\tilde{\varphi}$. Hence, all $\mathfrak{su}(2)_\varphi$ for $\varphi \in V^\perp$ coincide.

Conversely, let $\omega_0, \omega_1, \omega_2$ be 2–forms in $\mathbb{R}^5$ defining a Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$ and let U be the subspace of these $\varphi \in \Delta$ such that $\omega_j \cdot \varphi = 0$. Clearly, U is complex.

Fix $\varphi \in U$ and denote by V the orthogonal complement of U, i.e., $V = U^\perp$. By Lemma 1.1, V_φ is complex 2–dimensional and orthogonal to φ. By Lemma 1.3, V_φ is admissible. Since $\tilde{\varphi} \in V_\varphi^\perp$, where $\tilde{\varphi}$ is as above, by the first part $\mathfrak{su}(2)_{\tilde{\varphi}} = \mathfrak{su}(2)_\varphi = \mathfrak{g}$, i.e., $\omega_j \cdot \tilde{\varphi} = 0$. Thus $\tilde{\varphi} \in U$. We have shown that $V_\varphi^\perp \subset U$. In other words, $V \subset V_\varphi \subset W_\varphi$. It suffices to show that $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V^\perp = 2$. Suppose $\dim_{\mathbb{C}} V = 1$ and let u be a unit vector in $\mathbb{R}^5$ orthogonal to y_φ and such that $\psi = u \cdot \varphi$ is orthogonal to V. Then $\psi \in V^\perp$, hence $\mathfrak{su}(2)_\psi = \mathfrak{g}$. Therefore

$$0 = u \cdot \omega_j \cdot \varphi = 2(u \lrcorner \omega_j) \cdot \varphi + \omega_j \cdot \psi = 2(u \lrcorner \omega_j) \cdot \varphi. \tag{3}$$

For $x \in D_\varphi$, by the fact that V_φ is a complex subspace, we have

$$(x \wedge y) \cdot \varphi = ix \cdot \varphi \in V_\varphi.$$

This implies $\mathbb{R}^4_\varphi \cdot \varphi = V_\varphi$. Since $\mathfrak{so}(5) \cdot \varphi = \langle \varphi \rangle^\perp$, it follows that $\mathfrak{g} \subset \mathfrak{so}(D_\varphi)$. By (3), we see that in fact $\mathfrak{g} \subset \mathfrak{so}(3)$, where $\mathfrak{so}(3)$ is taken with respect to the 3–dimensional subspace of D_φ orthogonal to $u \in D_\varphi$. Thus $\mathfrak{g} = \mathfrak{so}(3)$. This is impossible, since $\mathfrak{so}(3)$ contains pure elements $\omega = \alpha \wedge \beta$ and $\omega \cdot \varphi$ cannot vanish. Finally, $\dim_{\mathbb{C}} V = 2$. $\square$

Notice that by Theorem 1.4 we may write $\mathfrak{su}(2)_V$ for the Lie algebra induced by any spinor $\varphi \in V^\perp$, where V is an admissible space i.e., $\mathfrak{su}(2)_V = \mathfrak{su}(2)_\varphi$ for any $\varphi \in V^\perp$. Moreover, the subspace $\mathbb{R}^4_\varphi \subset \mathfrak{so}(5)$ is also independent of the choice of φ in the orthogonal complement of an admissible space V. Hence we may denote it by $\mathbb{R}^4_V$.

Let us now describe the decomposition of $\mathfrak{so}(5)$ into irreducible $\mathfrak{su}(2)$–modules. Fix an admissible space V. The orthogonal complement $V^\perp$ is again an admissible space (as will be seen below). Therefore we introduce the $\pm$–notation: $V^- = V$ and $V^+ = V^\perp$. Denote by $\mathfrak{su}(2)_-$ the Lie algebra corresponding to V^-, which is isomorphic to $\mathfrak{su}(2)$. Analogously, we write $\mathbb{R}^4_-$ instead of $\mathbb{R}^4_{V^-}$.

Lemma 1.5. *The following holds:*

1. *The orthogonal complement of an admissible space is again admissible.*

2. Let V^- be an admissible space. Denoting by $\mathfrak{su}(2)_+$ the Lie algebra corresponding to V^+ (isomorphic to $\mathfrak{su}(2)$) we have

$$\Delta = V^- \oplus V^+, \quad \mathfrak{so}(5) = \mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+ \oplus \mathbb{R}^4_-.$$

In particular, $\mathbb{R}^4_- = \mathbb{R}^4_+$.

Proof. Let V^- be admissible and choose $\psi \in V^-$. Choose any $\varphi \in V^+$. By Lemma 1.3 $V^- = V_\varphi$, hence, there is $x_0 \in \mathbb{R}^5$ such that $\psi = x_0 \cdot \varphi$. Therefore, $x_0 \cdot \psi = -\|x_0\|^2\varphi$, i.e., $\varphi \in W_\psi$. We have shown that $V^+ \subset W_\psi$, thus V^+ is admissible.

By Lemma 1.1, x_0 is orthogonal to y, where $y \cdot \varphi = i\varphi$. Thus

$$y \cdot \psi = y \cdot x_0 \cdot \varphi = -x_0 \cdot y \cdot \varphi = -i(x_0 \cdot \varphi) = -i\psi.$$

Hence $y_\psi = -y$. In particular, $D_\psi = D_\varphi$ for any $\varphi \in V^+$. By the proof of Theorem 1.4 we have $\mathfrak{su}(2)_\psi \subset \mathfrak{so}(D_\psi)$. For $\omega \in \mathfrak{su}(2)_\varphi$ we have

$$\omega \cdot \psi = \omega \cdot x_0 \cdot \varphi = 2(x_0 \lrcorner \omega) \cdot \varphi + x_0 \cdot \omega \cdot \varphi = 2(x_0 \lrcorner \omega) \cdot \varphi.$$

The right hand side vanishes only if $x_0 \lrcorner \omega = 0$. Choose a basis (e_j) in D_φ such that $e_1 = \frac{1}{\|x_0\|} x_0$ and a basis (u_j) as in the proof of Theorem 1.4. Then ω is a linear combination of $\omega_0, \omega_1, \omega_2$ given by (2), say $\omega = \sum_j a_j \omega_j$. If $\omega \neq 0$, then

$$x_0 \lrcorner \omega = a_0 u_1 + a_1 e_3 + a_2 u_3 \neq 0.$$

Thus $\omega \cdot \varphi \neq 0$. Therefore, $\mathfrak{su}(2)_\psi$ is transversal to $\mathfrak{su}(2)_\varphi$. This completes the proof. □

Corollary 1.6. *Let V^- be an admissible space. Then the following actions are surjective*

$$\mathfrak{su}(2)_- : V^+ \to V^+, \quad \mathfrak{su}(2)_+ : V^- \to V^-.$$

Proof. Follows from the fact that for a given spinor $\varphi \in V^+$ we have $\mathfrak{su}(2)_+ \cdot \varphi = V^+ \cap \langle \varphi \rangle^\perp$. □

Remark 1.7. From the considerations above we have a useful observation concerning the Clifford action. Namely, let V^- be admissible and let $\varphi \in (V^-)^\perp = V^+$ be unit. Then the right multiplication $R_\varphi : \mathfrak{so}(5) \to \Delta$ by φ satisfies the following restrictions

$$R_\varphi : \mathbb{R}^4_- \to V^+,$$
$$R_\varphi : \mathfrak{su}(2)_+ \to \langle \varphi \rangle^\perp \cap V^-,$$

which are isomorphisms.

Example 1.8 (The fundamental example). Denote by $s_1, \ldots, s_4$ the canonical $\mathbb{C}$–basis in $\Delta = \mathbb{C}^4$. Fix a spinor $\varphi = s_1$. Then

$$e_1\varphi = is_4, \quad e_2\varphi = s_4, \quad e_3\varphi = -is_3, \quad e_4\varphi = -s_3, \quad e_5\varphi = is_1.$$

Hence $V^- = \langle s_3, s_4 \rangle_\mathbb{C}$ and $V^+ = \langle s_1, s_2 \rangle_\mathbb{C}$. Indeed,

$$W_\varphi = \langle s_3, s_4 \rangle_\mathbb{C} \oplus \{i\varphi\}.$$

Now it suffices to apply Lemma 1.1. Moreover,

$$\begin{aligned}
&e_{12}\varphi = is_1, \quad e_{13}\varphi = s_2, \quad e_{14}\varphi = -is_2, \quad e_{15}\varphi = -s_4,\\
&e_{23}\varphi = -is_2, \quad e_{24}\varphi = -s_2, \quad e_{25}\varphi = is_4,\\
&e_{34}\varphi = is_1, \quad e_{35}\varphi = s_3,\\
&e_{45}\varphi = -is_3.
\end{aligned}$$

Here and further, e_{jk} denotes the two–form $e_j \wedge e_k$. Now, it is easy to see that the equation $\omega \cdot \varphi = 0$ is satisfied by the following 2–forms

$$\tilde{\omega}_1 = e_{12} - e_{34}, \quad \tilde{\omega}_2 = e_{13} + e_{24}, \quad \tilde{\omega}_3 = e_{14} - e_{23},$$

which define $\mathfrak{su}(2)_-$. Hence, $(\mathfrak{su}(2)_-)^\perp \subset \mathfrak{so}(5)$ is spanned by $\mathfrak{su}(2)_+$ generated by the elements

$$\omega_1 = e_{12} + e_{34}, \quad \omega_2 = e_{13} - e_{24}, \quad \omega_3 = e_{14} + e_{23},$$

and $\mathbb{R}^4_-$ (see the definition after proof of Theorem 1.4) is generated by the elements

$$e_{15}, \quad e_{25}, \quad e_{35}, \quad e_{45}.$$

Notice that for the spinor s_2 we have

$$e_1 \cdot s_2 = is_3, \quad e_2 \cdot s_2 = -s_3, \quad e_3 \cdot s_2 = is_4, \quad e_4 \cdot s_2 = -s_4, \quad e_5 \cdot s_2 = is_2,$$

which confirms that V^- and V^+ are exactly as stated above. Moreover, we have the following relations

$$e_1 \cdot s_1 = e_3 \cdot s_2, \quad e_2 \cdot s_1 = -e_4 \cdot s_2, \quad e_3 \cdot s_1 = -e_1 \cdot s_2, \quad e_4 \cdot s_1 = e_2 \cdot s_2. \tag{4}$$

1.3. Complex Structures

Firstly, we will show nonexistence of complex structures on Δ satisfying certain relations.

Assuming $j : \Delta \to \Delta$ is a complex structure on a real vector space Δ, there is a complex structure I_φ on D_φ for fixed φ in the orthogonal complement of the admissible space V_φ (compare [4,6]). Namely,

$$I_\varphi(x) \cdot \varphi = j(x \cdot \varphi), \quad x \in D_\varphi. \tag{5}$$

The definition of I_φ depends on φ. Moreover, in the definition (5) we only need the values of j on V_φ, since for any $x \in D_\varphi$ we have $x : V_\varphi^\perp \to V_\varphi$.

We want to find all complex structures j on V such that the induced complex structure I_φ is independent of the choice of $\varphi \in V^\perp$. We will prove below that there is no such complex structure. Even more, let $t : V \to V$ be an $\mathbb{R}$–linear map and define the induced linear map $T^\varphi : D_\varphi \to D_\varphi$, for $\varphi \in V^\perp$, analogously as in (5):

$$T_\varphi(x) \cdot \varphi = t(x \cdot \varphi), \quad x \in D_\varphi. \tag{6}$$

Theorem 1.9. *The only linear map $t : V \to V$ such that the induced linear map T_φ is independent of the choice of $\varphi \in V^\perp$ is a scalar multiple of the identity map.*

Proof. Without loss of generality we may take V to be $\langle s_3, s_4\rangle_{\mathbb{C}}$. Assume t induces a map T, which does not depend on the choice of the spinor in $V^\perp = \langle s_1, s_2\rangle_{\mathbb{C}}$. Then $D = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$. Let $T(e_1) = (a, b, c, d)$. Then

$$t(s_3) = -(a, b, c, d) \cdot (is_2), \qquad t(is_3) = (a, b, c, d) \cdot s_2,$$
$$t(s_4) = -(a, b, c, d) \cdot (is_1), \qquad t(is_4) = (a, b, c, d) \cdot s_1.$$

This implies that the matrix of $t : V \to V$ with respect to the basis s_3, is_3, s_4, is_4 is of the form

$$t = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}.$$

From this we conclude that T is represented by the same matrix with respect to the basis e_1, e_2, e_3, e_4.

Let us study the independence from φ. For $x, y \in D_\varphi$ and $\varphi, \psi \in V^\perp$ such that $x \cdot \varphi = y \cdot \psi$ it must hold

$$T(x) \cdot \varphi = T(y) \cdot \psi.$$

Substituting relations (4) we obtain $b = d = 0$. Moreover, for $\varphi \in V^\perp$ and $x \in D_\varphi$ there is y (depending on φ) such that $y \cdot \varphi = x \cdot i\varphi$. Considering this condition we conclude, as above, that $c = 0$. Hence, $t = a \cdot \mathrm{id}_V$. Finally, it is clear that such t satisfies the assumptions of the theorem. □

Corollary 1.10. *There is no complex structure I on D_φ induced from the complex structure j on V by the formula (5), which does not dependent on the choice of $\varphi \in V^\perp$.*

Secondly, we give a natural procedure on how to define an associated quaternionic structure on an $SU(2)$–structure via spinorial approach. We have already shown that an approach similar to the one considered in [4] (compare [6]) is not valid. Nevertheless, these is a nice description of the complex structures on D_φ with the spinorial (invariant) approach. The intuition has been already used in the proof of Theorem 1.9.

Let V be an admissible space. We know that all D_φ coincide for $\varphi \in V^\perp$. Denote this space by D. Fix $\varphi \in V^\perp$ and $x \in D$. Then there exists a unique element $J^\varphi(x) \in D$ such that

$$J^\varphi(x) \cdot \varphi = x \cdot (i\varphi) = i(x \cdot \varphi).$$

The second equality follows from the fact that the action of $x \in D$ on Δ is $\mathbb{C}$–linear.

Notice that the map $J^\varphi : D \to D$ is a complex structure,

$$J^\varphi(J^\varphi(x)) \cdot \varphi = i(J^\varphi(x) \cdot \varphi) = -x \cdot \varphi.$$

Hence $(J^\varphi)^2 = -\mathrm{id}_D$.

Moreover, we have $J^{\lambda\varphi} = J^\varphi$ for any complex number $\lambda \neq 0$. Thus we have the correspondence

$$\{\varphi \in V^\perp : \|\varphi\| = 1\} \mapsto \{\text{complex str. in D}\}, \quad \varphi \mapsto J^\varphi. \tag{7}$$

Proposition 1.11. *The correspondence (7) is in fact the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. In particular, the image $\{J^\varphi : \varphi \in D\}$ is a 2–sphere which defines a quaternionic structure on D.*

Proof. Without loss of generality we may take $V = \langle s_3, s_4\rangle_{\mathbb{C}}$. Then $V^\perp = \langle s_1, s_2\rangle_{\mathbb{C}}$. Any spinor $\varphi \in V^\perp$ equals $\varphi = as_1 + bis_1 + cs_2 + dis_2$ for some $a, b, c, d \in \mathbb{R}$. Moreover, let $J^\varphi(e_1) = \sum_j x_j e_j$. Substituting e_1 for x in the definition of J^φ we get

$$\begin{pmatrix} -d & -c & b & -a \\ c & -d & -a & -b \\ -b & a & -d & -c \\ a & b & c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -c \\ -d \\ -a \\ -b \end{pmatrix}$$

This implies $x_1 = 0$, $x_2 = -(a^2 + b^2 - c^2 - d^2)$, $x_3 = 2(ad - bc)$, $x_4 = 2(ac + db)$. We proceed in a similar way taking $x = e_2, e_3, e_4$, respectively. Finally, denoting

$$\alpha = a^2 + b^2 - c^2 - d^2, \quad \beta = 2(ad - bc), \quad \gamma = 2(ac + bd),$$

we obtain

$$J^\varphi = \begin{pmatrix} 0 & \alpha & -\beta & -\gamma \\ -\alpha & 0 & -\gamma & \beta \\ \beta & \gamma & 0 & \alpha \\ \gamma & -\beta & -\alpha & 0 \end{pmatrix}.$$

Notice that $\alpha^2 + \beta^2 + \gamma^2 = 1$ and that the map $(a, b, c, d) \mapsto (\alpha, \beta, \gamma)$ is the Hopf fibration of $\mathbb{S}^3$ onto $\mathbb{S}^2$.

Let us end by showing that the 2–sphere of complex structures defines a quaternionic structure on D. Denote by $(\alpha, \beta, \gamma) \in \mathbb{S}^2$ a point on the 2–sphere inducing the complex structure $J = J(\alpha, \beta, \gamma)$ from the image of the map (7). Then,

$$J(\alpha, \beta, \gamma)J(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = -J(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})J(\alpha, \beta, \gamma)$$

if and only if the vectors (α, β, γ) and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ are orthogonal (with respect to the standard inner product in $\mathbb{R}^3$). Now, it suffices to take two orthogonal vectors $u, v \in \mathbb{S}^3$ and define

$$J_1 = J(u), \quad J_2 = J(v), \quad J_3 = J_1 J_2.$$

It is easy to see that the triple (J_1, J_2, J_3) is a quaternionic structure. $\square$

1.4. Quaternionic Approach

Let us rewrite some of the results from subsection 1.1 with the quaternionic approach.

Let us begin with a choice of a quaternionic structure on Δ. The choice is not unique. We begin with the one considered in [6]. It can be shown [11] that in Δ there is a quaternionic structure $i_2 : \Delta \to \Delta$ which anticommutes with multiplication by vectors. Recall that a quaternionic structure j may be seen as an antilinear map such that $j^2 = -\mathrm{id}$. Let i_1 be the complex structure on $\Delta = \mathbb{C}^4$ induced by the volume element $\mathrm{vol} = e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot e_5$. It is easy to see that vol induces the standard complex structure given by the multiplication by i, which clearly commutes with the multiplication by vectors. Define $i_3 = i_1 \circ i_2$. i_3 anticommutes with multiplication. Then we have a triple (i_1, i_2, i_3) of complex structures on Δ. Each i_k is an isometry [6].

We will need the following useful fact.

Lemma 1.12 ([6]). *Fix a unit spinor $\varphi \in \Delta$. Then the subspace U generated by $\varphi, i_1\varphi, i_2\varphi, i_3\varphi$ and its orthogonal complement $V = U^\perp$ are i_k–invariant, $k = 1, 2, 3$. Moreover, the subspace $D \subset \mathbb{R}^5$ such that $D \cdot \varphi = V$ inherits a quaternionic structure induced by the complex structures I_k, $k = 1, 2, 3$, defined by $I_k(x) \cdot \varphi = i_k(x \cdot \varphi)$. In particular, each i_k leaves the decomposition $\Delta = V \oplus V^\perp$ invariant.*

This allows us to prove the main result of this subsection.

Theorem 1.13. *A real 4–dimensional subspace $V \subset \Delta$ is admissible if and only if it is a quaternionic subspace with respect to (i_1, i_2, i_3).*

Proof. Firstly, assume V is admissible. Choose any $\varphi \in V^\perp$. Then $V = V_\varphi$. By Lemma 1.12, it suffices to show that $V^\perp = \langle \varphi, i_1\varphi, i_2\varphi, i_3\varphi \rangle$. Since i_1 is the multiplication by i and $i_3\varphi = ii_2\varphi$ we need to show that $i_2\varphi$ is orthogonal to V. Any element in V is of the form $x \cdot \varphi$. We may assume x is unit. Hence

$$\langle i_2\varphi, x \cdot \varphi \rangle = -\langle \varphi, i_2(x \cdot \varphi) \rangle = \langle \varphi, x \cdot i_2\varphi \rangle = -\langle x \cdot \varphi, i_2\varphi \rangle.$$

Thus $\langle i_2\varphi, x \cdot \varphi \rangle = 0$.

Conversely, assume V is quaternionic. Thus its orthogonal complement $V^\perp$ is also quaternonic. Take $\varphi \in V^\perp$. By Lemma 1.3 it suffices to show that $V = V_\varphi$.

We have $V^\perp = \langle \varphi, i_1\varphi, i_2\varphi, i_3\varphi \rangle$. By Lemma 1.12 there is a subspace $D \subset \mathbb{R}^5$ such that $D \cdot \varphi = V$. Since V is complex, by Lemma 1.1, $V = V_\varphi$. □

For our approach, to make the description at least partially invariant, we need to modify the quaternionic structure (i_1, i_2, i_3) a little bit. The modification depends on the choice of an admissible space V. Let $j_1 = i_1$ and we define j_2 as follows

$$j_2 = i_2 \quad \text{on } V, \quad j_2 = -i_2 \quad \text{on } V^\perp.$$

Finally, let $j_3 = j_1 j_2$. Then (j_1, j_2, j_3) equals (i_1, i_2, i_3) on V and $(i_1, -i_2, -i_3)$ on $V^\perp$. The triple (j_1, j_2, j_3) is in fact a quaternionic structure (since V and $V^\perp$ are invariant with respect to (i_1, i_2, i_3)). Moreover,

$$j_k(x \cdot \varphi) = x \cdot j_k(\varphi), \quad \varphi \in \Delta.$$

Hence, all complex structures j_1, j_2 and j_3 commute with multiplication by vectors.

Now, we move to a description of a quaternionic structure on D. As discussed in the previous subsection take

$$J_1 = J(1,0,0), \quad J_2 = J(0,1,0), \quad J_3 = (0,0,-1).$$

Then $J_3 = J_1 J_2$ and $J_k J_l = -J_l J_k$ for distinct k, l. There are unit spinors (not unique) $\varphi_1, \varphi_2, \varphi_3 \in V^\perp$ such that

$$x \cdot i\varphi_k = J_k(x) \cdot \varphi_k.$$

Moreover, define three 2–forms ω_k by

$$\omega_k(x, y) = \langle J_k(x), y \rangle, \quad x, y \in D.$$

Then

$$\begin{aligned} x \cdot \omega_k \cdot \varphi_k &= \omega_k \cdot x \cdot \varphi_k + 2(x \lrcorner \omega_k) \cdot \varphi_k \\ &= 2(x \lrcorner \omega_k) \cdot \varphi_k = 2J_k(x) \cdot \varphi_k = x \cdot (i\varphi_k), \end{aligned}$$

which implies

$$\omega_k \cdot \varphi_k = 2i\varphi_k.$$

This relation shows that ω_k belongs to the Lie algebra dual to $\mathfrak{su}(2)$.

Remark 1.14. If $V = \langle s_3, s_4 \rangle_{\mathbb{C}}$, then we may take, for example,

$$\varphi_1 = s_1, \quad \varphi_2 = \frac{1}{\sqrt{2}}(s_1 + is_2), \quad \varphi_3 = \frac{1}{\sqrt{2}}(s_1 - s_2).$$

Moreover, the 2–forms ω_k are

$$\omega_1 = e_{12} + e_{34}, \quad \omega_2 = -e_{13} + e_{24}, \quad \omega_3 = e_{14} + e_{23}$$

and the corresponding complex structures J_1, J_2, J_3 are given by the following matrices

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We would like to add that a construction of a quaternionic structure from the given data was studied in [7] where the authors use the approach from [9]. Obtained (almost) complex structures agree with our approach.

1.5. Conjugacy Classes

In this subsection we find a condition for admissible spaces, to induce conjugate Lie algebras (equivalently, groups) isomorphic to $\mathfrak{su}(2)$. More precisely, we deal with the problem when $\mathfrak{su}(2)_V$ and $\mathfrak{su}(2)_{V'}$ are conjugate for V, V' admissible. Recall that $\mathfrak{su}(2)_V$ is the Lie algebra of all ω such that $\omega \cdot \varphi = 0$ for any $\varphi \in V^\perp$.

Lemma 1.15. *Assume $V \subset \Delta$ is an admissible space and let $g \in \mathrm{Spin}(5)$. Then gV is also admissible.*

Proof. Since g acts as a complex linear map, it follows that the space gV is complex. Moreover, let $\psi \in (gV)^\perp$ and let $\varphi \in V$. By the invariance of the Hermitian product, we see that $g^{-1}\psi \in V^\perp$. Since V is admissible, it follows that $V \subset W_{g^{-1}\psi}$. Thus $\varphi = x \cdot (g^{-1}\psi)$ for some $x \in \mathbb{R}^5$. Hence

$$g\varphi = (\mathrm{Ad}(g)x) \cdot \psi.$$

Since $\mathrm{Ad}(g)x$ is a vector in $\mathbb{R}^5$, we have $g\varphi \in W_\psi$. This proves admissibility of gV. □

Lemma 1.16. *The isotropy group of a fixed element of the action of* $\mathrm{Spin}(5)$ *on admissible subspaces is isomorphic to* $\mathrm{Spin}(4)$*.*

Proof. It suffices to take $\varphi = s_1$. Then $V = \langle s_3, s_4\rangle_{\mathbb{C}}$. It is easy to see that $gV = V$ if and only if $g \in \mathrm{Spin}(4)$ where we consider the spin group with respect to the first component in the decomposition $\mathbb{R}^5 = \mathbb{R}^4 \oplus \mathbb{R}$. □

Lemmas 1.15 and 1.16 imply the following interpretation of Spin(5) in terms of Grassmanians.

Corollary 1.17. $\mathrm{Spin}(5)$ *acts transitively on the space of admissible subspaces.*

Proof. By Theorem 1.13 any admissible space is of the form $V = \langle \psi, i_1\psi, i_2\psi, i_3\psi\rangle$ for some spinor ψ. Choose another admissible space V' induced in this way, by a spinor φ. Since the action of Spin(5) on Δ is transitive, there is $g \in \mathrm{Spin}(5)$ such that $g \cdot \varphi = \psi$. We know that i_1 acts by multiplication, i_2 commutes with the action of g, whereas i_3 is the composition of i_1 and i_3. Therefore

$$g \cdot j\varphi = jg\varphi = j\psi, \quad j \in \{\mathrm{id}, i_1, i_2, i_3\}.$$

Hence $gV = V'$. □

To justify, in a sense, the above fact let us count appropriate dimensions. We have $\dim \mathrm{Spin}(5) = 10$ and $\dim \mathrm{Spin}(4) = 6$. Since the space of all possible admissible subspaces is, by Theorem 1.13, the quaternionic Grassmanian $\mathrm{Gr}_1(\mathbb{H}^2)$, its dimension is $\dim \mathrm{Gr}_1(\mathbb{H}^2) = 4$.

Proposition 1.18. *Let $g \in \mathrm{Spin}(5)$. We have*

$$\mathrm{Ad}(g)\mathfrak{su}(2)_V = \mathfrak{su}(2)_{g^{-1}V}.$$

Proof. Let $\omega \in \mathfrak{su}(2)_V$, i.e., $\omega \cdot \varphi = 0$ for all $\varphi \in V^\perp$. Thus $0 = (\mathrm{Ad}(g)\omega)\cdot(g\varphi)$, which implies $\mathrm{Ad}(g)\mathfrak{su}(2)_V = \mathfrak{su}(2)_{g\varphi}$. Since $g\varphi \in (gV)^\perp$, by admissibility of gV, proposition follows. $\square$

Corollary 1.19. *Let $g \in \mathrm{Spin}(5)$. Then $\mathrm{Ad}(g)\mathfrak{su}(2)_V = \mathfrak{su}(2)_V$, for V admissible, if and only if $gV = V$. In particular, the stabilizer of $\mathfrak{su}(2)_V$ with respect to the adjoint action of* $\mathrm{Spin}(5)$ *is isomorphic to* $\mathrm{Spin}(4)$.

Let us compare the above considerations with a quaternionic approach. Choose a quaternionic structure (j_1, j_2, j_3) on Δ in the way such that each j_k commutes or anticommutes with the multiplication by vectors (see subsection 1.7). Consider the natural action of $\mathbb{H}$ on Δ: for $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ and $\varphi \in \Delta$ let

$$a \cdot \varphi = a_0\varphi + a_1 j_1\varphi + a_2 j_2\varphi + a_3 j_3\varphi.$$

This action commutes with the action of Spin(5). Moreover, by Theorem 1.13 each admissible space is of the form $V_\varphi = \{a \cdot \varphi \mid a \in \mathbb{H}\}$ for each $\varphi \in \Delta$. These arguments give another proof of Lemma 1.16.

The quotient space $\Delta/\mathbb{H}$ of this action, which is isomorphic to $\mathbb{R}^4$, is the space of all admissible spaces. In addition, for fixed φ and any $\psi \in V_\varphi^\perp$ the action of $\mathbb{H}$ on φ and ψ spans Δ.

2. Invariant Description of $SU(2)$–Structures

Let (M, g) be a spin 5–manifold with the corresponding Riemannian structure g. Denote by $\mathrm{Spin}(M)$ the spinor structure (with the structure group Spin(5)) and let $\mathbf{S}$ be the associated spinor bundle, $\mathbf{S} = \mathrm{Spin}(M) \times_{\mathrm{Spin}(5)} \Delta$, where $\Delta = \mathbb{C}^4$ is as in the first section. An $SU(2)$–structure on M is a reduction P of the frame bundle $SO(M)$ to the structure group $SU(2) \subset SO(5)$. We can extend P to $P_{\mathrm{Spin}(5)} = P \times_{SU(2)} \mathrm{Spin}(5)$. Alternatively, as shown by Conti and Salamon [9], an $SU(2)$–structure is given by a quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ consisting of a 1–form α and 2–forms $\omega_1, \omega_2, \omega_3$ such that

$$\omega_k \wedge \omega_l = \delta_{kl} v \quad \text{and} \quad (X \lrcorner \omega_1 = Y \lrcorner \omega_2 \quad \Rightarrow \quad \omega_3(X, Y) > 0)$$

for some 4–form v satisfying $\alpha \wedge v \neq 0$. The third approach is the following [9]. Fix a unit spinor field φ. Then we define a subundle P as the set of all frames u such that $\varphi(x) = [u, \varphi_0]$, where u is a frame over $x \in M$ and $\varphi_0 \in \Delta$ is a fixed unit spinor. Then P is an $SU(2)$ structure with $SU(2) = \mathrm{Stab}(\varphi_0)$.

Motivated by the final approach and the discussions in the first section we may consider the following approach to $SU(2)$–structures: Fix an admissible space $V \subset \Delta$, denote the corresponding Lie algebra by $\mathfrak{su}(2)_V$ and its Lie group by $SU(2)_V$. In other words, $\mathfrak{su}(2)_V = \mathfrak{su}(2)_{\varphi_0}$ for any $\varphi_0 \in V^\perp$ (see Theorem 1.4). If P is an $SU(2)_V$–structure we may define the space (of real dimension 4) of certain spinor fields

$$\mathbf{S}_P = \{\varphi \in \mathbf{S} \mid \text{ there exists } \varphi_0 \in V^\perp \text{ such that } \varphi = [u, \varphi_0] \text{ for any } u \in P\}.$$

Since spinors from $V^\perp$ are the fixed points of the action of $SU(2)_V$, it follows that, in fact, $\mathbf{S}$ is a real 4–dimensional subspace. Directly from the definition we see that $\mathbf{S}_P$ is isomorphic to $V^\perp$ (and V). Any spinor field in $\mathbf{S}_P$ is said to *induce* given $SU(2)$–structure P.

There is a natural subbundle in the spinor bundle $\mathbf{S}$ over the $SU(2)$–structure.

Definition. Assume P is an $SU(2)$–structure on a spin manifold M. We say that a subbundle $\mathbf{V}$ in the spinor bundle $\mathbf{S}$ is *adapted* to P if it is of the form $\mathbf{V} = P \times_{SU(2)} V$, where V is an admissible space in Δ such that $SU(2) = SU(2)_V$.

Consider the almost complex structure on $\mathbf{S}$ induced by j_1 on Δ (see also discussion on a quaternionic structure below). From the definition it follows that the adapted subbundle is complex 2–dimensional.

Notice that two spinor fields φ and ψ, which are sections of $\mathbf{V}^\perp$, i.e., an orthogonal complement of the adapted subbundle, do not in general induce the same $SU(2)$–structure. Indeed, assume φ defines the underlying $SU(2)$–structure P, i.e., $\varphi = [u, \varphi_0]$, $u \in P$, where $SU(2) = \mathrm{Stab}(\varphi_0)$. Then ψ defines the same structure if and only if $\psi = [u, \psi_0]$, $u \in P$, for some ψ_0 in the admissible space V. In other words, φ and ψ must lie in the space $\mathbf{S}_P$ for some P (which they induce).

We may consider the quaternionic structure i_2 on $\mathbf{S}$ induced from the quaternionic structure i_2 on Δ. Therefore, $i_3 = i_1 \circ i_2$, where $i_1 = j_1$, defines an additional almost complex structure on $\mathbf{S}$ and (i_1, i_2, i_3) forms a triple of almost complex structures [6]. By Theorem 1.13 we have the following result.

Corollary 2.1. *An admissible subbundle* $\mathbf{V}$ *is quaternionic with respect to the quaternionic structure* (i_1, i_2, i_3).

If an admissible subbundle $\mathbf{V}$ is fixed, we have, additionally, a quaternionic structure (j_1, j_2, j_3) induced from (i_1, i_2, i_3) on Δ. These two structures differ only by a sign (for i_2 and i_3 on the orthogonal complement $\mathbf{V}^\perp$ of the admissible distribution).

Corollary 2.2. *An admissible distribution is quaternionic with respect to a quaternionic structure* (j_1, j_2, j_3).

It is important to notice that the spinor fields φ, $j_1\varphi$, $j_2\varphi$ and $j_3\varphi$ (equivalently, φ, $i_1\varphi, i_2\varphi, i_3\varphi$) induce the same $SU(2)$–structure. In other words, if $\varphi \in \mathbf{S}_P$, then $j_k\varphi \in \mathbf{S}_P$ for any $k = 1, 2, 3$. Indeed, each complex structure j_k on Δ commutes with the action of $\mathrm{Spin}(5)$, i.e., $j_k(gs) = g(j_k s)$, $g \in \mathrm{Spin}(5)$, $s \in \Delta$. Therefore, if φ is induced by a spinor $\varphi_0 \in \Delta$, then $j_k\varphi$ is induced by $j_k\varphi_0$. Since the quaternionic structure (j_1, j_2, j_3) leaves the admissible space V and its orthogonal complement $V^\perp$ invariant, it follows that $j_k\varphi$ induces the same $SU(2)$–structure as φ. Therefore if φ is fixed, any spinor field ψ defining the same $SU(2)$–structure is given by

$$\psi = a_0\varphi + a_1 j_1\varphi + a_2 j_2\varphi + a_3 j_3\varphi = \sum_{k=0}^{3} a_k j_k \varphi, \quad a_0, a_1, a_2, a_3 \in \mathbb{R},$$

where j_0 denotes the identity. Form these considerations, we also see that

$$\nabla j_k = 0, \quad k = 1, 2, 3,$$

where ∇ is a connection in the spinor bundle induced from the Levi-Civita connection on M. This follows from the fact that each j_k commutes with the action on Spin(5). Let us gather these observations in the proposition below.

Proposition 2.3. *For any two spinor fields φ, ψ defining the same $SU(2)$–structure, i.e., $\varphi, \psi \in \mathbf{S}_P$, there is a quaternion (a_0, a_1, a_2, a_3) such that $\psi = \sum_{k=0}^{3} a_k j_k \varphi$.*

We may describe the above arguments on the level of principal bundles. Namely, fix a group $SU(2) \subset \mathrm{Spin}(5)$. We may assume that $SU(2)$ is induced by a unit spinor $\varphi_0 \in V^\perp$ for some admissible space V. There is a bijection between $SU(2)$ structures on M and sections σ of the associated homogeneous bundle $N = \mathrm{Spin}(M) \times_{\mathrm{Spin}(5)} (\mathrm{Spin}(5)/SU(2))$ – reduction $P \subset \mathrm{Spin}(M)$ to a subgroup $SU(2)$ defines $\sigma_P \in \Gamma(N)$ as follows

$$\sigma_P(x) = [u, e], \quad u \in P,$$

where e is the coset induced by the neutral element. We wish to show that there is a bijection between $\Gamma(N)$ and $\Gamma(\mathbf{V}_1^\perp)/\mathbb{H}_1$, where $\mathbf{V}_1^\perp$ denotes the bundle of unit spinors in $\mathbf{V}^\perp$ and the right action of unit quaternions on sections φ of this bundle is given by conjugation

$$\varphi \cdot a = a_0\varphi - a_1 j_1\varphi - a_2 j_2\varphi - a_3 j_3\varphi, \quad a = (a_0, a_1, a_2, a_3) \in \mathbb{H}_1.$$

The correspondence is the following: Let $\sigma \in \Gamma(N)$. Since $\mathrm{Spin}(5)/SU(2)$ is isomorphic to $\mathbb{S}^7 \subset \Delta$, the isomorphism depending on φ_0, section σ induces a spinor field $\varphi_\sigma \in \Gamma(S)$. It suffices to show that $\varphi_\sigma \in \mathbf{V}_1^\perp$ and changing φ_0 by $\varphi_0 a$, $a \in \mathbb{H}_1$, leads to $\varphi_\sigma a$. Indeed, σ may be treated as an equivariant function $\sigma : \mathrm{Spin}(M) \to \mathrm{Spin}(5)/SU(2)$. Hence, the induced spinor at $x \in M$ is given by $\varphi(x)\sigma(u) \cdot \varphi_0$, $\pi(u) = x$. Thus $\varphi(x)$ belongs to the same admissible space as φ_0. Secondly, substituting φ_0 by $\varphi_0 a$, $a \in \mathbb{H}_1$, we get

$$\sigma(u)(\varphi_0 a) = (\sigma(u)\varphi_0)a = \varphi(x)a.$$

Let us relate how to derive the quadruplet (α, ω_i) defining an $SU(2)$–structure in the sense of [9] from the admissible distribution $\mathbf{V}$. A choice of an $SU(2)$–structure P gives existence of a codimension one distribution $\mathbf{D}$, defined as $\mathbf{D} = P \times_{SU(2)} D$, where the existence of D follows from the first section, and a unit orthogonal vector field ζ called *Reeb*. ζ is induced by a vector $y \in \mathbb{R}^5$ in Lemma 1.1. Fix an admissible distribution $\mathbf{V}$ and consider the induced $SU(2)$–structure P. Denote by $\mathfrak{su}(2)_+$ the Lie algebra dual to $\mathfrak{su}(2)$ (compare the first section for details). Since the adjoint representation of SU(2) on $\mathfrak{su}(2)_+$ is trivial, the bundle

$$\mathfrak{su}_+(M) = P \times_{\mathrm{SU}(2)} \mathfrak{su}(2)_+$$

of 2–forms is trivial. Hence, there are global linearly independent three 2–forms $\omega_1, \omega_2, \omega_3$. Consider, moreover, the quaternionic structure (J_1, J_2, J_3) on the distribution $\mathbf{D}$ described as follows (see subsection 1.7):

$$J_k(X) \cdot \varphi_k = X \cdot j_1 \varphi_k,$$

where φ_k are three $\mathbb{R}$–linearly independent spinor fields in $\mathbf{V}^\perp$ (the representation of $SU(2)$ on $\mathbf{V}^\perp$ is trivial).

Proposition 2.4. *The forms α and ω_k defining an $SU(2)$–structure in the sense of [9] may be given by the following relations*

$$\alpha = \zeta^\flat, \quad \omega_k(X, Y) = g(J_k(X), Y).$$

Proof. We may choose a local section of the orthonormal frame $(e_1, e_2, e_3, e_4, e_5)$ such that the quadruplet (α, ω_k) defining an $SU(2)$–structure in the sense of [9] is given by (see [9])

$$\alpha = e_5^\flat, \quad \omega_1 = e_{12} + e_{34}, \quad \omega_2 = e_{13} - e_{24}, \quad \omega_3 = e_{14} + e_{23}.$$

Then locally $\mathbf{V}^\perp = \langle s_1, s_2 \rangle_{\mathbb{C}}$ (compare the fundamental example in subsect. 1.4). It suffices to choose $\varphi_1, \varphi_2, \varphi_3$ as in Remark 1.14. $\square$

Remark 2.5. The relations contained in Proposition 2.4, adapted to the considered setting, have been already obtained in [6].

3. Characterization of the Intrinsic Torsion and its Modules

In this section we want to derive a decomposition of the module of all possible intrinsic torsions via spinorial approach.

Let M be a spin 5–manifold with the corresponding Riemannian structure g. Let $SO(M)$ be a frame bundle of oriented orthonornal frames and $\mathrm{Spin}(M) \supset SO(M)$ the induced spin structure with the structure group Spin(5). The Levi–Civita connection ∇ on M induces a connection form ω on $SO(M)$ and $\tilde{\omega}$ on $\mathrm{Spin}(M)$. Let $P \subset SO(M)$ be an $SU(2)$–structure. Identifying 2–forms with skew–symmetric endomorphism, i.e., considering the isomorpism $\Lambda^2((\mathbb{R}^5)^*) \equiv \mathfrak{so}(5)$ and taking into account the decomposition $\mathfrak{so}(5) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)^\perp = \mathfrak{su}(2) \oplus \mathfrak{su}(2)_+ \oplus \mathbb{R}^4$ described in the previous sections, we have the following subbundles of 2–forms

$$\mathfrak{su}(M) = P \times_{SU(2)} \mathfrak{su}(2),$$
$$\mathfrak{su}^\perp(M) = P \times_{SU(2)} \mathfrak{su}(2)^\perp,$$
$$\mathfrak{su}_+(M) = P \times_{SU(2)} \mathfrak{su}(2)_+.$$

Since the adjoint action on $\mathfrak{su}(2)_+$ is trivial, the bundle $\mathfrak{su}_+(M)$ admits, as mentioned earlier, three global linearly independent 2–forms.

An $\mathfrak{su}(2)$–component of ω induces a Riemannian connection ∇^P on M. The *intrinsic torsion* of the considered $SU(2)$–structure is a $(1,2)$–tensor field ξ of the form

$$\xi_X Y = \nabla^P_X Y - \nabla_X Y.$$

From the definition of ξ is follows that $\xi \in T^*M \otimes \mathfrak{su}^\perp(M)$.

Moreover, $\tilde{\omega}$ and its $\mathfrak{su}(2)$–component induce connections on the spinor bundle $\mathbf{S} = \mathrm{Spin}(M) \times_{\mathrm{Spin}(5)} \Delta = P \times_{SU(2)} \Delta$. Denote them by ∇ and ∇^P (as on M), respectively. If $\varphi \in \mathbf{S}$ is a spinor field defining P, it follows that $\nabla^P \varphi = 0$.

Let $\mathbf{V}$ be a subbundle adapted to P. Let V be the corresponding admissible space, $V^\perp$ is its orthogonal complement in Δ. It is clear from the previous considerations that with respect to the map $\omega \mapsto \omega \cdot \varphi_0$ for a fixed spinor $\varphi_0 \in V^\perp$, we have an isomorphism of $\mathfrak{su}(2)^\perp$ onto $\langle \varphi_0 \rangle^\perp$. It can be shown [9] that with respect to this isomorphism

$$\frac{1}{2}\xi_X \cdot \varphi = -\nabla_X \varphi,$$

where we consider ξ_X as an element of $\mathfrak{su}(2)^\perp$. More precisely, ξ_X is treated as an invariant function from P to $\mathfrak{su}(2)^\perp$.

Denote by $\mathcal{T}$ the space $T^*(M) \otimes \mathfrak{su}^\perp(M)$ of all possible intrinsic torsions. This space splits into irreducible modules under the action of the group $SU(2)$. In [6], the authors, applying the intuition developed in [4], show how to rewrite $\nabla\varphi$ for a fixed unit spinor field φ inducing the considered $SU(2)$–structure into components lying in each irreducible component of $\mathcal{T}$. Let us recall this approach. Since for a unit spinor field φ, $\nabla_X \varphi$ is orthogonal to φ is follows that there is a linear map $S^\varphi : TM \to \mathbf{D}$ and three one–forms β^φ_k on M such that

$$\nabla_X \varphi = S^\varphi(X) \cdot \varphi + \sum_k \beta^\varphi_k(X) i_k \varphi.$$

Here, (i_1, i_2, i_3) is a quaternionic structure on $\mathbf{S}$, i_2 anticommutes with the multiplication by vectors, i_1 is induced by multiplication by the volume element [6]. Let $S^\varphi(\zeta) = V^\varphi$, where ζ is the Reeb field. Thus, we may write,

$$S^\varphi = S^\varphi_{\mathbf{D}} + \alpha \otimes V^\varphi,$$

where α is a one–form dual to ζ and $S_{\mathbf{D}}$ is an endomorphism of $\mathbf{D}$. Analogously, we may "decompose" each β_k with respect to the splitting $\mathbf{D} \oplus \langle \zeta \rangle$ as

$$\beta^\varphi_k = \beta^{\varphi,\mathbf{D}}_k + f^\varphi_k \alpha$$

for some function f_k^φ. Finally, $S^\varphi_{\mathbf{D}}$ splits as (we skip writing indices φ and $\mathbf{D}$ to make the formula more readable) [6]

$$S^\varphi_{\mathbf{D}} = \lambda_0 \mathrm{Id}_{\mathbf{D}} + S_0 + \sum_k \lambda_k J_k + \sum_k \sigma_k,$$

where $S_0 \in \mathfrak{su}(M) = P \times_{SU(2)} \mathfrak{su}(2)$ and σ_k is traceless, symmetric and such that $J_l \sigma_k = (-1)^{\delta_{kl}+1} \sigma_k J_l$, $l = 1,2,3$. Thus elements $\lambda_0, \lambda_k, f_k, S_0, \sigma_k, \beta_k^{\varphi,\mathbf{D}}, V^\varphi$ $(k = 1,2,3)$ are the components with respect to the splitting of $\mathcal{T}$ into irreducible modules [6,9]

$$\mathcal{T} = 7\mathbb{R} \oplus 4\mathfrak{su}(2) \oplus 4(\mathbb{R}^4)^* \quad \text{pointwise}.$$

3.1. Partial Invariance

The aim is to make the above decomposition independent of the choice of $\varphi \in \mathbf{S}_P$. Let $\mathbf{V}$ be a subbundle in $\mathbf{S}$ adapted to P and let $V \subset \Delta$ be corresponding admissible subspace. We introduce a slight modification. Instead of considering the quaternionic structure (i_1, i_2, i_3) and elements $\varphi, i_1\varphi, i_2\varphi, i_3\varphi$ spanning $\mathbf{V}^\perp$ we consider the quaternionic structure (j_1, j_2, j_3). For a unit spinor field φ we may write

$$\nabla_X \varphi = S^\varphi(X) \cdot \varphi + \sum_k \beta_k^\varphi(X) j_k \varphi. \tag{8}$$

The first main result of this section gives partial invariance.

Proposition 3.1. *S^φ is independent on the choice of $\varphi \in \mathbf{S}_P$. Moreover, the one–forms β_k^φ change with respect to the following formula: if $\psi = \sum_{k=0}^3 a_k j_k \varphi$, where j_0 is the identity, then*

$$\begin{aligned}
\beta_1^\psi &= (a_0^2 + a_1^2 - a_2^2 - a_3^2)\beta_1^\varphi + 2(a_1a_2 - a_0a_3)\beta_2^\varphi + 2(a_0a_2 + a_1a_3)\beta_3^\varphi \\
\beta_2^\psi &= 2(a_1a_2 + a_0a_3)\beta_1^\varphi + (a_0^2 - a_1^2 + a_2^2 - a_3^2)\beta_2^\varphi + 2(a_2a_3 - a_0a_1)\beta_3^\varphi \\
\beta_3^\psi &= 2(a_1a_3 - a_0a_2)\beta_1^\varphi + 2(a_2a_3 + a_0a_1)\beta_2^\varphi + (a_0^2 - a_1^2 - a_2^2 + a_3^2)\beta_3^\varphi.
\end{aligned}$$

Proof. For $\varphi_0 \in V$ denote by $(\varphi_0)_{\mathbb{H}}$ the following quadruplet of elements spanning V:

$$(\varphi_0)_{\mathbb{H}} = (\varphi, j_1\varphi, j_2\varphi, j_3\varphi).$$

We have a natural action of $a = (a_0, a_1, a_2, a_3) \in \mathbb{H}$ on V (compare subsection 1.8), namely

$$a \cdot \varphi_0 = \sum_k a_k j_k \varphi_0,$$

where j_0 is the identity. We have

$$(a \cdot \varphi_0)_{\mathbb{H}}^\top = \rho(a) \cdot (\varphi_0)_{\mathbb{H}}^\top, \quad \rho(a) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$$

Notice that for $a \neq 0$, $\rho(a)^{-1} = \frac{1}{\|a\|^2}\rho(\bar{a})$ (ρ is one of possible inclusions of $\mathbb{H}$ into $\mathfrak{gl}(4,\mathbb{R})$).

Take two unit spinor fields φ, ψ defining the same $SU(2)$–structure. Let S^φ, S^ψ and $\beta_k^\varphi, \beta_k^\psi$ be the corresponding elements with respect to the decomposition (8). Firstly, we will show that $S^\varphi = S^\psi$. We have

$$\begin{aligned}\nabla_X\psi &= a_0\nabla_X\varphi + \sum_k a_k j_k(\nabla_X\varphi)\\ &= a_0 S^\varphi(X)\cdot\varphi + a_0\sum_l \beta_l^\varphi(X) j_l\varphi\\ &\quad + \sum_k a_k j_k(S^\varphi(X)\cdot\varphi) + \sum_{k,l} a_k\beta_l^\varphi(X) j_k(j_l\varphi).\end{aligned}$$

Since, by the definition of j_k, $j_k(S^\varphi(X)\cdot\varphi) = S^\varphi(X)\cdot j_k\varphi$, we get

$$\begin{aligned}\nabla_X\psi = S^\varphi(X)\cdot\psi &- \sum_k a_k\beta_k^\varphi(X)\varphi\\ &+ (a_0\beta_1^\varphi(X) + a_2\beta_3^\varphi(X) - a_3\beta_2^\varphi(X))j_1\varphi\\ &+ (a_0\beta_2^\varphi(X) + a_3\beta_1^\varphi(X) - a_1\beta_3^\varphi(X))j_2\varphi\\ &+ (a_0\beta_3^\varphi(X) + a_1\beta_2^\varphi(X) - a_2\beta_1^\varphi(X))j_3\varphi.\end{aligned}$$

Hence, $S^\varphi = S^\psi$. Moreover, by the considerations at the beginning of the proof

$$\varphi_{\mathbb{H}} = \rho(\bar{a})\psi_{\mathbb{H}},$$

where $a = (a_0, a_1, a_2, a_3)$. Substituting this relation we get the desired formula for the change of β_k^φ. $\square$

Remark 3.2. Notice that the components of each β_k^ψ constitute the Hopf fibration (compare Proposition 1.11 and its proof). Moreover, treating $(\beta_1^\varphi, \beta_2^\varphi, \beta_3^\varphi)$ as a trivialization of $\mathfrak{su}(2)_+$ isomorphic to $\mathfrak{su}(2)$, transformation for the one-forms β_k^φ restricted to unit quaternions is just the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$.

3.2. Full Invariance

Let us now show how to decompose $\nabla_X\varphi$ to obtain all "components" independent of φ. Recall that the multiplication of two–forms in $\mathfrak{so}(5)$ by $\varphi_0 \in \Delta$ is a surjective map onto $\langle\varphi_0\rangle^\perp$ with the kernel $\mathfrak{su}(2)$. Moreover, restricted to the $\mathfrak{su}(2)_+$ (the dual to $\mathfrak{su}(2)$) it is an isomorphism onto $V^\perp \cap \langle\varphi_0\rangle^\perp$, where V is admissible space such that $\varphi_0 \in V^\perp$. Hence, the component $\sum_{k=0}^3 \beta_k^\varphi(X) j_k\varphi$ in (8) can be written as

$$\sum_k \beta_k^\varphi(X) j_k\varphi = \omega_X^\varphi \cdot \varphi,$$

where $\omega_X^\varphi \in \mathfrak{su}_+(M)$ is a 2–form. Therefore, for a unit spinor field φ we have

$$\nabla_X\varphi = S^\varphi(X)\cdot\varphi + \omega_X^\varphi\cdot\varphi, \tag{9}$$

The advantage of the use of ω^φ is that we do not specify the "coordinates" $(j_1\varphi, j_2\varphi, j_3\varphi)$ on the space $\mathbf{V}^\perp \cap \langle\varphi\rangle^\perp$. Moreover, S^φ and ω^φ do not depend on the choice of φ in $\mathbf{S}_P$.

Proposition 3.3. *There exists an endomorphism $S : TM \to \mathbf{D}$ and an element $\omega \in T^*(M)\otimes\mathfrak{su}_+(M)$, such that*

$$\nabla_X\varphi = S(X)\cdot\varphi + \omega_X\cdot\varphi, \tag{10}$$

for any unit spinor field $\varphi \in \mathbf{S}_P$.

Proof. Fix a spinor field φ and let $\psi = \sum_{k=0}^3 a_k j_k\varphi \in \mathbf{S}_P$. Then (9) holds. We will show that $S^\psi = S^\varphi$ and $\omega_X^\varphi = \omega_X^\psi$. We have

$$\begin{aligned}\nabla_X\psi &= a_0\nabla_X\varphi + \sum_k a_k j_k \nabla_X\varphi \\ &= a_0 S^\varphi(X)\cdot\varphi + a_0\omega_X^\varphi\cdot\varphi + \sum_k a_k j_k(S^\varphi(X)\cdot\varphi) + \sum_k a_k j_k(\omega_X^\varphi\cdot\varphi) \\ &= a_0 S^\varphi(X)\cdot\varphi + \sum_k a_k S^\varphi(X)\cdot j_k\varphi + a_0\omega_X^\varphi\cdot\varphi + \sum_k a_k\omega_X^\varphi\cdot j_k\varphi \\ &= S^\varphi(X)\cdot\psi + \omega_X^\varphi\cdot\psi.\end{aligned}$$

We used the fact that for any k, $j_k(Z\cdot\varphi) = Z\cdot j_k\varphi$, where $X \in TM$, and $j_k(\eta\cdot\varphi) = \eta\cdot j_k\varphi$, where η is a two–form. □

Notice that ω_X splits as

$$\omega_X = \omega_{X_\mathbf{D}} + \alpha(X)\omega_\zeta.$$

3.3. Intrinsic Torsion Explicitly

Let us now provide an explicit formula for the intrinsic torsion. We proceed analogously as in [4]. Fix an $SU(2)$–structure on M. Take any $\varphi \in \mathbf{S}_P$ and define a 3–form Ψ as follows

$$\Psi(X,Y,Z) = -\langle XYZ\cdot\varphi,\varphi\rangle, \quad X,Y,Z \in TM.$$

An easy calculation shows that Ψ does not depend on the choice of $\varphi \in S_P$.

Lemma 3.4. *For any $\varphi \in S_P$ the following relation holds*

$$(p(X)\lrcorner\Psi)\cdot\varphi = X\cdot\varphi, \quad X \in TM,$$

where $p : TM \to TM$ is given by $p(X) = X - \frac{1}{2}\alpha(X)\zeta$. In particular, if $X_\mathbf{D} \in \mathbf{D}$, then

$$(X_\mathbf{D}\lrcorner\Psi)\cdot\varphi = X_\mathbf{D}\cdot\varphi.$$

Proof. We may assume that $\varphi = s_1$. Then it can be verified that

$$\Psi = e_{125} + e_{345} \quad \text{and} \quad \alpha = e_5.$$

Let $X = \sum_{j=1}^{5} x_j e_j$. Then

$$p(X) = \sum_{j=1}^{4} x_j e_j + \frac{1}{2} x_5 e_5,$$

which implies

$$\begin{aligned}(p(X) \lrcorner \Psi) \cdot \varphi &= x_1 e_{25} \cdot \varphi - x_2 e_{15} \cdot \varphi + x_3 e_{45} \cdot \varphi - x_4 e_{35} \cdot \varphi + \frac{1}{2} x_5 (e_{12} + e_{34}) \cdot \varphi \\ &= x_1 i s_4 + x_2 s_4 - x_3 i s_3 - x_4 s_3 + x_5 i s_1 \\ &= X \cdot \varphi.\end{aligned}$$

□

Now we may state an explicit formula for the intrinsic torsion.

Proposition 3.5. *The intrinsic torsion ξ of an $SU(2)$–structure P is given by*

$$\xi_X = -2S(X) \lrcorner \Psi - 2\omega_X,$$

where S and ω_X are given by the formula (10).

Proof. Firstly notice that $\omega_X \in \mathfrak{su}(2)_+(M) \subset \mathfrak{su}(2)^\perp(M)$. Secondly, by the proof of Lemma 3.4 we see that $p(X) \lrcorner \Psi \in \mathfrak{su}(2)^\perp(M)$. Moreover, by Lemma 3.4 we have

$$-\frac{1}{2} \xi_X \cdot \varphi = \nabla_X \varphi = S(X) \cdot \varphi + \omega_X \cdot \varphi = (p(S(X)) \lrcorner \Psi + \omega_X) \cdot \varphi$$

for any $\varphi \in \mathbf{S}_P$. This ends the proof. □

Considering the decomposition $S = S_{\mathbf{D}} + \alpha \otimes V$, where $S_{\mathbf{D}} \in \mathrm{End}(\mathbf{D})$ is a restriction of S and $V = S(\zeta) \in \mathbf{D}$, we have

$$\xi_X = -2S_{\mathbf{D}}(X_{\mathbf{D}}) \lrcorner \Psi - 2\alpha(X) V \lrcorner \Psi - 2\omega_X.$$

In particular, for $X = \zeta$,

$$\xi_\zeta = -2V \lrcorner \Psi - 2\omega_\zeta.$$

3.4. Characteristic Connection

We end this section by providing a formula and conditions for existence of the characteristic connection ∇^c [2]. Recall, that ∇^c is a metric connection which has totally skew–symmetric torsion T^c and which is also an $SU(2)$–connection, i.e., $\nabla^c \varphi = 0$ for any unit spinor $\varphi \in \mathbf{S}_P$ defining the underlying $SU(2)$–structure P.

In order to do so, we need some additional definitions. We rely on the approach developed in [4]. We will derive a formula for ∇^c using the formula for characteristic connection for $U(2) \subset SO(5)$ structures [12]. An $U(2)$–structure on M, i.e., an almost contact metric structure, is defined by a 1–form α and an almost complex structure J on $\mathbf{D} = \ker\alpha$ (compatible with the metric). Fixing an $SU(2)$–structure on M the choice of an $U(2)$–structure is not unique, in fact, it can be any J taken from a quaternionic structure (we extend J to the Reeb field trivially). We fix any such J and denote by $F \in \mathfrak{su}_+(M)$ the induced 2–form. There is a natural choice of a spinor φ associated with J. Namely, there is unique, up to multiplication by complex numbers, unit $\varphi \in V^\perp$ such that $F \cdot \varphi = 2j \cdot \varphi = 2i\varphi$, where j is an almost complex structure on $\mathbf{S}$ (compare Lemma 6.2 in [10]). Moreover, j, J and φ are also related by the formula (compare subsection 1.3).

$$JX \cdot \varphi = j(X \cdot \varphi).$$

In fact, since j commutes with multiplication by vectors,

$$j(X \cdot \varphi) = X \cdot j\varphi = \frac{1}{2} X \cdot F \cdot \varphi = \frac{1}{2}(F \cdot X \cdot \varphi - 2(X \lrcorner F) \cdot \varphi) = J(X) \cdot \varphi.$$

This implies

$$F(X,Y) = g(X, JY) = \langle X \cdot \varphi, JY \cdot \varphi \rangle = \langle X \cdot \varphi, j(X \cdot \varphi) \rangle, \quad X, Y \in TM.$$

Moreover, let Ψ^J be a 3–form defined by

$$\Psi^J(X,Y,Z) = \Psi(JX, JY, JZ) = -\langle XYZ \cdot \varphi, j\varphi \rangle.$$

and Φ a 3–tensor in $T^*M \otimes \Lambda^2(T^*M)$ given by

$$\Phi(X,Y,Z) = \frac{1}{2}\langle \omega_X \cdot \varphi, (ZY - YZ) \cdot j\varphi \rangle$$

Lemma 3.6. *The following formula holds*

$$(\nabla_X F)(Y,Z) = 2\Psi^J(S(X), Y, Z) + 2\Phi(X,Y,Z),$$

where S is given by the formula (9).

Proof. Since j is ∇-parallel, we have

$$\begin{aligned}(\nabla_X F)(Y,Z) &= XF(Y,Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z)\\ &= \langle Y \nabla_X \varphi, j(Z \cdot \varphi) \rangle + \langle Y \cdot \varphi, j(Z \cdot \nabla_X \varphi) \rangle.\end{aligned}$$

Using the formula $\nabla_X \varphi = S(X) \cdot \varphi + \omega_X \cdot \varphi$ we obtain

$$\begin{aligned}(\nabla_X F)(Y,Z) &= \langle YS(X) \cdot \varphi, j(Z \cdot \varphi) \rangle + \langle Y \cdot \varphi, j(ZS(X) \cdot \varphi) \rangle\\ &\quad + \langle Y\omega_X \cdot \varphi, Z \cdot j\varphi \rangle + \langle Y \cdot \varphi, Z\omega_X \cdot j\varphi \rangle\\ &= 2\Psi^J(S(X), Y, Z) + 2\Phi(X,Y,Z).\end{aligned}$$

$\square$

Consider now the Nijenhuis tensor of an almost contact metric structure (M, g, α, J)

$$\begin{aligned}N(X,Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + (\nabla_X J)(JY) - (\nabla_Y J)(JX)\\ -\alpha(Y)\nabla_X \zeta + \alpha(X)\nabla_Y \zeta\end{aligned} \tag{11}$$

and treat it as a 3-tensor contracting it with the metric.

Lemma 3.7. *The Nijenhuis tensor may be described as follows*

$$\begin{aligned} N(X,Y,Z) &= 2\Psi^J((SJ+JS)X,Y,Z) - 2\Psi^J((SJ+JS)Y,X,Z)) \\ &\quad + 2\Phi(JX,Y,Z) - 2\Phi(JY,X,Z) + 2\Phi(X,JY,Z) - 2\Phi(Y,JX,Z) \\ &\quad - \alpha(Y)(\nabla_X\alpha)Z + \alpha(X)(\nabla_Y\alpha)Z. \end{aligned}$$

Proof. Firstly notice that

$$\Psi^J(X,Y,Z) = \Psi(JX,JY,JZ) = \Psi(JX,Y,Z) = \Psi(X,JY,Z) = \Psi(X,Y,JZ).$$

Now, the formula follows by (11) and Lemma 3.6. □

We are almost ready to describe the characteristic connection ∇^c for an $SU(2)$–structure. Since an $SU(2)$–structure is a $U(2)$–structure, it follows that ∇^c is also the characteristic connection for a $U(2)$–structure. By the results in [12] it follows that such a connection is unique and its torsion T^c equals

$$T^c = \alpha \wedge d\alpha + d^J F + N - \alpha \wedge (\zeta \lrcorner N), \tag{12}$$

where $d^J F(X,Y,Z) = -(dF)(JX,JY,JZ)$. The existence of ∇^c implies that N is a 3–form and the Reeb field ζ is Killing (with respect to the Levi–Civita connection) [12].

Theorem 3.8. *An $SU(2)$–structure on a spin 5–manifold M admits a characteristic connection if and only if the Nijenhuis tensor is totally skewsymmetric, the Reeb field is Killing and $4F(\omega_X) = \delta F$.*

Proof. The proof follows the same lines as the proof of Theorem 3.18 in [4]. Assume ∇^c is a characteristic connection for an $U(2)$–structure on M. Then, in particular, $\nabla^c F = 0$. Let φ be a unit spinor considered above, i.e., such that $j(X\cdot\varphi) = J(X)\cdot\varphi$. By the formula for F and the fact that j is ∇^c–parallel we obtain

$$\begin{aligned} 0 = (\nabla^c_X F)(Y,Z) &= XF(Y,Z) - F(\nabla^c_X Y, Z) - F(Yy, \nabla^c_X Z) \\ &= \langle Y\cdot\nabla^c_X\varphi, j(Z\cdot\varphi)\rangle + \langle Y\cdot\varphi, j(Z\cdot\nabla^c_X\varphi)\rangle \\ &= -2\langle\nabla^c_X\varphi, YZ\cdot j\varphi\rangle - 2g(Y,Z)\langle\nabla^c_X\varphi, j\varphi\rangle. \end{aligned}$$

Thus $\langle\nabla^c_X\varphi, YZ\cdot j\varphi\rangle = 0$ for Y orthogonal to Z. This implies that $\nabla^c_X\varphi$ is in the span of $j\varphi$. Concluding, a $U(2)$ characteristic connection ∇^c is in fact an $SU(2)$ connection, i.e. $\nabla^c\varphi = 0$, if and only if $\langle\nabla^c_X\varphi, j\varphi\rangle = 0$ for all X. Since (see for example [4])

$$\nabla^c_X\varphi = \nabla_X\varphi + \frac{1}{4}(X\lrcorner T)\cdot\varphi,$$

we have

$$0 = \langle \nabla^c_X \varphi, j\varphi\rangle = \langle S(X)\cdot\varphi + \omega_X\cdot\varphi + \frac{1}{4}(X \lrcorner T)\cdot\varphi, j\varphi\rangle$$
$$= \langle \omega_X\cdot\varphi, j\varphi\rangle + \frac{1}{4}F(X\lrcorner T) = F(\omega_X) + \frac{1}{4}T(F,X)$$
$$= F(\omega_X) + \frac{1}{4}\delta F.$$

Thus $4F(\omega_X) = -\delta F$. □

Theorem 3.9. *Consider an $U(2)$–structure (M, g, F, ζ, α) on a 5–dimensional spin manifold induced from an $SU(2)$–structure. Assume that the Nijenhuis tensor N is skew–symmetric, the Reeb field ζ is Killing and $4F(\omega_X) = -\delta F(X)$, where ω_X is given by (9). Then the torsion T^c of the characteristic connection ∇^c is given by the formula*

$$\begin{aligned} T^c(X,Y,Z) = {} & 2\mathfrak{S}_{XYZ}(\Psi(S(X),Y,Z) - \Phi(JX,JY,JZ)) \\ & + 2\Psi^J((SJ+JS)X,Y,Z) - 2\Psi^J((SJ+JS)Y,X,Z) \\ & + 2\Phi(JX,Y,Z) - 2\Phi(JY,X,Z) + 2\Phi(X,JY,Z) - 2\Phi(Y,JX,Z) \\ & + \frac{5}{2}\alpha(X)d\alpha(Y,Z) - \frac{5}{2}\alpha(Y)d\alpha(X,Z) + \frac{1}{2}\alpha(Z)d\alpha(X,Y) \\ & - \alpha(X)d\alpha(JY,JZ) + \alpha(Y)d\alpha(JX,JZ) - \alpha(Z)d\alpha(JX,JY). \end{aligned} \tag{13}$$

Proof. Follows by Lemma 3.7, formula (12) and the facts that $\nabla_\zeta\zeta = 0$ and $(\zeta\lrcorner N)(Y,Z) = -d\alpha(Y,Z) + d\alpha(JY,JZ)$ [12]. □

Notice that some components in the formula (13) as well as in the formula for the Nijenhuis tensor in Lemma 3.7 may be derived and rewritten in a different way. Firstly, we state and prove a formula analogous to the one contained in Lemma 3.4. Namely, we have the following simple observation.

Lemma 3.10. *For a fixed unit spinor φ, a vector $X \in TM$ and the corresponding almost complex structure J the following formula holds*

$$-(\alpha\wedge(JX)^\flat)\cdot\varphi = X\cdot\varphi,$$

where $\alpha\wedge(JX)^\flat$ lies in the $\mathbb{R}^4$–component of $\mathfrak{so}(5)$.

Proof. By the definition of J, noticing that $\zeta\cdot\varphi = j\varphi$, we have

$$-(\alpha\wedge(JX)^\flat)\cdot\varphi = -\zeta(JX)\cdot\varphi = -j\cdot j(X\cdot\varphi) = X\cdot\varphi.$$

□

By the lemma above we have

$$0 = \nabla^c_X\varphi = S(X)\cdot\varphi + \omega_X\cdot\varphi + \frac{1}{4}(X\lrcorner T)\cdot\varphi$$
$$= -(\alpha\wedge(JS(X))^\flat)\cdot\varphi + \omega_X\cdot\varphi + \frac{1}{4}(X\lrcorner T)\cdot\varphi.$$

For an element $\eta \in \Lambda^2((\mathbb{R}^5)^*) \equiv \mathfrak{so}(5)$ denote its components with respect to the decomposition $\mathfrak{so}(5) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)_+ \oplus \mathbb{R}^4$ by η_-, η_+ and η_4, respectively. Then by the above

$$\begin{aligned}(X \lrcorner T)_4 \cdot \varphi &= -4(\alpha \wedge (JS(X))^\flat) \cdot \varphi,\\ (X \lrcorner T)_+ \cdot \varphi &= 4\omega_X \cdot \varphi.\end{aligned}$$

Applying the formula (12) and noticing that the projection to the $\mathbb{R}^4$ factor equals $\eta \mapsto \alpha \wedge (\zeta \lrcorner \eta)$, we get

$$\begin{aligned}2JV &= (\zeta \lrcorner d\alpha)^\flat = 0,\\ 4\omega_\zeta &= (d\alpha)_+,\\ 4S(X_{\mathbf{D}}) &= X_{\mathbf{D}} \lrcorner d\alpha,\\ 4\omega_{X_{\mathbf{D}}} &= (X_{\mathbf{D}} \lrcorner N)_+ + (X_{\mathbf{D}} \lrcorner d^J F)_+.\end{aligned}$$

where $V = S(\zeta)$ and $X_{\mathbf{D}} \in \mathbf{D}$. Here, we used the facts that $\zeta \lrcorner T = d\alpha$ and $\zeta \lrcorner d\alpha = 0$ [12]. This gives the partial correspondence already established in [6] between the approach by differential forms [9] and the spinorial approach [6] in the case of the existence of the characteristic connection.

Remark 3.11. The formulas for the Nijenhuis tensor N and the torsion T^c of the characteristic connection contained in Lemma 3.7 and Theorem 3.9 depend on the almost complex structure J. Since J was chosen arbitrary from a corresponding quaternionic structure associated with an $SU(2)$–structure, these formulas are valid for any such choice. In the approach in [9] and [6] the considered almost complex structure is induced by a 2–form ω_1 or by the almost complex structure j on $\mathbf{S}$ induced from the volume form and a fixed unit spinor defining $SU(2)$–structure.

3.5. Intrinsic Torsion of a $U(2)$–Structure

Any $SU(2)$–structure in dimension 5 is a $U(2)$–structure, hence an almost contact structure. Let us compare the intrinsic torsion of these two structures. For the formula for the intrinsic torsion of a $U(2)$–structure, or more generally, an $U(n)$–structure on a $(2n+1)$–dimensional Riemannian manifold we refer to [13].

Let M be a spin 5 dimensional manifold with the corresponding Riemannian structure g. Consider an $SU(2)$–structure on M. Then, as we have seen in the previous subsection, we may choose a $U(2)$–structure on M corresponding to the choice of an almost complex structure J on the distribution $\mathbf{D} = \ker\alpha$, where α is a 1–form, whose existence follows from the choice of an $SU(2)$–structure on M. Then J defines an endomorphism (denoted by the same letter) $J : TM \to TM$ which is J in $\mathbf{D}$ and zero on $\zeta = \alpha^\sharp$. Moreover, let F be the 2–form associated with J, $F(X,Y) = g(X, JY)$. The intrinsic torsion of the $U(2)$–structure is given by [13]

$$g(\xi_X^{U(2)}Y, Z) = -\frac{1}{2}g(J(\nabla_X J)Y), Z) + \alpha(Z)(\nabla_X\alpha)Y - \frac{1}{2}\alpha(Y)(\nabla_X\alpha)Z.$$

Let us rewrite $\xi^{U(2)}$, with the use of the objects defined in the previous section, which are induced by the spinors defining the $SU(2)$–structure. In fact, we have to deal with the first component. By Lemma 3.6 we have

$$g(J((\nabla_X J)Y), Z) = F(\nabla_X F)(Y, JZ) = 2\Psi^J(S(X), Y, JZ) + 2\Phi(X, Y, JZ).$$

Moreover, by the fact that $(\nabla_X \alpha)Y = (\nabla_X F)(\zeta, JY)$ [8], we get

$$(\nabla_X \alpha)Y = -2\Phi(X, JY, \zeta).$$

Finally,

$$\begin{aligned} g(\xi_X^{U(2)} Y, Z) = \Psi(S(X), Y, Z) - \Phi(X, Y, JZ) \\ - 2\alpha(Z)\Phi(X, JY, \zeta) + \alpha(Y)\Phi(X, JZ, \zeta) \end{aligned}$$

Recall that the intrinsic torsion of an $SU(2)$–structure is given by

$$g(\xi_X Y, Z) = -2\Psi(S(X), Y, Z) - 2\omega_X(Y, Z).$$

The difference of ξ and $\xi^{U(2)}$ should be of the form $\gamma(X)F(Y, Z)$ for some 1–form γ. We believe this is the case and leave the details to the reader. This follows from the fact that we have the splitting $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathbb{R}$, where the second component is induced by the element J.

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