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Results in Mathematics

Extension of Hoffman's Combinatorial Identity via Specific Zeta-Like Series

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Abstract. The goal of the paper is twofold. First, we present an analytic method leading to a class of combinatorial identities with Bernoulli, Euler and Catalan numbers based on considering specific multiple zeta-like series and infinite products. The developed method allows us to naturally extend Hoffman's combinatorial identity that led to the famous evaluation of the multiple zeta value $\zeta(\{2\}_k)$ in 1992. Second, we present new evaluations of two multiple zeta-like series with their consequences to combinatorial identities, and, as a by-product of our technical considerations, we establish two combinatorial identities with a trinomial coefficient and Stirling numbers respectively.

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1. Introduction

In 1992, Hoffman [8, Proposition 2.4] discovered the remarkable identity

$$\sum_{\substack{\sum_{i=1}^{k} im_i=k \\ m_i\geq 0}} \frac{1}{m_1!\cdots m_k!}\cdot\left(\frac{B_2}{2\cdot 2!}\right)^{m_1}\cdots\left(\frac{B_{2k}}{2k\cdot(2k)!}\right)^{m_k}=\frac{1}{4^k\cdot(2k+1)!}, \tag{1.1}$$

where m_i's are non-negative integers satisfying the diophantine equation $\sum_{i=1}^{k} im_i = k$ with $k\in\mathbb{N}$ and B_k denoting the k-th Bernoulli number. Hoffman found the above identity (1.1) in order to evaluate the famous values $\zeta(\{2\}_k) := \zeta(2,2,\ldots,2)$ with k repetitions of the argument 2, where

$\zeta(s_1,\dots,s_k)$ denotes the so-called *multiple zeta value* defined by

$$\zeta(s_1,\dots,s_k) := \sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}, \tag{1.2}$$

with $s_i \in \mathbb{N}$ and $s_1 \geq 2$ for sake of the convergence of the infinite series in (1.2). The multiple zeta values $\zeta(s_1,\dots,s_k)$ were introduced by Hoffman [8] and Zagier [14] in 1992 as a generalization of the single Riemann zeta values $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$, $s \in \mathbb{N}$, $s \geq 2$. It is worth mentioning that multiple zeta values play an important role in quantum physics by calculations with Feynman diagrams, or in knot theory. For more information on these attractive values, we refer interested readers to the books Gil and Fresán [1] and Zhao [15] providing an interesting exposition of this branch of mathematics. In addition to the value $\zeta(s_1,\dots,s_k)$ defined above, we also define the corresponding *multiple zeta-star value* $\zeta^\star(s_1,\dots,s_k)$ by changing the strict inequalities below the sum in (1.2) to non-strict ones, i.e.

$$\zeta^\star(s_1,\dots,s_k) := \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}, \tag{1.3}$$

with identical conditions on the arguments s_i as by $\zeta(s_1,\dots,s_k)$.

Identity (1.1) is closely connected with the formula

$$\zeta(\{2\}_k) = \frac{\pi^{2k}}{(2k+1)!}, \tag{1.4}$$

and Hoffman's [8] original proof of (1.4) requires (1.1) which he deduced by means of analytical methods. However, the connection of $\zeta(\{2\}_k)$ and (1.1) presented in [8] subsume combinatorial ideas and concepts from abstract algebra.

1.1. Objectives

It is generally known that relations between multiple zeta values imply combinatorial identities, see Eie [5] for an overview of such results. However, we feel that there is still space for further investigations in this direction.

In contrast to Hoffman's approach, the technical objective of our paper is to provide a fully analytical link between the combinatorial identities of the type like in (1.1) on the one hand and of the general nested zeta-like infinite series

$$\varphi_k(a_n) := \sum_{n_1>\cdots>n_k\geq 1} \frac{1}{a_{n_1}\cdots a_{n_k}}, \tag{1.5}$$

$$\varphi_k^\star(a_n) := \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{a_{n_1}\cdots a_{n_k}}, \tag{1.6}$$

on the other hand. This is the purpose of the auxiliary Lemma 3.1 and of the identities in (3.5) and in (3.6) that form the technical base of our method.

Secondly, Hoffman [8] proved identity (1.1) as a counterpart to (1.4), i.e. to the closed-form evaluation of $\varphi_k(n^2)$. Hence, there is a natural question about the corresponding combinatorial counterpart relating to the evaluation of $\varphi_k^\star(n^2) = \zeta^\star(\{2\}_k)$. To our best knowledge, we are not aware of any published counterpart corresponding to

$$\zeta^\star(\{2\}_k) = (-1)^{k-1} \cdot \left(4^k - 2\right) \cdot \frac{B_{2k}}{(2k)!} \cdot \pi^{2k}. \tag{1.7}$$

We present such combinatorial identity in Theorem 2.1. Nevertheless, we can pose analogous questions in case of other suitable a_n's, say $a_n = (2n-1)^2$ or $a_n = n(n+1)$. While the identities generated by the 'odd variant' $a_n = (2n-1)^2$ are stated in Theorem 2.2, the identities generated by $a_n = n \cdot (n+1)$ are presented in Theorems 2.5 and 2.6. Based on Lemma 3.1, the proofs of Theorems 2.1 and 2.2 will be almost self-evident. In contrast to this, the proofs of Theorems 2.5 and 2.6 require more effort.

For the purpose of obtaining any combinatorial identities by means of the tools presented in this work, we always need the closed forms of both $\varphi_k(a_n)$ and $\varphi_k^\star(a_n)$. For $a_n = n^2$, these evaluations are given above in (1.4) and (1.7), for $a_n = (2n-1)^2$ these are also known and we will present them later. Yet, the closed forms of $\varphi_k(n \cdot (n+1))$ and $\varphi_k^\star(n \cdot (n+1))$ stated in Theorem 2.3 and Theorem 2.4 respectively are new. Moreover, we apply these theorems in a slightly different way compared to how we apply the evaluations of $\varphi_k(a_n)$ and $\varphi_k^\star(a_n)$ with $a_n = n^2$ or $a_n = (2n-1)^2$. Our considerations thus lead to the combinatorial identities presented in Theorems 2.5 and 2.6, where Catalan numbers occur. Hence, the part devoted to the analysis of $\varphi_k(n \cdot (n+1))$ and $\varphi_k^\star(n \cdot (n+1))$ and their consequences can be considered as main.

Lastly, we bring up two combinatorial identities presented in Corollaries 4.1 and 4.2. Both these results follow as a by-product of the auxiliary Lemma 3.2. Even if the mentioned corollaries do not belong to our main goals, and therefore postponed to Sect. 4, we include them in this paper with regard to their visual attractiveness.

2. Results

In connection with the objectives described above, we can formally split this section into two parts. While the simpler part is formed by Theorems 2.1 and 2.2, the main part concerns Theorems 2.3–2.6.

We start with a pair of identities including Hoffman's identity (1.1) corresponding to $\varepsilon = 1$. Notice that changing the involved parameter to $\varepsilon = -1$ causes a change on the right-hand side, where a Bernoulli number appears.

Theorem 2.1. *Suppose that* $k \in \mathbb{N}_0$. *Then*

$$\sum_{\substack{\sum_{i=1}^{k} i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{\varepsilon \cdot B_{2i}}{2i \cdot (2i)!} \right)^{m_i} = \frac{1}{4^k \cdot (2k)!} \cdot \begin{cases} \dfrac{1}{2k+1} & \text{if} \quad \varepsilon = 1, \\ \left(2 - 4^k\right) \cdot B_{2k} & \text{if} \quad \varepsilon = -1. \end{cases}$$

A similar pair of identities is stated in the following theorem. We believe that the formula corresponding to $\varepsilon = -1$ is especially interesting since its left--hand side involves Bernoulli numbers whereas its right-hand side is a simple expression with the Euler number E_{2k}.

Theorem 2.2. *Suppose that* $k \in \mathbb{N}_0$. *Then*

$$\sum_{\substack{\sum_{i=1}^{k} i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left((4^i - 1) \cdot \frac{\varepsilon \cdot B_{2i}}{2i \cdot (2i)!} \right)^{m_i} = \frac{1}{4^k \cdot (2k)!} \cdot \begin{cases} 1 & \text{if} \quad \varepsilon = 1, \\ E_{2k} & \text{if} \quad \varepsilon = -1. \end{cases}$$

To obtain the identities in Theorems 2.5 and 2.6, we utilize the following two theorems dealing with the evaluation of the nested series $\varphi_k(n \cdot (n+1))$ and $\varphi_k^\star(n \cdot (n+1))$ respectively.

Theorem 2.3. *Suppose that* $k \in \mathbb{N}$. *Then*

$$\sum_{n_1 > \cdots > n_k \geq 1} \prod_{j=1}^{k} \frac{1}{n_j \cdot (n_j + 1)} = \frac{1}{k+1} \cdot \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \cdot \binom{2k - 2i}{k} \cdot \frac{\pi^{2i}}{(2i)!}.$$

The closed form for the non-strict variant $\varphi_k^\star(n \cdot (n+1))$ presented in Theorem 2.4 is slightly different than the evaluation of $\varphi_k(n \cdot (n+1))$ presented in Theorem 2.3 as it is involving the so-called Dirichlet η-function $\eta(s) := \sum_{n=1}^{\infty} (-1)^{n-1}/n^s$, $s > 0$.

Theorem 2.4. *Suppose that* $k \in \mathbb{N}$, $k \geq 2$. *Then*

$$\sum_{n_1 \geq \cdots \geq n_k \geq 1} \prod_{j=1}^{k} \frac{1}{n_j \cdot (n_j + 1)} = \frac{2 \cdot (-1)^k}{k-1} \cdot \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (2i - 1) \cdot \binom{2k - 2i - 2}{k - 2} \cdot \eta(2i),$$

where $\eta(0) := 1/2$.

Remark. Notice that the above formula holds true for integers $k \geq 2$ only. Evidently, the missing case relating to $k = 1$ can be calculated immediately since we have $\varphi_1^\star(n \cdot (n+1)) = \sum_{n=1}^{\infty} 1/(n \cdot (n+1)) = 1$ by telescoping. Beside this, we can also apply Theorem 2.3 due to $\varphi_1(a_n) = \varphi_1^\star(a_n)$.

The last two theorems imply two pairs of elegant combinatorial identities involving Catalan numbers.

Theorem 2.5. *Suppose that $k \in \mathbb{N}$. Then*

$$\sum_{\substack{\sum_{i=1}^k im_i = k\\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{\varepsilon}{2i} \cdot \binom{2i}{i}\right)^{m_i} = \begin{cases} C_k & \textit{if} \quad \varepsilon = 1,\\ -C_{k-1} & \textit{if} \quad \varepsilon = -1. \end{cases}$$

Finally, we state the last pair of combinatorial identities that represents a kind of weighted variant of Theorem 2.5.

Theorem 2.6. *Suppose that $k \in \mathbb{N}$, $k \geq 2$. Then*

$$\sum_{\substack{\sum_{i=1}^k im_i = k\\ m_i \geq 0}} W_k(\boldsymbol{m}) \cdot \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{\varepsilon}{2i} \cdot \binom{2i}{i}\right)^{m_i} = \begin{cases} C_k - C_{k-1} & \textit{if} \quad \varepsilon = 1,\\ -C_{k-2} & \textit{if} \quad \varepsilon = -1, \end{cases}$$

where $W_k(\boldsymbol{m}) = W_k(m_1, \ldots, m_k)$ is the weight function defined by

$$W_k(\boldsymbol{m}) := \sum_{t=2}^k \frac{t \cdot m_t}{2t-1}. \tag{2.1}$$

3. Proofs

3.1. Auxiliary Lemma and Its Consequences

We first state and prove the following simple technical tool which forms the base for the combinatorial identities described in this paper.

Lemma 3.1. *Let $\{a_n\}_{n=1}^\infty$ be an arbitrary sequence of real numbers such that the infinite product $\prod_{n=1}^\infty (1 - x/a_n)$ converges, where x is an arbitrary real number satisfying the condition $|x| < \max |a_n|$. Choose a fixed parameter $\varepsilon = 1$ or $\varepsilon = -1$. Then*

$$\prod_{n=1}^\infty \left(1 - \frac{x}{a_n}\right)^\varepsilon = \sum_{k=0}^\infty x^k \cdot \sum_{\substack{\sum_{i=1}^k im_i = k\\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-\varepsilon \cdot S_i}{i}\right)^{m_i}, \tag{3.1}$$

where

$$S_i := \sum_{n=1}^\infty \frac{1}{a_n^i}. \tag{3.2}$$

Remark. Even if we restricted the possible values for the parameter ε to ± 1 only, one can easily generalize Lemma 3.1 accordingly. Since the cases $\varepsilon \neq \pm 1$ will not be important for our later considerations, we omit them.

Proof of Lemma 3.1. Let $\varepsilon = 1$ or $\varepsilon = -1$ be fixed according to the assumptions of the lemma. Since x satisfies the condition $|x| < \max |a_n|$, we can rewrite the convergent infinite product on the left-hand side of (3.1) in the following way:

$$\begin{aligned} P_\varepsilon(x) &:= \prod_{n=1}^{\infty}\left(1-\frac{x}{a_n}\right)^{\varepsilon} \\ &= \exp\left(\sum_{n=1}^{\infty}\varepsilon\cdot\ln\left(1-\frac{x}{a_n}\right)\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{-\varepsilon}{k}\cdot\frac{x^k}{a_n^k}\right) \\ &= \sum_{j=0}^{\infty}\frac{1}{j!}\cdot\left(\sum_{k=1}^{\infty}x^k\cdot\frac{-\varepsilon}{k}\cdot\sum_{n=1}^{\infty}\frac{1}{a_n^k}\right)^{j}. \end{aligned}$$

Further, we employ the notation introduced in (3.2) and calculate also the j-th power of the middle infinite series:

$$\begin{aligned} P_\varepsilon(x) &= \sum_{j=0}^{\infty}\frac{1}{j!}\cdot\left(\sum_{k=1}^{\infty}x^k\cdot\frac{-\varepsilon\cdot S_k}{k}\right)^{j} \\ &= \sum_{j=0}^{\infty}\frac{1}{j!}\cdot\sum_{k=j}^{\infty}x^k\cdot\sum_{\substack{\sum_{i=1}^{k} im_i=k\\ \sum_{i=1}^{k} m_i=j\\ m_i\geq 0}}\binom{j}{m_1,\ldots,m_k}\cdot\prod_{i=1}^{k}\left(\frac{-\varepsilon\cdot S_i}{i}\right)^{m_i}. \end{aligned}$$

Interchanging the summation order and canclelling $j!$ imply

$$\begin{aligned} P_\varepsilon(x) &= \sum_{k=0}^{\infty}x^k\cdot\sum_{j=0}^{k}\sum_{\substack{\sum_{i=1}^{k} im_i=k\\ \sum_{i=1}^{k} m_i=j\\ m_i\geq 0}}\prod_{i=1}^{k}\frac{1}{m_i!}\cdot\left(\frac{-\varepsilon\cdot S_i}{i}\right)^{m_i} \\ &= \sum_{k=0}^{\infty}x^k\cdot\sum_{\substack{\sum_{i=1}^{k} im_i=k\\ m_i\geq 0}}\prod_{i=1}^{k}\frac{1}{m_i!}\cdot\left(\frac{-\varepsilon\cdot S_i}{i}\right)^{m_i}. \end{aligned}$$

This concludes the proof. □

The importance of the previous lemma consists in its connection with the nested infinite series $\varphi_k(a_n)$ and $\varphi_k^\star(a_n)$ defined in (1.5) and in (1.6). Actually, it is well-known that

$$\prod_{n=1}^{\infty}\left(1-\frac{x}{a_n}\right) = 1+\sum_{k=1}^{\infty}(-x)^k\cdot\varphi_k(a_n), \tag{3.3}$$

$$\prod_{n=1}^{\infty}\left(1-\frac{x}{a_n}\right)^{-1} = 1+\sum_{k=1}^{\infty}x^k\cdot\varphi_k^\star(a_n), \tag{3.4}$$

see, e.g. Genčev [6, Lemma 2.1 and Eq. (24)]. Therefore, comparing the coefficients of both power series representations for the infinite products $P_{\pm 1}(x)$ in Lemma 3.1 with the above expansions (3.3) and (3.4) gives apparently

$$\varphi_k(a_n) = (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i}\right)^{m_i}, \tag{3.5}$$

$$\varphi_k^\star(a_n) = \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{S_i}{i}\right)^{m_i}. \tag{3.6}$$

We would like to point out that if the series S_i can be calculated in the closed form for every $i \in \mathbb{N}$ then the above relations provide a (somewhat cumbersome) way for calculating the closed forms of $\varphi_k(a_n)$ and $\varphi_k^\star(a_n)$.

The above facts, however, do not imply specific combinatorial identities immediately. To obtain our results presented in Sect. 2 with the help of Lemma 3.1, it is necessary to find another suitable closed-form evaluations of the corresponding series $\varphi_k(a_n)$ and $\varphi_k^\star(a_n)$. The standard way is based on finding the closed form of the infinite products $P_{\pm 1}(x)$ in (3.1) and, consequently, on finding their expansions into Maclaurin series. The second alternative way consists in considering a specific transformation of the nested series $\varphi_k(a_n)$, $\varphi_k^\star(a_n)$ into a simple infinite series. This usually comes into focus when the standard way is difficult or inconvenient. As we will see later, we will make use of such alternative when $a_n = n \cdot (n+1)$ and $\varepsilon = -1$. Further details are described in the next sections that are devoted to the proofs of our theorems.

3.2. Proof of Theorem 2.1

We split this subsection into two parts. In the first part, we present a simple proof of Hoffman's identity (1.1) corresponding to $\varepsilon = 1$ in Theorem 2.1. In the second part, we deal with the proof of the case $\varepsilon = -1$.

Proof of Hoffman's identity. Setting $a_n = n^2$ in (1.5) yields $\varphi_k(n^2) = \zeta(\{2\}_k)$. Since (see [9, p. 88, Theorem 2.6.1] for the infinite product evaluation)

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin(\pi x)}{\pi x} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{(\pi x)^{2k}}{(2k+1)!},$$

the relation in (3.3) implies that

$$\varphi_k\left(n^2\right) = \frac{\pi^{2k}}{(2k+1)!} \tag{3.7}$$

by equating the coefficients of the same powers of x. Indeed, this is equivalent to formula (1.4).

On the other hand, we have $S_i = \sum_{n=1}^{\infty} 1/n^{2i} = \zeta(2i)$. Thus, in virtue of (3.5), (3.7) and by Euler's evaluation

$$\zeta(2i) = (-1)^{i-1} \cdot \frac{(2\pi)^{2i}}{2 \cdot (2i)!} \cdot B_{2i}, \quad i \in \mathbb{N}, \tag{3.8}$$

we find that

$$\begin{aligned}\frac{\pi^{2k}}{(2k+1)!} &= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i=k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-\zeta(2i)}{i}\right)^{m_i} \\ &= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i=k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{(-1)^i}{i} \cdot \frac{(2\pi)^{2i}}{2 \cdot (2i)!} \cdot B_{2i}\right)^{m_i} \\ &= (2\pi)^{2k} \cdot \sum_{\substack{\sum_{i=1}^k im_i=k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{B_{2i}}{2i \cdot (2i)!}\right)^{m_i}.\end{aligned}$$

Comparing the left-hand side at the beginning with the last expression and dividing by $(2\pi)^{2k}$ imply Hoffman's combinatorial formula immediately. □

Proof of Theorem 2.1 for $\varepsilon = -1$. The proof is, in essence, the same as the proof of Hoffman's identity except that we apply the evaluation

$$\begin{aligned}\varphi_k^{\star}\left(n^2\right) &= \zeta^{\star}(\{2\}_k) \\ &= (-1)^{k-1} \cdot \pi^{2k} \cdot \left(4^k - 2\right) \cdot \frac{B_{2k}}{(2k)!}\end{aligned} \tag{3.9}$$

following from (3.4) and from the relation (see also the remark following immediately after the end of this proof)

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)^{-1} &= \frac{\pi x}{\sin(\pi x)} \\ &= \sum_{k=0}^{\infty} (-1)^{k-1} \cdot \left(4^k - 2\right) \cdot \frac{B_{2k}}{(2k)!} \cdot (\pi x)^{2k}.\end{aligned} \tag{3.10}$$

Hence, utilizing (3.6) with $S_i = \zeta(2i)$, and (3.9), one deduces that

$$\begin{aligned}&(-1)^{k-1} \cdot \pi^{2k} \cdot \left(4^k - 2\right) \cdot \frac{B_{2k}}{(2k)!} \\ &= \sum_{\substack{\sum_{i=1}^k im_i=k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{\zeta(2i)}{i}\right)^{m_i}\end{aligned}$$

$$= (-1)^k \cdot (2\pi)^{2k} \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{-B_{2i}}{2i \cdot (2i)!} \right)^{m_i}.$$

The proof of Theorem 2.1 for $\varepsilon = -1$ follows now easily by comparing the left-hand side with the last expression and by dividing by $(2\pi)^{2k}$. □

Remark. The presented expansion of the function $\pi x / \sin(\pi x)$ into the Maclaurin series is well-known. We refer readers to the paper Chen et al. [4, p. 828], where a simple proof of the identity

$$\csc(x) = \frac{2}{x} \cdot \sum_{k=0}^{\infty} (-1)^{k-1} \cdot \left(4^k - 2\right) \cdot \frac{B_{2k}}{(2k)!} \cdot x^{2k}, \quad |x| < \pi,$$

is given. Of course, multiplying this relation by x and changing the variable x to πx implies the Maclaurin expansion on the right-hand side of (3.10).

3.3. Proof of Theorem 2.2

Proof. Assume first that $\varepsilon = 1$. Since (see [9, p. 88, Theorem 2.6.1] for the infinite product evaluation)

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n-1)^2} \right) = \cos\left(\frac{\pi x}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot \left(\frac{\pi x}{2}\right)^{2k},$$

we deduce by this and by (3.3) that

$$\varphi_k\left((2n-1)^2\right) = \frac{\pi^{2k}}{4^k \cdot (2k)!}.$$

Moreover, we have

$$\begin{aligned} S_i &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2i}} \\ &= \left(1 - 4^{-i}\right) \cdot \zeta(2i) \\ &= (-1)^{i-1} \cdot \left(4^i - 1\right) \cdot \frac{\pi^{2i}}{2 \cdot (2i)!} \cdot B_{2i}, \end{aligned}$$

where the last equality follows from (3.8). Hence, using the above relations and (3.5), we obtain

$$\begin{aligned} \frac{\pi^{2k}}{4^k \cdot (2k)!} &= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left((-1)^i \cdot \left(4^i - 1\right) \cdot \frac{\pi^{2i}}{2i \cdot (2i)!} \cdot B_{2i} \right)^{m_i} \\ &= \pi^{2k} \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\left(4^i - 1\right) \cdot \frac{B_{2i}}{2i \cdot (2i)!} \right)^{m_i}. \end{aligned}$$

Dividing the last relation by π^{2k} implies the validity of Theorem 2.2 for $\varepsilon = 1$.

The proof for $\varepsilon = -1$ proceeds analogously with the help of the well--known expansion

$$\frac{1}{\cos(x)} = \sum_{k=0}^{\infty} \left(-x^2\right)^k \cdot \frac{E_{2k}}{(2k)!},$$

where E_{2k} denotes the $2k$-th Euler number. We leave the proof details to readers. $\square$

3.4. Proof of Theorem 2.3

3.4.1. Technical Lemma. Before we approach the proof of Theorem 2.3, we prove a technical statement in Lemma 3.2 that will be important later. In addition to this, Lemma 3.2 implies also two specific combinatorial identities. In order not to distract attention from the main exposition line, we present these corollaries not until in Sect. 4.

Lemma 3.2. *Suppose that $k \in \mathbb{N}_0$ and $n \in \mathbb{R}$. Then*

$$\prod_{i=0}^{k}(n-i) = \sum_{i=0}^{k} c_{i,k} \cdot \prod_{t=0}^{i}(2n-t), \tag{3.11}$$

where

$$c_{i,k} := \frac{(-1)^{i+k}}{2^{2k-i+1} \cdot i!} \cdot \prod_{t=i}^{i+k-1}(2k-t). \tag{3.12}$$

Proof. We will proceed by induction with respect to k. Of course, for $k = 0$ formula (3.11) holds trivially. Next, assume that (3.11) is true for a fixed $k \in \mathbb{N}_0$. Then

$$\prod_{i=0}^{k+1}(n-i) = (n-k-1) \cdot \sum_{i=0}^{k} c_{i,k} \cdot \prod_{t=0}^{i}(2n-t) \tag{3.13}$$

by the inductive assumption. Since

$$n-k-1 = \frac{2n-i-1}{2} - \frac{2k-i+1}{2},$$

we deduce by this and by (3.13) that

$$\begin{aligned}
\prod_{i=0}^{k+1}(n-i) &= \sum_{i=0}^{k} \frac{c_{i,k}}{2} \cdot \prod_{t=0}^{i+1}(2n-t) - \sum_{i=0}^{k} \frac{c_{i,k}}{2} \cdot (2k-i+1) \cdot \prod_{t=0}^{i}(2n-t) \\
&= \sum_{i=1}^{k+1} \frac{c_{i-1,k}}{2} \cdot \prod_{t=0}^{i}(2n-t) - \sum_{i=0}^{k} \frac{c_{i,k}}{2} \cdot (2k-i+1) \cdot \prod_{t=0}^{i}(2n-t).
\end{aligned}$$

Next, we separate the first summand from the last sum corresponding to $i = 0$ and the last term from the first sum corresponding to $i = k+1$. Moreover, this and merging the remaining summands of both sums into a single sum allow us to write

$$\begin{aligned}\prod_{i=0}^{k+1}(n-i) &= -\frac{2k+1}{2}\cdot c_{0,k}\cdot 2n + \frac{c_{k,k}}{2}\cdot\prod_{t=0}^{k+1}(2n-t)\\ &\quad + \sum_{i=1}^{k}\Big(\frac{c_{i-1,k}}{2} - \frac{c_{i,k}}{2}\cdot(2k-i+1)\Big)\cdot\prod_{t=0}^{i}(2n-t)\\ &= \sum_{i=0}^{k+1}\gamma_{i,k+1}\cdot\prod_{t=0}^{i}(2n-t),\end{aligned}$$

where

$$\begin{aligned}\gamma_{0,k+1} &:= -\frac{2k+1}{2}\cdot c_{0,k},\\ \gamma_{i,k+1} &:= \frac{c_{i-1,k}}{2} - (2k-i+1)\cdot\frac{c_{i,k}}{2}, \quad i=1,\dots,k,\\ \gamma_{k+1,k+1} &:= \frac{c_{k,k}}{2}.\end{aligned}$$

To finish the induction step, it suffices to show that $\gamma_{i,k+1} = c_{i,k+1}$ for $i = 0,\dots,k+1$, where the form of $c_{i,k+1}$ corresponds to $c_{i,k}$ defined in (3.12) with $k+1$ instead of k. Thus, by the definitions of $\gamma_{i,k+1}$ above and by (3.12), we deduce that

$$\begin{aligned}\gamma_{0,k+1} &= -\frac{2k+1}{2}\cdot\frac{(-1)^k}{2^{2k+1}}\cdot\prod_{t=0}^{k-1}(2k-t)\\ &= (2k+1)\cdot\frac{(-1)^{k+1}}{2^{2k+2}}\cdot\prod_{t=0}^{k-1}\big(2(k+1)-(t+2)\big)\\ &= \frac{(-1)^{k+1}}{2^{2k+2}}\cdot\prod_{t=1}^{k+1}\big(2(k+1)-t\big)\\ &= \frac{(-1)^{k+1}}{2^{2k+3}}\cdot\prod_{t=0}^{k}\big(2(k+1)-t\big) = c_{0,k+1},\end{aligned}$$

as expected. Similarly for $\gamma_{i,k+1}$ with $i = 1,\dots,k$:

$$\begin{aligned}\gamma_{i,k+1} &= \frac{(-1)^{i+k-1}}{2\cdot 2^{2k-i+2}\cdot(i-1)!}\cdot\prod_{t=i-1}^{i+k-2}(2k-t)\\ &\quad - (2k-i+1)\cdot\frac{(-1)^{i+k}}{2\cdot 2^{2k-i+1}\cdot i!}\cdot\prod_{t=i}^{i+k-1}(2k-t)\\ &= \frac{(-1)^{i+k+1}}{2^{2k-i+3}\cdot i!}\cdot\left(\prod_{t=i-1}^{i+k-2}(2k-t)\right)\cdot\big(i+2(k-i+1)\big)\end{aligned}$$

$$= \frac{(-1)^{i+k+1}}{2^{2k-i+3} \cdot i!} \cdot \prod_{t=i-2}^{i+k-2} (2k - t)$$
$$= \frac{(-1)^{i+k+1}}{2^{2k-i+3} \cdot i!} \cdot \prod_{t=i}^{i+k} (2(k+1) - t) = c_{i,k+1}.$$

Finally, due to the fact that $c_{k,k} = 1/2^{k+1}$ following directly from (3.12), we easily see that $\gamma_{k+1,k+1} = c_{k,k}/2 = 1/2^{k+2} = c_{k+1,k+1}$. This concludes the proof by induction. □

3.4.2. The Proof. The proof of Theorem 2.3 presented below makes use of the gamma function defined by

$$\Gamma(z) := \lim_{n\to\infty} \frac{n^z \cdot n!}{\prod_{i=0}^{n}(i+z)}, \quad z \in \mathbb{C}\backslash\mathbb{Z}_0^-.$$

Since $\Gamma(z+1) = z \cdot \Gamma(z)$ as follows from the definition above, we readily see that

$$\prod_{n=1}^{N} (n+z) = \frac{\Gamma(N+z+1)}{\Gamma(z+1)}, \quad N \in \mathbb{N}_0, \tag{3.14}$$

whenever the right-hand side exists. See [9, Chap. 3] or [13, Sect. 1.1] for more details on the gamma function.

Proof of Theorem 2.3. For calculating the value of the series $\varphi_k(n \cdot (n+1))$, we take into account its generating function in (3.3) and express the closed form of the corresponding infinite product. We obtain

$$P_1(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{n \cdot (n+1)}\right)$$
$$= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{\left(n + \frac{1}{2} - \sqrt{x + \frac{1}{4}}\right) \cdot \left(n + \frac{1}{2} + \sqrt{x + \frac{1}{4}}\right)}{n \cdot (n+1)},$$

where x is sufficiently small. Thanks to (3.14), we immediately infer that

$$P_1(x) = \lim_{N\to\infty} \frac{\frac{\Gamma\left(N + \frac{3}{2} - \sqrt{x+\frac{1}{4}}\right)}{\Gamma\left(\frac{3}{2} - \sqrt{x+\frac{1}{4}}\right)} \cdot \frac{\Gamma\left(N + \frac{3}{2} + \sqrt{x+\frac{1}{4}}\right)}{\Gamma\left(\frac{3}{2} + \sqrt{x+\frac{1}{4}}\right)}}{\Gamma(N+1) \cdot \Gamma(N+2)}.$$

Let us now consider the limit of the terms with the variable N. Indeed, invoking the formula (see [13, p. 117])

$$\lim_{N\to\infty} \prod_{s=1}^{t} \frac{\Gamma(N+a_s)}{\Gamma(N+b_s)} = 1, \quad \sum_{s=1}^{t} a_s = \sum_{s=1}^{t} b_s,$$

with $t=2$, $a_{1,2}=\frac{3}{2}\pm\sqrt{x+\frac{1}{4}}$ and $b_1=1$, $b_2=2$, we deduce that

$$\begin{aligned} P_1(x) &= \frac{1}{\Gamma\left(\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)\cdot\Gamma\left(\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)} \\ &= -\frac{1}{\pi x}\cdot\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\sqrt{1+4x}\right) \\ &= -\frac{\cos\left(\frac{\pi}{2}\sqrt{1+4x}\right)}{\pi x}, \end{aligned} \tag{3.15}$$

where we utilized the relation $\Gamma(z+1)=z\cdot\Gamma(z)$ and Euler's reflection formula $\Gamma(z)\cdot\Gamma(1-z)=\pi/\sin(\pi z)$, $z\notin\mathbb{Z}$.

Next, we expand the cosine function in (3.15) into the Maclaurin series and extract the coefficients of the powers of x:

$$\begin{aligned} P_1(x) &= \frac{1}{\pi x}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot(1+4x)^n \\ &= \frac{1}{\pi x}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot\sum_{k=0}^{n}\binom{n}{k}\cdot(4x)^k \\ &= \frac{1}{\pi x}\cdot\sum_{k=0}^{\infty}(4x)^k\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot\binom{n}{k} \\ &= \frac{1}{\pi x}\cdot\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot\prod_{i=0}^{k-1}(n-i), \end{aligned}$$

where the last outer sum is extended over $k\in\mathbb{N}$ since for $k=0$, the corresponding term is $\cos(\pi/2)=0$. Further, we apply Lemma 3.2 on the last finite product which yields

$$P_1(x)=\frac{1}{\pi x}\cdot\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\cdot\sum_{i=0}^{k-1}c_{i,k-1}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot\prod_{t=0}^{i}(2n-t), \tag{3.16}$$

with $c_{i,k}$ defined in (3.12). Let us further denote the inner infinite series in (3.16) by R_i, i.e.

$$R_i:=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\cdot\left(\frac{\pi}{2}\right)^{2n}\cdot\prod_{t=0}^{i}(2n-t),\quad i\in\mathbb{N}_0.$$

To finish the proof, it is enough to express the closed form of R_i which can be realized as follows:

$$R_i=\sum_{n=\lceil\frac{i+1}{2}\rceil}^{\infty}\frac{(-1)^{n-1}}{(2n-i-1)!}\cdot\left(\frac{\pi}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+\lceil \frac{i+1}{2} \rceil - 1}}{\left(2\left(n+\lceil \frac{i+1}{2}\rceil\right) - i - 1\right)!} \cdot \left(\frac{\pi}{2}\right)^{2\left(n+\lceil \frac{i+1}{2}\rceil\right)}$$
$$= (-1)^{\lceil \frac{i+1}{2} \rceil - 1} \cdot \left(\frac{\pi}{2}\right)^{2\lceil \frac{i+1}{2} \rceil - \delta_i} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+\delta_i)!} \cdot \left(\frac{\pi}{2}\right)^{2n+\delta_i},$$

where we set

$$\delta_i := 2 \cdot \left\lceil \frac{i+1}{2} \right\rceil - i - 1$$
$$= \frac{1+(-1)^i}{2}, \quad i \in \mathbb{N}_0.$$

Consequently, it is not difficult to observe that this leads to

$$R_i = (-1)^{\lceil \frac{i+1}{2} \rceil - 1} \cdot \left(\frac{\pi}{2}\right)^{i+1} \cdot \begin{cases} \sin\left(\frac{\pi}{2}\right) & \text{for} \quad i \text{ even}, \\ \cos\left(\frac{\pi}{2}\right) & \text{for} \quad i \text{ odd} \end{cases}$$
$$= (-1)^{\lceil \frac{i+1}{2} \rceil - 1} \cdot \left(\frac{\pi}{2}\right)^{i+1} \cdot \delta_i.$$

These calculations and (3.16) imply now that

$$P_1(x) = \frac{1}{\pi x} \cdot \sum_{k=1}^{\infty} \frac{(4x)^k}{k!} \cdot \sum_{i=0}^{k-1} c_{i,k-1} \cdot (-1)^{\lceil \frac{i+1}{2} \rceil - 1} \cdot \left(\frac{\pi}{2}\right)^{i+1} \cdot \delta_i$$
$$= \frac{1}{\pi x} \cdot \sum_{k=1}^{\infty} \frac{(4x)^k}{k!} \cdot \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} c_{2i,k-1} \cdot (-1)^i \cdot \left(\frac{\pi}{2}\right)^{2i+1}$$
$$= \sum_{k=1}^{\infty} x^{k-1} \cdot \frac{4^k}{k!} \cdot \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^{i+k-1} \cdot \pi^{2i}}{2^{2k} \cdot (2i)!} \cdot \prod_{t=2i}^{2i+k-2} (2k-t-2)$$
$$= \sum_{k=0}^{\infty} (-x)^k \cdot \frac{1}{(k+1)!} \cdot \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \cdot \frac{\pi^{2i}}{(2i)!} \cdot \prod_{t=2i}^{2i+k-1} (2k-t)$$
$$= \sum_{k=0}^{\infty} (-x)^k \cdot \frac{1}{k+1} \cdot \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \cdot \frac{\pi^{2i}}{(2i)!} \cdot \binom{2k-2i}{k}.$$

If we compare the coefficients of the same powers of x in the last infinite series and in (3.3), we conclude the proof of Theorem 2.3. □

3.5. Proof of Theorem 2.4

3.5.1. Initial Discussion. According to (3.3) and (3.4), it would seem likely that we utilize (3.15) and find the closed form of $\varphi_k^{\star}(n \cdot (n+1))$ by means of the relation

$$\frac{-\pi x}{\cos\left(\frac{\pi}{2}\sqrt{1+4x}\right)} = 1 + \sum_{k=1}^{\infty} x^k \cdot \varphi_k^\star(n \cdot (n+1)).$$

Unfortunately, after making several attempts, we were currently unable to find a suitable way for arriving at a compact form of coefficients of the expansion with the involved secant function into Laurent series centered at $x = 0$.

This is the reason why we opt for another approach based on an effective transformation of the series $\varphi_k^\star(n \cdot (n+1))$. In order to realize this transformation, we apply the following general theorem proved by the author in [7, Theorem 2.1].

Theorem 3.1. *Let $p(n)$, $q(n)$ be real polynomials. Put $R(n) = p(n)/q(n)$ and suppose that the following conditions are satisfied:*

(C1) *the infinite series $\sum_{n=1}^{\infty} R(n)$ converges,*
(C2) *$R(n) \neq 0$ for all $n \in \mathbb{N}$,*
(C3) *$R(m) \neq R(n)$ for all pairs $(m, n) \in \mathbb{N}^2$ with $m \neq n$,*
(C4) *the derivative $R'(n) \neq 0$ for all $n \in \mathbb{N}$,*
(C5) *there exists $l_0 \in \mathbb{N}$ such that $R(n) > 0$ for all $n > l_0$.*

Set $Q := \deg q(n)$. Then for every $k \in \mathbb{N}$ we have

$$\sum_{n_1 \geq \cdots \geq n_k \geq 1} \prod_{j=1}^{k} R(n_j) = \sum_{l=1}^{\infty} \frac{(-1)^l}{(l-1)!} \cdot R'(l) \cdot R^{k-1}(l) \cdot \frac{\prod_{d=2}^{Q} \Gamma(1-\beta_d(l))}{\prod_{d=1}^{Q} \Gamma(1-\alpha_d)} \tag{3.17}$$

whenever the right-hand side converges with α_d's defined by the unique factorization

$$q(n) = a_0 \cdot \prod_{d=1}^{Q} (n - \alpha_d),$$

β_d's are all roots of the equation $R(n) = R(l)$, with the unknown variable n and a discrete parameter l, and $\beta_1(l) := l$ denotes its trivial solution.

Remark. It is not hard to see that for $p(n) = 1$ and $q(n) = n^2$, Theorem 3.1 immediately implies the identity

$$\zeta^\star(\{2\}_k) = 2\,\eta(2k), \quad k \in \mathbb{N},$$

where η denotes the Dirichlet η-function. Consequently, according to the relation $\eta(s) = (1 - 2^{1-s}) \cdot \zeta(s)$, Theorem 3.1 enables us to prove the evaluation of $\zeta^\star(\{2\}_k)$ in (3.9) by applying (3.8).

3.5.2. The Proof. We are now ready to present the proof of Theorem 2.4 based on the transformation identity (3.17).

Proof of Theorem 2.4. If we set $p(n) = 1$ and $q(n) = n\cdot(n+1)$ in Theorem 3.1, then $R(n) = 1/(n\cdot(n+1))$ and the conditions (C1)–(C5) are obviously satisfied. Indeed, the left-hand side of (3.17) coincides with the investigated series $\varphi_k^\star(n\cdot(n+1))$ and, as follows from the form of $q(n)$, we have $a_0 = 1$, $\alpha_1 = 0$ and $\alpha_2 = -1$. Solving the equation $R(n) = R(l)$ with respect to the unknown n implies its solutions $\beta_1(l) = l$ and $\beta_2(l) = -l-1$. Consequently, by applying the transformation (3.17), we get for every $k \in \mathbb{N}$

$$\begin{aligned}\varphi_k^\star(n\cdot(n+1)) &= \sum_{n_1\geq\cdots\geq n_k\geq 1}\prod_{j=1}^{k}\frac{1}{n_j\cdot(n_j+1)}\\ &= \sum_{l=1}^{\infty}\frac{(-1)^l}{(l-1)!}\cdot\frac{-2l-1}{(l\cdot(l+1))^{k+1}}\cdot(l+1)!\\ &= \sum_{l=1}^{\infty}(-1)^{l-1}\cdot\frac{2l+1}{l^k\cdot(l+1)^k}.\end{aligned} \tag{3.18}$$

To finish the proof, we will first decompose the rational part of the summand in (3.18) into partial fractions and then we perform the summation. To realize the decomposition, we employ

$$\frac{1}{l^k\cdot(l+1)^k} = (-1)^k\cdot\sum_{i=1}^{k}\binom{2k-i-1}{k-1}\cdot\left(\frac{(-1)^i}{l^i}+\frac{1}{(l+1)^i}\right), \quad k\in\mathbb{N}, \tag{3.19}$$

see, e.g. Smirnov [12, p. 38]. Differentiating this identity with respect to l yields

$$\frac{-k\cdot(2l+1)}{l^{k+1}\cdot(l+1)^{k+1}} = (-1)^k\cdot\sum_{i=1}^{k}\binom{2k-i-1}{k-1}\cdot\left(\frac{i\cdot(-1)^{i-1}}{l^{i+1}}-\frac{i}{(l+1)^{i+1}}\right),$$

and, after evident simplifications, we arrive at the sought decomposition

$$\frac{2l+1}{l^{k+1}\cdot(l+1)^{k+1}} = \frac{(-1)^k}{k}\cdot\sum_{i=2}^{k+1}(i-1)\cdot\binom{2k-i}{k-1}\cdot\left(\frac{(-1)^{i-1}}{l^i}+\frac{1}{(l+1)^i}\right).$$

Reducing the parameter k to $k-1$ in the last identity and applying it on (3.18) allow us to evaluate $\varphi_k^\star := \varphi_k^\star(n\cdot(n+1))$ for $k\geq 2$ as follows:

$$\begin{aligned}\varphi_k^\star = {}& \frac{(-1)^{k-1}}{k-1}\cdot\sum_{i=2}^{k}(i-1)\cdot\binom{2k-i-2}{k-2}\\ &\times\left((-1)^{i-1}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l^i}+\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{(l+1)^i}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{k-1}}{k-1} \cdot \sum_{i=2}^{k} (i-1) \cdot \binom{2k-i-2}{k-2} \cdot \big(((-1)^{i-1} - 1) \cdot \eta(i) + 1 \big) \\
&= \frac{(-1)^{k}}{k-1} \cdot \Bigg(\sum_{i=2}^{k} (i-1) \cdot \binom{2k-i-2}{k-2} \cdot \big(((-1)^{i} + 1) \cdot \eta(i) \big) \\
&\quad - \sum_{i=2}^{k} (i-1) \cdot \binom{2k-i-2}{k-2} \Bigg) \\
&= \frac{(-1)^{k}}{k-1} \cdot \Bigg(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} (2i-1) \cdot \binom{2k-2i-2}{k-2} \cdot 2\,\eta(2i) - \binom{2k-2}{k-2} \Bigg) \\
&= \frac{2 \cdot (-1)^{k}}{k-1} \cdot \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (2i-1) \cdot \binom{2k-2i-2}{k-2} \cdot \eta(2i), \qquad (3.20)
\end{aligned}
$$

where we used $\eta(0) = 1/2$ and the identity $\sum_{i=2}^{k}(i-1) \cdot \binom{2k-i-2}{k-2} = \binom{2k-2}{k-2}$ whose standard proof we leave to readers. Finally, we find out that the form in (3.20) completely coincides with the formula in Theorem 2.4. The proof is complete. □

3.6. Proof of Theorem 2.5

For the proof of Theorem 2.5, we need the closed-form evaluation of the series S_i defined in (3.2) with $a_n = n \cdot (n+1)$. The proof of Theorem 2.5 is therefore preceded by the following lemma, where we provide the closed form for S_i.

3.6.1. Auxiliary Summation Result.

Lemma 3.3. *Suppose that $i \in \mathbb{N}$. Then*

$$
\sum_{n=1}^{\infty} \frac{1}{n^{i} \cdot (n+1)^{i}} = \frac{(-1)^{i-1}}{2} \cdot \binom{2i}{i} + 2 \cdot (-1)^{i} \cdot \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \cdot \zeta(2j). \qquad (3.21)
$$

Proof. We evaluate the investigated series by employing the partial fraction decomposition in (3.19). Denoting the series on the left-hand side in (3.21) as S_i, we deduce by (3.19) that

$$
S_i = \sum_{n=1}^{\infty} (-1)^{i} \cdot \sum_{j=1}^{i} \binom{2i-j-1}{i-1} \cdot \left(\frac{(-1)^{j}}{n^{j}} + \frac{1}{(n+1)^{j}} \right).
$$

Splitting the inner finite sum into two sums with respect to the parity of j implies

$$
S_i = (-1)^{i} \cdot \sum_{j=1}^{\lceil \frac{i}{2} \rceil} \binom{2i-2j}{i-1} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^{2j-1}} - \frac{1}{n^{2j-1}} \right)
$$

$$
\begin{aligned}
&+ (-1)^i \cdot \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^{2j}} + \frac{1}{n^{2j}} \right) \\
&= (-1)^{i-1} \cdot \sum_{j=1}^{\lceil \frac{i}{2} \rceil} \binom{2i-2j}{i-1} + (-1)^i \cdot \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \cdot \big(2\,\zeta(2j) - 1\big) \\
&= (-1)^{i-1} \cdot \left(\sum_{j=1}^{\lceil \frac{i}{2} \rceil} \binom{2i-2j}{i-1} + \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \right) \\
&\quad + 2 \cdot (-1)^i \cdot \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \cdot \zeta(2j) \\
&= (-1)^{i-1} \cdot \sum_{j=1}^{i} \binom{2i-j-1}{i-1} + 2 \cdot (-1)^i \cdot \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \binom{2i-2j-1}{i-1} \cdot \zeta(2j).
\end{aligned}
$$

Since for the first sum we have $\sum_{j=1}^{i} \binom{2i-j-1}{i-1} = \frac{1}{2} \cdot \binom{2i}{i}$, the identity in (3.21) now follows. This completes the proof. □

3.6.2. The Proof.

Proof of Theorem 2.5. The basis of the proof consists in the crucial Eq. (3.5) with $a_n = n \cdot (n+1)$. However, instead of comparing whole expressions on the left- and right-hand sides of (3.5), we only compare specific coefficients of powers of π occurring in them. This is certainly possible since π is transcendental. For the proof of Theorem 2.5, it suffices to consider the coefficients of π^0, i.e.

$$
\left[\pi^0\right]\ \varphi_k(n \cdot (n+1)) = \left[\pi^0\right]\ (-1)^k \cdot \sum_{\substack{\sum_{i=1}^{k} i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i} \right)^{m_i},
$$

where we used the standard notation with $[\pi^p] \sum_{i=0}^{m} c_i \cdot \pi^i := c_p$.

Assume now that $\varepsilon = 1$. Then, by Theorem 2.3, we immediately obtain

$$
\left[\pi^0\right]\ \varphi_k(n \cdot (n+1)) = \frac{1}{k+1} \cdot \binom{2k}{k} = C_k, \tag{3.22}
$$

where C_k denotes the k-th Catalan number. Similarly, we can find the corresponding coefficient on the right-hand side. For this purpose, it suffices to consider the coefficients $[\pi^0]\, S_i$ that can readily be obtained by Lemma 3.3. Hence,

$$
\left[\pi^0\right]\ (-1)^k \cdot \sum_{\substack{\sum_{i=1}^{k} i m_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i} \right)^{m_i}
$$

$$= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-\left[\pi^0\right] S_i}{i} \right)^{m_i}$$

$$= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{(-1)^i}{2i} \cdot \binom{2i}{i} \right)^{m_i}$$

$$= \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{1}{2i} \cdot \binom{2i}{i} \right)^{m_i}. \tag{3.23}$$

Finally, by equating the obtained coefficients in (3.22) and in (3.23), we conclude the proof of Theorem 2.5 for $\varepsilon = 1$.

For the case $\varepsilon = -1$, it suffices to apply Theorem 2.4 accordingly. In analogy with (3.22), we deduce that

$$\left[\pi^0\right] \varphi_k^\star(n \cdot (n+1)) = \begin{cases} 1 & \text{for } k = 1, \\ \dfrac{(-1)^{k-1}}{k-1} \cdot \binom{2k-2}{k-2} & \text{for } k \geq 2 \end{cases}$$
$$= (-1)^{k-1} \cdot C_{k-1}.$$

Further, the proof is almost identical with the considerations in the previous part taking into account that it is necessary to apply (3.6). The details are left to readers. □

3.7. Proof of Theorem 2.6

Proof. The proof of Theorem 2.6 is similar to the proof of Theorem 2.5. First, assume that $\varepsilon = 1$. The idea consists in considering the equality

$$\left[\pi^2\right] \varphi_k(n \cdot (n+1)) = \left[\pi^2\right] (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k im_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i} \right)^{m_i}$$

following from (3.5), i.e. we equate the coefficients of π^2 only. For the left-hand side, we deduce from Theorem 2.3 that

$$\left[\pi^2\right] \varphi_k(n \cdot (n+1)) = \frac{-1}{2(k+1)} \cdot \binom{2k-2}{k} = \frac{C_k - C_{k-1}}{-6}.$$

To analyse the corresponding coefficient of π^2 of the right-hand side of the above equality, we first investigate the coefficients $[\pi^0]S_i^{m_i}$, $i \geq 1$, and $[\pi^2]S_i^{m_i}$, $i \geq 2$. For short, we write the linear combination for S_i, see Lemma 3.3, as $S_i = \sum_{j=0}^{\lfloor i/2 \rfloor} \alpha_{j,i} \cdot \pi^{2j}$. Consequently,

$$\alpha_{0,i} = \frac{(-1)^{i-1}}{2} \cdot \binom{2i}{i},$$

$$\alpha_{1,i} = \frac{(-1)^i}{3} \cdot \binom{2i-3}{i-1},$$

and, accordingly,

$$\left[\pi^0\right] S_i^{m_i} = \alpha_{0,i}^{m_i}, \tag{3.24}$$

$$\left[\pi^2\right] S_i^{m_i} = m_i \cdot \alpha_{0,i}^{m_i - 1} \cdot \alpha_{1,i}. \tag{3.25}$$

These facts imply

$$\begin{aligned}
&\left[\pi^2\right] (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i}\right)^{m_i} \\
&= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0, \\ m_1 < k}} \left(\prod_{i=1}^k \frac{\left(\frac{-1}{i}\right)^{m_i}}{m_i!}\right) \cdot \left[\pi^2\right] \left(S_1^{m_1} \cdot S_2^{m_2} \cdots S_k^{m_k}\right),
\end{aligned}$$

where it was necessary to append the condition $m_1 < k$ for excluding the k-tuple $(m_1, \ldots, m_k) = (k, 0, \ldots, 0)$ from the summation range since the corresponding summand does not involve any π^2-term due to $S_1 = 1$. Therefore, by this and by (3.24) and (3.25), we obtain

$$\begin{aligned}
&\left[\pi^2\right] (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i}\right)^{m_i} \\
&= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0, \\ m_1 < k}} \left(\prod_{i=1}^k \frac{\left(\frac{-1}{i}\right)^{m_i}}{m_i!}\right) \cdot \sum_{\substack{\sum_{i=2}^k t_i = 1 \\ t_i \geq 0}} \prod_{i=2}^k \left[\pi^{2t_i}\right] S_i^{m_i} \\
&= (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0, \\ m_1 < k}} \left(\prod_{i=1}^k \frac{\left(\frac{-\alpha_{0,i}}{i}\right)^{m_i}}{m_i!}\right) \cdot \sum_{t=2}^k m_t \cdot \frac{\alpha_{1,t}}{\alpha_{0,t}}.
\end{aligned}$$

Since $\frac{\alpha_{1,t}}{\alpha_{0,t}} = -\frac{t}{6(2t-1)}$, we finally get after performing usual simplifications that

$$\begin{aligned}
&\left[\pi^2\right] (-1)^k \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{-S_i}{i}\right)^{m_i} \\
&= -\frac{1}{6} \cdot \sum_{\substack{\sum_{i=1}^k i m_i = k \\ m_i \geq 0}} \left(\prod_{i=1}^k \frac{1}{m_i!} \cdot \left(\frac{1}{2i} \cdot \binom{2i}{i}\right)^{m_i}\right) \cdot \sum_{t=2}^k \frac{t \cdot m_t}{2t-1},
\end{aligned}$$

where we dropped the summation condition $m_1 < k$ corresponding to the excluded k-tuple $(m_1, m_2, \ldots, m_k) = (k, 0, \ldots, 0)$. Namely, for this choice the involved finite sum $\sum_{t=2}^{k} t \cdot m_t/(2t-1)$ vanishes. Equating the above coefficient with $\left[\pi^2\right] \varphi_k(n \cdot (n+1))$ and multiplying by -6 yield

$$\sum_{\substack{\sum_{i=1}^{k} i m_i = k \\ m_i \geq 0}} \left(\prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{1}{2i} \cdot \binom{2i}{i} \right)^{m_i} \right) \cdot \sum_{t=2}^{k} \frac{t \cdot m_t}{2t-1} = C_k - C_{k-1}$$

which coincides with the statement of Theorem 2.6 for $\varepsilon = 1$ since the inner finite sum involved on the left-hand side of the last relation can be identified as the weight $W_k(\boldsymbol{m})$ introduced in (2.1).

For $\varepsilon = -1$, the proof is conducted analogously. We omit the details and leave them to readers. $\square$

4. Remarks on Lemma 3.2

Even if Lemma 3.2 is rather a technical tool with respect to the main objectives of this paper, the following two combinatorial identities can be of particular interest.

4.1. Identity with a Trinomial

The relation in (3.11) can be treated as a class of identities with the parameters k and n. For instance, setting $n = k + p$ with $k \in \mathbb{N}_0$ and $p \in \mathbb{N}$ results in the following formula with a trinomial coefficient on the right-hand side.

Corollary 4.1. *Suppose that $k \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$\sum_{i=0}^{k} (-2)^i \cdot \frac{\binom{k}{i}}{\binom{2k+2p-i-1}{2p-1}} = (-4)^k \cdot \frac{\binom{2p}{p}}{\binom{2k+2p}{k,p,k+p}}.$$

Outline of the proof. Set $n = k + p$ with $k \in \mathbb{N}_0$, $p \in \mathbb{N}$, in (3.11) and perform standard simplifications. $\square$

Specifically, by setting $p = k$, $k \in \mathbb{N}$, the identity in the above corollary reduces after simple manipulations to the refined form

$$\sum_{i=0}^{k} \frac{1}{(-2)^i} \cdot \frac{\binom{k}{i}}{\binom{3k+i-1}{2k-1}} = \frac{2^k}{\binom{4k}{2k}}.$$

4.2. Identity with Stirling Numbers of the First Kind

The second identity following from (3.11) arises when considering the powers n^p, $p = 1, \ldots, k+1$, on both sides of it.

Corollary 4.2. *For $k \in \mathbb{N}_0$ and $p \in \mathbb{N}$, we have*

$$\sum_{i=p-1}^{k} \frac{(-2)^i}{i!} \cdot \begin{bmatrix} i+1 \\ p \end{bmatrix} \cdot \binom{2k-i}{k} = 2^{k-p+1} \cdot \frac{(-2)^k}{k!} \cdot \begin{bmatrix} k+1 \\ p \end{bmatrix},$$

where $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ denotes the signed Stirling number of the first kind.

Proof. From the definition of the Stirling number of the first kind (see [13, p. 76], e.g.), we immediately deduce that

$$\prod_{i=0}^{k}(n-i) = \sum_{t=1}^{k+1} n^t \cdot \begin{bmatrix} k+1 \\ t \end{bmatrix},$$
$$\prod_{t=0}^{i}(2n-t) = \sum_{t=1}^{i+1} (2n)^t \cdot \begin{bmatrix} i+1 \\ t \end{bmatrix}.$$

Therefore, for $p = 1, \ldots, k+1$ we conclude that

$$[n^p]\prod_{i=0}^{k}(n-i) = \begin{bmatrix} k+1 \\ p \end{bmatrix}, \tag{4.1}$$

and, taking into account the definition of $c_{i,k}$ in (3.12),

$$\begin{aligned} [n^p]\sum_{i=0}^{k} c_{i,k} \cdot \prod_{t=0}^{i}(2n-t) &= [n^p]\sum_{i=0}^{k} \frac{(-1)^{i+k}}{2^{2k-i+1} \cdot i!} \cdot \frac{(2k-i)!}{(k-i)!} \cdot \sum_{t=1}^{i+1}(2n)^t \begin{bmatrix} i+1 \\ t \end{bmatrix} \\ &= \sum_{i=p-1}^{k} \frac{(-1)^{i+k} \cdot k!}{2^{2k-i+1} \cdot i!} \cdot \binom{2k-i}{k} \cdot 2^p \cdot \begin{bmatrix} i+1 \\ p \end{bmatrix} \\ &= \frac{(-1)^k \cdot k!}{2^{2k-p+1}} \cdot \sum_{i=p-1}^{k} \frac{(-2)^i}{i!} \cdot \binom{2k-i}{k} \cdot \begin{bmatrix} i+1 \\ p \end{bmatrix}. \end{aligned} \tag{4.2}$$

Consequently, equating the expressions in (4.1) and in (4.2), we arrive at

$$\begin{bmatrix} k+1 \\ p \end{bmatrix} = \frac{(-1)^k \cdot k!}{2^{2k-p+1}} \cdot \sum_{i=p-1}^{k} \frac{(-2)^i}{i!} \cdot \binom{2k-i}{k} \cdot \begin{bmatrix} i+1 \\ p \end{bmatrix}$$

which is equivalent to the formula in Corollary 4.2. The proof is complete. □

5. Concluding Remarks

5.1. Specific Generalizations

It is worth pointing out that Hoffman's formula (1.1) can be generalized in light of the known formulas for the multiple zeta values $\zeta(\{2i\}_k)$ and $\zeta^\star(\{2i\}_k)$, $i \in \mathbb{N}$, see Chen and Chung [2, p. 7], Chen et al. [3, Theorem 1, Proposition 3] for the closed-form evaluations of these values. For instance, for $a_n = n^4$ and $\varepsilon = 1$ we get

$$\sum_{\substack{\sum_{i=1}^{k} im_i = k \\ m_i \geq 0}} \prod_{i=1}^{k} \frac{1}{m_i!} \cdot \left(\frac{B_{4i}}{2i \cdot (4i)!} \right)^{m_i} = \frac{2 \cdot (-4)^k}{(4k+2)!}$$

with a slightly different structure than the form in Theorem 2.1 for $\varepsilon = 1$.

We notice, however, that the approach with evaluating the infinite product $P_1(x)$ is not very suitable for obtaining the above formula since $P_1(x) = \sin(\pi x) \sinh(\pi x)/(\pi^2 x^2)$. Therefore, expanding the obtained function for $P_1(x)$ into Maclaurin series directly gives its coefficients in the form of a convolution. To obtain the coefficients of the Maclaurin expansion of $P_1(x)$ for the considered choice of a_n and ε, it is advisable to apply stronger methods described in the afore-mentioned papers. Moreover, for $\varepsilon = -1$ and $a_n = n^4$, the formula corresponding to the relation above seems to be inevitably more complicated in the sense that its right-hand side involves a finite sum which follows from the closed form of $\varphi_k^\star(n^4) = \zeta^\star(\{4\}_k)$, see [2, Eq. (2.1)].

5.2. Results on Similar Sums

We remark that Merca [10,11] studied specific sums over unrestricted integer partitions with relations to Bernoulli numbers and values of the Riemann zeta function. However, as far as we can see, none of Merca's results seem to be related to the results presented in this work.

Also, Zriaa and Mouçouf [16] investigated specific combinatorial sums in their very recent research. Their method makes use of sums over partitions in connection with the Faà di Bruno formula and derivatives of rational functions. See [16, Theorem 3.3] for an identity involving Bell polynomials. But also these results do not seem to be connected with our findings.

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Declarations

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