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Results in Mathematics

A Universal Space for the Bourbaki-Complete Spaces and Further Examples

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Abstract. In this paper it is provided an explicit embedding for complete metric spaces having a star-finite base of its uniformity into the universal space $\kappa^{\omega_0} \times \mathbb{R}^{\omega_0}$, where $\kappa \geq \omega_0$ is the cellularity of the space and κ^{ω_0} is the Baire space of weight κ. From this result, similar embeddings for general Bourbaki-complete uniform spaces as well as for metric spaces are derived. In particular, these embeddings partially preserve the uniform structure of the original space. Several examples of Bourbaki-complete metric spaces having no star-finite base for its metric uniformity are constructed.

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Keywords. Bourbaki-complete uniform spaces, star-finite covers, strongly metrizable, uniform embedding.

1. Introduction

The central topic of this paper is the uniform notion of Bourbaki-completeness which was introduced in [12] in the frame of metric spaces and that later was generalized to uniform spaces in [13].

This notion is stronger than usual completeness in metric and uniform spaces. Basic examples of Bourbaki-complete spaces are finite dimensional Banach space and uniformly discrete spaces. On the other hand, not every complete uniform space is Bourbaki-complete. For instance, any infinite dimensional Banach space and any metric hedgehog $J(\kappa)$, $\kappa \geq \omega_0$, (see [7, Example 4.1.5]) are complete metric spaces failing Bourbaki-completeness. Richer examples can be found in [12, 17].

In this paper we collect several classical results concerning Bourbaki-completeness. These are classical in the sense that parallel results are known for the complete uniform spaces and the complete metric spaces (see for instance [7]), as well as, for the cofinally complete uniform spaces and cofinally complete metric spaces (see for instance [16,18,29]).

More precisely, we find respective universal spaces for the Bourbaki-complete uniform and metric spaces (Theorems 4.3 and 5.8) and from them we derived results about the Bourbaki-complete uniformization (Theorem 4.8) and Bourbaki-complete metrization (Theorem 6.5). These results on uniformization and metrization were known (see [13,17]) and are, respectively, related to the topological properties of δ-completeness ([8]) and strong-metrizability ([23]).

The novelty here is the use of a universal space, which preserves partially the uniformity of the embedded Bourbaki-complete spaces. The main technical tool used in the embedding results is the star-finite modification of a uniformity (Theorems 3.2, 4.2, 5.7, 6.3 and 7.1), since Bourbaki-complete uniform spaces and complete uniform spaces having a base of star-finite covers for their uniformity are strongly related (Theorem 2.4, see also [14]). On the other hand, in order to show the independence of this two kinds on uniform concepts we construct two examples (Examples 7.3 and 7.5) of Bourbaki-complete spaces do not having a base of star-finite covers for their respective uniformities.

At the end of this paper we study the stronger notion of cofinal Bourbaki-completeness, also originally introduced in [12,13]. The main objective of addressing cofinal Bourbaki-completeness is to give an example (Example 8.16) of a metric space which is cofinally complete and Bourbaki-complete metric at the same time but which is not metrizable by any cofinally Bourbaki-complete metric because, precisely, it is not strongly paracompact but only strongly metrizable. In particular, this space is another example of Bourbaki-complete metric space which does not have a star-finite base for its uniformity (Theorem 8.3). Besides, we characterize the metric spaces which are cofinally complete and Bourbaki-complete at the same time through an intermediate property called uniform complete paracompactness (Theorem 8.13).

Please, notice that the main part of the results of this paper is included in the Ph.D. Thesis of the author [22] which was defended in 2019 at the University Complutense of Madrid (UCM).

2. Preliminaries

The aim of this section is to explain the relation between the stronger uniform notion of Bourbaki-completeness and the uniformities having a star-finite base. These notions are defined next. Basic facts about uniform spaces can be found in [5,18,32].

The following notation is useful in order to define Bourbaki-completeness. Let A be a subset of a uniform space (X, μ) and $\mathcal{U} \in \mu$, then:

- $St\,(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$;
- $St^0\,(A,\mathcal{U}) = A$;
- $St^m\,(A,\mathcal{U}) = St\left(St^{m-1}\,(A,\mathcal{U})\,,\mathcal{U}\right),\ \forall m \in \mathbb{N}$;
- $St^\infty\,(A,\mathcal{U}) = \bigcup_{m\in\mathbb{N}} St^m\,(A,\mathcal{U})$;
- if $A = \{x\}$, instead of $St^m\,(\{x\}\,,\mathcal{U})$ write $St^m\,(x,\mathcal{U})\,,\ \forall m \in \mathbb{N} \cup \{\infty\}$.

Observe that, if $U \in \mathcal{U}$ satisfies that $U \subset St^m\,(A,\mathcal{U})$ for some $m \in \mathbb{N}$ and some $A \subset X$ then there exists a finite *chain* $U_0, U_1, ..., U_i \in \mathcal{U}$ of length $i \leq m$ such that $A \cap U_0 \neq \emptyset$, $U_i = U$ and $U_{j-1} \cap U_j \neq \emptyset$ for every $j = 1, ..., i$.

In the frame of metric spaces (X, d), $\mathcal{B}_\varepsilon$ denotes the cover of all the open balls $B_d\,(x, \varepsilon)$ of centre $x \in X$ and radius $\varepsilon > 0$. In this case, write:

- $B_d^m\,(x, \varepsilon) = St^{m-1}\,(B_d(x, \varepsilon)\,, \mathcal{B}_\varepsilon),\ \forall m \in \mathbb{N}$;
- $B_d^\infty\,(x, \varepsilon) = \bigcup_{m\in\mathbb{N}} B_d^m\,(x, \varepsilon)$.

Definition 2.1 ([12]). A filter $\mathcal{F}$ of a uniform space (X, μ) is said to be *Bourbaki-Cauchy* if

$$\forall \mathcal{U} \in \mu\ \exists m \in \mathbb{N},\ \exists U \in \mathcal{U} \text{ s.t. } St^m\,(U,\mathcal{U}) \in \mathcal{F}.$$

A uniform space (X, μ) is *Bourbaki-complete* if every Bourbaki-Cauchy filter clusters.

Clearly Bourbaki-completeness implies usual completeness as every Cauchy filter is a Bourbaki-Cauchy filter, and in fact this property is strictly stronger as not every complete uniform space is Bourbaki-complete.

Example 2.2 [12, Example 16]. *The metric hedgehog $J\,(\kappa)$, $\kappa \geq \omega_0$, is a complete metric space which is not Bourbaki-complete.* Let A be a set of cardinal $\kappa \geq \omega_0$. The metric hedgehog $J\,(\kappa)$ of κ spininess is defined as follows. Let $I = [0, 1] \subset \mathbb{R}$ and consider the product $I \times A$. Next, take the equivalence relation $\sim$ on $I \times A$ defined by

$$(x_1, \alpha_1) \sim (x_2, \alpha_2) \text{ if and only if } x_1 = x_2 = 0.$$

Then the quotient $I \times A/\sim$ is the set of points of $J\,(\kappa)$ and we endowed this set with the following metric $\gamma : J\,(\kappa) \times J\,(\kappa) \to [0, \infty)$

$$\gamma\left(\left[(x_1, \alpha_1)\right], \left[(x_2, \alpha_2)\right]\right) = \begin{cases} |x_1 - x_2| & \text{if } \alpha_1 = \alpha_2; \\ x_1 + x_2 & \text{if } \alpha_1 \neq \alpha_2. \end{cases}$$

To prove the completeness of $(J\,(\kappa)\,, \gamma)$ is easy. Moreover, it is not difficult to see that every filter of $(J\,(\kappa)\,, \gamma)$ is Bourbaki-Cauchy since for every $\varepsilon > 0$,

$$J\,(\kappa) = B_\gamma^m\,([0, \alpha]\,, \varepsilon)$$

where $m = \left[\varepsilon^{-1}\right] + 1$ and $\left[\varepsilon^{-1}\right]$ denotes the integer part of ε^{-1}. Since $J\,(\kappa)$ is not compact, the space cannot be Bourbaki-complete.

In spite of this strength, Bourbaki-completeness and completeness are actually closely related through the *star-finite modification* of the uniformity defined next.

Definition 2.3. A cover $\mathcal{C}$ of a set X is *star-finite* if every $C \in \mathcal{C}$ meets at most finitely many $C' \in \mathcal{C}$.

It is well-known that if μ is a uniformity on a Tychonoff space X, then the family of all the star-finite covers from μ is a base for a compatible uniformity, that is, it generates the topology of X. Moreover, the elements of this base can be chosen being open covers ([20, Proposition 28, Chapter IV]). This uniformity, called the star-finite modification of μ, is denoted by $sf\mu$. For instance, any *weak uniformity*, that is, any uniformity induced by a collection of real-valued continuous functions ([19]), has a star-finite base.

The relation of Bourbaki-completeness, completeness and the star-finite modification of a uniformity, is stated in the following result which is proved in [14].

Theorem 2.4 ([14, Theorem 16]). *A uniform space (X, μ) is Bourbaki-complete if and only if $(X, sf\mu)$ is complete.*

On a product $X = \prod_{i\in I} X_i$ of uniform spaces we always denote by π the product uniformity. Observe that if each uniformity μ_i on the respective factor X_i satisfies that $\mu_i = sf\mu_i$, then the product uniformity π on X also has a star-finite base. On the other hand, if Y is a subspace of (X, μ) and $\mu = sf\mu$ then the subspace uniformity on Y also has a star-finite base. Similarly, it is easy to see that Bourbaki-completeness is also a productive property and it is also inherited by closed subspaces (see [14]).

Theorem 2.5. *The uniformity induced by the norm on a Banach space has a star-finite base if and only if the space is finite dimensional.*

Proof. If the space is infinite dimensional then it is not Bourbaki-complete as it is proved in [12, Corollary 10]. Therefore, by Theorem 2.4, the uniformity induced by the metric cannot have a star-finite base because the space is complete.

On the other hand, consider the real-line $(\mathbb{R}, |\cdot|)$ endowed with the usual Euclidean norm and let d_u be the usual Euclidean metric induced by the norm. It is well-known, that the uniformity induced by the norm is exactly the weak uniformity induced by all the real-valued Lipschitz functions on $(\mathbb{R}, d_u)$. Indeed, fixed $x \in X$,

$$B_{d_u}(x, \varepsilon) = \{y \in X : |x - y| < \varepsilon\} = \{y \in X : |Id(x) - Id(y)| < \varepsilon\}$$

where the identity function $Id : (\mathbb{R}, d_u) \to (\mathbb{R}, |\cdot|)$ is certainly a Lipschitz function. Therefore the uniformity induced by the norm has a star-finite base.

Any finite dimensional Banach space is uniformly homeomorphic to a product $(\mathbb{R}^n, \pi)$, $n \in \mathbb{N}$ where on each factor $\mathbb{R}$ we consider the uniformity induced by $|\cdot|$. Hence the result follows by productivity. □

Example 2.6. Every uniformly zero-dimensional space has a star-finite base for its uniformity. This is clear since, by definition, every uniformly zero-dimensional has a base of partitions for its uniformity. Anyway, as we recall in

Remark 4.4, every uniformly zero-dimensional space is uniformly homeomorphic to a subspace of the product space (κ^α, π), with κ, $\alpha \geq \omega_0$. This space is defined as follows. Given a discrete space A of cardinal $\kappa \geq \omega_0$ we identify it with its cardinal and we endowed it with the 0-1 metric

$$\chi(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

More precisely, in this paper we will in general identify a cardinal κ with the above uniformly discrete space. In this case, π denotes the product uniformity on κ^α where on each factor κ we consider the uniformity induced by the metric χ. By productivity it follows that on κ^α the product uniformity π has a star-finite base. In particular, any uniform subspace of (κ^α, π) has a star-finite base for the inherited uniformity.

In the following sections it is shown that a universal space for the complete uniform spaces having a star-finite base, as well as, for the Bourbaki-complete spaces, is provided by a product of uniformly discrete spaces and of real-lines, that is,

$$\left(\left(\prod_{i \in I} \kappa_i\right) \times \mathbb{R}^\alpha, \pi\right), \ \alpha \geq \omega_0.$$

By all the foregoing, this space is Bourbaki-complete and π satisfies that $sf\pi = \pi$.

3. Embedding of a Bourbaki-Complete Metric Space Whose Metric Uniformity has a Star-Finite Base

Let (X, d) be a complete metric space satisfying that $sf\mu_d = \mu_d$. By Theorem 2.4, (X, d) is in particular Bourbaki-complete. In this section we prove that there is an embedding

$$\varphi : (X, d) \to \left(\left(\prod_{n \in \mathbb{N}} \kappa_n\right) \times \mathbb{R}^{\omega_0}, \pi\right),$$

where $\{\kappa_n : n \in \mathbb{N}\}$ is a countable family of cardinals, which preserves partially the uniform structure of the metric space (X, d). Observe that in this case, the product uniformity π is metrizable by the the metric $\rho + t$, where ρ denotes the "first difference metric" on the uniformly zero-dimensional space $\prod_{n \in \mathbb{N}} \kappa_n$, that is,

$$\rho(\langle x_n \rangle_n, \langle y_n \rangle_n) = \begin{cases} 0 & \text{if } x_n = y_n \text{ for every } n \in \mathbb{N} \\ 1/n & \text{if } x_j = y_j \text{ for every } j = 1, ..., n-1 \text{ and } x_n \neq y_n \end{cases}$$

and with t denoting the product metric on $\mathbb{R}^{\omega_0}$, that is,

$$t\left(\langle x_n\rangle_n, \langle y_n\rangle_n\right) = \sum_{n=1}^{\infty}\left(d_u\left(x_n, y_n\right) \wedge 1\right)/2^n$$

where $\wedge$ denotes the infimum. In later sections, this embedding's result will be extended to the general case of the Bourbaki-complete uniform and metric spaces.

Observe that for every $\mathcal{U} \in \mu$ it is always possible to choose a family of points $\{x_i : i \in I\}$ of X such that the family of sets $\{St^{\infty}\left(x_i, \mathcal{U}\right) : i \in I\}$ is a uniform cover of (X, μ) satisfying that

$$St^{\infty}\left(x_i, \mathcal{U}\right) \cap St^{\infty}\left(x_j, \mathcal{U}\right) = \emptyset,\ \forall i \neq j.$$

This cover is called *the family of all the chained components induced by* $\mathcal{U}$, where each set $St^{\infty}\left(x_i, \mathcal{U}\right)$ is a chained component. Moreover, if $\mathcal{U} < \mathcal{P}$ and $\mathcal{P}$ is a uniform partition of (X, μ), then

$$\{St^{\infty}\left(x_i, \mathcal{U}\right) : i \in I\} < \mathcal{P}.$$

Clearly, the family of all the chained components induced by a uniform cover is always a uniform partition.

Next, let (X, μ) be a uniform space and let $\mathscr{P}$ the family of all the uniform partitions of (X, μ). Define

$$\vartheta\left(X, \mu\right) = \sup\left\{|\mathcal{P}| : \mathcal{P} \in \mathscr{P}\right\}.$$

For a connected uniform space, or in general, for a uniformly connected uniform space (X, μ), it is satisfied that $\vartheta\left(X, \mu\right) = 1$. Recall that a *uniformly connected space* (or *well-chained* space) is a uniform space (X, μ) such that for every $\mathcal{U} \in \mu$, $X = St^{\infty}\left(x, \mathcal{U}\right)$ for any $x \in X$.

The following technical lemma will be useful.

Lemma 3.1. *Let $\mathcal{U}$ be a uniform cover of a uniform space (X, μ). Let $\{St^{\infty}\left(x_i, \mathcal{U}\right), i \in I\}$ be the family of all the chained components induced by $\mathcal{U}$. Write:*

- $A_1^i = St^2\left(x_i, \mathcal{U}\right),\ \forall i \in I;$
- $A_n^i = \bigcup\left\{U \in \mathcal{U} : U \subset St^{n+1}\left(x_i, \mathcal{U}\right), U \cap \left(X \backslash St^{n-1}\left(x_i, \mathcal{U}\right)\right) \neq \emptyset\right\},\ \forall n \in \mathbb{N},\ \forall i \in I.$

Let $\mathcal{A}\left(\mathcal{U}\right) = \left\{A_n^i : A_n^i \neq \emptyset,\ n \in \mathbb{N},\ i \in I\right\}$. Then $\mathcal{A}\left(\mathcal{U}\right)$ is a uniform cover satisfying that

$$A_n^i \cap A_m^j = \emptyset,\ \forall i \neq j \text{ and } A_n^i \cap A_m^i = \emptyset \text{ if } |n - m| > 1.$$

In particular, $\mathcal{U}$ refines $\mathcal{A}\left(\mathcal{U}\right)$ and $\mathcal{A}\left(\mathcal{U}\right) \in sf\mu$.

Proof. It is clear. □

Theorem 3.2. *Let* (X,d) *be a complete metric space such that* $\mu_d = sf\mu_d$. *Then there exists an embedding*

$$\varphi : (X,d) \to \left(\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times \mathbb{R}^{\omega_0}, \rho + t\right)$$

where each κ_n *is a cardinal,* φ *is uniformly continuous and* $\varphi(X)$ *is a closed subspace of the arrival space* $\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times \mathbb{R}^{\omega_0}$. *Moreover,* $\vartheta(X,d) \leq \sup \left\{\prod_{j=1}^{n} \kappa_j : n \in \mathbb{N}\right\}$.

Proof. (For similar techniques used in this proof see first [2] and after [10].) Take $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a family of star-finite open covers being a base for the metric uniformity μ_d and such that $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every $n \in \mathbb{N}$. Without loss of generality assume that for every $n \in \mathbb{N}$, $\mathcal{U}_n$ refines $\mathcal{B}_{1/n} = \{B_d(x, 1/n) : x \in X\}$. Next, for every $n \in \mathbb{N}$, let $\mathcal{P}_n = \{St^{\infty}(x_{i_n}, \mathcal{U}_n) : i_n \in I_n\}$ be the family of all the chained components of X induced by $\mathcal{U}_n$. Notice that $\mathcal{P}_{n+1} < \mathcal{P}_n$ for every $n \in \mathbb{N}$. Take the cardinal $\kappa_1 = |I_1|$ and order the elements of the partition $\mathcal{P}_1$ by writing

$$\mathcal{P}_1 := \left\{P_{(\alpha_1)} : \alpha_1 < \kappa_1\right\} \text{ (where } \alpha_1 < \kappa_1 \text{ means } 0 \leq \alpha_1 < \kappa_1).$$

Then, for every $\alpha_1 < \kappa_1$ let $\mathcal{P}_{(\alpha_1)} = \left\{P \in \mathcal{P}_2 : P \subset P_{(\alpha_1)}\right\}$ and $\kappa_{(\alpha_1)} = \left|\mathcal{P}_{(\alpha_1)}\right|$. Next, put $\kappa_2 = \sup\left\{\kappa_{(\alpha_1)} : \alpha_1 < \kappa_1\right\}$. In particular, it is clear that

$$\kappa_1 \times \kappa_2 \geq \left|\left\{(\alpha_1, \alpha_2) : \alpha_1 < \kappa_1, \alpha_2 < \kappa_{(\alpha_1)}\right\}\right| \geq \left|\bigcup_{\alpha_1 < \kappa_1} \mathcal{P}_{(\alpha_1)}\right| = |I_2|$$

Moreover, order each $\mathcal{P}_{(\alpha_1)}$ as follows:

$$\mathcal{P}_{(\alpha_1)} := \left\{P_{(\alpha_1,\alpha_2)} : \alpha_2 < \kappa_{(\alpha_1)}\right\}.$$

Next, suppose that for $n \in \mathbb{N}$ we have defined the families of sets

$$\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} = \left\{P \in \mathcal{P}_n : P \subset P_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}\right\},$$

where $\alpha_1 < \kappa_1$ and $\alpha_j < \kappa_{(\alpha_1,\ldots,\alpha_{j-1})}$ for every $j = 2, \ldots, n-1$, and let $\kappa_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} = \left|\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}\right|$. Also, suppose that the family $\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}$ is ordered as follows:

$$\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})} := \left\{P_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n)} : \alpha_n < \kappa_{(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}\right\}.$$

Then, by induction, define $\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \left\{P \in \mathcal{P}_{n+1} : P \subset P_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right\}$ and $\kappa_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \left|\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right|$. Finally, order each family of sets $\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ as before:

$$\mathcal{P}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} := \left\{P_{(\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1})} : \alpha_{n+1} < \kappa_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right\}.$$

Now, for every $n \in \mathbb{N}$ put

$$\kappa_{n+1} = \sup\left\{\kappa_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} : \alpha_1 < \kappa_1, \alpha_j < k_{(\alpha_1,\ldots,\alpha_{j-1})}, j = 2, \ldots, n\right\}.$$

Observe that

$$\begin{aligned}&\kappa_1\times\kappa_2\times\ldots\times\kappa_n\times\kappa_{n+1}\\&\quad\geq\left|\left\{(\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1}):\alpha_1<\kappa_1,\alpha_j<k_{(\alpha_1,\ldots,\alpha_{j-1})},j=2,\ldots,n+1\right\}\right|\\&\quad\geq\left|\bigcup\left\{\mathcal{P}_{(\alpha_1,\ldots,\alpha_n)}:\alpha_1<\kappa_1,\alpha_j<k_{(\alpha_1,\ldots,\alpha_{j-1})},j=2,\ldots,n\right\}\right|=\left|I_{n+1}\right|.\end{aligned}$$

Moreover, as $\{\mathcal{U}_n : n\in\mathbb{N}\}$ is a base for the uniformity μ_d,

$$\vartheta\left(X,d\right)\leq\sup\left\{\left|I_n\right|:n\in\mathbb{N}\right\}\leq\sup\left\{\prod_{j=1}^{n}\kappa_j:n\in\mathbb{N}\right\}.$$

Notice that for every $n\in\mathbb{N}$ there exists a unique $(\alpha_1,\ldots,\alpha_n)\in\prod_{j=1}^{n}\kappa_j$ such that $x\in\mathcal{P}_{(\alpha_1,\ldots,\alpha_n)}$. Besides, $(\alpha_1,\ldots,\alpha_n,\alpha_{n+1})$ extends $(\alpha_1,\ldots,\alpha_n)$, so there exists a unique $\sigma(x)\in\prod_{n\in\mathbb{N}}\kappa_n$ such that the restriction $\sigma(x)|n$ of $\sigma(x)$ over the first n's coordinates is exactly $(\alpha_1,\ldots,\alpha_n)$. Therefore, it is possible to define the map

$$\begin{aligned}\sigma:(X,d)&\rightarrow\left(\prod_{n\in\mathbb{N}}\kappa_n,\rho\right)\\x&\mapsto\sigma\left(x\right).\end{aligned}$$

Recall, that for every $x\in X$ and every $n\in\mathbb{N}$ there exists a unique $i_n\in I_n$ such that $P_{\sigma(x)|n}=St^{\infty}\left(x_{i_n},\mathcal{U}_n\right)=P_{\sigma(x_{i_n})|n}$.

Now, for every $n\in\mathbb{N}$ let $\mathcal{A}\left(\mathcal{U}_n\right)=\left\{A_{m,i_n}:m\in\mathbb{N},i_n\in I_n\right\}$ the open cover from Lemma 3.1 induced by $\mathcal{U}_n$, and define the sets $A_m^n=\bigcup\left\{A_{m,i_n}:i_n\in I_n\right\}$, $m\in\mathbb{N}$. Then the cover $\mathcal{A}_n=\left\{A_m^n:m\in\mathbb{N}\right\}$ is uniform and linear, that is, $\mathcal{A}_n$ is a uniform countable cover satisfying that

$$A_m^n\cap A_l^n=\emptyset\text{ whenever }\left|m-l\right|>1.$$

Take $\varepsilon_n>0$ such that $\left\{B_d\left(x,\varepsilon_n\right):x\in X\right\}<\mathcal{U}_n<\mathcal{A}_n$. Applying the same techniques than in [10, Lemma 1.2] there exists a uniformly continuous function $h_n:(X,d)\rightarrow(\mathbb{R},d_u)$ such that $h_n^{-1}\left((m-1,m+1)\right)=A_m^n$ for every $m\in\mathbb{N}$. Moreover, the following is always satisfied (see [10, Lemma 1.2] again):

$$\text{if } d\left(x,y\right)\leq\varepsilon_n\text{ then }\left|h_n\left(x\right)-h_n\left(y\right)\right|\leq\tfrac{10}{\varepsilon_n^2}\cdot d\left(x,y\right)$$

Next, recall that it is possible to write $\mathcal{U}_n=\left\{U_{j,i_n},j\in\mathbb{N},i_n\in I_n\right\}$ where $U_{j,i_n}\cap U_{j',i'_n}=\emptyset$ if $i_n\neq i'_n$ as the covers $\mathcal{U}_n$ are star-finite and then every chained component $\mathcal{P}_{\sigma(x_{i_n})|_n}$, $i_n\in I_n$, contains at most countable many $U\in\mathcal{U}_n$.

Since for every $x \in X$ and for every $n \in \mathbb{N}$, there exists a unique $i_n \in I_n$ such that $\sigma(x)|n = \sigma(x_{i_n})|n$, we can define the map

$$\varphi : (X,d) \to \left(\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times (\mathbb{R}\times\mathbb{R}^{\omega_0})^{\omega_0}, \pi\right)$$
$$x \mapsto \varphi(x) = \left(\sigma(x), \left\langle h_n(x), \langle d(x, X\backslash U_{j,i_n})\rangle_{j\in\mathbb{N}}\right\rangle_{n\in\mathbb{N}}\right).$$

• The map φ is injective. Let $x, y \in X$, $x \neq y$ and take some $\varepsilon < d(x,y)$ such that $y \notin B_d(x,\varepsilon)$. Since $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a base for the uniformity inducing the topology on X, for some $n \in \mathbb{N}$ it is possible to choose $U_{j,i_n} \in \mathcal{U}_n$ such that $x \in U_{j,i_n} \subset B_d(x,\varepsilon)$. Then $d(x, X\backslash U_{j,i_n}) > 0$ and $d(y, X\backslash U_{j,i_n}) = 0$. Therefore, $\varphi(x) \neq \varphi(y)$ and φ is an injective map.

• The map φ is uniformly continuous. The map φ will be uniformly continuous if it is uniformly continuous when we compose it with the projections. So first, see that σ is a uniformly continuous map since whenever $d(x,y) < \varepsilon_n$ then $P_{\sigma(x)|n} = P_{\sigma(x_{i_n})|n} = P_{\sigma(y)|n}$ for a unique $i_n \in I_n$. Therefore, $\sigma(x)|n = \sigma(y)|n$ and then $\rho(x,y) < \frac{1}{n+1}$. Next, let $d(x,y) < \varepsilon_n$ again, then

$$\begin{aligned}
&d_u(h_n(x), h_n(y)) + t\left(\langle d(x, X\backslash U_{j,i_n})\rangle_{j\in\mathbb{N}}, \langle d(y, X\backslash U_{j,i_n})\rangle_{j\in\mathbb{N}}\right)\\
&= |h_n(x) - h_n(y)| + \sum_{j=1}^{\infty} |d(x, X\backslash U_{j,i_n}) - d(y, X\backslash U_{j,i_n})| \cdot 2^{-j}\\
&\le |h_n(x) - h_n(y)| + \sum_{j=1}^{\infty} d(x,y) \cdot 2^{-j}\\
&\le \tfrac{10}{\varepsilon_n^2} \cdot d(x,y) + d(x,y) = \left(\tfrac{10}{\varepsilon_n^2} + 1\right) \cdot d(x,y).
\end{aligned}$$

• The map φ is closed. Before proving that φ is closed, we need the following claim.

Claim. Let $Y \subset \varphi(X)$ and $\mathcal{F}$ a Cauchy filter of the subspace $(Y, \pi|_Y)$. Then $\varphi^{-1}(\mathcal{F})$ is a Bourbaki-Cauchy filter of (X,d). □

Proof of the claim. Let $\mathcal{F}$ be a Cauchy filter of $(Y, \pi|_Y)$. Then, fixed $k \in \mathbb{N}$, since $\mathcal{F}$ is Cauchy, there is some $W \in \mathcal{F}$, $W = \left(V \times \left(\prod_{n\in\mathbb{N}} U_n\right)\right) \cap Y$, where, for some $x_0 \in X$ and some $i_k \in I_k$, $V = B_\rho(\sigma(x_0), 1/k)$, $\sigma(x_0)|k = \sigma(x_{i_k})|k$ and $U_k = B_{d_u}((h_k(x_0), 1/k) \times \mathbb{R}^{\omega_0})$, and $U_n = \mathbb{R} \times \mathbb{R}^{\omega_0}$ for every $n \neq k$.

As the fixed $i_k \in I_k$ such that $\sigma(x_0)|k = \sigma(x_{i_k})|k$ is unique, then

$$\begin{aligned}
\varphi^{-1}(W) &= \left\{x \in P_{\sigma\left(x_{i_k}\right)|k} : |h_k(x) - h_k(x_0)| < 1/k\right\}\\
&\subset h_k^{-1}((h_k(x_0) - 1/k, h_k(x_0) + 1/k)) \cap P_{\sigma\left(x_{i_k}\right)|k}.
\end{aligned}$$

By the construction of h_k there is some $m \in \mathbb{N}$ such that

$$h_k^{-1}((h_k(x_0) - 1/k, h_k(x_0) + 1/k)) \cap P_{\sigma\left(x_{i_k}\right)|k} \subset A_{m,i_k} \subset St^{m+1}(x_{i_k}, \mathcal{U}_k).$$

Therefore

$$\varphi^{-1}(W) \subset St^{m+1}(x_{i_k}, \mathcal{U}_k)$$

and we have proved that $\varphi^{-1}(\mathcal{F})$ is a Bourbaki-Cauchy filter in (X, d). $\square$

Continue with the proof that φ is a closed map. Let $C \subset X$ a closed subset and let $\mathcal{F}$ be an ultrafilter in $\varphi(C)$ which converges to some $z \in \left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times (\mathbb{R} \times \mathbb{R}^{\omega_0})^{\omega_0}$. Then $\mathcal{F}$ is a Cauchy ultrafilter of the subspace $\left(\varphi(C), \pi|_{\varphi(C)}\right)$. By maximality of $\mathcal{F}$ and the above claim, $\varphi^{-1}(\mathcal{F})$ is a Bourbaki-Cauchy ultrafilter of (X, d). Then, $\varphi^{-1}(\mathcal{F})$ converges in X because, by Theorem 2.4, (X, d) is in particular Bourbaki-complete. By continuity of φ, $\varphi\left(\varphi^{-1}(\mathcal{F})\right)$ converges in $\varphi(X)$. Since $\mathcal{F} = \varphi\left(\varphi^{-1}(\mathcal{F})\right)$, by maximality, $\mathcal{F}$ converges in $\varphi(X)$, that is, $z \in \varphi(C)$. Thus, $\varphi(C)$ is a closed subspace of $\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times (\mathbb{R} \times \mathbb{R}^{\omega_0})^{\omega_0}$.

- The image $\varphi(X)$ is a closed subspace of $\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times (\mathbb{R} \times \mathbb{R}^{\omega_0})^{\omega_0}$. Since φ is a closed map, this is clear.

Finally, observe that, by the results in Bourbaki [5, II.2.3 Prop 5 p. 177, p.180], the space $\left(\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times (\mathbb{R} \times \mathbb{R}^{\omega_0})^{\omega_0}, \pi\right)$ is uniformly equivalent to the space $\left(\left(\prod_{n\in\mathbb{N}} \kappa_n\right) \times \mathbb{R}^{\omega_0}, \rho + t\right)$ and this complete the proof. $\square$

Remark 3.3. The map φ of the above embedding is uniformly continuous. However, the inverse map

$$\varphi^{-1} : \left(\varphi(X), \pi|_{\varphi(X)}\right) \to (X, d)$$

is not necessarily uniformly continuous as it is shown next.

Let $f \in C(\mathbb{R})$ be any continuous real-valued function not being uniformly continuous for the metric d_u, that is, $f \notin U_{d_u}(\mathbb{R})$, where $U_{d_u}(\mathbb{R})$ denotes the family of all the uniformly continuous real-valued functions on $(\mathbb{R}, d_u)$. Put $d(x, y) = d_u(x, y) + d_u(f(x), f(y))$. It is well-known that d is compatible with the Euclidean topology on $\mathbb{R}$. Let $F = U_{d_u}(\mathbb{R}) \cup \{f\}$ and let w_F denote the weak uniformity on $\mathbb{R}$ induced by the family of functions ([32]). Then the uniformity induced by d is exactly w_F. Indeed, the identity map $i : (\mathbb{R}, d) \to (\mathbb{R}, w_F)$ is uniformly continuous because any function $g \in F$ is uniformly continuous for d. Conversely, the identity map $j : (\mathbb{R}, w_F) \to (\mathbb{R}, d)$ is also uniformly continuous because if $d_u(i(x), i(y)) < \frac{\epsilon}{2}$ and $d_u(f(x), f(y)) < \frac{\epsilon}{2}$ then $d(x, y) < \epsilon$. Observe that $i, f \in F$. Besides, the metric space $(\mathbb{R}, d)$ is complete because the identity map $id : (\mathbb{R}, d) \to (\mathbb{R}, d_u)$ is uniformly continuous.

Since $\mathbb{R}$ is connected then the first factor can be deleted and then it is only necessary to take $\mathbb{R}^{\omega_0}$ into account, that is, we can restrict the embedding φ from the previous result

$$\varphi : (\mathbb{R}, d) \to (\mathbb{R}^{\omega_0}, \pi)$$

and it is also uniformly continuous.

However the inverse homeomorphism $\varphi^{-1} : \left(\varphi(\mathbb{R}), \pi|_{\varphi(\mathbb{R})}\right) \to (\mathbb{R}, d)$ cannot be uniformly continuous because the real-valued function

$$f \circ \varphi^{-1} : \left(\varphi(\mathbb{R}), \pi|_{\varphi(\mathbb{R})}\right) \to (\mathbb{R}, d_u)$$

cannot be uniformly continuous for the weak uniformity π on $\varphi(\mathbb{R})$ inherited from $(\mathbb{R}^{\omega_0}, \pi)$, as $f \notin U_{d_u}(\mathbb{R})$. Recall that the product uniformity π on $\mathbb{R}^{\omega_0}$ obtained as the product of the weak uniformities induced by d_u on each factor, and it is exactly the weak uniformity induced by all the projections maps $p_k : (\mathbb{R}^{\omega_0}, \pi) \to (\mathbb{R}, d_u)$, $k < \omega_0$ ([32]).

4. Universal Space for Bourbaki-Complete Uniform Spaces

From the above result Theorem 3.2, we deduce an embedding for the complete uniform spaces having a star-finite base for their uniformity (Theorem 4.2) and for the general case of the Bourbaki-complete uniform spaces (Theorem 4.3). Besides, from this last result, we also obtain the characterization of the Tychonoff spaces which are uniformizable by a Bourbaki-complete uniformity.

Lemma 4.1 ([32, Theorem 23.4], [28, Prop 1.1.4, Prop 2.2.3]). *Let (X, μ) be a uniform space. Let $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ be a normal sequence of open (uniform) covers of X. Then there exists a (uniformly) continuous pseudometric $d : X \times X \to [0, \infty)$ such that*

$$\mathcal{B}_{1/2^{n+1}} < \mathcal{G}_n < \mathcal{B}_{1/2^{n-1}} \text{ for every } n \in \mathbb{N}.$$

In addition, if $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is a base for the topology of X, then ρ is compatible with the topology of X, and if $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a base for the uniformity of μ, then ρ is compatible with the uniformity μ. When the topology of X is Hausdorff, ρ is in fact a metric.

Theorem 4.2. *Let (X, μ) be a complete uniform space such that $\mu = sf\mu$. Then there exists an embedding*

$$\varphi : (X, \mu) \to \left(\left(\prod_{i\in I, n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^\alpha, \pi\right)$$

where each κ_n^i is a cardinal endowed with the uniformly discrete metric χ, $\alpha \geq \omega_0$, φ is uniformly continuous and $\varphi(X)$ is a closed subspace of $\left(\prod_{i\in I, n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^\alpha$. Moreover, $\vartheta(X, \mu) \leq \sup\left\{\left(\prod_{j=1}^n \kappa_j^i\right) : n \in \mathbb{N}, i \in I\right\}$.

Proof. Let (X, μ) be a uniform space having a base of star-finite open covers for the uniformity, and let $\{\mathcal{U}^i : i \in I\}$ be a base for μ. Then, for every $\mathcal{U}^i \in \mu$, $i \in I$ there is a normal sequence $\langle \mathcal{U}_n^i \rangle_{n\in\mathbb{N}}$ of star-finite uniform open covers such that $\mathcal{U}_1^i < \mathcal{U}^i$. This can be obtained applying the axioms of uniformity and [20, Proposition 8, Chapter IV].

For every $i \in I$, let ρ_i be the pseudometric on X from Lemma 4.1 generated by the normal sequence $\langle \mathcal{U}_n^i \rangle_{n\in\mathbb{N}}$. Then, the family of covers $\left\{\mathcal{U}_n^i : n \in \mathbb{N}\right\}$ is a base of star-finite open covers of the space (X, ρ_i).

Let $(Y_i, \widehat{\rho_i})$ be the metric space obtained by doing the usual metric identification $\sim$ on (X, ρ_i):

$$x_1 \sim x_2 \text{ if and only if } \rho_i(x_1, x_2) = 0.$$

If $\psi_i : (X, \rho_i) \to (Y_i, \widehat{\rho_i})$ denotes the quotient map induced by $\sim$, then

$$\psi_i^{-1}(\widehat{x}) = \{z \in X : \rho_i(x, z) = 0\},$$

$$\widehat{A} := \psi_i(A) = \{\widehat{x} : x \in A\}$$

and

$$\psi_i^{-1}\left(B_{\widehat{\rho_i}}(\widehat{x}, \varepsilon)\right) = B_{\rho_i}(x, \varepsilon).$$

Hence, the family of covers $\left\{\widehat{\mathcal{U}_n^i} : n \in \mathbb{N}\right\}$ is a base of star-finite covers for the metric uniformity on Y_i induced by $\widehat{\rho_i}$. In addition, the map ψ_i preserves the uniform partitions induced by the covers $\mathcal{U}_n^i$, $n \in \mathbb{N}$.

Let (Z_i, d_i) denote the completion of $(Y_i, \widehat{\rho_i})$ and $\mathcal{V}_n^i$ denote the extension to (Z_i, d_i) of the covers $\widehat{\mathcal{U}_n^i}$. Then $\left\{\mathcal{V}_n^i : n \in \mathbb{N}\right\}$ is a base of star-finite open covers for the metric uniformity of (Z_i, d_i) ([26, Lemma p. 370]).

By [32, Theorems 39.11 and 39.12] (X, μ) is uniformly homeomorphic to a subspace of the product $\prod_{i\in I}(Z_i, d_i)$. In particular it is closed by completeness. Denote by φ_i the embedding of (Z_i, d_i) into $\left(\left(\prod_{n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^{\omega_0}, \rho + t\right)$ from Theorem 3.2, and let

$$\varphi : \prod_{i\in I}(Z_i, d_i) \to \prod_{i\in I}\left(\left(\prod_{n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^{\omega_0}, t + \rho\right)$$

be the product map $\varphi = \prod_{i\in I} \varphi_i$. Then, the restriction of φ over the uniform homeomorphic image of (X, μ) in $\prod_{i\in I}(Z_i, d_i)$ is the desired map. Indeed, notice that $\varphi(X)$ is closed in $\varphi\left(\prod_{i\in I} Z_i\right) = \prod_{i\in I} \varphi_i(Z_i)$. Moreover, by [5, II.2.3 Prop 5 p.177, p.180], the spaces $\prod_{i\in I}\left(\left(\prod_{n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^{\omega_0}\right)$ and $\left(\prod_{i\in I, n\in\mathbb{N}} \kappa_n^i\right) \times \mathbb{R}^{\alpha}$ for $\alpha = \sup\{|I|, \omega_0\}$ are uniformly equivalent when they are endowed with their respective product uniformities. Finally, by Theorem 3.2,

$$\begin{aligned}\vartheta(X, \mu) \leq \sup\{\vartheta(X, \rho_i) : i \in I\} &= \sup\{\vartheta(Z_i, d_i) ; i \in I\} \\ &\leq \sup\left\{\prod_{j=1}^{n} \kappa_j^i : n \in \mathbb{N}, i \in I\right\}\end{aligned}$$

as the quotient map ψ_i and the operation of completion preserve uniform partitions. $\square$

Theorem 4.3. *Let* (X, μ) *be a Bourbaki-complete uniform space. Then there exists an embedding*

$$\varphi : (X, \mu) \to \left(\left(\prod_{i \in I, n \in \mathbb{N}} \kappa_n^i \right) \times \mathbb{R}^\alpha, \pi \right),$$

where each κ_n^i *is a cardinal endowed with the uniformly discrete metric* χ, $\alpha \geq \omega_0$, φ *is uniformly continuous and* $\varphi(X)$ *is a closed subspace of* $\left(\prod_{i \in I, n \in \mathbb{N}} \kappa_n^i \right) \times \mathbb{R}^\alpha$. *Moreover,* $\vartheta(X, \mu) \leq \sup \left\{ \left(\prod_{j=1}^n \kappa_j^i \right) : n \in \mathbb{N}, i \in I \right\}$.

Proof. Recall that by Theorem 2.4, $(X, sf\mu)$ is complete if and only if (X, μ) is Bourbaki-complete. Therefore, if we compose the embedding

$$\varphi : (X, sf\mu) \to \left(\left(\prod_{i \in I, n \in \mathbb{N}} \kappa_n^i \right) \times \mathbb{R}^\alpha, \pi \right)$$

from Theorem 4.2, with the identity map $id : (X, \mu) \to (X, sf\mu)$, we obtain a universal space for the Bourbaki-complete uniform spaces immediately. Finally, observe that $\vartheta(X, \mu) = \vartheta(X, sf\mu)$. □

Remark 4.4. Whenever (X, μ) is a connected or uniformly connected space then every uniform partition has cardinal 1 and hence, in the above embeddings, the discourse on the chained components is clearly not needed. That is, one can straightly embed the Bourbaki-complete uniform space in a product of real-lines, as in Remark 3.3.

On the other hand, uniformly zero-dimensional spaces represent the opposite situation. Observe that from Theorem 4.2 any complete uniformly zero-dimensional space can be uniformly embedded, as a closed subspace, in a product of uniformly discrete spaces where the embedding is given by the map $\sigma : (X, \mu) \to \left(\prod_{i \in I, n \in \mathbb{N}} \kappa_n^i, \pi \right)$ in the proof of Theorem 4.2 because the inverse map σ^{-1} from Theorem 3.2 is uniformly continuous. Indeed, observe that, since (X, d) is uniformly zero-dimensional the family of all the chained components $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a base for the uniformity of (X, d). Fix $n \in \mathbb{N}$ and suppose that for $x, y \in X$, $\rho(\sigma(x), \sigma(y)) < 1/(n+1)$. Then $\sigma(x)|n = \sigma(y)|n$ and this implies that x, y belong to the same chained component of $\mathcal{P}_k$ for every $k = 1, ..., n$. Therefore the map σ^{-1} is uniformly continuous. Finally, the general case of the uniformly zero-dimensional uniform spaces proceeds like in Theorem 4.2 and taking into the account that the product map of uniformly continuous functions is uniformly continuous.

From the above Theorem 4.3 it is possible to characterize the Tychonoff spaces that are uniformizable by a Bourbaki-complete uniformity. These are exactly the δ-complete spaces of García-Máynez (see [8] and the comments after [13, Definition 2]).

Definition 4.5. A Tychonoff space X is *δ-complete* if $(X, sf\mathtt{u})$ is complete, where $\mathtt{u}$ denotes the fine uniformity on X.

Next result, essentially contained in [13], is now clear.

Theorem 4.6 ([13, Theorem 10 and Theorem 13]) *Let X be a space. The following statements are equivalent:*

(1) X is δ-complete;
(2) $(X, sf\mathtt{u})$ is Bourbaki-complete;
(3) $(X, \mathtt{u})$ is Bourbaki-complete.

In the next result, we apply Theorem 4.3 to a Tychonoff space endowed the fine uniformity $\mathtt{u}$.

Definition 4.7. The *cellularity* of a space X is the supremum of the cardinal of all the partitions by open sets of the space.

Theorem 4.8. *For a space X the following statements are equivalent:*

(1) X is uniformizable by a Bourbaki-complete uniformity;
(2) X is δ-complete;
(3) X is homeomorphic to a closed subspace of $\kappa^\alpha \times \mathbb{R}^\alpha$ where κ is a discrete space of cardinality the cellularity of X;
(4) X is homeomorphic to a closed subspace of a product of locally compact metric spaces.

Proof. (1) $\Rightarrow$ (2). This implication follows from the easy fact that every Bourbaki-complete uniform space is δ-complete.

(2) $\Rightarrow$ (3) Since $(X, sf\mathtt{u})$ is complete then, by Theorem 4.2, X can be embedded as a closed subspace of $\prod_{i\in I, n\in\mathbb{N}} \kappa_n^i \times \mathbb{R}^\alpha$ where each κ_n^i is a cardinal endowed with the discrete topology. Moreover, if κ is the cellularity of X then $\kappa \geq \sup\left\{\kappa_n^i : i \in I, n \in \mathbb{N}\right\}$. Thus, each κ_n can be identified with a subset of κ and $\prod_{i\in I, n\in\mathbb{N}} \kappa_n^i$ is a closed subspace of the product space κ^α.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (1) Every locally compact metrizable space is metrizable by a uniformly locally compact metric. Indeed, for every $x \in X$ let V^x be an open neighbourhood of x such that $\mathrm{cl}_X V^x$ is compact. Put $\mathcal{V} = \{V^x : x \in X\}$. By [6, Chap. IX, 9.4], X is metrizable by a metric d such that $\{B_d(x, 1) : x \in X\} < \mathcal{V}$. In particular $\mathrm{cl}_X B_d(x, 1)$ is compact for every $x \in X$. Now, every uniformly locally compact metric space is Bourbaki-complete by [12, Theorem 14] and since Bourbaki-completeness is a productive property and hereditary by closed subspaces (see [14]), the result follows. □

The equivalence of 2), 3) and 4) in the previous theorem was well-known by García-Máynez (see [8,9]). Moreover, the metric hedgehog $J(\kappa)$ where κ is an Ulam-measurable cardinal is an example of topological complete space (it is a complete metric space) which is not δ-complete. This follows from the following result.

Theorem 4.9. *A connected Tychonoff space is realcompact if and only if it is δ-complete.*

Proof. Recall that a Tychonoff is realcompact if and only if $(X, e\mathtt{u})$ is complete (see for instance [18]) where $e\mathtt{u}$ denotes the well-known countable modification (see [15]) of the fine uniformity $\mathtt{u}$. Moreover, by the result of Morita that states *that every countable cover of cozero sets has a star-finite countable refinement of cozero sets* ([21] or [7, Lemma 5.2.4]) it follows that $e\mathtt{u} \subset sf\mathtt{u}$. Then realcompactness implies, in general, δ-completeness. Since in addition X is connected, every star-finite cozero cover of X is countable (see [7, Lemma 5.3.9]), that is, $e\mathtt{u} = sf\mathtt{u}$. Therefore the result follows. □

5. Universal Metric Space for Bourbaki-Complete Metric Spaces

Trivially Theorem 4.3 provides also a universal space for Bourbaki-complete metric spaces. However, in the metrizable case, it would be preferable that this universal space is also metric. Observe that for a Bourbaki-complete metric space (X, d), the star-finite modification $s\mu_d$ is not metrizable in general and therefore the universal space obtained from Theorem 4.3 is not metric. More precisely, the following result holds.

Theorem 5.1. *Let (X, d) be a Bourbaki-complete metric space. Then the uniform space $(X, sf\mu_d)$ is metrizable if and only if $sf\mu_d = \mu_d$.*

Proof. One implication is clear, so let ρ be a metric on X such that $\mu_\rho = sf\mu_d$. In particular (X, ρ) is complete. Let s_dX and $s_\rho X$ to denote the Samuel compactification of (X, d) and (X, ρ), respectively ([33]). By [14, Theorem 3] s_dX and $s_\rho X$ are equivalent compactifications ([7]). By [11, Corollary 3], it is known that (X, d) and (X, ρ) are uniformly homeomorphic. Therefore, since both are complete, by [11, Corollary 3], $sf\mu_d = \mu_\rho = \mu_d$. □

In order to find a universal metric space for the Bourbaki-complete metric spaces some technical results are needed. For the following definitions see [7,18]

Definition 5.2. A sequence $\langle \mathcal{A}_n \rangle_{n\in\mathbb{N}}$ of open covers of a topological space X is a *complete sequence of covers* if, for every filter $\mathcal{F}$ of X satisfying that $\mathcal{F} \cap \mathcal{A}_n \neq \emptyset$ for every $n \in \mathbb{N}$, then $\mathcal{F}$ has a cluster point.

Definition 5.3. A sequence of covers $\langle \mathcal{C}_n \rangle_{n\in\mathbb{N}}$ of a set X is a *decreasing sequence of covers* if for every $n \in \mathbb{N}$, $\mathcal{C}_{n+1} < \mathcal{C}_n$ and for each $C \in \mathcal{C}_n$, we have that $C = \bigcup \{C' \in \mathcal{C}_{n+1} : C' \subset C\}$.

For (open) covers $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$ of a (space) set X, let $\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge ... \wedge \mathcal{G}_n$ be the (open) cover

$$\{G_1 \cap G_2 \cap ... \cap G_n : G_i \in \mathcal{G}_i, i = 1, 2, ..., n\}.$$

In particular $\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge ... \wedge \mathcal{G}_n$ refines $\mathcal{G}_i$ for each $i = 1, ..., n$. If $\langle \mathcal{A}_n \rangle_{n\in\mathbb{N}}$ is a sequence of covers of X and for every $n \in \mathbb{N}$, put $\mathcal{C}_n = \mathcal{A}_1 \wedge ... \wedge \mathcal{A}_n$ then, $\langle \mathcal{C}_n \rangle_{n\in\mathbb{N}}$ is a decreasing sequence of covers.

Definition 5.4. A a family of sets $\mathcal{L}$ of sets is *directed* provided that, for all, $L, M \in \mathcal{L}$, there exists $N \in \mathcal{L}$ such that $L \cup M \subset N$.

Note that, for any cover $\mathcal{A}$ of X, the family $\{\bigcup \mathcal{E} : \mathcal{E} \subset \mathcal{A} : \mathcal{E} \text{ is finite}\}$ is a directed cover.

Lemma 5.5 ([17, Lemma 2.8]). *Let $\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$ be a decreasing complete sequence of covers of a topological space X. Then the family $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ contains a refinement of every directed open cover of X.*

Observe that the next result is a uniform extension of [17, Theorem 2.16].

Theorem 5.6. *Let (X, d) be a Bourbaki-complete metric space. Then there exists a complete sequence $\langle \mathcal{V}_n \rangle_{n\in\mathbb{N}}$ of uniform star-finite open covers of (X, d) such that $\bigcup \mathcal{V}_n$ is a base of the topology of X.*

Proof. For every $n \in \mathbb{N}$, let $\mathcal{A}_n := \mathcal{A}\left(\mathcal{B}_{1/n}\right) = \{A_{m,i_n} : m \in \mathbb{N}, i_n \in \mathbb{N}\}$ the cover from Lemma 3.1 induced by the cover of open balls $\mathcal{B}_{1/n} = \left\{B_d(x, \frac{1}{n}) : x \in X\right\}$.

Next, define $\mathcal{U}_n = \mathcal{A}_1 \wedge \mathcal{A}_2 \wedge ... \wedge \mathcal{A}_n$, $n \in \mathbb{N}$. It is clear that $\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$ is a decreasing sequence of star-finite uniform open covers of X since finite intersection of star-finite uniform open covers is again star-finite, open and uniform. Then $\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$ is a complete sequence. Indeed, let $\mathcal{F}$ be a filter in X such that for every $n \in \mathbb{N}$ there exists some $U \in \mathcal{U}_n$ such that $F \subset U$ for some $F \in \mathcal{F}$. In particular, $\mathcal{F}$ is a Bourbaki-Cauchy filter because, if $U \in \mathcal{F}$ for some $U \in \mathcal{U}_n$, then

$$U \subset A_{m,i_n} \subset B_d^{m+1}\left(x_{i_n}, \tfrac{1}{n}\right)$$

for some $m \in \mathbb{N}$ and $i_n \in I_n$. Therefore, $\mathcal{F}$ clusters in X and $\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$ is a complete sequence.

Next, let $\mathcal{G}$ be an open cover of X and $\mathcal{G}^f$ be the directed open cover given by finite unions of elements of $\mathcal{G}$. By all the foregoing and by Lemma 5.5, there exists a cover $\mathcal{U} \subset \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ such that $\mathcal{U} < \mathcal{G}^f$. Now, for every $U \in \mathcal{U}$ fix $\mathcal{G}_U$ a finite subfamily of $\mathcal{G}$ such that $U \subset \bigcup \mathcal{G}_U$. Note that for each $n \in \mathbb{N}$, the family $\mathcal{U}_n(\mathcal{G}) = \mathcal{U}_n \cup \{U \cap G : U \in \mathcal{U} \cap \mathcal{U}_n \text{ and } G \in \mathcal{G}_U\}$ is a star-finite open cover of X. Moreover, it is also uniform since $\mathcal{U}_n \subset \mathcal{U}_n(\mathcal{G})$. Therefore, the cover $\mathcal{G}$ has a refinement which is contained in $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n(\mathcal{G})$.

Finally, let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection, and, for every $n, j \in \mathbb{N}$, define the covers $\mathcal{V}_{f((n,j))} = \mathcal{U}_n\left(\mathcal{B}_{1/j}\right)$. The family of sets $\bigcup_{(n,j)\in\mathbb{N}\times\mathbb{N}} \mathcal{V}_{f((n,j))}$ is a base for the topology of X. Indeed, let G be an open set of X and $x \in G$. Then it is possible to choose some $k \in \mathbb{N}$ such that $x \in B_d\left(x, \frac{1}{k}\right) \subset G$. Consider the open cover of balls $\mathcal{B}_{1/2k}$. By all the foregoing, $\mathcal{B}_{1/2k}$ has a refinement contained in $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n\left(\mathcal{B}_{1/2k}\right)=\bigcup_{n\in\mathbb{N}}\mathcal{V}_{f((n,2k))}$. Choose $V\in\bigcup_{n\in\mathbb{N}}\mathcal{V}_{f((n,2k))}$ such that $x\in V$. Then $x\in V\subset B_d\left(y,\frac{1}{2k}\right)$ for some $y\in X$. Since $y\in B_d\left(x,\frac{1}{2k}\right)$ then

$$x\in V\subset B_d\left(y,\tfrac{1}{2k}\right)\subset B_d\left(x,\tfrac{1}{k}\right)\subset G.$$

Thus, $\bigcup_{(n,j)\in\mathbb{N}\times\mathbb{N}}\mathcal{V}_{f((n,j))}$ is a base for the topology of X. □

Theorem 5.7. *Let (X,d) be a Bourbaki-complete metric space. Then there exists a complete metric d' on X which is compatible with the topology of X such that the metric uniformity $\mu_{d'}$ has a base of star-finite covers and $\mu_d\geq\mu_{d'}$. Moreover, $\vartheta(X,d)=\vartheta(X,d')$.*

Proof. By Theorem 5.6, there exists a complete sequence $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ of uniform star-finite open covers of (X,d) such that $\bigcup\mathcal{V}_n$ is a base of the topology of X. Observe that it is possible to take a complete normal sequence $\{\mathcal{W}_n : n\in\mathbb{N}\}$ of open covers from $sf\mu_d$ such that $\mathcal{W}_n<\mathcal{V}_n$ for every $n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}\mathcal{W}_n$ is a base for the topology of X. Indeed, take $\mathcal{W}_1=\mathcal{V}_1$. By [20, Proposition 8, Chapter IV] there is an open cover $\mathcal{A}_1\in sf\mu_d$ such that $\mathcal{A}_1^*<\mathcal{W}_1$. Put $\mathcal{W}_2=\mathcal{V}_1\wedge\mathcal{V}_2\wedge\mathcal{A}_1$. Then $\mathcal{W}_2^*<\mathcal{W}_1$, $\mathcal{W}_2<\mathcal{V}_2$ and clearly, $\mathcal{W}_2$ is a uniform star-finite open cover. Again, by [20, Proposition 8, Chapter IV], there is an open cover $\mathcal{A}_2\in sf\mu_d$ such that $\mathcal{A}_2^*<\mathcal{W}_2$. Put $\mathcal{W}_3=\mathcal{V}_1\wedge\mathcal{V}_2\wedge\mathcal{V}_3\wedge\mathcal{A}_2$. Thus, proceeding by induction we obtain the desired normal sequence.

By Lemma 4.1, there exists a uniformly continuous pseudometric d' on X such that (X,d') has a base of star-finite open covers for the pseudometric uniformity. Moreover, (X,d') is complete because the sequence $\langle\mathcal{W}_n\rangle_{n\in\mathbb{N}}$ is complete. By the same lemma, since $\bigcup_{n\in\mathbb{N}}\mathcal{W}_n$ is a base for the topology of X which is in addition Hausdorff, then d' is a metric compatible with the topology of X. Finally, $\mu_d\geq\mu_{d'}$ since $\mathcal{W}_n\in sf\mu_d$ for every $n\in\mathbb{N}$.

Now, since $\mu_d\geq\mu_{d'}$ then $\vartheta(X,d)\geq\vartheta(X,d')$. To check that $\vartheta(X,d)\leq\vartheta(X,d')$ is also satisfied let $\mathcal{P}_n$, $n\in\mathbb{N}$ denote the families of all the chained components induced by the covers $\left\{B_d(x,\frac{1}{n}) : x\in X\right\}$, and let $\mathcal{Q}_m$, $m\in\mathbb{N}$, the family of all the chained components induced by the above covers $\mathcal{W}_m$. Fix $n\in\mathbb{N}$ and consider $\mathcal{P}_n$. Looking into the end of the proof of Theorem 5.6, there exists some $m\in\mathbb{N}$ such that the cover $\mathcal{V}_m:=\mathcal{V}_{f((n,j))}$, from the beginning of this proof, induces the same chained components than the cover $\{B_d(x,\frac{1}{n}) : x\in X\}$, $n\in\mathbb{N}$, that is, precisely the family $\mathcal{P}_n$. Since $\mathcal{W}_m<\mathcal{V}_m$ then $\mathcal{Q}_m<\mathcal{P}_n$. Therefore for every $n\in\mathbb{N}$ there exists some $m\in\mathbb{N}$ such that $|\mathcal{P}_n|\leq|\mathcal{Q}_m|$ which implies that $\vartheta(X,d)\leq\vartheta(X,d')$. □

Theorem 5.8. *Let (X,d) be a Bourbaki-complete metric space. Then, there exists an embedding*

$$\varphi:(X,d)\to\left(\left(\prod_{n\in\mathbb{N}}\kappa_n\right)\times\mathbb{R}^{\omega_0},\rho+t\right)$$

where each κ_n *is a cardinal,* φ *is uniformly continuous and* $\varphi(X)$ *is a closed subspace of the arrival space* $\left(\prod_{n\in\mathbb{N}}\kappa_n\right)\times\mathbb{R}^{\omega_0}$*. Moreover,* $\vartheta(X,d)\leq$ $\sup\left\{\prod_{j=1}^{n}\kappa_j : n\in\mathbb{N}\right\}$.

Proof. By Theorem 5.7 there exists a compatible metric d' on X such that (X,d') is complete, the metric uniformity $\mu_{d'}$ has a star-finite base, the identity map $id:(X,d)\to(X,d')$ is uniformly continuous and $\vartheta(X,d)=\vartheta(X,d')$. On the other hand, consider the embedding

$$\varphi:(X,d')\to\left(\left(\prod_{n\in\mathbb{N}}\kappa_n\right)\times\mathbb{R}^{\omega_0},\rho+t\right)$$

form Theorem 3.2. Then the composition $\varphi\circ id=\varphi$ is the desired embedding. Finally, as $\vartheta(X,d)=\vartheta(X,d')$, by Theorem 3.2 it follows that $\vartheta(X,d)\leq$ $\sup\left\{\prod_{j=1}^{n}\kappa_j : n\in\mathbb{N}\right\}$. □

6. Metrization Results

The main theorem of this section is Theorem 6.5 which includes many equivalent characterization of the metrizable spaces which are metrizable by a Bourbaki-complete metric. Many of these were already stated in [17]. However, the proofs given here are more direct. The central property that link all the characterizations is strong-metrizability. Basic facts and bibliography about strongly metrizable spaces can be also found in [17].

Definition 6.1. A space is *strongly metrizable* if it has a base for the topology which consists of the union of countably many star-finite open covers.

Definition 6.2. A metrizable space is said to be *completely metrizable* if it is metrizable by a complete metric.

From Theorem 5.6 it is immediate that *every Bourbaki-complete metric space is completely metrizable and strongly metrizable.* The following result is the reciprocal.

Theorem 6.3. *Let* X *be a completely metrizable and strongly metrizable space. Then* X *is metrizable by a complete metric* ζ *such that* $\mu_\zeta=sf\mu_\zeta$*. In particular,* (X,ζ) *is Bourbaki-complete.*

Proof. Since X is strongly metrizable there exists a countable family $\{\mathcal{V}_n : n\in\mathbb{N}\}$ of star-finite open covers of X such that $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n$ is a base for the topology of X. In particular, by paracompactness of X, the family of all the star-finite open covers of X form a base for the uniformity $sf\mathtt{u}$, where $sf\mathtt{u}$ denotes the star-finite modification of the fine uniformity $\mathtt{u}$. Hence, it is possible to apply the axioms of uniformity to the countable family of star-finite open covers $\{\mathcal{V}_n : n\in\mathbb{N}\}$ and to find a normal sequence of star-finite open covers

$\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$, such that $\mathcal{U}_n < \mathcal{V}_n$ for every $n \in \mathbb{N}$ (as in the proof of Theorem 5.6). Since $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a base for the topology of X, then $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is also a base for the topology of X.

Indeed, let G an open set of X and $x \in X$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a base, there exists some $V \in \mathcal{V}_n$ for some $n \in \mathbb{N}$, such that $x \in V \subset G$. Next, let ρ be any metric on X, and choose $k \in \mathbb{N}$ such that $B_\rho\left(x, \frac{1}{k}\right) \subset V$. Again by the base condition of $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, there exists some $m \in \mathbb{N}$, such that for some $V' \in \mathcal{V}_m$

$$x \in V' \subset B_\rho\left(x, \tfrac{1}{2k}\right) \subset B_\rho\left(x, \tfrac{1}{k}\right) \subset V \subset G.$$

Next, consider the cover $\mathcal{U}_m$ and choose some $U \in \mathcal{U}_m$, such that $x \in U$. Since $\mathcal{U}_m < \mathcal{V}_m$ then

$$x \in U \subset St^2\left(V', \mathcal{V}_m\right) \subset B_\rho\left(x, \tfrac{1}{k}\right) \subset V \subset G.$$

Thus, $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is also a base for the topology of X.

Next, apply Lemma 4.1 to $\langle \mathcal{U}_n \rangle_{n\in\mathbb{N}}$ and let d be the pseudometric obtained. Then d is compatible with the topology X and, in particular, d is a metric. Moreover, the uniformity induced by d has a star-finite base, that is, $\mu_d = sf\mu_d$.

Consider the completion $\left(\widetilde{X}, \widetilde{d}\right)$ of (X, d). Then, $\left(\widetilde{X}, \widetilde{d}\right)$ is complete and has a star-finite base by [26, Lemma p. 370], that is, $\mu_{\widetilde{d}} = sf\mu_{\widetilde{d}}$. Now, since X is completely metrizable, by [7, Theorem 4.3.24], X is a G_δ-set of $\widetilde{X}$. Thus, by [7, Theorem 4.3.22], X is homeomorphic to a closed subspace of $\left(\widetilde{X} \times \mathbb{R}^{\omega_0}, \widetilde{d} + t\right)$. The restriction of $\widetilde{d} + t$ over X is a metric ζ on X satisfying that $\mu_\zeta = sf\mu_\zeta$ which is, in particular, Bourbaki-complete.

□

Now, take into the account that it is possible to write, in the above definition of strongly metrizable space, star-countable instead of star-finite ([17]), where a star-countable cover is naturally defined as a star-finite cover but changing finite by countable. This follows from the well-known fact that *every star-countable cover of cozero sets has a star-finite refinement of cozero sets* which derives from the already cited Morita's result (see [21] or [7, Lemma 5.2.4, Lemma 5.3.9]).

Remark 6.4. Contrarily to the case of the star-finite covers, the family of all the star-countable uniform covers of a uniform space (X, μ), denoted by $sc\mu$, is not necessarily a base for a uniformity but just for a quasi-uniformity [26, p. 368]. However, in some cases, it is a base for a uniformity. For instance, as we have noticed in the previous paragraph, for any Tychonoff space X, $sf\mathtt{u} = sc\mathtt{u}$ because every star-countable cover of cozero sets has a star-finite refinement of cozero sets. Another example are the uniform spaces satisfying that $\mu = e\mu$, where $e\mu$ denotes the countable modification of μ (see [15]). For these, it is clear that $sc\mu$ is a uniformity since $sc\mu = e\mu = \mu$. Moreover, Theorem 6.5

shows that the strongly metrizable spaces are exactly the metrizable spaces by a metric such that its metric uniformity has a star-countable base.

Theorem 6.5. *Let X be a space. The following statements are equivalent:*

(1) X is metrizable by a Bourbaki-complete metric;
(2) X is metrizable by a complete metric d, such that $\mu_d = sf\mu_d$;
(3) X is homeomorphic to a closed subspace of $\kappa^{\omega_0} \times \mathbb{R}^{\omega_0}$ where κ is the cellularity of X;
(4) X is homeomorphic to a closed subspace of a countable product of locally compact metric spaces;
(5) X is Čech-complete (completely metrizable) and strongly metrizable;
(6) X is metrizable by a complete metric d such that $\mu_d = sc\mu_d$

Proof. 1) $\Rightarrow$ 2) This is Theorem 5.7.

2) $\Rightarrow$ 3) Let κ be the cellularity of X. Then, it is clear that $\kappa \geq \sup\{\kappa_n : n \in \mathbb{N}\}$ where the κ_n's are the cardinals from Theorem 3.2 such that X can be embedded as a closed subspace of $\prod_{n\in\mathbb{N}} \kappa_n \times \mathbb{R}^{\omega_0}$. Then for every $n \in \mathbb{N}$, κ_n can be identified with a subspace of κ and thus $\prod_{n\in\mathbb{N}} \kappa_n$ is a closed subspace of the Baire space κ^{ω_0}. Then, the result follows from Theorem 3.2.

3) $\Rightarrow$ 4) This is trivial since κ, when it is endowed with the discrete topology, and $\mathbb{R}$ are locally compact spaces.

4) $\Rightarrow$ 1) Like in the proof of implication 4) $\Rightarrow$ 1) of Theorem 4.8 every locally compact metrizable space is metrizable by a uniformly locally compact metric. Since every uniformly locally compact space is Bourbaki-complete, a countable product of Bourbaki-complete metric spaces is a Bourbaki-complete metric space, endowed with the usual product metric, and Bourbaki-completeness is inherited by closed subspaces ([12]), the result follows.

1) $\Rightarrow$ 5) It follows at once from Theorem 5.6.

5) $\Rightarrow$ 1) This is Theorem 6.3.

2) $\Rightarrow$ 6) This is immediate.

6 $\Rightarrow$ 5) It is enough to prove strong metrizability. If $\mu_d = sc\mu_d$ then, for every $n \in \mathbb{N}$ it is possible to take a uniform star-countable cover $\mathcal{V}_n$ such that $\mathcal{V}_n$ refines the uniform cover $\left\{B_d(x, \frac{1}{n}) : x \in X\right\}$. Then $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a base of for the topology of X. It follows that X is strongly-metrizable since, in metrizable spaces, every star-countable open cover has an open star-finite refinement. □

The following result states several characterizations of the property of strong metrizability that can be deduced from the above result on Bourbaki-complete metrization (these can also be found in [12,17]). The link between strongly metrizable spaces and Bourbaki-complete metrizable spaces lies in the spaces having a Bourbaki-complete completion. To understand these spaces we need the following notion of uniform boundedness.

Definition 6.6. A subset B of a uniform space (X, μ) is *Bourbaki-bounded in* X if

$$\forall \mathcal{U} \in \mu \ \exists m \in \mathbb{N}, \ \exists U_1, ..., U_k \in \mathcal{U} \text{ s.t. } B \subset \bigcup_{i=1}^{k} St^m (U_i, \mathcal{U}).$$

If X is Bourbaki-bounded in itself then X is a *Bourbaki-bounded space*. If for every $\mathcal{U} \in \mu$ we can take always $m = 1$ then B is a *totally bounded* subset of X.

Theorem 6.7 ([12, Theorem 9]). *A metric space* (X, d) *is Bourbaki-complete if and only if the closure of every Bourbaki-bounded subset in* X *is compact.*

Theorem 6.8. *Let* X *be a space. The following statements are equivalent:*

(1) X *is metrizable by a metric* d *such that every Bourbaki-bounded subsets in* (X, d) *is totally bounded;*
(2) X *is metrizable by a metric* d *such that the completion of* (X, d) *is Bourbaki-complete;*
(3) X *is metrizable by a metric* d *such that* $\mu_d = sf\mu_d$*;*
(4) X *is homeomorphic to a subspace of* $\kappa^{\omega_0} \times \mathbb{R}^{\omega_0}$ *where* κ *is the cellularity of* X*;*
(5) X *is homeomorphic to a subspace of a countable product of locally compact metric spaces;*
(6) X *is strongly metrizable;*
(7) X *is metrizable by a metric* d *such that* $\mu_d = sc\mu_d$*.*

Proof. (1) $\Leftrightarrow$ (2) This equivalence is evident.

2) $\Rightarrow$ 3) If the completion $\widetilde{X}$ of (X, d) is Bourbaki-complete then by Theorem 5.7, $\widetilde{X}$ is metrizable by a complete metric ρ such that $\left(\widetilde{X}, \rho\right)$ satisfies that $\mu_\rho = sf\mu_\rho$. Since this last property is clearly hereditary, the restriction of ρ over X is the desired metric.

3) $\Rightarrow$ 4) If (X, d) satisfies that $\mu_d = sf\mu_d$ then its completion $\left(\widetilde{X}, \widetilde{d}\right)$ too by [26, Lemma p. 370]. Therefore, $\widetilde{X}$ is homeomorphic to a closed subspace of $\kappa^{\omega_0} \times \mathbb{R}^{\omega_0}$ by Theorem 3.2, and X is also homeomorphic to a subspace of $\kappa^{\omega_0} \times \mathbb{R}^{\omega_0}$.

4) $\Rightarrow$ 5) This is trivial.

5) $\Rightarrow$ 2) Consider the closure of X in the countable product of locally compact spaces and apply Theorem 6.5.

2) $\Rightarrow$ 6) This implication follows from Theorem 5.6 and from the fact that strong metrizability is an hereditary property.

6) $\Rightarrow$ 2) Let X be a strongly metrizable space. Then for any compatible metric d on X, if $\left(\widetilde{X}, \widetilde{d}\right)$ denotes its completion, then $\widetilde{X}$ is strongly metrizable. In fact the open covers of X are extended to $\left(\widetilde{X}, \widetilde{d}\right)$. Thus $\left(\widetilde{X}, \widetilde{d}\right)$ is strongly metrizable and complete. The result is then immediate from Theorem 6.3.

3) ⇒ 7) This is immediate.

7) ⇒ 6) This is like the proof of the implication 6) ⇒ 5) in Theorem 6.5. □

7. Bourbaki-Complete Metric Spaces do not Having a Star-Finite Base for their Uniformity

The Theorem 2.4 is not very surprising (but not immediate to prove). In fact, the key is in the next result which relates the star-finite covers with sets of the form $St^m(U,\mathcal{U})$.

Theorem 7.1. *A uniform space (X,μ) has a star-finite base for its uniformity, that is, $\mu = sf\mu$, if and only if it satisfies the following property:*

$$(\star)\ \forall \mathcal{U}\in\mu\ \exists \mathcal{V}\in\mu\ s.t.\ \forall V\in\mathcal{V},\ \forall n\in\mathbb{N}\ \exists U_1,...,U_k\in\mathcal{U}\ s.t.\ St^n(V,\mathcal{V})\subset\bigcup_{i=1}^{k}U_i.$$

In particular, it is possible to choose $\mathcal{U}$ and $\mathcal{V}$ belonging to a base of μ.

Proof. ⇒) Let $\mathcal{U}\in\mu$ and $\mathcal{V}\in\mu$ star-finite such that $\mathcal{V}<\mathcal{U}$. By the star-finite property, for every $V\in\mathcal{V}$ and every $n\in\mathbb{N}$ there exist at most finitely many $V'\in\mathcal{V}$ such that $V'\cap St^n(V,\mathcal{V})\neq\emptyset$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$ the property $(\star)$ follows.

⇐) Conversely, let $\mathcal{U}\in\mu$ and select $\mathcal{V}\in\mu$ such that $\mathcal{V}^*<\mathcal{U}$. By hypothesis there is some $\mathcal{W}\in\mu$ such that for every $n\in\mathbb{N}$ and every $W\in\mathcal{W}$ there exists finitely many $V_i\in\mathcal{V}$, $i=1,...,k$ such that $St^n(W,\mathcal{W})\subset\bigcup_{i=1}^{k}V_i$. Without loss of generality it is possible to take $\mathcal{W}$ refining $\mathcal{V}$. Let $\mathcal{A}(\mathcal{W})=\{A_n^i : n\in\mathbb{N}, i\in I\}$ the cover from Lemma 3.1 induced by $\mathcal{W}$. Clearly $\mathcal{A}(\mathcal{W})$ belongs to $sf\mu$. By hypothesis, for every $i\in I$ and every $n\in\mathbb{N}$ it is possible to fix a finite family $\mathcal{V}_{i,n}\subset\mathcal{V}$ such that

$$A_n^i\subset St^{n+1}(x_i,\mathcal{W})\subset St^{n+1}(W_i,\mathcal{W})\subset\bigcup\{V : V\in\mathcal{V}_{i,n}\}$$

where $W_i\in\mathcal{W}$ is some set such that $W_i\subset St(x_i,\mathcal{W})$. Define

$$\mathcal{G}=\{A_n^i\cap St(V,\mathcal{V}) : V\in\mathcal{V}_{i,n},\ i\in I,\ n\in\mathbb{N}\}.$$

Then $\mathcal{A}(\mathcal{W})\wedge\mathcal{V}=\{A\cap V : A\in\mathcal{A}(\mathcal{W}), V\in\mathcal{V}\}<\mathcal{G}<\mathcal{V}^*<\mathcal{U}$, and it is easy to check that $\mathcal{G}$ is also star-finite. □

By the above result and the examples in the second section, one could think that every Bourbaki-complete uniform space (X,μ) satisfies that $\mu = sf\mu$. But this is not true as both examples of this section show. The first example given not only fails that $sf\mu=\mu$ but, more precisely, its uniformity do not has a *point-finite* base.

Definition 7.2. A cover $\mathcal{C}$ of a set X is *point-finite* if every $x\in X$ belongs to at most finitely many $C\in\mathcal{C}$.

If $\mathcal{U}$ is a compatible uniformity of a Tychonoff space then the family of all the point-finite uniform covers from μ is a base for a compatible uniformity on X ([20]), as it happens with the star-finite covers. This uniformity is called the *point-finite modification* of μ and it is denoted by $pf\mu$. Since every star-finite cover is point-finite then every uniformity having a star-finite base also has a point-finite base.

The next example is constructed from the following result that can be found in [1,25]. Consider the Banach space $(\ell_\infty(\omega_1), ||\cdot||_\infty)$ of all the bounded real-valued functions $f:\omega_1 \to \mathbb{R}$ over a set having ω_1 as a cardinal, endowed with the norm of the supremum $||\cdot||_\infty$. Let d_∞ the metric induced by $||\cdot||_\infty$. Then $(\ell_\infty(\omega_1), d_\infty)$ is a complete metric space which does not have a point-finite base for its uniformity.

Example 7.3. There exists a Bourbaki-complete metric space which does not have a point-finite base for its uniformity. Therefore, the uniformity does not have a star-finite base either.

Proof. Construction. For every $n \in \mathbb{N}$, let $X_n = \ell_\infty(\omega_1)$, be the set of all bounded real-valued functions over a set of cardinality ω_1, but now endowed with the metric

$$s_n(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^n} + min\{1, ||x-y||_\infty\} & \text{if } x \neq y. \end{cases}$$

Then the metric space (X_n, s_n) is Bourbaki-complete and every uniform cover of it has a point-finite uniform refinement since it is a uniformly discrete metric space.

Now, let $X = \biguplus_{n\in\mathbb{N}} X_n$ be the set given by the disjoint union of the above spaces, and endowed X with the metric

$$s(x,y) = \begin{cases} s_n(x,y) & \text{if for some } n \in \mathbb{N},\ x,y \in X_n \\ 2 & \text{otherwise.} \end{cases}$$

Then (X,s) is a Bourbaki-complete metric space because it is a disjoint union of uniformly separated Bourbaki-complete metric spaces. However its metric uniformity does not has a base of star-finite covers as we show next.

Let $d_\infty(x,y) = ||x-y||_\infty$ the usual metric on $\ell_\infty(\omega_1)$. Since this space does not has a base of point-finite covers for its metric uniformity then it is possible to choose some $N \geq 2$ such that every uniform refinement $\mathcal{V}$, for the metric uniformity induced by d_∞, of the cover $\left\{B_{d_\infty}\left(x, 1/2^N\right) : x \in \ell_\infty(\omega_1)\right\}$ is not point-finite. In particular, the same is true for the space $(\ell_\infty(\omega_1), \eta)$, where $\eta(x,y) = min\{1, d_\infty(x,y)\}$, by uniform equivalence of the metrics η and d_∞.

Now, observe that for every $n \in \mathbb{N}$ and every $x \in \ell_\infty(\omega_1)$,

$$(\clubsuit) \quad B_{s_n}\left(x, \tfrac{1}{2^{n-1}}\right) = B_\eta\left(x, \tfrac{1}{2^n}\right).$$

Moreover, for every $n, k \in \mathbb{N}$, $n > k$, and every $x \in \ell_\infty(\omega_1)$

$$(\heartsuit) \quad B_{s_n}\left(x, \tfrac{1}{2^k}\right) \subset B_\eta\left(x, \tfrac{1}{2^k}\right).$$

Take the uniform cover $\mathcal{B} = \left\{B_s\left(x, \frac{1}{2^N}\right) : x \in X\right\}$ of X. Then,

$$\mathcal{B} = \biguplus_{n\in\mathbb{N}} \left\{B_{s_n}\left(x, \tfrac{1}{2^N}\right) : x \in X_n\right\}$$

as $N \geq 2$. We are going to prove that every uniform refinement $\mathcal{V}$ of $\mathcal{B}$ fails to be point-finite.

Indeed, since $\mathcal{V}$ is uniform we can choose some $m \in \mathbb{N}$, $m > N$ such that the cover

$$\left\{B_s\left(x, \tfrac{1}{2^m}\right) : x \in X\right\} = \biguplus_{n\in\mathbb{N}} \left\{B_{s_n}\left(x, \tfrac{1}{2^m}\right) : x \in X_n\right\}$$

refines $\mathcal{V}$. Thus, we can write also that$\mathcal{V} = \biguplus_{n\in\mathbb{N}} \mathcal{V}_n$.where each $\mathcal{V}_n$ is a uniform cover of (X_n, s_n) that refines $\left\{B_{s_n}\left(x, \frac{1}{2^N}\right) : x \in X_n\right\}$.

Now, by ($\clubsuit$), whenever $n > N$, $\mathcal{V}_n$ is a uniform cover of $\ell_\infty(\omega_1)$ and, by ($\heartsuit$), it also refines $\left\{B_\rho\left(x, \frac{1}{2^N}\right) : x \in \ell_\infty(\omega_1)\right\}$. Then $\mathcal{V}_n$ fails to be point-finite for every $n > N$ which means that $\mathcal{V}$ is not point-finite. Finally, we can conclude that (X, s) does not have a point-finite base, nor a star-finite base. □

Open Problem 7.4. It would be convenient to find a family of filters, similarly to the Bourbaki-Cauchy filters, that characterizes those uniform space spaces (X, μ) satisfying that $(X, pf\mu)$ is complete. Of course, the Cauchy filters of $(X, pf\mu)$ will work, but notice that not every Bourbaki-Cauchy filter of (X, μ) is a Cauchy filter of $(X, sf\mu)$ (see [14, Lemma 14]). Therefore the searched family of filters might be wider than the family of Cauchy filters for $(X, pf\mu)$.

The real line $\mathbb{R}$ endowed with the usual Euclidean metric d_u has a star-finite base for its metric uniformity. This is a very strong uniform condition. Moreover, the metric d_u also has very strong metric condition. For instance $(\mathbb{R}, d_u)$ satisfies the *Heine-Borel property*, that is, every closed and bounded subset of it is compact. In [17] it is proved that $\mathbb{R}$, is metrizable by a complete (Bourbaki-complete) metric d satisfying that for every $\varepsilon > 0$ each open ball $B_d(x, \varepsilon)$ meets at most 27 open balls of radius ε, which is even a stronger condition than the Heine-Borel property.

Now, instead of looking for metrics on $\mathbb{R}$ satisfying strong metric properties, the aim is to find metrics over $\mathbb{R}$, compatible with the usual topology, that preserve Bourbaki-completeness but that have weaker uniform pro- perties than the Euclidean metric d_u. More precisely, in the next example, which is in the same line than Example 7.3, we obtain a Bourbaki-complete metric on $\mathbb{R}$ which does not have a star-finite base for its metric uniformity. Notice that since $\mathbb{R}$ is Lindelöf and by [31, Theorem 1], any compatible uniformity μ on $\mathbb{R}$ satisfy that $\mu = e\mu = sc\mu = pf\mu$.

Example 7.5. There is a compatible Bourbaki-complete metric on $\mathbb{R}$*, endowed with Euclidean topology, such that the metric uniformity does not have a star-finite base.*

Proof. Construction. Start from $(\mathbb{R}, d_u)$. First, let $\{N_j : j \in \mathbb{N}\}$ be an infinite partition of $\mathbb{N}$ satisfying the following conditions:

(1) $1 \in N_1$;
(2) each subset N_j is infinite;
(3) if $n \in N_j$ then $n+1 \notin N_j$ for every $j, n \in \mathbb{N}$.

Next, for every $i \in \mathbb{N}$ consider the countable open cover $\mathcal{U}_i$ containing all the following sets:

(1) if $q \in \mathbb{Q} \cap (-\infty, 1)$ take $B_{d_u}\left(q, \frac{1}{2^{i+1}}\right)$;
(2) for every $n \in \mathbb{N}$, take the open balls $B_{d_u}\left(q, \frac{1}{2^{i+n}}\right)$ for every $q \in \mathbb{Q} \cap (n, n+1)$;
(3) if $j < i$ and $n \in N_j$ also take $B_{d_u}\left(n, \frac{1}{2^{i+n}}\right)$;
(4) for every $j \geq i$ take $A_i^j = \bigcup_{n \in N_j} B_{d_u}\left(n, \frac{1}{2^{i+n}}\right)$.

We first prove that the family of covers $\langle \mathcal{U}_{4i} \rangle_{i \in \mathbb{N}}$ is a normal sequence, that is, for every $i \in \mathbb{N}$, $\mathcal{U}_{4(i+1)}{}^* < \mathcal{U}_{4i}$. Fix $i \in \mathbb{N}$. There are several cases.

- If $x = n \in N_j$ and $j \geq 4(i+1) > 4i$ then $x \in A^j_{4(i+1)} \in \mathcal{U}_{4(i+1)}$ and

$$St\left(A^j_{4(i+1)}, \mathcal{U}_{4(i+1)}\right) \subset A^j_{4i} \in \mathcal{U}_{4i}.$$

- If $x = n \in N_j$ and $j < 4i < 4(i+1)$ then $x \in B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \in \mathcal{U}_{4(i+1)}$ and

$$\begin{aligned}
&St\left(B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right), \mathcal{U}_{4(i+1)}\right) \\
&= \bigcup_{q \in (n-1,n)} \left\{B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n-1}}\right) : B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n-1}}\right) \cap B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right) \neq \emptyset\right\} \\
&\cup \bigcup_{q \in [n,n+1)} \left\{B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n}}\right) : B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n}}\right) \cap B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right) \neq \emptyset\right\} \\
&\subset B_{d_u}\left(n, \tfrac{1}{2^{4i+1}}\right) \in \mathcal{U}_{4i}
\end{aligned}$$

because, if

$$y \in \bigcup_{q \in (n-1,n)} \left\{B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n-1}}\right) : B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n-1}}\right) \cap B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right) \neq \emptyset\right\}$$

then $d_u(y, n) \leq 2 \cdot \frac{1}{2^{4(i+1)+n-1}} < \frac{1}{2^{4i+n}}$, and if

$$y \in \bigcup_{q \in [n,n+1)} \left\{B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n}}\right) : B_{d_u}\left(q, \tfrac{1}{2^{4(i+1)+n}}\right) \cap B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right) \neq \emptyset\right\}$$

then $d_u(y,n) \leq 3 \cdot \frac{1}{2^{4(i+1)+n}} < \frac{1}{2^{4i+n}}$.

• If $x = n \in N_{4i}$ then $x \in B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \in \mathcal{U}_{4(i+1)}$ and, similarly to the previous case,

$$St\left(B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right), \mathcal{U}_{4(i+1)}\right) \subset B_{d_u}\left(n, \tfrac{1}{2^{4i+1}}\right) \subset A_{4i}^{4i} \in \mathcal{U}_{4i}.$$

• If $x \notin \mathbb{N}$ then $x \in B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \in \mathcal{U}_{4(i+1)}$, and if $B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \cap A_{4(i+1)}^{j} = \emptyset$ for every $j \geq 4(i+1)$, then

$$St\left(B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right), \mathcal{U}_{4(i+1)}\right) \subset B_{d_u}\left(x, \tfrac{1}{2^{4i+n}}\right) \in \mathcal{U}_{4i}.$$

• If $x \notin \mathbb{N}$ then $x \in B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \in \mathcal{U}_{4(i+1)}$, and if $B_{d_u}\left(x, \frac{1}{2^{4(i+1)+n}}\right) \cap A_{4(i+1)}^{j} \neq \emptyset$ for some $j \geq 4(i+1)$, then

$$St\left(B_{d_u}\left(x, \tfrac{1}{2^{4(i+1)+n}}\right), \mathcal{U}_{4(i+1)}\right) \subset St\left(A_{4(i+1)}^{j}, \mathcal{U}_{4(i+1)}\right) \subset A_{4i}^{j} \in \mathcal{U}_{4i}$$

as in the first case.

Now, we prove that $\bigcup_{i\in\mathbb{N}} \mathcal{U}_{4i}$ is a base for the topology of $\mathbb{R}$. Indeed let $r \in \mathbb{R}$ and $x \in B_{d_u}(r, \varepsilon)$. If $x < 1$, by density of the rationals, there exists some $q \in \mathbb{Q} \cap (-\infty, 1)$ for which is possible to take $i \in \mathbb{N}$ bigger enough such that

$$x \in B_{d_u}\left(q, \tfrac{1}{2^{4i+1}}\right) \subset B_{d_u}(r, \varepsilon) \text{ and } B_{d_u}\left(q, \tfrac{1}{2^{4i+1}}\right) \in \mathcal{U}_{4i}.$$

If $x \geq 1$ then, by density of the rationals, it is possible to choose some $q \in (\mathbb{Q}\backslash\mathbb{N}) \cap (1, \infty)$ such that for some $i \in \mathbb{N}$ bigger enough, if $[q]$ denotes the integer part of q,

$$x \in B_{d_u}\left(q, \tfrac{1}{2^{4i+[q]}}\right) \subset B_{d_u}(r, \varepsilon) \text{ and } B_{d_u}\left(q, \tfrac{1}{2^{4i+[q]}}\right) \in \mathcal{U}_{4i}.$$

By all the foregoing, applying Lemma 4.1, since the topology is Hausdorff, there exists a compatible metric ς on $\mathbb{R}$ such tat

$$\mathcal{B}_{1/2^{i+1}} < \mathcal{U}_{4i} < \mathcal{B}_{1/2^{i-1}} \text{ for every } i \in \mathbb{N}.$$

With this metric, $(\mathbb{R}, \varsigma)$ is Bourbaki-complete but fails to have a base of star-finite covers for the metric uniformity.

First, let us check that the metric uniformity does not have such a base by contradicting property $(\star)$ from Theorem 7.1. Indeed, fix any $\mathcal{U}_{4i}$. The next reasoning proves that for every $l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the set $St^m\left(A_{4l}^{4l}, \mathcal{U}_{4l}\right)$ cannot be covered by finitely many $U \in \mathcal{U}_{4i}$. In particular, $A_{4l}^{4l} \in \mathcal{U}_{4l}$. Hence the contradiction of property $(\star)$.

If $l < i$ the set $St^m\left(A_{4l}^{4l}, \mathcal{U}_{4l}\right)$, with $m = 0$, cannot be covered by finitely many $U \in \mathcal{U}_{4i}$. Indeed,

$$\bigcup_{n\in N_{4l}} B_{d_u}\left(n, \tfrac{1}{2^{4i+n}}\right) \subsetneq A_{4l}^{4l}$$

where $B_{d_u}\left(n, \frac{1}{2^{4i+n}}\right) \in \mathcal{U}_{4i}$ for every $n \in N_{4\,l}$ because $l < i$. So, in any case, we need infinitely many balls $B_{d_u}\left(q, \frac{1}{2^{4i+[q]}}\right) \in \mathcal{U}_{4i}$, $q \in \mathbb{Q}$, in order to cover

$St^0\left(A_{4l}^{4l},\mathcal{U}_{4l}\right)$. Besides, observe that there is no set of the form A_{4i}^{j}, $j \geq 4i$, which covers the set A_{4l}^{4l}.

Now if $l \geq i$, $A_{4i}^{4l} \in \mathcal{U}_{4i}$ but

$$A_{4i}^{4l} \subsetneq St^{2^{4l-4i}+1}\left(A_{4l}^{4l}, \mathcal{U}_{4l}\right)$$

since it is not difficult check that in $\mathbb{R}$,

$$B_{d_u}\left(n, \tfrac{1}{2^{4l+n}}\right) \subsetneq B_{d_u}^{2^{4l-4i}+1}\left(n, \tfrac{1}{2^{4i+n}}\right)$$

for every $n \in N_{4l}$. So we need again infinitely many balls $B_{d_u}\left(q, \frac{1}{2^{4i+[q]}}\right) \in \mathcal{U}_{4i}$, $q \in \mathbb{Q}$, in order to cover the set $St^m\left(A_{4l}^{4l}, \mathcal{U}_{4l}\right)$ where $m = 2^{4l-4i}+1$. Observe that there is no set of the form A_{4i}^{j}, $j \geq 4i$, which covers $St^{2^{4l-4i}+1}\left(A_{4l}^{4l}, \mathcal{U}_{4l}\right)$.

In order to show that $(\mathbb{R},\varsigma)$ is Bourbaki-complete, we prove that $B \subset \mathbb{R}$ is Bourbaki-bounded in $(\mathbb{R},\varsigma)$ if and only if B is a bounded subset for the metric d_u. Therefore the closure of B will be compact, satisfying in this way Theorem 6.7. Notice that $B \cap (\infty, 0]$ is bounded by the metric d_u if and only if it is bounded by the metric ς. Therefore we just need to check that $B \cap (0,\infty)$ is a bounded subset for d_u.

First, observe that $(\mathbb{R},\varsigma)$ is well-chained because $\mathbb{R}$ is connected. Therefore, if B is Bourbaki-bounded then for every $\mathcal{U}_{4i}$ and for any $U \in \mathcal{U}_{4i}$ there exists $m \in \mathbb{N}$ such that

$$(\spadesuit) \qquad\qquad B \subset St^m\left(U,\mathcal{U}_{4i}\right).$$

In addition, observe that the following is always satisfied in $\mathbb{R}$:
for every $i, n \in \mathbb{N}$,

$$\begin{aligned}
&[n,n+1] \subset St^{2^{4i+n}+1}\left(B_{d_u}\left(n, \tfrac{1}{2^{4i+n}}\right), \mathcal{U}_{4i}\right)\\
&\qquad\qquad \text{and } [n-1,n] \subset St^{2^{4i+n-1}+1}\left(B_{d_u}\left(n, \tfrac{1}{2^{4i+n}}\right), \mathcal{U}_{4i}\right),\\
&[n,n+1] \not\subset St^{2^{4i+n}}\left(B_{d_u}\left(n, \tfrac{1}{2^{4i+n}}\right), \mathcal{U}_{4i}\right)\\
&\qquad\qquad \text{and } [n-1,n] \not\subset St^{2^{4i+n-1}}\left(B_{d_u}\left(n, \tfrac{1}{2^{4i+n}}\right), \mathcal{U}_{4i}\right).
\end{aligned}$$

In particular, since $\sup\{n \in N_j\} = \infty$, $(\diamondsuit)$ there is no $m \in \mathbb{N} \cup \{0\}$ such that for every $n \in N_j$,

$$[n,n+1] \subset St^m\left(A_{4i}^{j},\mathcal{U}_{4i}\right),\ j \geq 4i,$$

and $(\diamondsuit\diamondsuit)$ there is no $m \in \mathbb{N} \cup \{0\}$ such that for every $n \in N_j$,

$$[n-1,n] \subset St^m\left(A_{4i}^{j},\mathcal{U}_{4i}\right),\ j \geq 4i.$$

Now, without lose of generality suppose that $B \subset (0,\infty)$ is Bourbaki-bounded and assume, on the contrary, that B is not bounded by d_u, that is, we have that $B \cap [n,n+1) \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Let $N = \{n \in \mathbb{N} : [n,n+1) \cap B \neq \emptyset\}$. For every $n \in N$ take just one $x_n \in [n,n+1) \cap$

B and put $S = \{x_n : n \in N\}$. If B is Bourbaki-bounded in $(\mathbb{R}, \varsigma)$ then S is also Bourbaki-bounded in $(\mathbb{R}, \varsigma)$ because a subset of a Bourbaki-bounded subset is also a Bourbaki-bounded subset. However, we show next that S is not Bourbaki-bounded.

Put $M = \{n+1 : n \in N\}$. There are three cases.

• Suppose that $N \cap N_j \neq \emptyset$ for infinitely many $j \in \mathbb{N}$ and that $M \cap N_j \neq \emptyset$ for infinitely many $j \in \mathbb{N}$ too. Choose any $i, j \in \mathbb{N}$ such that $j \geq 4i$. If S is Bourbaki-bounded then, by (♠) there exists some $m \in \mathbb{N}$ such that

$$S \subset St^m\left(A^j_{4i}, \mathcal{U}_{4i}\right).$$

But there are infinitely many $n \in N$ such that $n \notin N_j$ and $n+1 \notin N_j$, so the only possibility to join every $x_n \in S$ with the set A^j_{4i} through a chain in $\mathcal{U}_{4i}$ of lenght at most m is that this chain crosses through infinitely many $n \in N$ or infinitely many $n+1 \in M$. But this no possible by (♢) and (♢♢).

• Without loss of generality, suppose that $N \cap N_j \neq \emptyset$ for only finitely many $j \in \mathbb{N}$ but that $M \cap N_j \neq \emptyset$ for infinitely many $j \in \mathbb{N}$. Let $k, i \in \mathbb{N}$ such that $k \geq 4i$ and $k > j$ for every $j \in \mathbb{N}$ satisfying that $N \cap N_j \neq \emptyset$. Again, if S is Bourbaki-bounded, by (♠) there exists some $m \in \mathbb{N}$ such that

$$S \subset St^m\left(A^k_{4i}, \mathcal{U}_{4i}\right).$$

But there are infinitely many $n \in N$ such that $n \notin N_k$ and $n+1 \notin N_k$, so we can conclude as in the previous case.

• Finally, suppose that $N \cap N_j \neq \emptyset$ and $M \cap N_j \neq \emptyset$ for only finitely many $j \in \mathbb{N}$. Let $k, i \in \mathbb{N}$ such that $k \geq 4i$ and $k > j$ for every $j \in \mathbb{N}$ satisfying that $N \cap N_j \neq \emptyset$ and $M \cap N_j \neq \emptyset$. If S is Bourbaki-bounded, by (♠) there exists some $m \in \mathbb{N}$ such that

$$S \subset St^m\left(A^k_{4i}, \mathcal{U}_{4i}\right).$$

But there are infinitely many $n \in N$ such that $n \notin N_k$ and $n+1 \notin N_k$, so we can conclude as in the previous cases. □

Remark 7.6. Let (X, μ) be a uniform space. In the previous example we use several times the fact that a subset of a Bourbaki-bounded subset in X is also a Bourbaki-bounded subset in X. However, a Bourbaki-bounded subset Y in X is not necessarily a Bourbaki-bounded space. For example, consider any infinite uniformly discrete subset of a Bourbaki-bounded space as the metric hedgehog.

But even if the subset Y satisfies strong connectedness properties as a subspace of X, for instance, path connectedness, Bourbaki-boundedness is not hereditary either. Indeed, it is well-known that the real-line $(\mathbb{R}, d_u)$ endowed with the usual Euclidean metric is uniformly homeomorphic to a (closed) subspace of a product of metric hedgehogs ([28, pag. 138-139]). Clearly with the new metric inherited from the product of hedgehogs, $\mathbb{R}$ is not a Bourbaki-bounded space even if the product of metric hedgehogs is a Bourbaki-bounded space.

8. Metric Spaces that are Bourbaki-Complete and Cofinally Complete at the Same Time

Besides the Bourbaki-completeness, there exist other completeness notions in the frame of uniform and metric spaces (see for instance [3,4,12] for these notions and bibliography). This section focus on cofinal completeness and cofinal Bourbaki-completeness which are defined next.

Definition 8.1. A filter $\mathcal{F}$ in a uniform space is *cofinally Bourbaki-Cauchy* if

$$\forall \mathcal{U} \in \mu \ \exists m \in \mathbb{N}, \ \exists U \in \mathcal{U} \text{ s.t. } F \cap St^m (U,\mathcal{U}) \neq \emptyset \ \forall F \in \mathcal{F}.$$

The filter $\mathcal{F}$ is *cofinally Cauchy* if

$$\forall \mathcal{U} \in \mu \ \exists U \in \mathcal{U} \text{ s.t. } F \cap U \neq \emptyset \ \forall F \in \mathcal{F}.$$

Note that every Bourbaki-Cauchy (respect. Cauchy) filter is cofinally Bourbaki-Cauchy (respect. cofinally Cauchy).

Definition 8.2. A uniform space (X, μ) is *cofinally Bourbaki-complete* if every *cofinally Bourbaki-Cauchy* filter clusters. It is said to be *cofinally complete* if every cofinally Cauchy filter clusters.

Clearly, every cofinally Bourbaki-complete uniform space is cofinally complete and Bourbaki-complete. Moreover, the next theorem, which is parallel to Theorem 2.4 but clearly stronger, shows the relation between cofinal completeness and cofinal Bourbaki-completeness.

Theorem 8.3. *A uniform space (X, μ) is cofinally Bourbaki-complete if and only if it is cofinally complete and $\mu = sf\mu$.*

Proof. If (X, μ) is cofinally Bourbaki-complete then it is cofinally complete. Suppose on the contrary that it does not satisfies the property $(\star)$ from Theorem 7.1. Then there exists some $\mathcal{U}_0 \in \mu$, that we can take open, such that for every $\mathcal{V} \in \mu$ there exists $V_0 \in \mathcal{V}$ and $m_0 \in \mathbb{N}$ for which there is no finite subfamily in $\mathcal{U}_0$ covering $St^{m_0} (V_0, \mathcal{V})$. Let $\mathcal{U}_0^f$ be the cover obtained by taking finite unions of elements of $\mathcal{U}_0$. Then $\mathcal{U}_0^f$ is a directed open cover of X and $\left\{X \backslash A : A \in \mathcal{U}^f\right\}$ is a filter base of a filter $\mathcal{F}$ in X (note that $X \notin \mathcal{U}_0^f$). In particular, $\mathcal{F}$ is cofinally Bourbaki-Cauchy since for every $\mathcal{V} \in \mu$ there exists $V_0 \in \mathcal{V}$ such that $F \cap St^{m_0} (V_0, \mathcal{V}) \neq \emptyset$ for every $F \in \mathcal{F}$. Therefore, $\mathcal{F}$ clusters contradicting that $\mathcal{U}_0^f$ is a cover.

Conversely, suppose that (X, μ) is cofinally complete and that $\mu = sf\mu$. Let $\mathcal{F}$ be a cofinally Bourbaki-Cauchy filter of (X, μ). We prove next that $\mathcal{F}$ is also cofinal Cauchy. Therefore, the filter $\mathcal{F}$ clusters and (X, μ) is cofinally Bourbaki-complete.

Let $\mathcal{U} \in \mu$ and $\mathcal{V}$ a star-finite uniform refinement of it. This is possible because $\mu = sf\mu$. Since $\mathcal{F}$ is cofinally Bourbaki-Cauchy, for some $m \in \mathbb{N}$ and $V \in \mathcal{V}$, $St^m (V, \mathcal{V}) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. But $\mathcal{V}$ is star-finite, therefore it is possible to choose finitely many $V_i \in \mathcal{V}$, $i = 1, ..., k$ such that $St^m(V, \mathcal{V}) \subset \bigcup_{i=1}^k V_i$. In particular, for some $i \in \{1, ..., k\}$, $V_i \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Otherwise, for every $i = 1, ..., k$ there is some $F_i \in \mathcal{F}$ such that $V_i \cap F_i = \emptyset$. But then $F' = \bigcap_{i=1}^k F_i$ is an element of the filter $\mathcal{F}$ such that $F' \cap St^m(V, \mathcal{V}) = \emptyset$, which is a contradiction. Hence $\mathcal{F}$ is a cofinally Cauchy filter of (X, μ). □

Nevertheless, cofinal completeness and Bourbaki-completeness are not related properties.

Example 8.4 ([12, Example 16]) *The metric hedgehog* $(J(\kappa), \gamma)$, $\kappa \geq \omega_0$, *is cofinally complete but not Bourbaki-complete.*

Example 8.5. There is a Bourbaki-complete metric space which is not cofinally complete. For instance, Example 7.5. The space $(\mathbb{R}, \varsigma)$ in Example 7.5 is Bourbaki-complete, locally compact but not uniformly locally compact. Since every locally compact uniform space which is in addition cofinally complete must be uniformly locally compact (see [27, Theorem 4.4] and [18, Theorem 4.6]), $(\mathbb{R}, \varsigma)$ cannot be cofinally complete.

As odd as it may look, not every uniform space being cofinally complete and Bourbaki-complete at the same time is cofinally Bourbaki-complete. The main purpose of this section is to give such example, a metric one (Example 8.16), as it was asked in [12]. This example is possible because we already know which metric spaces are metrizable by a Bourbaki-complete metric (Theorem 6.5), which by a cofinally complete metric (see the next Theorem 8.6) and which by a cofinally Bourbaki-complete metric (see the next Theorem 8.8).

Theorem 8.6 ([29] and [3, Theorem 4.1]) *A metrizable space is metrizable by a cofinally complete metric if and only if the subset* $nlc(X)$ *of points of* X *without a locally compact neighborhood, is compact.*

Definition 8.7. A space is *strongly paracompact* if every open cover has an open star-finite refinement.

Theorem 8.8 ([12, Theorem 33]) *Let* X *be a metrizable space. The following statements are equivalent:*

(1) X *is metrizable by a cofinally Bourbaki-complete metric;*
(2) X *is strongly paracompact and* $nlc(X)$ *is compact.*

Observe that every metrizable strongly paracompact space is strongly metrizable [24], so completely metrizable spaces which are strongly paracompact are metrizable by a Bourbaki-complete metric. On the other hand, not every Bourbaki-complete metric space is strongly paracompact, as not every strongly metrizable space is strongly paracompact.

Example 8.9. Not every Bourbaki-complete metric spaces is strongly paracompact. Consider the Bourbaki-complete metric space $(\omega_1^{\omega_0} \times \mathbb{R}, \rho + d_u)$ (Bourbaki completeness is a productive property). This space is not strongly paracompact because it is homeomorphic to $\omega_1^{\omega_0} \times (0,1)$ which is not strongly paracompact as Nagata proved in [23, Remark p. 169]. In particular, $(\omega_1^{\omega_0} \times \mathbb{R}, \rho + d_u)$ is not metrizable by a cofinally complete (or cofinally Bourbaki-complete) metric because by Romaguera's result the factor $\omega_1^{\omega_0}$ is not metrizable but such metric either.

By the above results, it is expected that a metrizable space is metrizable by a metric which is cofinally complete and Bourbaki-complete at the same time if and only if the space satisfies Theorem 8.6 and it is strongly metrizable. Before we prove this fact in Theorem 8.13 we introduce the following paracompactness type property.

Definition 8.10. An open cover $\mathcal{U}$ of a space X is *σ-star-finite* if there exists a countable family $\mathcal{U}_n$, $n \in \mathbb{N}$, of star-finite open covers of X such that $\mathcal{U} \subset \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$.

Definition 8.11. A Tychonoff space X is *completely paracompact* if every open cover $\mathcal{G}$ has an open refinement $\mathcal{V}$ which is σ-star-finite.

Complete paracompactness is a property that lies between strong paracompactness and paracompactness (see [24, Ch. 2.2]). Moreover, Zarelua proved in [34, Lemma 5] that *a metrizable space is strongly metrizable if and only if it is completely paracompact*.

Next, we extend complete paracompactness to a uniform property in the same line than the properties of *uniform paracompactness* and *uniform strong paracompactness*. Recall, that cofinal completeness and cofinal Bourbaki-completeness are respectively equivalent to these notions, due to Rice [27] and o Hohti [16, Section 6] (see also [13]).

Definition 8.12. A uniform space (X, μ) is *uniformly completely paracompact* if it is cofinally complete and the uniformity μ has a base of σ-star-finite open covers.

Let σ-$sf\mu$ denote the family of all the uniform covers from a uniform space (X, μ) having a σ-star-finite uniform open refinement. It is not known if this family of uniform covers is in general a base for some compatible uniformity on X. However, in some cases, σ-$sf\mu$ is in fact a uniformity as we will see in the next result and examples.

Now, we are ready to characterize those metrizable spaces which are metrizable by a metric which is Bourbaki-complete metric and cofinal complete at the same time.

Theorem 8.13. *Let X be a metrizable space. The following statements are equivalent:*

(1) X is metrizable by a uniformly completely paracompact metric;
(2) X is strongly metrizable and $nlc(X)$ is compact;
(3) X is metrizable by a metric which is Bourbaki-complete and cofinally complete at the same time.

Proof. (1) $\Rightarrow$ (2) Let (X,d) being uniformly completely paracompact and for every $n\in\mathbb{N}$ let $\left\{\mathcal{U}_n^j : j\in\mathbb{N}\right\}$, a countable family of star-finite open covers of X containing a uniform refinement of the cover of open balls $\mathcal{B}_{1/n}$. Then, it is clear that $\bigcup_{n,j\in\mathbb{N}}\mathcal{U}_n^j$ is a base of the topology of X. Thus, the space X is strongly metrizable. Moreover, by Theorem 8.6, $nlc(X)$ is compact.

(2) $\Rightarrow$ (1) Suppose first that $nlc\,(X)=\emptyset$. Then X is locally compact and, as in the proof (4) $\Rightarrow$ (1) of Theorem 6.5, X is metrizable by a uniformly locally compact metric d. Therefore by Theorem [12, Theorem 14], (X,d) is cofinally Bourbaki-complete. In particular, it is uniformly completely paracompact by Theorem 8.3.

Otherwise, assume that $nlc\,(X)\neq\emptyset$ and let ρ a metric on X. Since $nlc(X)$ is compact, there exists a countable family of open sets $\{W_1,...,W_k,...\}$ in X such that for every open subset A of X containing $nlc\,(X)$ there exists $k\in\mathbb{N}$ satisfying that $nlc\,(X)\subset W_k\subset A$. For instance, consider $W_k=St\left(nlc\,(X)\,,\mathcal{B}_{1/k}\right)$, $k\in\mathbb{N}$, for the family of open balls $\mathcal{B}_{1/k}$ of radius $1/k$ in (X,ρ). Now for every $x\notin nlc\,(X)$ take V^x an open neighborhood of $x\in X$ with compact closure. For every $k\in\mathbb{N}$, let $\mathcal{G}_k=\{V^x : x\notin W_k\}\cup\{W_k\}$.

Let us start by $k=1$. By strong metrizability (equivalently, complete paracompactness), the open cover $\mathcal{B}_1\wedge\mathcal{G}_1$ has a σ-star-finite open refinement $\mathcal{U}_1$. Next, consider the open cover $\mathcal{U}_1\wedge\mathcal{B}_{1/2}\wedge\mathcal{G}_2$, and take an open cover $\mathcal{A}_2$ such that

$$\mathcal{A}_2^*<\mathcal{U}_1\wedge\mathcal{B}_{1/2}\wedge\mathcal{G}_2.$$

Again, by complete paracompactness we can take an open refinement $\mathcal{U}_2$ of $\mathcal{A}_2$ being σ-star-finite. Next, for every $k\geq 2$ take the open covers $\mathcal{A}_{k+1}$ and $\mathcal{U}_{k+1}$ such that

$$\mathcal{A}_{k+1}^*<\mathcal{U}_k\wedge\mathcal{B}_{1/(k+1)}\wedge\mathcal{G}_{k+1},$$

$\mathcal{U}_{k+1}<\mathcal{A}_{k+1}$ and $\mathcal{U}_{k+1}$ is σ-star-finite.

Proceeding in this way, we obtain a normal sequence $\langle\mathcal{U}_k\rangle$ of σ-star-finite open covers such that $\mathcal{U}_k<\mathcal{G}_k$ for every $k\in\mathbb{N}$. Let us prove that the sequence is compatible, that is, $\bigcup_{k\in\mathbb{N}}\mathcal{U}_k$ is a base for the topology of X. Indeed let $x\in A$, A being any open set in X then

$$x\in B_\rho^2\left(x,\tfrac{1}{2k}\right)\in B_\rho\left(x,\tfrac{1}{k}\right)\subset A$$

for some $k\in\mathbb{N}$. Take $U\in\mathcal{U}_{2k+1}$ such that $x\in U$. Then $U\subset B_\rho^2\left(x,\frac{1}{2k}\right)$ since $\mathcal{U}_{2k+1}<\mathcal{B}_{1/2k}$.

Now, applying Lemma 4.1, there exists a compatible pseudometric d on X such that

$$\mathcal{B}_{1/2^{k+1}}<\mathcal{U}_k<\mathcal{B}_{1/2^{k-1}}\text{ for every }k\in\mathbb{N}$$

where $\mathcal{B}_{1/2^k}$, $k \in \mathbb{N}$, denotes now the cover of open balls of radious $1/2^k$ in (X, d). Since the space X is Hausdorff, d is in fact a metric. Therefore, the metric uniformity μ_d has a base of σ-star-finite open covers.

Next, in order to prove that (X, d) is cofinally complete, let $\mathcal{F}$ be a cofinally Cauchy filter in (X, d). If for some $\kappa \in \mathbb{N}$, there is some $V^x \in \mathcal{G}_k$, where V^x is one of the above sets with compact closure, such that $F \cap V^x \neq \emptyset$ for every $F \in \mathcal{F}$, then $\mathcal{F}$ clusters in $\mathrm{cl}_X V^x$ by compactness. Otherwise, it follows that for every $k \in \mathbb{N}$, $F \subset W_k$ for some $F \in \mathcal{F}$. Indeed, by cofinal Cauchyness, as $\mathcal{B}_{1/2^{k+1}} < \mathcal{U}_k < \mathcal{G}_k$, then for every $k \in \mathbb{N}$ there must be some $G_k \in \mathcal{G}_k$ such that $F \cap G_k \neq \emptyset$ for every $F \in \mathcal{F}$.

Suppose that $\mathcal{F}$ does not cluster. Thus, $\mathcal{A} = \{X \backslash \mathrm{cl}_X F : F \in \mathcal{F}\}$ is an open cover of X and in particular it is an open cover of $nlc\,(X)$. Since $nlc\,(X)$ is compact, there exists a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that $nlc\,(X) \subset \bigcup \mathcal{A}'$. Hence, for some $k \in \mathbb{N}$ we have that $nlc\,(X) \subset W_k \subset \bigcup \mathcal{A}'$, and, since $\mathcal{A}'$ is finite, this implies that for some $A \in \mathcal{A}'$, $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. But this is a contradiction. Thus, the filter $\mathcal{F}$ clusters and the space is cofinally complete.

(2) $\Rightarrow$ (3). Let X be a strongly metrizable such that $nlc(X)$ is compact. In particular X is completely metrizable. Therefore, by Theorem 8.6 and Theorem 6.5, X is metrizable by a metric ρ and a metric t, which are cofinally complete and Bourbaki-complete respectively. Define the metric $d : X \times X \to [0, \infty)$ by $d\,(x, y) = \max\{\rho\,(x, y), t\,(x, y)\}$. Then it is easy to check that d is a metric compatible with the topology of X which is cofinally complete and Bourbaki-complete, as for every $x \in X$ and every $\varepsilon > 0$, $B_d\,(x, \varepsilon) = B_\rho\,(x, \varepsilon) \cap B_t\,(x, \varepsilon)$.

(3) $\Rightarrow$ (2) This follows from Theorem 6.3 and Theorem 8.6. □

Example 8.14. The metric hedgehog $(J\,(\omega_0), \gamma)$ is an example of cofinally complete metric space having a base of σ-star-finite covers for his metric uniformity which is not Bourbaki-complete. Indeed, since $J\,(\omega_0)$ is separable, it follows that $\mu_\gamma = e\mu_\gamma = \sigma\text{-}sf\mu_\gamma$. In general, if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a countable uniform cover then each cover $\mathcal{V}_n = \left\{U_n, \bigcup_{j \neq n} U_j\right\}$ is uniform and star-finite and $\mathcal{U} \subset \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, that is, $\mathcal{U}$ is σ-star-finite.

Example 8.15. There is a Bourbaki-complete metric space which does not have a base of σ-star-finite open covers for its metric uniformity. Observe that every σ-star-finite open cover is in particular σ-discrete. Indeed, let $\mathcal{A}_n$, $n \in \mathbb{N}$ be a family of star-finite open covers of X such that $\mathcal{A} \subset \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is a cover of X. Since each star-finite open cover is a σ-discrete open cover (see [28, Prop 13.2.6]), then for every $n \in \mathbb{N}$ there exist countably many families of open sets $\mathcal{A}_{n,j}$, $j \in \mathbb{N}$ such that $\mathcal{A}_n = \bigcup_{j \in \mathbb{N}} \mathcal{A}_{n,j}$ and for every x and every $n, j \in \mathbb{N}$ there exists an open set G such that $x \in G$ and G only meets finitely many $A \in \mathcal{A}_{n,j}$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection. In particular

$$\mathcal{A} \subset \bigcup_{f((n,j)) \in \mathbb{N}} \mathcal{A}_{f((n,j))},$$

that is, $\mathcal{A}$ is σ-discrete. Now, since any σ-discrete open cover is point-finite (see [28, Prop 13.2.4]), the Example 7.3 shows a Bourbaki-complete metric space which does not have a base of σ-star-finite open covers for its metric uniformity.

The following example answer the question that motivated this section.

Example 8.16. There exists a cofinally complete and strongly metrizable metric space (X, d) *which is not Bourbaki-complete and not strongly paracompact. In particular,* (X, d) *is not metrizable by a cofinally Bourbaki-complete metric, even if, by Theorem 8.13, it is metrizable by a metric which is Bourbaki-complete and cofinally complete at the same time. Besides, with this last metric,* X *does not have a star-finite base for its metric uniformity.*

Proof. Construction. ([30]) This is a subspace of the metric hedgehog $(J(\omega_1), \gamma)$. Let $\{A_n : n \in \mathbb{N}\}$ be a partition of ω_1 such that $|A_n| = \omega_1$ for every $n \in \mathbb{N}$. Let $L = \{0\} \cup \bigcup_{n\in\mathbb{N}} E_n$ where $E_n = \left\{[(x, \alpha)] : \frac{1}{n} \leq x \leq 1, \alpha \in A_n\right\}$. Then (L, γ) is a cofinally complete metric space since it is a closed subspace of $J(\omega_1)$. However, it is not Bourbaki-complete because the Bourbaki-bounded subset given by taking just one point of the form $[(1, \alpha)]$ in each E_n is closed but not compact.

On the other hand L is strongly metrizable. Indeed, fixed $k \in \mathbb{N}$, the open cover $\mathcal{B}_{1/k}$ has the following σ-star-finite refinement. For every $n \in \mathbb{N}$ and every $\alpha \in A_n$, in the spine $S_{n,\alpha} = \left\{[(x, \alpha)] : \frac{1}{n} \leq x \leq \alpha\right\}$ it is possible to choose, by compactness, a finite cover $\mathcal{G}_{n,\alpha}$ of open balls of radius $\varepsilon_n = min\left\{\frac{1}{2k}, \frac{1}{2n}\right\}$ and centre in $S_{n,\alpha}$. For every $n \in \mathbb{N}$ define the open covers

$$\mathcal{A}_n = \{B_\gamma(0, \varepsilon_n)\} \cup \left(\bigcup_{j\leq n} \bigcup_{\alpha\in A_j} \mathcal{G}_{j,\alpha}\right) \cup \left\{\bigcup_{j>n} E_j\right\}.$$

Then, it is clear that each $\mathcal{A}_n$ is open and star-finite. Moreover, $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ contains a refinement $\mathcal{A}$ of $\mathcal{B}_{1/k}$,

$$\mathcal{A} = \left\{B_\gamma\left(0, \tfrac{1}{2k}\right)\right\} \cup \bigcup_{j\in\mathbb{N}} \bigcup_{\alpha\in A_j} \mathcal{G}_{j,\alpha}.$$

However, X is not strongly paracompact. For instance, take again the open cover $\mathcal{B}_{1/k}$. Then for every $0 < \varepsilon \leq \frac{1}{k}$, the open ball $B_\gamma(0, \varepsilon)$ meets always uncountably many pairwise disjoint open balls $B_\gamma(x, \delta)$, of centre $x \notin B_\gamma(0, \varepsilon)$, for any $0 < \delta \leq \varepsilon$. Therefore, it is easy to deduce that $\mathcal{B}_{1/k}$ cannot have a star-finite, or star-countable (see [7, Theorem 5.3.10]), open refinement.

By Theorem 8.13 and Theorem 8.8, L is metrizable by a metric which is Bourbaki-complete and cofinally complete at the same time but not cofinally Bourbaki-complete. In addition, by Theorem 8.3 this last metric on L does not have a star-finite base for its uniformity. □

Open Problem 8.17. Recall that Hohti [16, pp. 31–32] proved that every cofinally complete uniform space has a point-finite base for its uniformity. Similarly, Theorem 8.3 shows that every cofinally Bourbaki-complete uniform spaces have a star-finite base for its uniformity. Moreover, from the definition, every uniformly completely paracompact uniform space has a σ-star-finite base for it uniformity. However, it would be better to give a definition of uniform complete paracompactness by means of some special family of covers as it is done for uniform paracompactness or uniform strong paracompactness, or even, to give a definition trough filters like it is done for cofinal Bourbaki-completeness.

Open Problem 8.18. We wonder if every metric space which is cofinally complete and Bourbaki-complete at the same time has always a σ-star-finite base base for its uniformity. Observe that Example 8.15 is not cofinally complete by the above cited result of Hohti.

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