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Convergence of Perturbed Sampling Kantorovich Operators in Modular Spaces

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Abstract. In the present paper we study the perturbed sampling Kantorovich operators in the general context of the modular spaces. After proving a convergence result for continuous functions with compact support, by using both a modular inequality and a density approach, we establish the main result of modular convergence for these operators. Further, we show several instances of modular spaces in which these results can be applied. In particular, we show some applications in Musielak–Orlicz spaces and in Orlicz spaces and we also consider the case of a modular functional that does not have an integral representation generating a space, which can not be reduced to previous mentioned ones.

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1. Introduction

In [12] the authors have introduced the generalized sampling Kantorovich operators perturbed by multiplicative noise, in order to model the presence of a possible perturbation in the reconstruction process of a given function. A typical example of noise source is a "Speckle" type noise: this is a disturbance that typically affects SAR (Synthetic Aperture Radar) remote sensing systems (see, e.g., [25,28]), but many others can be described.

In details, the family of operators above mentioned are of the form:

$$\left(K_w^{\chi,\mathcal{G}} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k)\frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ f(u)\ du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ du}, \qquad (I)$$

where $g_{k,w}$, $k\in\mathbb{Z}$, $w>0$ are locally integrable functions which represent multiplicative noise sources.

For the above operators (I), pointwise, uniform and modular convergence in Orlicz spaces have been studied in [12], where the last ones include, as particular cases, the L^p, the Zygmund, the exponential spaces, and others. It is easy to see that in case that $g_{k,w}$, $k\in\mathbb{Z}$, $w>0$ are constant functions, then (I) reduces to the well-known sampling Kantorovich operators of the form:

$$\left(K_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k)\int_{k/w}^{(k+1)/w} f(u)\ du,$$

which can be considered as an L^p version of the generalized sampling operators, introduced by P.L. Butzer and his school at RWTH-Aachen (see [6,7,9]).

For references about the approximation properties of sampling Kantorovich operators, see, e.g., [2,10,11,13–21,33,35], while for operators of Kantorovich type, see, e.g., [22,24].

The aim of this work is to extend the study in the more general context of modular spaces.

The theory of modular spaces has been introduced by Nakano in [32], and then extensively studied by Musielak and Orlicz [29,31]. This setting, among the Orlicz spaces, includes also the weighted Orlicz spaces, the Musielak–Orlicz spaces and the case in which the modular has not an integral representation, as it happens instead in the previous mentioned cases (see, e.g., [3]). Although modular spaces were first introduced by Nakano, a systematic study of these spaces has been carried forward by Musielak and Orlicz [31], by Musielak [29,30], and successively by other authors (see, e.g., [1,5,26,27]).

This study is motivated by the goal of formulating a unifying theory for the convergence of operators (I) allowing us to obtain, as special cases, instances not examined up to now, such as Musielak–Orlicz spaces, weighted Orlicz spaces and others, above mentioned. For references of this topic, the reader can see the monographs [3,23,29,34].

However, the elegance and the generality of this unifying structure has a price to pay. In fact, since the modular functional on which the modular spaces are built has very weak properties (the concept of modular is in fact much more general than the concept of norm), it is necessary to make a series of assumptions, which are satisfied in several particular cases, or alternatively, we will provide sufficient conditions that guarantee their validity; this is completely natural working in this abstract setting. On the other hand, being able to recover from the general theory the results obtained in a variety of settings (such as L^p, Zygmund, exponential, Orlicz, Musielak–Orlicz spaces, etc), is not only very beneficial, but also unifying.

The paper is organized as follows. In Sect. 2, we introduce some notations and preliminaries useful for what follows and we define the operators $\left(K_w^{\chi,\mathcal{G}}\right)_{w>0}$. In Sect. 3, we obtain the convergence results. Namely, as first step in order to prove the main convergence result for the operators under study, we prove a modular convergence theorem for continuous functions with compact support; then by using a modular-type inequality here obtained, and a density result of the space of continuous functions with compact support in the modular spaces (with respect to the modular topology), we establish the desired convergence result in modular spaces. In the last Sect. 4, we provide several examples of modular spaces where the theory holds, discussing in details the various setting, including the case in which the modular has not an integral representation. Finally, we briefly mention some examples of kernels satisfying the assumptions used in the results obtained along the paper.

2. Notations and Preliminaries

Let $\Omega = (\Omega, \Sigma_\Omega, \mu_\Omega)$ be an arbitrary measure space and let $X(\Omega)$ be the corresponding vector space of all Σ_Ω-measurable real-valued functions on Ω. A functional $\rho : X(\Omega) \to \widetilde{\mathbb{R}}_0^+$ is said to be a *modular* on $X(\Omega)$ if the following conditions hold:

($\rho 1$) $\rho(f) = 0$ if and only if $f = 0$ μ_Ω-a.e. in Ω;
($\rho 2$) $\rho(-f) = \rho(f)$ for every $f \in X(\Omega)$;
($\rho 3$) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(\Omega)$ and $\alpha, \beta \in \mathbb{R}_0^+$ with $\alpha + \beta = 1$.

The functional ρ generates the *modular space* $L_\rho(\Omega)$ defined as follows:

$$L_\rho(\Omega) := \left\{ f \in X(\Omega) : \lim_{\lambda \to 0} \rho(\lambda f) = 0 \right\}.$$

We note that $L_\rho(\Omega)$ is a vector subspace of $X(\Omega)$, and we can define the following notion of *modular convergence*:

a net of functions $(f_w)_{w>0} \subset L_\rho(\Omega)$ is *modularly convergent* to a function $f \in L_\rho(\Omega)$, if there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} \rho\big(\lambda(f_w - f)\big) = 0. \tag{2.1}$$

This convergence induces a topology on $L_\rho(\Omega)$, called *modular topology.*

Moreover if (2.1) holds for every $\lambda > 0$ and if the modular ρ is convex, we will say that the convergence is with respect to the *Luxemburg-norm*, defined as:

$$\|f\|_\rho = \inf\{u > 0 : \ \rho(f/u) \leq 1\}.$$

We can also introduce a subspace of $X(\Omega)$, that is the *space of finite elements*, denoted by $E_\rho(\Omega)$ and defined as:

$$E_\rho(\Omega) := \big\{ f \in X(\Omega) : \rho(\lambda f) < +\infty \text{ for every } \lambda > 0 \big\}.$$

It is well-known that if the modular ρ is a convex functional, the following inclusion is true:

$$E_\rho(\Omega) \subset L_\rho(\Omega).$$

For the above concepts and further notions of modular spaces, see, e.g., [3,24,29,34].

Now we recall the following important properties concerning modular functionals.

We say that the modular ρ is:

(a) *monotone* if $|f| \leq |g|$ implies $\rho(f) \leq \rho(g)$, for every $f, g \in X(\Omega)$;
(b) *finite* if the characteristic function $\mathbf{1}_X$ of every measurable set X of finite μ_Ω-measure belongs to $L_\rho(\Omega)$;
(c) *strongly finite* if every $\mathbf{1}_X$ as above belongs to $E_\rho(\Omega)$;
(d) *absolutely finite* if ρ is finite and if for every ε, $\lambda_0 > 0$ there exists a $\delta > 0$ such that $\rho(\lambda_0 \mathbf{1}_Y) < \varepsilon$, for every $Y \in \Sigma_\Omega$ with $\mu_\Omega(Y) < \delta$;
(e) *absolutely continuous* if there is an $\alpha > 0$ such that for every $f \in X(\Omega)$ with $\rho(f) < +\infty$, the following two conditions hold:
   (i) for every $\varepsilon > 0$ there exists a measurable set $X \subset \Omega$ with $\mu_\Omega(X) < +\infty$ such that $\rho(\alpha f \mathbf{1}_{\Omega \setminus X}) < \varepsilon$;
   (ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(\alpha f \mathbf{1}_Y) < \varepsilon$, for all measurable sets $Y \subset \Omega$ with $\mu_\Omega(Y) < \delta$.

We note that if ρ is convex, then any strongly finite modular is finite.

In this paper we consider the spaces $\mathbb{R} = (\mathbb{R}, \Sigma_\mathbb{R}, \mu_\mathbb{R})$ and $\mathbb{Z} = (\mathbb{Z}, \Sigma_\mathbb{Z}, \mu_\mathbb{Z})$, where $\mu_\mathbb{R}$ and $\mu_\mathbb{Z}$ are the Lebesgue and the counting measures respectively, and $\Sigma_\mathbb{R}$ and $\Sigma_\mathbb{Z}$ are the corresponding σ-algebras. Moreover we denote by $\rho_\mathbb{R}$ and $\rho_\mathbb{Z}$ two modulars on $X(\mathbb{R})$ and $X(\mathbb{Z})$.

In the following we give the definition of kernel, used in order to define the operators we deal with.

A function $\chi : \mathbb{R} \to \mathbb{R}$ is called *kernel* if it satisfies the following assumptions:

(χ1) $\chi \in X(\mathbb{R})$ and it is bounded in a neighbourhood of the origin;
(χ2) for every $u \in \mathbb{R}$, there holds:

$$\sum_{k \in \mathbb{Z}} \chi(u - k) = 1;$$

(χ3) there exists $\beta > 0$ such that

$$m_\beta(\chi) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u - k)| |u - k|^\beta < +\infty.$$

We recall that from the definition of the kernel, it is possible to prove the following properties (see [2]):

(i) $m_0(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)| < +\infty$;
(ii) for every $\gamma > 0$ we have:

$$\lim_{w\to+\infty} \sum_{|u-k|>\gamma w} |\chi(u-k)| = 0,$$

uniformly with respect to $u \in \mathbb{R}$;
(iii) for every $\gamma > 0$ and $\varepsilon > 0$, there exists a constant $M > 0$ such that

$$\int_{|x|>M} w|\chi(wx-k)|\, dx < \varepsilon,$$

for sufficiently large $w > 0$ and $k \in \mathbb{Z}$ such that $k/w \in [-\gamma, +\gamma]$.

Now we introduce the following compatibility condition between the kernel χ and the modulars $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$.

We will say that the kernel χ is *compatible* with the modulars $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$, if there exist two positive constants D_1, D_2 and a net $(b_w)_{w>0} \subset \mathbb{R}_0^+$ with $b_w \to 0$ as $w \to +\infty$, such that

$$\rho_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}} h_k |\chi(w\cdot -k)|\right) \le \frac{1}{w} D_1 \rho_{\mathbb{Z}}(D_2 h) + b_w, \tag{2.2}$$

for any non-negative $h \in X(\mathbb{Z}),\ h = (h_k)_{k\in\mathbb{Z}}$ and for sufficiently large $w > 0$.

Remark 2.1. We note that the above compatibility condition is a particular case of the general condition introduced in [27], and it is often used in order to study approximation results for operators in modular spaces.

In order to prove the convergence results of the next section, we also need to introduce an additional assumption which relates the kernel χ with the modular $\rho_{\mathbb{R}}$.

We assume that for any fixed $\nu > 0$ and $a > 0$, there exist a positive constant L and a measurable set $\mathcal{S} \subset \mathbb{R},$ with $\mu_{\mathbb{R}}(\mathcal{S}) < +\infty$ such that

$$\rho_{\mathbb{R}}\left(a\mathbf{1}_{\mathbb{R}\setminus\mathcal{S}}(\cdot) \sum_{k\in[-\nu w, +\nu w]} |\chi(w\cdot -k)|\right) \le L, \tag{2.3}$$

for sufficiently large $w > 0$, where $\mathbf{1}_{\mathbb{R}\setminus\mathcal{S}}$ denotes the characteristic function of the set $\mathbb{R}\setminus\mathcal{S}$.

Remark 2.2. Note that if χ has compact support, it is easy to prove that the assumption (2.3) is satisfied (see, e.g., [20]).

In the following, we denote by $C(\mathbb{R})$ the space of all bounded and uniformly continuous functions, equipped by the usual sup-norm $\|\cdot\|_\infty$ and by $C_c(\mathbb{R})$ the space of all continuous functions with compact support.

Now we recall here two known theorems. The first one represents a version of the Lebesgue dominated convergence theorem in the setting of modular spaces.

Theorem 2.3 [30]. *Let $\rho_{\mathbb{R}}$ be a monotone, finite and absolutely continuous modular on $X(\mathbb{R})$. Let $(f_w)_{w>0} \subset X(\mathbb{R})$ be a net of functions such that $f_w \to 0$, a.e. in $\mathbb{R}$, as $w \to +\infty$. Suppose in addition that there exixts a function $g \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ such that $\rho_{\mathbb{R}}(3g) < +\infty$ and $|f_w(x)| \leq g(x)$, a.e. in $\mathbb{R}$, for every $w > 0$. Then $\rho_{\mathbb{R}}(f_w) \to 0$, as $w \to +\infty$.*

The next theorem is instead a density result.

Theorem 2.4 [26]. *Let $\rho_{\mathbb{R}}$ be an absolutely continuous, monotone and absolutely finite modular on $X(\mathbb{R})$. Then*

$$\overline{C_c(\mathbb{R})} = L_{\rho_{\mathbb{R}}}(\mathbb{R}),$$

where the bar represents the closure with respect to the modular topology on $L_{\rho_{\mathbb{R}}}(\mathbb{R})$.

Now we are able to recall the definition of the class of operators introduced in [12].

Definition 2.5. We define by $\left(K_w^{\chi,\mathcal{G}}\right)_{w>0}$ the family of generalized sampling Kantorovich operators perturbed by multiplicative noise, such that

$$\left(K_w^{\chi,\mathcal{G}} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k) \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ f(u)\ du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ du},$$

where $\mathcal{G} := (\mathcal{G}_w)_{w>0}$ is a family of noise sequences, with $\mathcal{G}_w = (g_{k,w})_{k\in\mathbb{Z}}$, $g_{k,w} : \mathbb{R} \to \mathbb{R}^+$ are locally integrable noise functions, while $f : \mathbb{R} \to \mathbb{R}$ is such that $g_{k,w} f$ are locally integrable and the series above is convergent for every $x \in \mathbb{R}$. We simply call the operators $K_w^{\chi,\mathcal{G}}$ as the *perturbed sampling Kantorovich operators*.

It is easy to see that the above operators $K_w^{\chi,\mathcal{G}}$ are well-defined if, e.g., f is a bounded function (see, e.g., [12] again).

Now we recall the following theorem concerning the pointwise and the uniform convergence of the above operators.

Theorem 2.6 [12]. *Let $f \in X(\mathbb{R})$ be a bounded function which is continuous at $x \in \mathbb{R}$. Then*

$$\lim_{w\to+\infty} \left(K_w^{\chi,\mathcal{G}} f\right)(x) = f(x).$$

Furthermore, if $f \in C(\mathbb{R})$, then

$$\lim_{w\to+\infty} \|K_w^{\chi,\mathcal{G}} f - f\|_\infty = 0.$$

3. Convergence Results

We now prove a modular convergence theorem for continuous functions with compact support on $\mathbb{R}$.

Theorem 3.1. *Let $\rho_{\mathbb{R}}$ be a convex, monotone, strongly finite and absolutely continuous modular on $X(\mathbb{R})$. Moreover let χ be a kernel which satisfies assumption (2.3) together with $\rho_{\mathbb{R}}$. Then for any $f \in C_c(\mathbb{R})$ and for every $\lambda \in \mathbb{R}$, with $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolute continuity of $\rho_{\mathbb{R}}$, there holds:*

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\big(\lambda(K_w^{\chi,\mathcal{G}} f - f)\big) = 0.$$

Proof. Since $f \in C_c(\mathbb{R})$, let $\bar{\nu} > 0$ such that $\operatorname{supp} f \subset [-\bar{\nu}, +\bar{\nu}] =: T$. Now we can fix $\nu > \bar{\nu} + 1$ and let us consider the interval $[-\nu, +\nu]$. Then if $k \notin [-\nu w, +\nu w]$, we have for sufficiently large $w > 0$ that $[-\nu, +\nu] \cap [-k/w, +k/w] = \emptyset$, and so:

$$\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du = 0.$$

By the arguments above, the definition of the perturbed sampling Kantorovich operators reduces to:

$$\left(K_w^{\chi,\mathcal{G}} f\right)(x) = \sum_{k\in[-\nu w,+\nu w]} \chi(wx-k) \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du}, \; x \in \mathbb{R}.$$

Now since $f \in C_c(\mathbb{R})$, obviously $f \in X(\mathbb{R})$ and this condition, together with $(\chi 1)$ and the hypothesis on $\mathcal{G}$, implies that $K_w^{\chi,\mathcal{G}} f \in X(\mathbb{R})$ and so $K_w^{\chi,\mathcal{G}} f - f \in X(\mathbb{R})$. Further, by using the monotonicity of $\rho_{\mathbb{R}}$, property $(\rho 3)$ of the modulars and the condition $g_{k,w}(u) \geq 0$ for every $u \in \mathbb{R},\ k \in \mathbb{Z}$ and $w > 0$, we can write what follows:

$$\begin{aligned}
\rho_{\mathbb{R}}\big(K_w^{\chi,\mathcal{G}} f - f\big) &= \rho_{\mathbb{R}}\left(\sum_{k\in[-\nu w,+\nu w]} \chi(w\cdot -k) \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du} - f(\cdot) \right) \\
&\leq \rho_{\mathbb{R}}\left(\sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)| \; \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) |f(u)|\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du} + |f| \right) \\
&\leq \rho_{\mathbb{R}}\left(2\|f\|_\infty \sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)| \right) + \rho_{\mathbb{R}}(2|f|)
\end{aligned}$$

$$\leq \rho_{\mathbb{R}}\left(2\|f\|_\infty \sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)|\right) + \rho_{\mathbb{R}}(2\|f\|_\infty \mathbf{1}_T),$$

where $\mathbf{1}_T$ is the characteristic function of the interval T with $\mu_{\mathbb{R}}(T) = 2\bar{\nu} < +\infty$.

For the first term by applying assumption (2.3) with ν fixed above and $a := 4\|f\|_\infty$, there exists $L > 0$ and a measurable set $\mathcal{S} \subset \mathbb{R}$ with $\mu_{\mathbb{R}}(\mathcal{S}) < +\infty$, such that

$$\rho_{\mathbb{R}}\left(4\|f\|_\infty \mathbf{1}_{\mathbb{R}\setminus\mathcal{S}}(\cdot) \sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)|\right) \leq L, \tag{3.1}$$

for sufficiently large $w > 0$. Recalling that $\mathbf{1}_{\mathcal{S}},\ \mathbf{1}_T \in E_{\rho_{\mathbb{R}}}(\mathbb{R})$ since $\rho_{\mathbb{R}}$ is strongly finite, by using again ($\rho 3$) and the monotonicity of $\rho_{\mathbb{R}}$, property (i) of the kernel and (3.1), we finally obtain:

$$\begin{aligned}\rho_{\mathbb{R}}(K_w^{\chi,\mathcal{G}} f - f) &\leq \rho_{\mathbb{R}}\left(4\|f\|_\infty \mathbf{1}_{\mathcal{S}}(\cdot) \sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)|\right)\\ &\quad + \rho_{\mathbb{R}}\left(4\|f\|_\infty \mathbf{1}_{\mathbb{R}\setminus\mathcal{S}}(\cdot) \sum_{k\in[-\nu w,+\nu w]} |\chi(w\cdot -k)|\right) + \rho_{\mathbb{R}}(2\|f\|_\infty \mathbf{1}_T)\\ &\leq \rho_{\mathbb{R}}\big(4\|f\|_\infty \mathbf{1}_{\mathcal{S}}(\cdot) m_0(\chi)\big) + L + \rho_{\mathbb{R}}(2\|f\|_\infty \mathbf{1}_T) < +\infty,\end{aligned}$$

for sufficiently large $w > 0$.

Now we denote by $\alpha > 0$ the constant of the absolute continuity of $\rho_{\mathbb{R}}$ and let $\varepsilon > 0$ be fixed. In correspondence to $\varepsilon/2$, from property (i) of condition (e) of the absolute continuity of the modular, we obtain that there exists a measurable subset $X \subset \mathbb{R}$, with $\mu_{\mathbb{R}}(X) < +\infty$, such that

$$\rho_{\mathbb{R}}\big(\alpha(K_w^{\chi,\mathcal{G}} f - f)\mathbf{1}_{\mathbb{R}\setminus X}\big) < \varepsilon/2. \tag{3.2}$$

Since X is such that $\mu_{\mathbb{R}}(X) < +\infty$ and $\rho_{\mathbb{R}}$ is convex and strongly finite (so it is finite), hence one has that $\mathbf{1}_X \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ and so:

$$\lim_{\lambda\to 0} \rho_{\mathbb{R}}(\lambda \mathbf{1}_X) = 0.$$

Then in correspondence to $\varepsilon/2$, there exists a sufficiently small $\lambda_\varepsilon > 0$ such that

$$\rho_{\mathbb{R}}(\lambda_\varepsilon \mathbf{1}_X) < \varepsilon/2. \tag{3.3}$$

Moreover since $f \in C_c(\mathbb{R})$ and Theorem 2.6 holds, we have:

$$\lim_{w\to+\infty} \|K_w^{\chi,\mathcal{G}} f - f\|_\infty = 0$$

and so in correspondence to $\lambda_\varepsilon/\alpha$, there exists $\overline{w} > 0$ such that for every $w \geq \overline{w}$ one has:

$$\|K_w^{\chi,\mathcal{G}} f - f\|_\infty \leq \lambda_\varepsilon/\alpha,$$

and since $\alpha > 0$, we can conclude:

$$\alpha\|K_w^{\chi,\mathcal{G}} f - f\|_\infty \leq \lambda_\varepsilon, \tag{3.4}$$

for sufficiently large $w > 0$.

Now let $0 < \lambda \leq \alpha/2$ be fixed. We can write what follows:

$$\begin{aligned}\lambda\Big|\big(K_w^{\chi,\mathcal{G}}f\big)(x) - f(x)\Big| &\leq \frac{\alpha}{2}\Big|\big(K_w^{\chi,\mathcal{G}}f\big)(x) - f(x)\Big| \\ &= \frac{1}{2}\alpha\Big|\big(K_w^{\chi,\mathcal{G}}f\big)(x) - f(x)\Big|\mathbf{1}_{\mathbb{R}\setminus X}(x) \\ &\quad + \frac{1}{2}\alpha\Big|\big(K_w^{\chi,\mathcal{G}}f\big)(x) - f(x)\Big|\mathbf{1}_X(x),\ x \in \mathbb{R}.\end{aligned}$$

By using the monotonicity of $\rho_{\mathbb{R}}$, property ($\rho 3$) and the conditions (3.2), (3.4), (3.3), we obtain:

$$\begin{aligned}\rho_{\mathbb{R}}\Big(\lambda\big|K_w^{\chi,\mathcal{G}}f - f\big|\Big) &\leq \rho_{\mathbb{R}}\Big(\frac{1}{2}\ \alpha\big|K_w^{\chi,\mathcal{G}}f - f\big|\mathbf{1}_{\mathbb{R}\setminus X} + \frac{1}{2}\ \alpha\big|K_w^{\chi,\mathcal{G}}f - f\big|\mathbf{1}_X\Big) \\ &\leq \rho_{\mathbb{R}}\Big(\alpha\big|K_w^{\chi,\mathcal{G}}f - f\big|\mathbf{1}_{\mathbb{R}\setminus X}\Big) + \rho_{\mathbb{R}}\Big(\alpha\big|K_w^{\chi,\mathcal{G}}f - f\big|\mathbf{1}_X\Big) \\ &\leq \frac{\varepsilon}{2} + \rho_{\mathbb{R}}\Big(\alpha\|K_w^{\chi,\mathcal{G}}f - f\|_\infty\mathbf{1}_X\Big) \\ &\leq \frac{\varepsilon}{2} + \rho_{\mathbb{R}}(\lambda_\varepsilon\mathbf{1}_X) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for $w > 0$ sufficiently large and so the proof follows. □

Now we can prove a convergence result with respect to the Luxemburg norm for kernels χ with compact support.

Theorem 3.2. *Let $\rho_{\mathbb{R}}$ be a convex, monotone, strongly finite and absolutely continuous modular on $X(\mathbb{R})$ and χ be a kernel with compact support. Then for any $f \in C_c(\mathbb{R})$ and for every $\lambda > 0$, there holds:*

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\big(\lambda(K_w^{\chi,\mathcal{G}}f - f)\big) = 0.$$

Proof. As in Theorem 3.1, we can write:

$$\Big(K_w^{\chi,\mathcal{G}}f\Big)(x) = \sum_{k\in[-\nu w,+\nu w]} \chi(wx-k)\frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)f(u)\ du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ du},\ x \in \mathbb{R}.$$

Now, since χ has compact support, $\operatorname{supp}\chi \subset [-M, M]$, $M > 0$ and from Remark 2.2, we obtain:

$$\chi(wx-k) = 0,\ \text{for every } x \notin [-\nu - M, \nu + M],\ k \in [-\nu w, +\nu w],\ w \geq 1,$$

so $\operatorname{supp} K_w^{\chi,\mathcal{G}}f \subset [-\nu - M, \nu + M]$, for every $w \geq 1$. Now by observing that $\operatorname{supp} f \subset [-\bar{\nu}, +\bar{\nu}] \subset [-\nu - M, \nu + M]$, we denote by $\mathcal{M} := [-\nu - M, \nu + M]$ and we have for every $\lambda > 0$:

$$\lambda\Big|\Big(K_w^{\chi,\mathcal{G}}f\Big)(x) - f(x)\Big| = \lambda\Big|\Big(K_w^{\chi,\mathcal{G}}f\Big)(x) - f(x)\Big|\mathbf{1}_{\mathcal{M}}(x)$$

$$\leq \lambda \left[\|f\|_\infty \sum_{k \in [-\nu w, \nu w]} |\chi(wx-k)| + |f(x)| \right] \mathbf{1}_{\mathcal{M}}(x)$$
$$\leq \lambda \Big[\|f\|_\infty m_0(\chi) + \|f\|_\infty \Big] \mathbf{1}_{\mathcal{M}}(x)$$
$$\leq \lambda \Big[\Big(1 + m_0(\chi)\Big) \|f\|_\infty \Big] \mathbf{1}_{\mathcal{M}}(x) := g(x), \tag{3.5}$$

for every $x \in \mathbb{R}$ and $w > 0$ sufficiently large. In order to apply Theorem 2.3 to $\lambda(K_w^{\chi,\mathcal{G}} f - f)$, we recall that $\rho_{\mathbb{R}}$ is convex, monotone, strongly finite and absolutely continuous. Moreover by denoting with $f_w = \lambda(K_w^{\chi,\mathcal{G}} f - f)$, we know that $f_w \subset X(\mathbb{R})$ and from Theorem 2.6 we deduce that:

$$\left| \Big(K_w^{\chi,\mathcal{G}} f\Big)(x) - f(x) \right| \to 0, \ x \in \mathbb{R}, \ \text{as } w \to \infty$$

and so also $f_w(x) \to 0, \ x \in \mathbb{R}, \ \text{as } w \to \infty$. Further since $\mathbf{1}_{\mathcal{M}} \in E_{\rho_{\mathbb{R}}}(\mathbb{R}) \subset L_{\rho_{\mathbb{R}}}(\mathbb{R})$, we have:

$$\rho_{\mathbb{R}}(\mu g) \to 0, \ \text{as } \mu \to 0,$$

i.e., $g \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ and also $\rho_{\mathbb{R}}(3g) < +\infty$. Finally by using also (3.5), we can apply Theorem 2.3 to conclude that:

$$\rho_{\mathbb{R}}(f_w) = \ \rho_{\mathbb{R}}\big(\lambda(K_w^{\chi,\mathcal{G}} f - f)\big) \to 0, \ \text{as } w \to +\infty, \ \text{for every } \lambda > 0.$$

□

Now, in order to establish a useful inequality for the operators $K_w^{\chi,\mathcal{G}}$, we firstly introduce the following subset of $X(\mathbb{R})$.

Given the constants $E, K > 0$ and the modulars $\rho_{\mathbb{R}}$, $\rho_{\mathbb{Z}}$ on $X(\mathbb{R})$ and on $X(\mathbb{Z})$ respectively, we define the subset $\mathcal{L}_{E,K}(\mathbb{R})$ of $X(\mathbb{R})$ whose f are such that $g_{k,w} f$ are locally absolutely integrable and satisfying the following inequality:

$$\limsup_{w \to +\infty} \frac{1}{w} \rho_{\mathbb{Z}}(\lambda H_w) \leq E \rho_{\mathbb{R}}(\lambda K f),$$

for every $\lambda > 0$, where $H_w = (h_{k,w})_{k \in \mathbb{Z}} \in X(\mathbb{Z}), \ w > 0$ with

$$h_{k,w} := \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) |f(u)| \, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) \, du}. \tag{3.6}$$

By using the above condition, we can prove the following theorem.

Theorem 3.3. *Let $\rho_{\mathbb{R}}$ be a monotone modular on $X(\mathbb{R})$, $\rho_{\mathbb{Z}}$ be a modular on $X(\mathbb{Z})$ and χ be a kernel which is compatible with $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$. Then, for any function $f \in \mathcal{L}_{E,K}(\mathbb{R})$ for some $E, \ K > 0$, there holds:*

$$\limsup_{w \to +\infty} \rho_{\mathbb{R}}(c \ K_w^{\chi,\mathcal{G}} f) \leq E D_1 \rho_{\mathbb{R}}(c D_2 K f),$$

for every $c > 0$, where D_1 and D_2 are the constants of the compatibility condition among χ, $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$. Moreover if $f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$, it turns out that $K_w^{\chi,\mathcal{G}} f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$, for sufficiently large $w > 0$.

Proof. For every $c > 0$, by using the monotonicity of $\rho_{\mathbb{R}}$ and the compatibility condition (2.2) we can write:

$$\rho_{\mathbb{R}}(c\ K_w^{\chi,\mathcal{G}} f) \leq \rho_{\mathbb{R}}\left(c \sum_{k\in\mathbb{Z}} |\chi(w\cdot -k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)|\ du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\ du} \right)$$
$$\leq \frac{1}{w} D_1 \rho_{\mathbb{Z}}(D_2 c H_w) + b_w,$$

for $w > 0$ sufficiently large, with $D_1,\ D_2 > 0$ and H_w defined as in (3.6). Now recalling that the net $(b_w)_{w>0}$ is such that $b_w \to 0$, as $w \to +\infty$ and the definition of the subset $\mathcal{L}_{E,K}(\mathbb{R})$, by passing to the upper limit we obtain:

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}}(c\ K_w^{\chi,\mathcal{G}} f) \leq \limsup_{w\to+\infty} \left[\frac{1}{w} D_1 \rho_{\mathbb{Z}}(D_2 c H_w) + b_w\right]$$
$$\leq E D_1 \rho_{\mathbb{R}}(D_2 c K f).$$

For the second part of the theorem, arguing as before, from $f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ we have that $\rho_{\mathbb{R}}(D_2 \lambda K f) \to 0$, as $\lambda \to 0$ and then we conclude that $\rho_{\mathbb{R}}(\lambda\ K_w^{\chi,\mathcal{G}} f) \to 0$, as $\lambda \to 0$, for $w > 0$ sufficiently large, i.e., $K_w^{\chi,\mathcal{G}} f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$. □

The following thorem represents the main result of this section.

Theorem 3.4. *Let $\rho_{\mathbb{R}}$ be a convex, monotone, strongly finite, absolutely finite and absolutely continuous modular on $X(\mathbb{R})$, $\rho_{\mathbb{Z}}$ be a modular on $X(\mathbb{Z})$ and χ be a kernel compatible with $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$ which satisfies condition (2.3) together with $\rho_{\mathbb{R}}$. Then for every $f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ such that $f - C_c(\mathbb{R}) \subset \mathcal{L}_{E,K}(\mathbb{R})$ for some $E,\ K > 0$, there is a constant $c > 0$:*

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\big(c(K_w^{\chi,\mathcal{G}} f - f)\big) = 0.$$

Proof. Let $f \in L_{\rho_{\mathbb{R}}}(\mathbb{R})$ be such that $f - C_c(\mathbb{R}) \subset \mathcal{L}_{E,K}(\mathbb{R})$, for some $E,\ K > 0$. From Theorem 2.4, there exists $\lambda > 0$ such that for every $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R})$ with

$$\rho_{\mathbb{R}}(\lambda(f - g)) < \varepsilon. \tag{3.7}$$

Since $g \in C_c(\mathbb{R})$, by Theorem 3.1, for every $\tilde{\lambda} > 0$, with $\tilde{\lambda} \leq \alpha/2$, where α is the parameter of the absolute continuity of the modular $\rho_{\mathbb{R}}$, it turns out:

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\big(\tilde{\lambda}(K_w^{\chi,\mathcal{G}} g - g)\big) = 0. \tag{3.8}$$

Now we choose a positive constant c such that $c \leq \min\{\frac{\lambda}{3D_2K}, \frac{\alpha}{6}, \frac{\lambda}{3}\}$, where D_2 is the constant of the compatibility condition among $\chi, \rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$ and we can write what follows using (ρ3) of the modulars:

$$\begin{aligned}\rho_{\mathbb{R}}\big(c(K_w^{\chi,\mathcal{G}}f - f)\big) &\leq \rho_{\mathbb{R}}\left(\frac{1}{3}\ 3c\ (K_w^{\chi,\mathcal{G}}f - K_w^{\chi,\mathcal{G}}g) + \frac{1}{3}\ 3c\ (K_w^{\chi,\mathcal{G}}g - g) + \frac{1}{3}\ 3c(g-f)\right)\\ &\leq \rho_{\mathbb{R}}\big(3c\ (K_w^{\chi,\mathcal{G}}f - K_w^{\chi,\mathcal{G}}g)\big) + \rho_{\mathbb{R}}\big(3c\ (K_w^{\chi,\mathcal{G}}g - g)\big) + \rho_{\mathbb{R}}\big(3c\ (g-f)\big)\\ &=: J_1 + J_2 + J_3, \text{ for } w > 0.\end{aligned}$$

We analyze $J_i,\ i = 1, 2, 3$ separately, where we will use the monotonicity of $\rho_{\mathbb{R}}$ and the constant c as above.

Regarding J_1, we recall that the operator $K_w^{\chi,\mathcal{G}}$ is linear and that $f - g \in \mathcal{L}_{E,K}(\mathbb{R})$, so we can apply Theorem 3.3 to obtain that there exist $E, K > 0$ such that

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}}\big(3c\ K_w^{\chi,\mathcal{G}}(f-g)\big) \leq ED_1\rho_{\mathbb{R}}\big(3c\ D_2K(f-g)\big) \leq ED_1\rho_{\mathbb{R}}\big(\lambda(f-g)\big). \tag{3.9}$$

For what concerns J_2, by the choice of c and by (3.8) we have:

$$\rho_{\mathbb{R}}\left(3c(K_w^{\chi,\mathcal{G}}g - g)\right) \leq \rho_{\mathbb{R}}\left(\frac{\alpha}{2}(K_w^{\chi,\mathcal{G}}g - g)\right) < \varepsilon, \tag{3.10}$$

for $w > 0$ sufficiently large. For J_3, by property (ρ2) of modulars and by (3.7), we obtain:

$$\rho_{\mathbb{R}}\big(3c(g-f)\big) = \rho_{\mathbb{R}}\big(3c(f-g)\big) \leq \rho_{\mathbb{R}}\big(\lambda(f-g)\big) < \varepsilon. \tag{3.11}$$

Now from (3.9), (3.10) and (3.11), we conclude that:

$$\rho_{\mathbb{R}}\big(c(K_w^{\chi,\mathcal{G}}f - f)\big) \leq ED_1\rho_{\mathbb{R}}\big(\lambda(f-g)\big) + \varepsilon + \rho_{\mathbb{R}}\big(\lambda(f-g)\big) \leq (2 + ED_1)\varepsilon,$$

for $w > 0$ sufficiently large and this completes the proof. □

4. Applications to Special Cases

Below, we will discuss in detail some remarkable examples of modular spaces.

4.1. Musielak–Orlicz Spaces

In order to recall the definition of Musielak–Orlicz spaces, we need the following:

Definition 4.1. A function $\varphi : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is is said to be a *τ-bounded φ-function* if the following conditions hold:

(φ1) $\varphi(\cdot, u)$ is measurable and locally integrable on $\mathbb{R}$, for every $u \in \mathbb{R}_0^+$;

(φ2) for every $s \in \mathbb{R}$, $\varphi(s, \cdot)$ is convex on $\mathbb{R}_0^+$ with $\varphi(s, 0) = 0$ and $\varphi(s, u) > 0$, for $u > 0$;

(φ3) φ is τ-bounded, i.e., there exist a constant $C \geq 1$ and a measurable function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ such that for every $t, s \in \mathbb{R}$ and $u \geq 0$ there holds:

$$\varphi(s - t, u) \leq \varphi(s, Cu) + F(s, t).$$

We can observe that assumption ($\varphi 2$) implies that the function φ is continuous and non decreasing with respect to the second variable $u \in \mathbb{R}_0^+$.

For a sake of semplicity, from now on, we will consider τ-bounded φ-functions which satisfy condition ($\varphi 3$) with $F \equiv 0$ and we note that examples of τ-bounded φ-functions φ with $F \neq 0$ can be constructed as in [4].

Firstly, we consider the non-negative functionals below defined:

$$\rho_{\mathbb{R}}(f) := \int_{\mathbb{R}} \varphi\big(t, |f(t)|\big)\, dt, \quad \rho_{\mathbb{Z}}(h) := \sum_{k\in\mathbb{Z}} \varphi\big(k, |h_k|\big),$$

where $f \in X(\mathbb{R})$ and $h = (h_k)_{k\in\mathbb{Z}} \in X(\mathbb{Z})$, respectively.

We can easily note that $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$ are modulars on $X(\mathbb{R})$ and on $X(\mathbb{Z})$ respectively and they satisfy properties (a)–(e) of Sect. 2 (see, e.g., [3]).

The modular spaces generated by $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{Z}}$ are respectively,

$$L^{\varphi}(\mathbb{R}) := L_{\rho_{\mathbb{R}}}(\mathbb{R}) = \left\{ f \in X(\mathbb{R}) : \lim_{\lambda\to 0} \rho_{\mathbb{R}}(\lambda f) = 0 \right\}$$

and

$$L^{\varphi}(\mathbb{Z}) := L_{\rho_{\mathbb{Z}}}(\mathbb{Z}) = \left\{ f \in X(\mathbb{Z}) : \lim_{\lambda\to 0} \rho_{\mathbb{Z}}(\lambda f) = 0 \right\}$$

and they are called Musielak–Orlicz spaces.

In order to apply Theorem 3.3 in these particular spaces, we want to prove that condition (2.2) is satisfied. Hence, we want to find two positive constants D_1, D_2 and a net $(b_w)_{w>0} \subset \mathbb{R}_0^+$, with $b_w \to 0$ as $w \to +\infty$, such that

$$\int_{\mathbb{R}} \varphi\left(t, \left|\sum_{k\in\mathbb{Z}} h_k\ \chi(wt-k)\right|\right) dt \le \frac{1}{w} D_1 \sum_{k\in\mathbb{Z}} \varphi\Big(k, D_2|h_k|\Big) + b_w.$$

Since Jensen inequality and Fubini-Tonelli theorem hold and by using the property of the τ-boundedness of the function φ (with $C = 1$) and the change of variables $y = wt - k$, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi\left(t, \left|\sum_{k\in\mathbb{Z}} h_k\ \chi(wt-k)\right|\right) dt &\le \int_{\mathbb{R}} \varphi\left(t, \sum_{k\in\mathbb{Z}} |h_k||\chi(wt-k)|\right) dt \\
&= \int_{\mathbb{R}} \varphi\left(t, \sum_{k\in\mathbb{Z}} |h_k| \frac{1}{m_0(\chi)} m_0(\chi)|\chi(wt-k)|\right) dt \\
&\le \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} \varphi\Big(t, m_0(\chi)|h_k|\Big)|\chi(wt-k)|\right] dt \\
&\le \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} \varphi\Big(k, m_0(\chi)|h_k|\Big)|\chi(wt-k)|\right] dt \\
&= \frac{1}{m_0(\chi)} \sum_{k\in\mathbb{Z}} \varphi\Big(k, m_0(\chi)|h_k|\Big) \frac{1}{w} \int_{\mathbb{R}} |\chi(y)|\ dy \\
&= \frac{1}{w} \frac{1}{m_0(\chi)} \|\chi\|_1 \sum_{k\in\mathbb{Z}} \varphi\Big(k, m_0(\chi)|h_k|\Big),
\end{aligned}$$

i.e., condition (2.2) is true with $D_1 = \|\chi\|_1/m_0(\chi), D_2 = m_0(\chi)$ and $b_w = 0$.

The following lemmas represent sufficient conditions for the validity of (2.3) of Sect. 2 and they will be useful later.

Lemma 4.2 [20]. *Let χ be a kernel belonging to $L^1(\mathbb{R})$ and φ be a fixed τ-bounded φ-function which satisfies the following additional assumption:*

($\varphi 4$) for sufficiently large $M > 0$, there holds:

$$\sup_{|s|>M} \varphi(s,u) := L_u < +\infty, \text{ for every } u \in \mathbb{R}_0^+.$$

Then condition (2.3) results to be satisfied.

Proof. For the proof see Lemma 4.1 of [20] with $t_k = k$. □

Lemma 4.3 [20]. *Let χ be a kernel belonging to $L^1(\mathbb{R})$ and φ be a fixed τ-bounded φ-function which satisfies the following additional assumption:*

($\varphi 5$) $\varphi(\cdot, u) \in L^1(\mathbb{R})$, for every $u \in \mathbb{R}_0^+$.

Then condition (2.3) results to be satisfied.

Proof. For the proof see Lemma 4.2 of [20] with $t_k = k$. □

In order to have a result for the approximation problem we deal with, we can prove the below result concerning the space $\mathcal{L}_{E,K}(\mathbb{R})$.

Lemma 4.4. *Let $f \in X(\mathbb{R})$ be such that $g_{k,w} f$ are locally absolutely integrable functions and suppose that there exist two positive numbers δ, σ such that $0 < \delta \le g_{k,w}(u) \le \sigma$, for every $u \in \mathbb{R}$, $k \in \mathbb{Z}$, $w > 0$.*

Then $f \in \mathcal{L}_{E,K}(\mathbb{R})$ with $E := \sigma/\delta$ and $K := C$, where C is the constant of the τ-boundedness of φ.

Moreover if $f \in L^\varphi(\mathbb{R})$, it turns out that $K_w^{\chi,\mathcal{G}} f \in L^\varphi(\mathbb{R})$, for sufficiently large $w > 0$.

Proof. We want to prove that for every $\lambda > 0$ this inequality holds:

$$\limsup_{w\to+\infty} \frac{1}{w} \rho_{\mathbb{Z}}(\lambda H_w) \le E \rho_{\mathbb{R}}(\lambda K f), \tag{4.1}$$

with $H_w = (h_{k,w})_{k\in\mathbb{Z}},\ w > 0$ and $h_{k,w} = \dfrac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)|\,du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du}$,

and for some constants $E,\ K > 0$.

By assumption ($\varphi 2$) we have that for every $s \in \mathbb{R}$, $\varphi(s,\cdot)$ is non decreasing on $\mathbb{R}_0^+$, so by using Jensen inequality, the condition $0 < \delta \le g_{k,w}(u) \le \sigma$, for every $u \in \mathbb{R}$, $k \in \mathbb{Z}$, $w > 0$ and the τ-boundedness of φ we get:

$$\frac{1}{w}\rho_{\mathbb{Z}}(\lambda H_w) = \frac{1}{w}\sum_{k\in\mathbb{Z}} \varphi\left(k, \left|\lambda \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\,du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du}\right|\right)$$

$$
\leq \frac{1}{w} \sum_{k \in \mathbb{Z}} \varphi \left(k, \lambda \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)| \, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) \, du} \right)
$$

$$
\leq \frac{1}{w} \sum_{k \in \mathbb{Z}} \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) \varphi(k, \lambda |f(u)|) \, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) \, du}
$$

$$
\leq \frac{\sigma}{\delta} \sum_{k \in \mathbb{Z}} \int_{k/w}^{(k+1)/w} \varphi(k, \lambda |f(u)|) \, du
$$

$$
\leq \frac{\sigma}{\delta} \sum_{k \in \mathbb{Z}} \int_{k/w}^{(k+1)/w} \varphi(u, \lambda C |f(u)|) \, du
$$

$$
= \frac{\sigma}{\delta} \int_{\mathbb{R}} \varphi(u, \lambda C |f(u)|) \, du = \frac{\sigma}{\delta} \rho_{\mathbb{R}}(\lambda C f), \ \forall \ w > 0, \qquad (4.2)
$$

i.e., (4.1) is satisfied with $E = \sigma/\delta$ and $K = C$.

Now by passing to the lim sup as $w \to +\infty$ we obtain that f belongs to $\mathcal{L}_{E,K}(\mathbb{R})$.

Concerning the second part of the lemma, if $f \in L^{\varphi}(\mathbb{R})$ it turns out that $g_{k,w} f$ are locally absolutely integrable and hence, for the first part of this lemma, $f \in \mathcal{L}_{E,K}(\mathbb{R})$. Then, the proof immediately follows by Theorem 3.3. □

Now we are able to prove the following results.

Theorem 4.5. *Let χ be a kernel belonging to $L^1(\mathbb{R})$ and φ be a τ-bounded φ-function which satisfies at least one between conditions ($\varphi 4$) and ($\varphi 5$). Further suppose that there exist two positive numbers δ, σ such that $0 < \delta \leq g_{k,w}(u) \leq \sigma$, for every $u \in \mathbb{R}, \ k \in \mathbb{Z}, \ w > 0$. Hence the following statements hold:*

(1) for $f \in C_c(\mathbb{R})$:

$$
\lim_{w \to +\infty} \rho_{\mathbb{R}} \Big(\lambda (K_w^{\chi, \mathcal{G}} f - f) \Big) = 0,
$$

for every $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolute continuity of the modular $\rho_{\mathbb{R}}$.

Moreover if χ has compact support the convergence above holds for every $\lambda > 0$;

(2) for any function $f \in X(\mathbb{R})$ there holds:

$$
\limsup_{w \to +\infty} \rho_{\mathbb{R}}(c \ K_w^{\chi, \mathcal{G}} f) \leq \frac{\|\chi\|_1}{m_0(\chi)} \frac{\sigma}{\delta} \rho_{\mathbb{R}}(c \ m_0(\chi) \ C^2 f),
$$

for every $c > 0$, where C is the constant of the τ-boundedness of φ.

Moreover if $f \in L^{\varphi}(\mathbb{R})$, then it turns out that $K_w^{\chi,\mathcal{G}} f \in L^{\varphi}(\mathbb{R})$, for every $w > 0$;

(3) for any function $f \in L^{\varphi}(\mathbb{R})$, there is a constant $c > 0$ such that

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\Big(c(K_w^{\chi,\mathcal{G}} f - f)\Big) = 0.$$

Proof. (1) As we said above $\rho_{\mathbb{R}}$ is a modular with properties (a)-(e) of Sect. 2 and from Lemmas 4.2 and 4.3, φ is a φ-function which satisfies condition (2.3). So we can apply Theorem 3.1 to conclude that for every λ, with $0 < \lambda \leq \alpha/2$,

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\Big(\lambda(K_w^{\chi,\mathcal{G}} f - f)\Big) = 0.$$

If χ is a compact kernel we can use Theorem 3.2 to get the convergence above for every $\lambda > 0$.

(2) This part is given by Lemma 4.4. However, in order to make explicit the constants in the right term of the thesis, we can proceed as follows.

Since (φ2) and (φ3) are satisfied, Jensen inequality and Fubini-Tonelli theorem hold and the change of variables $wt - k = v$ is made, we can write:

$$\begin{aligned}
\rho_{\mathbb{R}}(c\, K_w^{\chi,\mathcal{G}} f) &= \int_{\mathbb{R}} \varphi\left(t, c\left|\sum_{k\in\mathbb{Z}} \chi(wt-k) \frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du}\right|\right) dt \\
&\leq \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \left[|\chi(wt-k)| \varphi\left(t, c\, m_0(\chi)\left|\frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du}\right|\right)\right] dt \\
&\leq \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \left[|\chi(wt-k)| \varphi\left(k, c\, m_0(\chi) C\left|\frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du}\right|\right)\right] dt \\
&= \frac{1}{m_0(\chi)} \frac{1}{w} \sum_{k\in\mathbb{Z}} \varphi\left(k, c\, m_0(\chi) C\left|\frac{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\, du}{\displaystyle\int_{k/w}^{(k+1)/w} g_{k,w}(u)\, du}\right|\right) \int_{\mathbb{R}} |\chi(v)|\, dv \\
&= \frac{\|\chi\|_1}{m_0(\chi)} \frac{1}{w} \rho_{\mathbb{Z}}(c\, m_0(\chi) C H_w),
\end{aligned}$$

for every $w > 0$, and by applying (4.2) of the proof of Lemma 4.4, we get:

$$\rho_{\mathbb{R}}(c\, K_w^{\chi,\mathcal{G}} f) \leq \frac{\|\chi\|_1}{m_0(\chi)} \frac{\sigma}{\delta} \rho_{\mathbb{R}}(c\, m_0(\chi) C^2 f),$$

for every $w > 0$.

(3) From Theorem 3.4 we have that for every $f \in L^\varphi(\mathbb{R})$ such that $f - C_c(\mathbb{R}) \subset \mathcal{L}_{E,K}(\mathbb{R})$ for some $E, K > 0$, there exists a constant $c > 0$ such that

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\Big(c(K_w^{\chi,\mathcal{G}} f - f)\Big) = 0. \tag{4.3}$$

But since $f \in L^\varphi(\mathbb{R})$, we know that $f \in X(\mathbb{R})$, hence also $f - h \in X(\mathbb{R})$, for $h \in C_c(\mathbb{R})$ and $g_{k,w}(f-h)$ are absolutely locally integrable (with $h \in C_c(\mathbb{R})$). So from Lemma 4.4, we deduce that $f - h \in \mathcal{L}_{E,K}(\mathbb{R})$, where $h \in C_c(\mathbb{R})$ and we conclude that for every $f \in L^\varphi(\mathbb{R})$, (4.3) holds.

□

Now we can discuss the case of φ-functions of the form $\varphi(s,u) = \varphi(u)$, $s \in \mathbb{R},\ u \in \mathbb{R}_0^+$. So the modulars on $X(\mathbb{R})$ and on $X(\mathbb{Z})$ reduce respectively to:

$$\rho_{\mathbb{R}}(f) := \int_{\mathbb{R}} \varphi\big(|f(t)|\big)\, dt, \quad \rho_{\mathbb{Z}} := \sum_{k\in\mathbb{Z}} \varphi\big(|(h_k|\big)$$

and the corresponding modular spaces $L^\varphi(\mathbb{R})$ and $L^\varphi(\mathbb{Z})$ are the so-called Orlicz spaces (see, e.g., [29,31]).

We note that condition (φ3) is trivially satisfied with $F \equiv 0$ and $C \equiv 1$, together with assumption (φ4) and so we can get the following theorem:

Theorem 4.6. *Let χ be a kernel belonging to $L^1(\mathbb{R})$ and φ be a fixed φ-function of the form $\varphi(s,u) = \varphi(u)$, $s \in \mathbb{R},\ u \in \mathbb{R}_0^+$. Further suppose that there exist two positive numbers δ, σ such that $0 < \delta \leq g_{k,w}(u) \leq \sigma$, for every $u \in \mathbb{R},\ k \in \mathbb{Z},\ w > 0$. Hence the following statements hold:*

(1) for $f \in C_c(\mathbb{R})$:

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\Big(\lambda(K_w^{\chi,\mathcal{G}} f - f)\Big) = 0,$$

for every $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolute continuity of the modular $\rho_{\mathbb{R}}$.

Moreover if χ has compact support the convergence above holds for every $\lambda > 0$;

(2) for any function $f \in X(\mathbb{R})$ there holds:

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}}(c\ K_w^{\chi,\mathcal{G}} f) \leq \frac{\|\chi\|_1}{m_0(\chi)} \frac{\sigma}{\delta} \rho_{\mathbb{R}}(c\ m_0(\chi) f),$$

for every $c > 0$.

Moreover if $f \in L^\varphi(\mathbb{R})$, then it turns out that $K_w^{\chi,\mathcal{G}} f \in L^\varphi(\mathbb{R})$, for every $w > 0$;

(3) for every function $f \in L^\varphi(\mathbb{R})$, there is a constant $c > 0$ such that

$$\lim_{w\to+\infty} \rho_{\mathbb{R}}\Big(c(K_w^{\chi,\mathcal{G}} f - f)\Big) = 0.$$

Proof. This theorem follows as a consequence of Theorem 4.5. □

Now we consider φ-functions (with respect to the second variable) of the product-type in the form:

$$\varphi(s,u) := \xi(s)\varphi(u),\ s \in \mathbb{R}, \quad u \in \mathbb{R}_0^+,$$

which satisfy the following conditions:

($\mathcal{F}1$) $\xi \in X(\mathbb{R})$ and there exist $M \geq m > 0$ such that $m \leq \xi(s) \leq M$, for every $s \in \mathbb{R}$;
($\mathcal{F}2$) $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a convex function such that $\varphi(0) = 0$ and $\varphi(u) > 0$, for $u > 0$;
($\mathcal{F}3$) for every $\lambda_1 > 0$ there exists $\lambda_2 \geq 1$ such that

$$\lambda_1 \varphi(u) \leq \varphi(\lambda_2 u),\ u \in \mathbb{R}_0^+.$$

These properties allow us to say that assumptions ($\varphi1$), ($\varphi2$) and ($\varphi4$) are trivially satisfied. Moreover from ($\mathcal{F}3$) with $\lambda_1 = M/m$ we can write

$$\varphi(s-t,u) = \xi(s-t)\varphi(u) \leq \frac{M}{m}\xi(s)\varphi(u) \leq \xi(s)\varphi(\lambda_2 u) = \varphi(s,\lambda_2 u),$$

for every $u \geq 0$, where $\lambda_2 \geq 1$ is the parameter associated to λ_1 fixed. The inequality above shows that this type of φ-functions satisfy also condition ($\varphi3$) with $F \equiv 0$ and $C = \lambda_2$. These φ-functions generate the so-called weighted Orlicz spaces. Therefore we can easily deduce the same result of Theorem 4.5 in this instance.

Now in order to show the validity of the previous results also for modular functionals which are not of integral type, we consider for instance the following modulars (see [3], p.7):

$$\rho_{\mathbb{R}}^{\Phi}(f) := \sup_{l \in W} \int_a^b a_l(x) \left[\int_{\mathbb{R}} \Phi\big(x, |f(t)|\big)\ dt \right] dm(x)$$

and

$$\rho_{\mathbb{Z}}^{\Phi}(h) := \sup_{l \in W} \int_a^b a_l(x) \left[\sum_{k \in \mathbb{Z}} \Phi\big(x, |h_k|\big) \right] dm(x),$$

with $f \in X(\mathbb{R}), h = (h_k)_{k \in \mathbb{Z}} \in X(\mathbb{Z})$ and where m is a measure on an interval $[a,b[\subset \mathbb{R}$ (b can be also equal to $+\infty$) defined on the σ-algebra of all Lebesgue measurable subsets of $[a,b[, W$ is a non-empty set of indices, $a_l : [a,b[\to \mathbb{R}_0^+$ are measurable functions for every $l \in W$ and $\Phi : [a,b[\times \mathbb{R}_0^+ \to \mathbb{R}_0^+$. If the function Φ is convex with respect to the second variable and it satisfies other suitable conditions (see [3]: 1-4 p.7, (b) of p.19 and (b) of p.23), it turns out that $\rho_{\mathbb{R}}^{\Phi}$ is a convex, monotone, strongly finite, absolutely finite and absolutely continuous modular on $X(\mathbb{R})$ and $\rho_{\mathbb{Z}}^{\Phi}$ is a modular on $X(\mathbb{Z})$.

Now we prove the compatibility condition (2.2), i.e., if there exist two positive constants D_1, D_2 and a net $(b_w)_{w>0} \subset \mathbb{R}_0^+$, with $b_w \to 0$ as $w \to +\infty$

such that

$$\sup_{l\in W}\int_a^b a_l(x)\left[\int_{\mathbb{R}}\Phi\Big(x,\sum_{k\in\mathbb{Z}}|h_k||\chi(wt-k)|\Big)dt\right]dm(x)$$
$$\leq \frac{1}{w}D_1\sup_{l\in W}\int_a^b a_l(x)\left[\sum_{k\in\mathbb{Z}}\Phi\Big(x,D_2|h_k|\Big)\right]dm(x)+b_w.$$

By using Jensen inequality, Fubini-Tonelli theorem and the change of variables $y = wt - k$, we get:

$$\begin{aligned}\int_{\mathbb{R}}\Phi\left(x,\sum_{k\in\mathbb{Z}}|h_k||\chi(wt-k)|\right)dt &= \int_{\mathbb{R}}\Phi\left(x,\sum_{k\in\mathbb{Z}}|h_k|\frac{1}{m_0(\chi)}m_0(\chi)|\chi(wt-k)|\right)dt\\ &\leq \frac{1}{m_0(\chi)}\int_{\mathbb{R}}\left[\sum_{k\in\mathbb{Z}}\Phi\Big(x,m_0(\chi)|h_k|\Big)|\chi(wt-k)|\right]dt\\ &= \frac{1}{m_0(\chi)}\sum_{k\in\mathbb{Z}}\Phi\Big(x,m_0(\chi)|h_k|\Big)\frac{1}{w}\int_{\mathbb{R}}|\chi(y)|\,dy\\ &= \frac{1}{w}\frac{1}{m_0(\chi)}\|\chi\|_1\sum_{k\in\mathbb{Z}}\Phi\Big(x,m_0(\chi)|h_k|\Big).\end{aligned}$$

Recalling that $a_l(x) \geq 0$ and passing to the supremum we obtain:

$$\sup_{l\in W}\int_a^b a_l(x)\left[\int_{\mathbb{R}}\Phi\Big(x,\sum_{k\in\mathbb{Z}}|h_k||\chi(wt-k)|\Big)dt\right]dm(x)$$
$$\leq \frac{1}{w}\frac{\|\chi\|_1}{m_0(\chi)}\sup_{l\in W}\int_a^b a_l(x)\left[\sum_{k\in\mathbb{Z}}\Phi\Big(x,m_0(\chi)|h_k|\Big)\right]dm(x),$$

so we have proved condition (2.2) with $D_1 = \|\chi\|_1/m_0(\chi)$, $D_2 = m_0(\chi)$ and $b_w = 0$.

Further, if we choose for instance a kernel χ with compact support, from Remark 2.2 we also have that (2.3) is fullfilled. So all the necessary assumptions are satisfied and we may easily obtain the previous results also in this setting.

As concerns kernels satisfying the assumptions mentioned before, one can easily verify that Fejér, de la Vallée Poussin, Jackson and B-splines (of order $n \in \mathbb{N}^+$) kernels are good examples. Moreover also radial kernels can be furnished as, e.g., the Bochner-Riesz kernel. For these and for other examples, the reader can see, e.g., [20].

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