Results Math (2023) 78:241

1422-6383/23/060001-19
published online September 28, 2023
https://doi.org/10.1007/s00025-023-02013-2

Results in Mathematics

Umbilics of Surfaces in the Lorentz–Minkowski 3-Space

Naoya Ando and Masaaki Umehara

Abstract. In this paper, we prove several fundamental properties on umbilics of a space-like or time-like surface in the Lorentz–Minkowski space $\mathbb{L}^3$. In particular, we show that the local behavior of the curvature line flows of the germ of a space-like surface in $\mathbb{L}^3$ is essentially the same as that of a surface in Euclidean space. As a consequence, for each positive integer m, there exists a germ of a space-like surface with an isolated C^∞-umbilic (resp. C^1-umbilic) of index $(3-m)/2$ (resp. $1+m/2$). We also show that the indices of isolated umbilics of time-like surfaces in $\mathbb{L}^3$ that are not the accumulation points of quasi-umbilics are always equal to zero. On the other hand, when quasi-umbilics accumulate, there exist countably many germs of time-like surfaces which admit an isolated umbilic with non-zero indices.

Mathematics Subject Classification. Primary 53A05; Secondary 53B30.

Keywords. Umbilic, curvature line flow, Ribaucour's parametrization, Carathéodory conjecture.

1. Introduction

We denote by $\mathbb{E}^3$ the Euclidean 3-space and by $\mathbb{L}^3$ the Lorentz–Minkowski 3-space of signature $(++-)$. An immersion $f : U \to \mathbb{L}^3$ defined on a neighborhood U of the origin $o := (0,0) \in \mathbb{R}^2$ is said to be *space-like* (resp. *time-like*) if its induced metric (i.e. the first fundamental form of f) is Riemannian (resp. Lorentzian). An *umbilic* (resp. a *quasi-umbilic*) is a point, where the shape operator A_f of f is a scalar multiple of the identity transformation (resp. has an eigen-equation with a double root but A_f is not diagonalizable). Quasi-umbilics never appear on a space-like surface, but may appear on a time-like surface. We note that the principal curvatures of a time-like surface may not be real-valued. The location of the umbilics on a surface S in $\mathbb{R}^3$ does depend on the choice of the ambient metric $\mathbb{R}^3 = \mathbb{E}^3$ or $\mathbb{R}^3 = \mathbb{L}^3$. In fact, even when p is an umbilic of a space-like surface S in $\mathbb{L}^3$, the point p may not be an umbilic point if we think of S as lying in $\mathbb{E}^3$ in general (see Example 3.1).

In this paper, we focus on the study of curvature line flows of surfaces in $\mathbb{L}^3$, and so we only consider space-like surfaces and time-like surfaces whose principal curvatures are real. An *index* of an isolated umbilic on a given regular surface is the index of one of the curvature line flows of the surface at that point, which takes values in the set $\frac{1}{2}\mathbb{Z}$ of half-integers. For a curvature line flow $\mathcal{F}$ around an isolated umbilic o in the domain of definition of a space-like or time-like surface in $\mathbb{L}^3$, we can construct another curvature line flow $\mathcal{F}^\perp$ associated with $\mathcal{F}$ satisfying $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ (see Proposition C and Remark 3.2). When the surface is space-like, the indices of these two flows at o coincide. In particular, for a given isolated umbilic on a space-like surface, its index is uniquely determined. On the other hand, when the surface is time-like, the two indices at o might be different (cf. Theorem F, see also Example 4.7).

Since the turn of the 21st century, Tari [4] proved an analogue of the Carathéodory conjecture for closed convex surfaces in $\mathbb{L}^3$, and Fontenele and Xavier [3] showed the non-positivity of the index of an isolated umbilic on a surface in $\mathbb{E}^3$ which is negatively curved except at the umbilic. The present article, inspired by these works, investigates the behavior of the curvature line flows on space-like surfaces and time-like surfaces.

In the authors' previous work [1] with Fujiyama, the existence of isolated C^1-differentiable umbilics with arbitrarily high indices was shown by two ways. One is to use the inversion sending the point at infinity in $\mathbb{E}^3$ to the origin as an umbilic, and the other is the method using the "Ribaucour parameter" described in the Appendix of authors' previous work [1] with Fujiyama, by which the pair of curvature line flows of a surface in $\mathbb{E}^3$ can be transformed into the pair of the eigen-flows of the Hessian of a certain smooth function (see [2] where Ribaucour's parameters are explained in terms of Ribaucour's transformations). We call this procedure *Ribaucour's reduction*, which is the keystone of the method of this paper. In fact, we first show that Ribaucour's reduction can be modified for space-like surfaces in $\mathbb{L}^3$. Using the modified reduction, we show that the pair of curvature line flows around a given umbilic of a space-like surface in $\mathbb{L}^3$ can be realized as a pair of the curvature line flows around the corresponding umbilic of a certain regular surface in $\mathbb{E}^3$. By this with the result in [1], we can deduce the following assertion:

Theorem A. *For each positive integer m, there exist a neighborhood U of the origin $o \in \mathbb{R}^2$ and a space-like C^∞-immersion* (*resp.* C^1*-immersion*) $g : U \to \mathbb{L}^3$ *satisfying the following properties:*

(1) *g is C^∞-differentiable and has no umbilics on $U \setminus \{o\}$,*

(2) *o is an isolated singular point of each of the curvature line flows of g with index $(3-m)/2$ (resp. $1+m/2$).*

Motivated by the main theorem of [3], we show the following:

Proposition B. *The index of an isolated umbilic o on a space-like surface in $\mathbb{L}^3$ whose Gaussian curvature is positive except at o is non-positive.*

We next consider time-like surfaces, and then introduce an analogue of Ribaucour's reduction as in the case of space-like surfaces in $\mathbb{L}^3$. Using it, we show the following:

Proposition C. *Let U be a neighborhood of the origin $o \in \mathbb{R}^2$ and $h : U \to \mathbb{L}^3$ a C^∞-differentiable time-like surface which has no umbilics on $U \backslash \{o\}$. If there exists a C^1-differentiable curvature line flow $\mathcal{F}$ on $U \backslash \{o\}$ whose index at o is $n/2 \in \frac{1}{2}\mathbb{Z}$, then $\mathcal{F}$ canonically induces another curvature line flow $\mathcal{F}^\perp$ (satisfying $(\mathcal{F}^\perp)^\perp = \mathcal{F}$) whose index at o is $-n/2$. Moreover, if $\mathcal{F}$ is C^r-differentiable $(r \geq 1)$, then so is $\mathcal{F}^\perp$.*

When o is not an accumulation point of quasi-umbilics, the following assertion holds:

Proposition D. *Let $h : U \to \mathbb{L}^3$ be a time-like surface as in Proposition C, which has a curvature line flow $\mathcal{F}$ on $U \backslash \{o\}$. If h has no quasi-umbilics on $U \backslash \{o\}$, then the index of the flow $\mathcal{F}$ at o is equal to zero. In particular, if the Gaussian curvature of h is negative on $U \backslash \{o\}$, then the index vanishes.*

In this proposition, the assumption that h has no quasi-umbilics on $U \backslash \{o\}$ is necessary. If it is dropped, more than two C^1-differentiable curvature line flows with non-zero indices might exist (cf. Example 4.7). In fact, we can show the following two assertions:

Theorem E. *Let $h : U \to \mathbb{L}^3$ be a C^∞-differentiable time-like surface whose principal curvatures are real-valued. If $o \in U$ is a point such that there are no umbilics on $U \backslash \{o\}$, then there exists a C^1-differentiable curvature line flow on $U \backslash \{o\}$ with zero index.*

Theorem F. *There exist countably many real analytic time-like surfaces with non-positive Gaussian curvature which admit a pair of real analytic curvature line flows with an isolated umbilic with indices ± 1.*

Even when the curvature line flow $\mathcal{F}$ is real analytic, there might be no real analytic curvature line flows other than $\mathcal{F}$ and $\mathcal{F}^\perp$ (cf. Remark 4.5 and Example 4.6). The authors do not know of any C^r-differentiable $(r \geq 2)$ curvature line flows on time-like surfaces having isolated umbilics whose indices I satisfy $|I| > 1$.

In Tari [4], the non-existence of umbilics on the time-like part of a convex surface in $\mathbb{L}^3$ was pointed out (see Remark 4.3).

2. Preliminaries

Let $\lambda(x,y)$ be a C^∞-function defined on a certain open neighborhood U of the origin $o := (0,0)$ of the xy-plane. Assuming $\lambda(o) = \lambda_x(o) = \lambda_y(o) = 0$, we consider the symmetric matrix

$$H_\lambda(x,y) := \begin{pmatrix} \lambda_{xx}(x,y) & \lambda_{xy}(x,y) \\ \lambda_{xy}(x,y) & \lambda_{yy}(x,y) \end{pmatrix},$$

which is the Hessian of λ. We suppose that o is a *scalar point* of H_λ, that is, $H_\lambda(0,0)$ is a scalar multiple of the 2×2 identity matrix. In addition, we also assume that there are no scalar points of H_λ on U other than o. We denote by I_λ the index of one of the eigen-flows of the matrix $H_\lambda(x,y)$ at o. We remark that the index of the other eigen-flow is equal to I_λ (see Fact A.1 in the Appendix). We set

$$f_\lambda(x,y) := (x,y,0) - \lambda(x,y)\nu(x,y) + \lambda(x,y)\mathbf{e}_3, \tag{2.1}$$

which gives a C^∞-regular surface in $\mathbb{E}^3$ defined on a neighborhood of the origin o, where $\mathbf{e}_3 := (0,0,1)$ and

$$\nu := \frac{1}{1+\lambda_x^2+\lambda_y^2}\left(2\lambda_x, 2\lambda_y, \lambda_x^2+\lambda_y^2-1\right) \tag{2.2}$$

is a unit normal vector field of f_λ. In this setting, (x,y) is called *Ribaucour's parametrization* of f_λ. The following fact is classically known (cf. [1, Appendix A]):

Fact 2.1. The origin of $\mathbb{E}^3$ is an isolated umbilic of f_λ whose index coincides with I_λ.

Remark 2.2. One can show that $p \in U$ is an umbilic of the surface f_λ if and only if $(\lambda_{xx} - \lambda_{yy}, \lambda_{xy})$ vanishes at p (see the proof of Ando et al. [1, Fact A.1]).

Here, we give elementary examples:

Example 2.3. We consider an ellipsoid

$$\frac{x^2}{a^2} + y^2 + \frac{z^2}{b^2} = 1 \quad (1 < a \le b). \tag{2.3}$$

When $a = b$, the ellipsoid has two umbilics of index 1, and when $a < b$, it has four umbilics of index 1/2 (cf. [5, Sect. 16]).

Example 2.4. We consider the function $\lambda(x,y) := x^4 - y^4$. Then H_λ is diagonal, and o is an isolated scalar point of H_λ. So, the associated regular surface f_λ [cf. (2.1)] has an isolated umbilic at o, whose index is equal to zero.

Example 2.5. We consider the polynomial

$$\lambda(x,y) := \mathrm{Re}(\zeta^n) \qquad (\zeta := x+iy,\ n \ge 3). \tag{2.4}$$

Since $(\lambda)_{\zeta\zeta} = n(n-1)\zeta^{n-2}/2$, the winding number of the complex-valued function $\lambda_{\zeta\zeta}$ with respect to a counterclockwise circle of small radius centered at the origin o is equal to $n-2$, and the associated regular surface f_λ has an isolated umbilic at o with index $1-n/2$ (cf. [1, (B-1)]).

In these examples, the following well-known fact can be observed.

Fact 2.6. For each positive integer m, there exists a C^∞-regular surface in $\mathbb{E}^3$ having an isolated umbilic whose index is equal to $(3-m)/2$.

3. Umbilics of Space-Like Surfaces in $\mathbb{L}^3$

In this section, "space-like surfaces" always mean space-like regular surfaces. As in the case of regular surfaces in $\mathbb{E}^3$, space-like surfaces in $\mathbb{L}^3$ has exactly two distinct principal directions at each non-umbilic point. So, around an isolated umbilic, a pair of curvature line flows is induced. In this section, we investigate them. More precisely, we modify Ribaucour's reduction given in [1, Appendix A] for space-like surfaces in $\mathbb{L}^3$ and prove Theorem A. We first give an example of a space-like surface with umbilics:

Example 3.1. We let E_a be the ellipsoid given in (2.3) by setting $a = b$. If $a > 1$, then umbilics of E_a as a surface in $\mathbb{E}^3$ are the two points $(0, \pm 1, 0)$. However, if we think of E_a as lying in $\mathbb{L}^3$, then these points $(0, \pm 1, 0)$ lie on the time-like part of E_a and cannot be umbilics in $\mathbb{L}^3$ (cf. Remark 4.3). In fact, by a straightforward calculation, one can check that the space-like part of E_a has exactly four space-like umbilics $\frac{1}{\sqrt{2}}(\pm\sqrt{a^2-1}, 0, \pm\sqrt{a^2+1})$ in $\mathbb{L}^3$.

We let $g : U \to \mathbb{L}^3$ be a space-like immersion defined on a neighborhood U of the origin o in the xy-plane. Since we are interested in local properties of surfaces, we may set

$$g(x, y) := (x, y, \phi(x, y)),$$

where ϕ is a C^∞-function satisfying $\phi(o) = \phi_x(o) = \phi_y(o) = 0$. We denote by "$\cdot$" the Lorentzian inner product on $\mathbb{L}^3$, and consider the point

$$Q(x, y) := (\xi(x, y), \eta(x, y), 0)$$

in the xy-plane satisfying

$$Q + \mu\mathbf{e}_3 = g + \mu\nu \quad (\mathbf{e}_3 := (0, 0, 1)), \tag{3.1}$$

where μ is a certain C^∞-function on U and

$$\nu(x, y) := \frac{-1}{\delta_+(x, y)}(\phi_x(x, y), \phi_y(x, y), 1) \quad \left(\delta_+ := \sqrt{1 - \phi_x^2 - \phi_y^2}\right)$$

is a unit normal vector field of g, that is, $|\nu \cdot \nu| = 1$. Comparing the third components of the both sides of (3.1), we obtain the relation

$$\mu = \frac{\phi\delta_+}{1 + \delta_+}.$$

Since $(0,0)$ is a critical point of the function ϕ, we have $\mu(0,0) = 0$ and $d\mu(0,0) = 0$ and so $dg = dQ$ holds at $(0,0)$. In particular, (ξ, η) can be taken as a new local coordinate system centered at o. Differentiating (3.1) by ξ and η respectively, and taking the Lorentzian inner products of the both sides of them with ν, we have

$$Q_\xi \cdot \nu - \mu_\xi \nu_3 = -\mu_\xi, \quad Q_\eta \cdot \nu - \mu_\eta \nu_3 = -\mu_\eta,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$. Since $Q_\xi = (1,0,0)$ and $Q_\eta = (0,1,0)$, we obtain the following:

$$\mu_\xi = \frac{-\nu_1}{1-\nu_3}, \quad \mu_\eta = \frac{-\nu_2}{1-\nu_3},$$

which correspond to the stereographic projection of the hyperbolic space

$$\{(x,y,z) \in \mathbb{L}^3;\ x^2 + y^2 - z^2 = -1,\ z < 0\}$$

to the xy-plane. By this, as an analogue of (2.2), we obtain

$$\nu(\xi,\eta) = \frac{1}{\mu_\xi^2 + \mu_\eta^2 - 1}(2\mu_\xi, 2\mu_\eta, \mu_\xi^2 + \mu_\eta^2 + 1). \tag{3.2}$$

So we have that

$$g(\xi,\eta) = (\xi,\eta,0) - \mu(\xi,\eta)\nu(\xi,\eta) + \mu(\xi,\eta)\mathbf{e}_3, \tag{3.3}$$

which is an analogue of (2.1). We call the above procedure *space-like Ribaucour's reduction* and the coordinate system (ξ,η) *space-like Ribaucour's parametrization* of g. Let $\gamma(t)$ be a regular curve in the $\xi\eta$-plane. This curve is an orbit of one of the curvature line flows of g if and only if $d(\nu\circ\gamma)(t)/dt$ and $d(g\circ\gamma)(t)/dt$ are linearly dependent. By the same argument as in [1, Appendix A], the equation $\det(\nu, dg, d\nu) = 0$ characterizes the curvature line flows of g, and (3.3) yields that

$$\det(\nu, dg, d\nu) = \det\begin{pmatrix} \nu_1 & d\xi & d\nu_1 \\ \nu_2 & d\eta & d\nu_2 \\ \nu_3 & \mu_\xi d\xi + \mu_\eta d\eta & d\nu_3 \end{pmatrix}, \tag{3.4}$$

where "det" means the determinant function. Since g is space-like, we have $\nu_1^2 + \nu_2^2 - \nu_3^2 = -1$ and

$$\mu_\xi\nu_1 + \mu_\eta\nu_2 = 1 + \nu_3. \tag{3.5}$$

By (3.2), it holds that

$$d\nu = -\frac{dk}{k}\nu + \frac{2}{k}(d\mu_\xi, d\mu_\eta, \mu_\xi d\mu_\xi + \mu_\eta d\mu_\eta), \tag{3.6}$$

where $k := \mu_\xi^2 + \mu_\eta^2 - 1$. By (3.4)–(3.6), we have (see again [1, Appendix A])

$$\det(\nu, dg, d\nu) = \frac{2}{k}\det\begin{pmatrix} \nu_1 & d\xi & d\mu_\xi \\ \nu_2 & d\eta & d\mu_\eta \\ \mu_\xi\nu_1 + \mu_\eta\nu_2 - 1 & \mu_\xi d\xi + \mu_\eta d\eta & \mu_\xi d\mu_\xi + \mu_\eta d\mu_\eta \end{pmatrix}$$

$$= \frac{2}{k} \det \begin{pmatrix} \nu_1 & d\xi & d\mu_\xi \\ \nu_2 & d\eta & d\mu_\eta \\ -1 & 0 & 0 \end{pmatrix} = \frac{-2}{k}(d\xi, d\eta) S_\mu \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}, \tag{3.7}$$

where

$$S_\mu := \begin{pmatrix} \mu_{\xi\eta} & (\mu_{\eta\eta} - \mu_{\xi\xi})/2 \\ (\mu_{\eta\eta} - \mu_{\xi\xi})/2 & -\mu_{\xi\eta} \end{pmatrix}.$$

Thus, the curvature line flows of g just coincide with the null direction flows of S_μ (see the Appendix in this paper for the definition of null directions).

Remark 3.2. If $\mathbf{v} := u(\partial/\partial\xi)_p + v(\partial/\partial\eta)_p$ is a tangent vector at $p \in U$ giving a null direction of S_μ, then $\mathbf{v}^\perp := -v(\partial/\partial\xi)_p + u(\partial/\partial\eta)_p$ is the 90°-rotation of $\mathbf{v}$ giving also a null direction of S_μ. So if $\mathcal{F}$ is a curvature line flow of the space-like surface g, then the 90°-rotation $\mathcal{F}^\perp$ in the $\xi\eta$-plane also gives a curvature line flow of g. This fact is one of the strengths of Ribaucour's parametrizations. As a consequence, the curvature line flows of g can be considered as a pair $(\mathcal{F}, \mathcal{F}^\perp)$.

The characteristic vector field $\mathbf{v}_{S_\mu}$ (cf. the Appendix) of S_μ is given by

$$\mathbf{v}_{S_\mu} = 2\mu_{\xi\eta}\frac{\partial}{\partial\xi} + (\mu_{\eta\eta} - \mu_{\xi\xi})\frac{\partial}{\partial\eta}.$$

The 90°-rotation of this vector field is $(\mu_{\xi\xi} - \mu_{\eta\eta})\frac{\partial}{\partial\xi} + 2\mu_{\xi\eta}\frac{\partial}{\partial\eta}$, which coincides with the characteristic vector field of the Hessian H_μ of the function μ. By Proposition A.3 in the Appendix, the null direction flows of S_μ are obtained by the 45°-rotation of the eigen-flows of S_μ in the $\xi\eta$-plane. Thus, the null direction flows of S_μ can be identified with the eigen-flows of H_μ. Using these discussions, we obtain the following:

Theorem 3.3. *For a given C^∞-function μ on a neighborhood U of the origin o in the $\xi\eta$-plane, the map $g : U \to \mathbb{L}^3$ given by (3.3) with (3.2) is a space-like immersion. Any congruence class of germs of space-like immersions in $\mathbb{L}^3$ is obtained in this manner. Moreover, $p \in U$ is an umbilic if and only if H_μ is a scalar matrix at p. If $p \in U$ is not an umbilic, the principal directions of g at p can be obtained by the eigen-directions of H_μ. In particular, if o is an isolated umbilic of g, then the pair of curvature line flows of g exists around o and they have the same index at o.*

Proof. It is sufficient to show that p is an umbilic point of g when S_μ vanishes at p, which follows from (3.7), since $\det(\nu, dg, d\nu)$ vanishes at p if and only if p is an umbilic. □

Using Theorem 3.3, we prove Theorem A in the introduction:

Proof of Theorem A. We fix a positive integer m. By Fact 2.6, there exists a regular surface f in $\mathbb{E}^3$ which has an isolated umbilic whose index is equal to $(3 - m)/2$. Then there exist a new local coordinate system (x, y) and a

C^∞-function $\lambda(x,y)$ such that f is expressed as (2.1) with (2.2). The pair of curvature line flows of f coincides with the pair of eigen-flows of H_λ, and the isolated umbilic of f corresponds to an isolated scalar point of H_λ.

By setting $\mu := \lambda$, we define a regular space-like surface g in $\mathbb{L}^3$ given by (3.3) with (3.2). By Theorem 3.3 [see also (3.7)], one of the two curvature line flows of g coincides with either of the eigen-flows of H_λ, and the isolated scalar point of H_λ corresponds to an isolated umbilic of g. So the index of the isolated umbilic of g is $(3-m)/2$.

We next consider the C^1-differentiable function

$$\lambda_m(\xi,\eta) = |z|^2 \tanh\left(|z|^{-a}\mathrm{Re}(z^m/|z|^m)\right) \qquad (z := \xi + \sqrt{-1}\eta,\ 0 < a < 1)$$

given in [1]. By setting $\xi = \rho\cos t$ and $\eta = \rho \sin t$ ($\rho > 0$), the function λ_m induces a function

$$\tilde{\lambda}_m(\rho,t) := \rho^2 \tanh(\rho^{-a}\cos mt)$$

of variables ρ, t. In [1, Sect. 6], it was proved that the indices of the eigen-flows of H_{λ_m} at o are equal to $1+m/2$. By setting $\mu := \lambda_m$, the immersion g_m defined by (3.3) with (3.2) is C^1-differentiable at o, and C^∞-differentiable on $V\setminus\{o\}$ for a sufficiently small neighborhood V of o. The indices of the curvature line flows of g_m at o are equal to $1+m/2$. Thus, Theorem A is obtained. □

It is well-known that the Gaussian curvature of a (space-like or time-like) surface in $\mathbb{L}^3$ has the opposite sign of that of the same surface in $\mathbb{E}^3$ (thinking of $\mathbb{E}^3$ as the space $\mathbb{R}^3$ with the canonical Euclidean metric). Regarding this, we now prove Proposition B:

Proof of Proposition B. Let $g(\xi,\eta)$ be the space-like immersion given by (3.2) and (3.3) using a C^∞-function μ defined on a neighborhood U of the origin of the $\xi\eta$-plane. We let $II_g := L_S d\xi^2 + 2M_S d\xi d\eta + N_S d\eta^2$ be the second fundamental form of g. We may assume that $(0,0)$ corresponds to the umbilic o. From now on, we will show the following equivalency:

$$II_g \text{ is negative definite} \iff H_\lambda \text{ is negative definite} \iff II_{f_\lambda} \text{ is negative definite},$$

where II_{f_λ} is the second fundamental form of the surface f_λ induced by $\lambda := \mu$ in the Euclidean 3-space. Then, by applying the theorem in [3], we can conclude that the indices of the curvature line flows of f_λ at o are non-positive, and so, the space-like surface g at o has the same property. By a straightforward computation, we have

$$L_S = \frac{2\mu_{\xi\xi}}{q_S} + \frac{4\mu(\mu_{\xi\eta}^2 + \mu_{\xi\xi}^2)}{q_S^2}, \quad M_S = \frac{2\mu_{\xi\eta}}{q_S} + \frac{4\mu\mu_{\xi\eta}(\mu_{\xi\xi}+\mu_{\eta\eta})}{q_S^2},$$
$$N_S = \frac{2\mu_{\eta\eta}}{q_S} + \frac{4\mu(\mu_{\xi\eta}^2 + \mu_{\eta\eta}^2)}{q_S^2},$$

where $q_S := 1 - \mu_\xi^2 - \mu_\eta^2$. Hence

$$L_S N_S - M_S^2 = \frac{4\left(\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2\right) D_S}{q_S^4}, \tag{3.8}$$

where

$$D_S := q_S^2 + 2\mu(\mu_{\xi\xi} + \mu_{\eta\eta})q_S + 4\mu^2(\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2).$$

Since

$$\mu(o) = \mu_\xi(o) = \mu_\eta(o) = 0, \tag{3.9}$$

we have $q_S > 0$ and $D_S > 0$ at $(\xi, \eta) = o$. So there exists a neighborhood $V(\subset U)$ of o such that q_S and D_S are positive on V. Then (3.8) implies that $L_S N_S - M_S^2 < 0$ is equivalent to

$$\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2 < 0 \tag{3.10}$$

on $V\backslash\{o\}$. We then set $\lambda := \mu$, and let $f_\lambda(\xi, \eta)$ be the regular surface given by (2.1) and (2.2). The second fundamental form II_{f_λ} of the surface f_λ in $\mathbb{E}^3$ can be written as $II_{f_\lambda} = L_E d\xi^2 + 2M_E d\xi d\eta + N_E d\eta^2$. Again, by a straightforward computation, we have

$$L_E N_E - M_E^2 = \frac{4\left(\lambda_{\xi\xi}\lambda_{\eta\eta} - \lambda_{\xi\eta}^2\right) D_E}{q_E^4}, \tag{3.11}$$

where $q_E := 1 + \lambda_\xi^2 + \lambda_\eta^2$ and

$$D_E := q_E^2 - 2\lambda(\lambda_{\xi\xi} + \lambda_{\eta\eta})q_E + 4\lambda^2(\lambda_{\xi\xi}\lambda_{\eta\eta} - \lambda_{\xi\eta}^2).$$

Since $\lambda = \mu$, the inequality (3.10) is equivalent to $L_E N_E - M_E^2 < 0$ (that is, the Gaussian curvature of f_λ is negative) on $W\backslash\{o\}$ for a sufficiently small neighborhood W of o. Thus, by the theorem in [3], the indices of the curvature line flows of f_λ at o are non-positive. So, we obtain the conclusion. □

4. Umbilics of Time-Like Surfaces in $\mathbb{L}^3$

In this section, "time-like surfaces" always mean time-like regular surfaces. For time-like surfaces in $\mathbb{L}^3$, even at non-umbilic points, principal directions might not exist in general, and even if the directions exist, the two principal directions might coincide (such cases happen when they coincide with a null direction of the first fundamental form). In this section, as in the case of space-like surfaces, we give an analogue of Ribaucour's parametrization for time-like surfaces and prove Proposition C. We let $h : U \to \mathbb{L}^3$ be a time-like immersion defined on a neighborhood U of the origin o in the yz-plane. We may set

$$h(y, z) := (\psi(y, z), y, z),$$

where ψ is a certain C^∞-function defined on U satisfying $\psi(0,0)=\psi_y(0,0)=\psi_z(0,0)=0$. We consider the point $Q(y,z):=(0,\xi(y,z),\eta(y,z))$ satisfying

$$Q+\mu\mathbf{e}_1=h+\mu\nu \quad \big(\mathbf{e}_1:=(1,0,0)\big), \tag{4.1}$$

where μ is a certain C^∞-function on U, and

$$\nu(y,z):=\frac{-1}{\delta_-(y,z)}(1,-\psi_y(y,z),\psi_z(y,z)) \quad \left(\delta_-:=\sqrt{1+\psi_y^2-\psi_z^2}\right)$$

is a unit normal vector field of h. Comparing the first components of the both sides of (4.1), we have

$$\mu=\frac{\psi\delta_-}{1+\delta_-}.$$

Since $(0,0)$ is a critical point of the function ψ, we have $\mu(0,0)=0$ and $d\mu(0,0)=0$. In particular, $dh=dQ$ holds at $(0,0)$, and (ξ,η) can be taken as a new local coordinate system centered at o. Differentiating (4.1) by ξ and η respectively, and taking the Lorentzian inner products of both sides of them with ν, we have

$$Q_\xi\cdot\nu+\mu_\xi\nu_1=\mu_\xi, \quad Q_\eta\cdot\nu+\mu_\eta\nu_1=\mu_\eta.$$

Since $Q_\xi=(0,1,0)$ and $Q_\eta=(0,0,1)$, we have

$$\mu_\xi=\frac{\nu_2}{1-\nu_1}, \quad \mu_\eta=\frac{-\nu_3}{1-\nu_1},$$

which correspond to the stereographic projection of the subset $\{(x,y,z)\in\mathbb{L}^3;\, x^2+y^2-z^2=1\}\setminus\{x=1\}$ of de Sitter plane to the yz-plane. So we have

$$\nu(\xi,\eta)=\frac{1}{-1-\mu_\xi^2+\mu_\eta^2}(1-\mu_\xi^2+\mu_\eta^2,-2\mu_\xi,2\mu_\eta) \tag{4.2}$$

and

$$h(\xi,\eta)=(0,\xi,\eta)-\mu(\xi,\eta)\nu(\xi,\eta)+\mu(\xi,\eta)\mathbf{e}_1 \tag{4.3}$$

analogous to (2.1) and (3.3). We call the procedure *time-like Ribaucour's reduction* and (ξ,η) *time-like Ribaucour's parametrization* of h. Since h is a time-like surface, $\nu_1^2+\nu_2^2-\nu_3^2=1$ holds. So we have $\mu_\xi\nu_2+\mu_\eta\nu_3=1+\nu_1$. By (4.2), we have

$$d\nu=-\frac{dk}{k}\nu+\frac{2}{k}(-\mu_\xi d\mu_\xi+\mu_\eta d\mu_\eta,-d\mu_\xi,d\mu_\eta),$$

where $k:=-1-\mu_\xi^2+\mu_\eta^2$. As in the case of space-like surfaces, we have [cf. (3.7)]

$$\det(\nu,dh,d\nu)=\det\begin{pmatrix}\nu_1 & \mu_\xi d\xi+\mu_\eta d\eta & d\nu_1\\ \nu_2 & d\xi & d\nu_2\\ \nu_3 & d\eta & d\nu_3\end{pmatrix}=\frac{-2}{k}(d\xi,d\eta)T_\mu\begin{pmatrix}d\xi\\ d\eta\end{pmatrix}, \tag{4.4}$$

where

$$T_\mu := \begin{pmatrix} \mu_{\xi\eta} & (\mu_{\eta\eta}+\mu_{\xi\xi})/2 \\ (\mu_{\eta\eta}+\mu_{\xi\xi})/2 & \mu_{\xi\eta} \end{pmatrix}. \tag{4.5}$$

Thus, if a curvature line flow of h on $U\setminus\{o\}$ exists, then it corresponds to a null direction flow (see the Appendix) of the symmetric matrix T_μ.

Remark 4.1. We consider two matrices

$$\begin{aligned} T &= \begin{pmatrix} c & (a+b)/2 \\ (a+b)/2 & c \end{pmatrix}, \quad \check{T} := E_2 \begin{pmatrix} a & c \\ c & b \end{pmatrix} \\ &= \begin{pmatrix} a & c \\ -c & -b \end{pmatrix} \quad \left(E_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \end{aligned}$$

Then the null vectors of T coincide with the eigenvectors of $\check{T}$. So the null vectors of T_μ coincide with the eigenvectors of the matrix $\check{T}_\mu := E_2 H_\mu$, where H_μ is the Hessian matrix of the function μ.

Using the above observation, we can show the following:

Theorem 4.2. *For a given C^∞-function μ on a neighborhood U of the origin in the $\xi\eta$-plane, the map $h : U \to \mathbb{L}^3$ given by (4.3) with (4.2) is a time-like immersion. Any congruence class of time-like immersions in $\mathbb{L}^3$ is obtained in this manner. Moreover, $p \in U$ is an umbilic (resp. an umbilic or a quasi-umbilic) if and only if T_μ (resp. $\det T_\mu$) vanishes at p. If p is not an umbilic, the principal directions of h at p can be characterized by the null vectors of T_μ of (4.5).*

Proof. It is sufficient to show the assertions on umbilics and quasi-umbilics. We first consider umbilics of h. It is sufficient to show that p is an umbilic point of h when T_μ vanishes at p, which follows by the same reason as the proof of Theorem 3.3.

We next consider quasi-umbilics of h. By (4.4), $p \in U$ is a quasi-umbilc precisely when T_μ has a unique null-dierction, that is, when $\det T_\mu = 0$ but $T_\mu \neq 0$ at p. So we obtain the conclusion. □

As an application of Theorem 4.2, we prove Proposition C in the introduction.

Proof of Proposition C. By Theorem 4.2, we may assume that the time-like immersion $h : U \to \mathbb{L}^3$ is given by (4.3) with (4.2) associated with a C^∞-function μ defined on a neighborhood U of the origin in the $\xi\eta$-plane. We let $\mathcal{F}$ be a C^1-curvature line flow of h on $U\setminus\{o\}$. Then $\mathcal{F}$ is generated locally by a null vector field

$$(\mathbf{v}(\xi,\eta) :=) u(\xi,\eta)\frac{\partial}{\partial\xi} + v(\xi,\eta)\frac{\partial}{\partial\eta}$$

of T_μ. For simplicity, we set

$$2T_\mu(\xi,\eta) = \begin{pmatrix} a(\xi,\eta) & b(\xi,\eta) \\ b(\xi,\eta) & a(\xi,\eta) \end{pmatrix} \qquad (a := 2\mu_{\xi\eta},\ b := \mu_{\eta\eta} + \mu_{\xi\xi}). \tag{4.6}$$

In this situation, $\mathbf{v}(\xi,\eta)$ is a null vector of $T_\mu(\xi,\eta)$ if and only if

$$0 = 2(u,v)T_\mu \begin{pmatrix} u \\ v \end{pmatrix} = (u,v)\begin{pmatrix} au+bv \\ bu+av \end{pmatrix} = a(u^2+v^2) + 2buv. \tag{4.7}$$

Since the right hand side of this equation is invariant when swapping u and v, the vector field

$$\mathbf{v}^\perp(\xi,\eta) := v(\xi,\eta)\frac{\partial}{\partial\xi} + u(\xi,\eta)\frac{\partial}{\partial\eta}$$

also yields a principal direction of h at each point $(\xi,\eta) \in U$. Then $\mathbf{v}^\perp$ generates another curvature line flow $\mathcal{F}^\perp$. By definition, $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ and $\mathcal{F}^\perp$ is C^r-differentiable ($r \geq 1$) if and only if $\mathcal{F}$ is C^r-differentiable. Since the transformation $(x,y) \mapsto (y,x)$ in $\mathbb{R}^2$ is orientation reversing, the index of $\mathcal{F}$ at o is $n/2$ ($n \in \mathbb{Z}$) if and only if the index of $\mathcal{F}^\perp$ at o is $-n/2$, proving the assertion. $\square$

Remark 4.3. If we denote by

$$E_T d\xi^2 + 2F_T d\xi d\eta + G_T d\eta^2, \quad L_T d\xi^2 + 2M_T d\xi d\eta + N_T d\eta^2$$

the first and the second fundamental forms of a time-like surface h, then as an analogue to (3.8) and (3.11), one can show that

$$L_T N_T - M_T^2 = \frac{4\left(\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2\right) D_T}{q_T^4}, \tag{4.8}$$

where $q_T := 1 + \mu_\xi^2 - \mu_\eta^2$ and

$$D_T := q_T^2 - 2\mu(\mu_{\xi\xi} - \mu_{\eta\eta})q_T - 4\mu^2\left(\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2\right).$$

If h is strictly locally convex, then $\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2$ is positive at $(\xi,\eta) = (0,0)$. In this case, the Gaussian curvature of h is negative. Since $E_T = -G_T = 1$ and $F_T = 0$ at $(\xi,\eta) = (0,0)$, the sign of the determinant of the shape operator is negative near $(0,0)$. So, h does not admit any umbilics on a neighborhood of $(0,0)$, which confirms the observation of Tari [4] mentioned in the last paragraph of the introduction.

Proof of Proposition D. For simplicity, we write T_μ as in (4.6). Since $h(\xi,\eta)$ has no quasi-umbilics on $U\setminus\{o\}$, there are two distinct principal directions at each $(\xi,\eta) \in U\setminus\{o\}$. By Theorem 4.2, T_μ has two null directions at (ξ,η). Hence

$$0 > 4\det T_\mu(\xi,\eta) = a(\xi,\eta)^2 - b(\xi,\eta)^2 \qquad ((\xi,\eta) \in U\setminus\{o\}).$$

We let $P(T_pU)$ be the projective space associated to the tangent space T_pU at each point p of the open submanifold U of $\mathbb{R}^2$. We denote by

$$T_pU \ni \mathbf{v} \mapsto [\mathbf{v}] \in P(T_pU) \tag{4.9}$$

the corresponding canonical projection. We set $\mathbf{v} = \pm\Big(\cos\theta(\xi,\eta), \sin\theta(\xi,\eta)\Big)^T$, and consider a null direction field $[\mathbf{v}]$ of T_μ defined on $U\backslash\{o\}$, where the superscript "T" denotes the transposition of the matrix. Then, the identity

$$0 = \mathbf{v}^T \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mathbf{v} = a + b\sin 2\theta \tag{4.10}$$

holds on $U\backslash\{o\}$. Since $|b| > |a|$, the function $\sin 2\theta$ never attains the values ± 1 on $U\backslash\{o\}$. So we conclude that the index of the flow at o induced by $[\mathbf{v}]$ is equal to zero, proving the first assertion.

We next consider the case that h is negatively curved on $U\backslash\{o\}$. Since h is a map into $\mathbb{L}^3$, matrix T_μ associated with h has two distinct null directions at each point of $U\backslash\{o\}$ (in this setting, $\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2$ is positive on a sufficiently small neighborhood of o). Thus, there are no quasi-umbilics on $U\backslash\{o\}$, and we obtain the second assertion. □

Example 4.4. If we set $\mu(\xi,\eta) := \xi^2 + \xi^4 - \eta^2 + \eta^4$, then the time-like surface $h(\xi,\eta)$ associated with μ (cf. (4.3)) has an isolated umbilic at $o := (0,0)$. Since T_μ has two distinct null directions away from o, the surface h has no quasi-umbilics. So, by Proposition D, the index at o is equal to zero. Since

$$\mu_{\xi\xi}\mu_{\eta\eta} - \mu_{\xi\eta}^2 = 4\left(6\xi^2 + 1\right)\left(6\eta^2 - 1\right) < 0,$$

(4.8) implies that the Gaussian curvature of this surface near the origin is positive.

We next prove Theorem E:

Proof of Theorem E. We set

$$a := 2\mu_{\xi\eta}, \quad b := \mu_{\xi\xi} + \mu_{\eta\eta}. \tag{4.11}$$

Since h has real principal curvatures at each point, the function $b^2 - a^2$ is non-negative on U, and so there exists a continuous function ϕ such that $\phi^2 = b^2 - a^2$. We set

$$\mathbf{v}_1 := (-b+\phi)\frac{\partial}{\partial\xi} + a\frac{\partial}{\partial\eta}, \qquad \mathbf{v}_2 := -a\frac{\partial}{\partial\xi} + (b+\phi)\frac{\partial}{\partial\eta}.$$

Then these two vector fields yield null vector fields of T_μ in the $\xi\eta$-plane. Since

$$\det(\mathbf{v}_1, \mathbf{v}_2) = \det\begin{pmatrix} -b+\phi & -a \\ a & b+\phi \end{pmatrix} = (b^2 - a^2) - b^2 + a^2 = 0,$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent at each point on U. Moreover, $\mathbf{v}_1$ and $\mathbf{v}_2$ do not have common zeros on $U\backslash\{o\}$: Assuming for a contradiction, $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$ at some point $p \in U\backslash\{o\}$. Then we deduce that $a = b = 0$ at p. Then, by

Theorem 4.2, p is an umbilic, contradicting our assumption that there are no umbilics on $U\backslash\{o\}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are continuous vector fields, they generate a C^1-curvature line flow $\mathcal{F}_0$ defined on $U\backslash\{o\}$.

As one of the possibilities of ϕ, we may set $\phi := \operatorname{sgn}(b)\sqrt{b^2 - a^2}$. Since

$$(b^2 - a^2 + b\phi)(b^2 - a^2 - b\phi) = -(b^2 - a^2)a^2 \leq 0, \tag{4.12}$$

we have $b^2 - a^2 - b\phi \leq 0$ and so

$$(\phi - b)^2 - a^2 = 2(b^2 - a^2 - b\phi) \leq 0.$$

Then we have

$$a^2 - (b + \varphi)^2 = -2(b^2 - a^2 + \varphi b) \leq 0.$$

So if we regard the $\xi\eta$-plane as the Lorentz–Minkowski plane of signature $(+-)$, then the vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$ cannot point in space-like directions. Consequently, the indices of the flows $\mathcal{F}_0$ and $\mathcal{F}_0^{\perp}$ are equal to zero. □

Remark 4.5. In the above proof, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are real analytic, then the function ϕ must be a real analytic function. For a given real analytic function germ Ψ, a real analytic function germ ψ satisfying $\psi^2 = \Psi$ is uniquely determined up to $\pm$-ambiguity whenever it exists. Thus, if a real analytic curvature line flow $\mathcal{F}$ exists, there are no real analytic curvature line flows other than $\mathcal{F}$ and $\mathcal{F}^{\perp}$.

Even when a time-like surface is real analytic, the corresponding curvature line flows might not exist:

Example 4.6. When $\mu(\xi,\eta) := \xi^3 + \xi\eta^3$, we have [cf. (4.11)]

$$T_\mu = \begin{pmatrix} 3\eta^2 & 3\xi(\eta+1) \\ 3\xi(\eta+1) & 3\eta^2 \end{pmatrix}, \quad E_2 H_\mu = 3\begin{pmatrix} 2\xi & \eta^2 \\ -\eta^2 & -2\xi\eta \end{pmatrix}.$$

So o is an isolated umbilic and the discriminant of the eigen-equation of $(1/3)E_2H_\mu$ is $4(\xi^2 + 2\xi^2\eta + \xi^2\eta^2 - \eta^4)$, which is a continuous real valued function but not C^1 can be negative on a sufficiently small neighborhood of o. So the resulting curvature line flows $\mathcal{F}$ and $\mathcal{F}^{\perp}$ exist only partially. C^1-C^2.

On the other hand, if $\mu(\xi,\eta) := \xi^3\eta + \xi\eta^3$, the origin o is an isolated umbilic and the equation $|\xi| = |\eta|$ $((\xi,\eta) \neq (0,0))$ gives the set of quasi-umbilics. Since the eigenvalues of E_2H_μ is $\pm 3i(-\xi^2 + \eta^2)$, the resulting curvature line flows do not exist.

Let h be a time-like surface whose principal curvatures are real-valued on a neighborhood of an isolated umbilic o. By Theorem E, the existence of a C^1-differentiable curvature line flow whose index is equal to zero at the umbilic is guaranteed. However, when quasi umbilics accumulate at o, there might exist another curvature line flow whose index does not vanish at o as the following example shows:

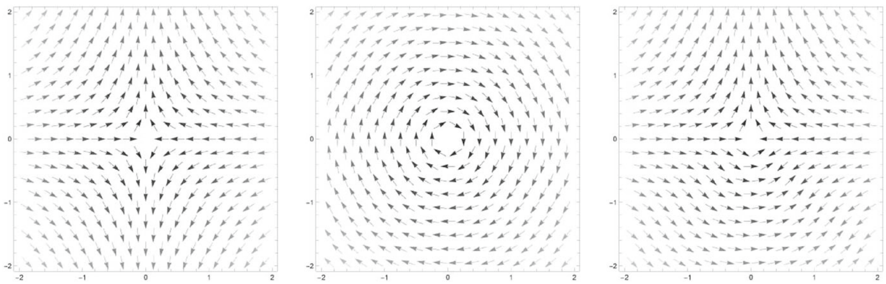

FIGURE 1. The flows generated by $\mathbf{v}_1$ (left), generated by $\mathbf{v}_2$ (center) and the flow $\mathcal{F}_{1/2}$ (right)

Example 4.7. If we set $\mu(\xi,\eta) := \xi^2\eta^2 + (\xi^4+\eta^4)/6$, then we have

$$T_\mu(\xi,\eta) = 2\begin{pmatrix} 2\xi\eta & \xi^2+\eta^2 \\ \xi^2+\eta^2 & 2\xi\eta \end{pmatrix}.$$

Hence, the two vector fields

$$\mathbf{v}_1 := -\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}, \quad \mathbf{v}_2 := \eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}$$

are both real analytic null vector fields of T_μ, and generate two curvature line flows $\mathcal{F}_1$ and $\mathcal{F}_1^\perp$ whose indices at o are -1 and 1, respectively (see the left and the center of Fig. 1).

On the other hand, we denote by $\mathcal{F}_{1/2}$ a C^1-differentiable curvature line flow induced by $[\mathbf{v}_2]$ if $\eta \le -|\xi|$ and $[\mathbf{v}_1]$ if otherwise. Then $\mathcal{F}_{1/2}$ has index $-1/2$ at o (see the right of Fig. 1). Hence taking into account the flows $\mathcal{F}_0$ and $\mathcal{F}_0^\perp$ given in the proof of Theorem E as well as $\mathcal{F}_{1/2}^\perp$, the time-like surface associated with μ admits curvature line flows whose indices take values $0, \pm 1/2$ and ± 1.

As a generalization of this example, we prove Theorem F:

Proof of Theorem F. We let j denote the imaginary digit of the para-complex numbers, that is, $j^2 = 1$ and any para-complex number can be written in the form $a + jb$ ($a, b \in \mathbb{R}$). The function μ in Example 4.7 can then be rewritten as $\mu(\xi,\eta) = \mathrm{Re}(\xi + j\eta)^4/6$. As a generalization, we set $\mu_m(\xi,\eta) := \mathrm{Re}(\xi + j\eta)^m$ for each positive integer $m(\ge 3)$. For the sake of simplicity, we set $\mu := \mu_m$, and denote by $h(\xi,\eta)$ the corresponding time-like surface [cf. (4.3)], which can be considered as an example analogous to Example 2.5. Then

$$T_\mu = m(m-1)\begin{pmatrix} \mathrm{Im}(\xi+j\eta)^{m-2} & \mathrm{Re}(\xi+j\eta)^{m-2} \\ \mathrm{Re}(\xi+j\eta)^{m-2} & \mathrm{Im}(\xi+j\eta)^{m-2} \end{pmatrix}$$

holds. We set $N_2[a+jb] := a^2 - b^2$ ($a, b \in \mathbb{R}$), which plays an analogue of the square of the norm of a complex number for a para-complex number. In fact, one can easily check the identity $N_2[(a+jb)(c+jd)] = N_2[a+jb]\,N_2[c+jd]$ for $a, b, c, d \in \mathbb{R}$. Using this, we obtain

$$\begin{aligned}\frac{\det T_\mu}{m^2(m-1)^2} &= \Big((\mathrm{Im}(\xi+j\eta)^{m-2})^2 - (\mathrm{Re}(\xi+j\eta)^{m-2})^2\Big)\\ &= -N_2[(\xi+j\eta)^{m-2}] = -N_2[\xi+j\eta]^{m-2} = -(\xi^2-\eta^2)^{m-2}.\end{aligned}$$

Moreover, we have

$$\frac{\mu_{\xi\xi}\mu_{\eta\eta}-\mu_{\xi\eta}^2}{m^2(m-1)^2} = \Big(\mathrm{Re}(\xi+j\eta)^{m-2}\Big)^2 - \Big(\mathrm{Im}(\xi+j\eta)^{m-2}\Big)^2 = -\frac{\det T_\mu}{m^2(m-1)^2}. \tag{4.13}$$

If m is an odd integer, then $\det T_\mu$ takes positive values when $|\eta| > |\xi|$, and near such a point (ξ, η), the curvature line flows of h do not exist.

So, we set $m = 2n$ $(n \geq 2)$. By (4.13), the Gaussian curvature of h is then non-positive. To simplify notations, we define two real-valued functions α, β of ξ, η by

$$\alpha + j\beta := (\xi + j\eta)^{n-1}. \tag{4.14}$$

Then we can write

$$\frac{T_\mu}{m(m-1)} = \begin{pmatrix}\mathrm{Im}(\alpha+j\beta)^2 & \mathrm{Re}(\alpha+j\beta)^2\\ \mathrm{Re}(\alpha+j\beta)^2 & \mathrm{Im}(\alpha+j\beta)^2\end{pmatrix} = \begin{pmatrix}2\alpha\beta & \alpha^2+\beta^2\\ \alpha^2+\beta^2 & 2\alpha\beta\end{pmatrix},$$

and

$$\mathbf{v}(\xi,\eta) := -\alpha(\xi,\eta)\frac{\partial}{\partial\xi} + \beta(\xi,\eta)\frac{\partial}{\partial\eta}, \quad \mathbf{w}(\xi,\eta) := \beta(\xi,\eta)\frac{\partial}{\partial\xi} - \alpha(\xi,\eta)\frac{\partial}{\partial\eta}$$

yield a pair of C^∞-differentiable null vector fields of T_μ. Obviously, the origin o is an isolated zero of these vector fields.

We first consider the case that n is odd and set $n = 2k+1$ $(k \geq 1)$. Then $\alpha = \mathrm{Re}(x+jy)^2$ with $x + jy := (\xi+j\eta)^k$. Since $\mathrm{Re}(x+jy)^2 = x^2+y^2$ for $x, y \in \mathbb{R}$, the function α is non-negative, and the indices of $\mathbf{v}$ and $\mathbf{w}$ at o are equal to zero.

We next consider the case that n is even. Each point $(\xi,\eta) \in \mathbb{L}^2$ can be identified with the para-complex number $\xi + j\eta$. Motivated by the definition of α and β [cf. (4.14)], we set $\Phi : \mathbb{L}^2 \ni (\xi,\eta) \mapsto (\xi+j\eta)^{n-1} \in \mathbb{L}^2$. Then, any space-like (resp. time-like) vector of $\mathbb{L}^2$ can be written as

$$(-1)^\sigma r E(jt) \quad (\text{resp. } (-1)^\sigma r j E(jt))$$

where $E(jt) := \cosh t + j \sinh t$, $\sigma \in \{0,1\}$, $r > 0$ and $t \in \mathbb{R}$. Since $n-1$ is odd, we have

$$\begin{aligned}&\Phi((-1)^\sigma r j^\tau E(jt)) = (-1)^\sigma r^{n-1} j^\tau E(j(n-1)t),\\ &\Phi((-1)^\sigma r(1\pm j)) = (-1)^\sigma 2^{n-1} r^{n-1}(1\pm j) \quad (t\in\mathbb{R}, \sigma,\tau \in \{0,1\}).\end{aligned} \tag{4.15}$$

There are four sectors separated by the two lines $\xi = \pm\eta$ in $\mathbb{L}^2$. From (4.15), we observe that Φ maps each of these sectors to itself as an orientation preserving diffeomorphism. So the indices of $\mathbf{v}$ and $\mathbf{w}$ at o are equal to -1 and 1, respectively. As this construction is applicable for each even integer $n(\geq 2)$, we obtain Theorem F. $\square$

Remark 4.8. In this paper, we used modified Ribaucour's parametrizations to prove all of the assertions and to construct examples. Instead one may use the method of (orthonormal) moving frames except for Theorems A, F and Proposition B. In this case, the Hessian matrix of the function associated with a modified Ribaucour's parametrization corresponds to the second fundamental matrix associated with the moving frame method (cf. Remark 4.1). However, when constructing umbilics with a given index, the method of our modified Ribaucour's parametrizations will be simpler than that using moving frames, as demonstrated in the proofs of Theorems A and F.

Acknowledgements

The second author thanks Professors Farid Tari and Federico Sanchez-Bringas for informative conversations during their stay in Japan in November 2022. The authors thank Professors Udo Hertrich Jeromin, Wayne Rossman for valuable comments.

Funding The first author was partially supported by JSPS KAKENHI Grant (C) No. 21K03228, and the second author was partially supported by JSPS KAKENHI Grant (B) No. 21H00981.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declaration

Conflict of interest The authors state that there is no conflict of interest.

Appendix A. Null-Direction Flows of Traceless Symmetric Tensors and Eigen-Flows of Symmetric Tensors

Let U be a neighborhood of the origin o in the xy-plane, and

$$S = s_{11}dx^2 + 2s_{12}dxdy + s_{22}dy^2$$

be a symmetric tensor field on U, which can be identified with the matrix-valued function expressed as $S = (s_{ij})_{i,j=1,2}$ ($s_{21} := s_{12}$). Suppose that o is an isolated scalar point of S. Then, at each point $p \in U \setminus \{o\}$, we may assume that there exists a pair of unit vectors $(\mathbf{v}_1, \mathbf{v}_2)$ satisfying $S\mathbf{v}_i = \lambda_i \mathbf{v}_i$ ($i = 1, 2$) and $\lambda_1(p) \neq \lambda_2(p)$ (see [5, Sect. 14]). Although $\mathbf{v}_i$ ($i = 1, 2$) have $\pm$-ambiguity, they define the elements $\xi_i := [\mathbf{v}_i]$ ($i = 1, 2$) in the projective space associated with the tangent space T_pU at each $p \in U \setminus \{o\}$ [cf. (4.9)]. Each of these two fields ξ_1 and ξ_2 is called an *eigen-directional field* of S (cf. [5, Sect. 14]). We remark that ξ_1 is perpendicular to ξ_2 at each point of $U \setminus \{o\}$. We define the vector field

$$\mathbf{v}_S := d_1 \frac{\partial}{\partial x} + d_2 \frac{\partial}{\partial y} \quad (d_1 := s_{11} - s_{22},\ d_2 := 2s_{12}) \tag{A.1}$$

on U, which is called the *characteristic vector field* of S. Since o is an isolated scalar point of S, o is an isolated zero point of $\mathbf{v}_S$. The following assertion is well-known (cf. [5, Proposition 15.4]):

Fact A.1. If o is an isolated scalar point of S, then the indices $\mathrm{Ind}_o(\xi_i)$ ($i = 1, 2$) of the eigen-directional fields (i.e. the indices of the eigen-flows) at o satisfy

$$\mathrm{Ind}_o(\xi_1) = \mathrm{Ind}_o(\xi_2) = \frac{1}{2}\mathrm{Ind}_o(\mathbf{v}_S) \in \frac{1}{2}\mathbb{Z},$$

where $\mathrm{Ind}_o(\mathbf{v}_S)$ is the index of the vector field $\mathbf{v}_S$ at o.

We next define "null directions":

Definition A.2. A tangent vector $\mathbf{v}$ at $p \in U$ is called a *null vector* if $S(\mathbf{v}, \mathbf{v}) = 0$, and the 1-dimensional vector subspace spanned by such a null vector is called a *null direction* of S at p.

If $\det S > 0$, then its null directions never exist. If $\det S < 0$ (resp. $\det S = 0$) at $p \in U$, then there exists a pair of null directions (resp. a unique null direction) at p. In particular, if $s_{11} + s_{22} = 0$, that is, S is a *traceless symmetric tensor*, then $\det S \leq 0$. Moreover, these directions are represented as vectors $\mathbf{v}_1 \pm \mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent unit eigenvectors of S, that is, S has the two null directions at each point of $U \setminus \{o\}$, which are obtained by $45°$-rotation of the eigen-directions if S is a traceless tensor. So, we obtain the following:

Proposition A.3. *Each of the null direction flows of a traceless symmetric tensor S is obtained by the $45°$-rotation of an eigen-flow of S.*

References

[1] Ando, N., Fujiyama, T., Umehara, M.: C^1-umbilics with arbitrarily high indices. Pac. J. Math. (2017). https://doi.org/10.2140/pjm.2017.288.1

[2] Burstall, F., Hertrich-Jeromin, U., Miro, M.L.: Ribaucour coordinates. Beitr. Algebra Geom. **60**, 39–55 (2019). https://doi.org/10.1007/s13366-018-0391-9

[3] Fontenele, F., Xavier, F.: The index of isolated umbilics on surfaces of non-positive curvature. L'Enseign. Math. **2**(61), 139–149 (2015)

[4] Tari, F.: Umbilics of surfaces in the Minkowski 3-space. J. Math. Soc. Jpn. **65**, 723–731 (2013)

[5] Umehara, M., Yamada, K.: Differential Geometry of Curves and Surfaces. World Scientific (2015)

Naoya Ando
Department of Mathematics, Faculty of Advanced Science and Technology
Kumamoto University
2-39-1 Kurokami, Chuo-ku
Kumamoto 860-8555
Japan
e-mail: andonaoya@kumamoto-u.ac.jp

Masaaki Umehara
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
2-12-1-W8-34, O-okayama, Meguro-ku
Tokyo 152-8552
Japan
e-mail: umehara@is.titech.ac.jp

Received: March 11, 2023.
Accepted: August 30, 2023.