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**Results in Mathematics**

# A Sufficient Condition for Uniform Convergence of Trigonometric Series with $p$-Bounded Variation Coefficients

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**Abstract.** In this paper we consider trigonometric series with $p$-bounded variation coefficients. We presented a sufficient condition for uniform convergance of such series in case $p > 1$. This condition is significantly weaker than these obtained in the results on this subject known in the literature.

**Mathematics Subject Classification.** 40A30, 42A10.

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## 1. Introduction

It is well known that there is a great number of interesting results in Fourier analysis established by assuming monotonicity of Fourier coefficients. The following classical convergence result can be found in many monographs (see for example [3,18] or [1]).

**Theorem 1.** *Suppose that $b_n \geq b_{n+1}$ and $b_n \to 0$ as $n \to \infty$. Then a necessary and sufficient condition for the uniform convergence of the series*

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

*is $nb_n \to 0$ as $n \to \infty$.*

This result has been generalized by weakening the monotonicity conditions of the coefficient (see for example [2,14]). We present below a historical outline of the generalizations of this theorem.

In 2001 Leindler defined (see [8] and [10]) a new class of sequences named as sequences of Rest Bounded Variation, briefly denoted by $RBVS$, i.e.,

$$RBVS=\left\{ a:=(a_n)\in\mathbb{C}:\sum_{n=m}^{\infty}|\Delta_1 a_n|\leq C\,|a_m| \text{ for all } m\in\mathbb{N}\right\},$$

where here and throughout the paper $C=C\,(a)$ always indicates a constant only depending on $a$ and $\Delta_r a_n = a_n - a_{n+r}$ for $r\in\mathbb{N}$.

Denote by $MS$ the class of monotone decreasing sequences, then it is clear that

$$MS\subsetneq RBVS.$$

Further, Tikhonov introduced a class of General Monotone Sequences $GMS$ defined as follows (see [16]):

$$GMS=\left\{ a:=(a_n)\in\mathbb{C}:\sum_{n=m}^{2m-1}|\Delta_1 a_n|\leq C\,|a_m| \text{ for all } m\in\mathbb{N}\right\}.$$

It is clear that

$$RBVS\subsetneq GMS.$$

The class of $GMS$ was generalized by Tikhonov (see [15]) and independently by Zhou, Zhou and Yu (see [17]) to the class of Mean Value Bounded Variation Sequences ($MVBVS$). We say that a sequence $a:=(a_n)$ of complex numbers is said to be $MVBVS$ if there exists $\lambda\geq 2$ such that

$$\sum_{k=n}^{2n-1}|\Delta_1 a_k|\leq\frac{C}{n}\sum_{k=[n/\lambda]}^{\lambda n}|a_k|$$

holds for $n\in\mathbb{N}$, where $[x]$ is the integer part of $x$. They proved also in [17] that

$$GMS\subsetneq MVBVS.$$

Theorem 1 was generalized for the class $RBVS$ in [8], for the class $GMS$ in [16] and for the class $MVBVS$ in [17].

Next, Tikhonov [13,15,16] and Leindler [9] defined the class of $\beta$—general monotone sequences as follows:

**Definition 1.** Let $\beta:=(\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a:=(a_n)$ is said to be $\beta-$ general monotone, or $a\in GM\,(\beta)$, if the relation

$$\sum_{n=m}^{2m-1}|\Delta_1 a_n|\leq C\beta_m$$

holds for all $m\in\mathbb{N}$.

In the paper [15] Tikhonov considered i.e. the following examples of the sequences $\beta_n$:

(1) ${}_1\beta_n = |a_n|$,
(2) ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$.

It is clear that $GM({}_1\beta) = GMS$. Moreover, Tikhonov showed in [15] that

$$GM({}_1\beta) \subsetneq GM({}_2\beta) \equiv \text{MVBVS}.$$

Tikhonov proved also in [15] the following theorem:

**Theorem 2.** *Let a sequence $(b_n) \in GM({}_2\beta)$. If $n|b_n| \to 0$ as $n \to \infty$, then the series (1.1) converges uniformly.*

Further, Szal defined a new class of sequences in the following way (see [11]):

**Definition 2.** Let $\beta := (\beta_n)$ be a nonnegative sequence and $r$ a natural number. The sequence of complex numbers $a := (a_n)$ is said to be $(\beta, r)$−general monotone, or $a \in GM(\beta, r)$, if the relation

$$\sum_{n=m}^{2m-1} |\Delta_r a_n| \leq C\beta_m$$

holds for all $m \in \mathbb{N}$.

It is clear that $GM(\beta, 1) \equiv GM(\beta)$. Moreover, it is easy to show that the sequence

$$a_n = \frac{(-1)^n}{n}$$

belongs to $GM({}_1\beta, 2)$ and does not belong to $GM({}_1\beta)$. This example shows that the class $GM({}_1\beta)$ is essentially wider than the class $GM({}_1\beta)$. In [11] Szal showed more general relations

$$GM({}_2\beta, 1) \subsetneq GM({}_2\beta, r)$$

for all $r > 1$.

In the paper [11] Szal generalized Theorem 1 by proving the following theorem.

**Theorem 3** [11]. *Let a sequence $(b_n) \in GM({}_2\beta, r)$, where $r \in \mathbb{N}$. If $n|b_n| \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{[r/2]} |b_{r \cdot n+k} - b_{r \cdot n+r-k}| < \infty \text{ for } r \geq 3,$$

*then the series (1.1) converges uniformly.*

In the paper [4] Kórus defined a new class of sequences in the following way:

**Definition 3.** The sequence of complex numbers $a := (a_n)$ is in the class $SBVS_2$ (Supremum Bounded Variation Sequence), if the relation

$$\sum_{n=m}^{2m-1} |\Delta_1 a_n| \le \frac{C}{n} \sup_{m \ge b(n)} \sum_{k=m}^{2m} |a_k|$$

holds for all $m \in \mathbb{N}$, where $(b(n))$ is a nonnegative sequence tending monotonically to infinity depending only on $a$.

In the paper [4] Kórus also proved the following theorem:

**Theorem 4.** *Let a sequence $(b_n) \in SBVS_2$. If $n|b_n| \to 0$ as $n \to \infty$, then the series (1.1) converges uniformly.*

Next Tikhonov and Liflyand defined a class of $GMS_p(\beta)$ in the following way (see [7], [6]):

**Definition 4.** Let $\beta = (\beta_n)$ be a nonnegative sequence and $p$ a positive real number. We say that a sequence of complex numbers $a = (a_n) \in GMS_p(\beta)$ if the relation

$$\left( \sum_{n=m}^{2m-1} |\Delta_1 a_n|^p \right)^{\frac{1}{p}} \le C\beta_m$$

holds for all $m \in \mathbb{N}$.

It is clear that $GMS_1(\beta) = GM(\beta)$.

The latest class of sequences was defined by Kubiak and Szal in [5] as follows:

**Definition 5.** Let $\beta := (\beta_n)$ be a nonnegative sequence, $r$ a natural number and $p$ a positive real number. The sequence of complex numbers $a := (a_n)$ is said to be $(p, \beta, r)-$ general monotone, or $a \in GM(p, \beta, r)$, if the relation

$$\left( \sum_{n=m}^{2m-1} |\Delta_r a_n|^p \right)^{\frac{1}{p}} \le C\beta_m$$

holds for all $m \in \mathbb{N}$.

It is clear that $GM(p, \beta, 1) = GMS_p(\beta)$ and $GM(1, \beta, r) = GM(\beta, r)$.

Further we will consider the following sequence:

$$_3\beta_n(q) = \frac{1}{n} \sup_{m \ge b(n)} m \left( \frac{1}{m} \sum_{k=m}^{2m} |a_k|^q \right)^{\frac{1}{q}},$$

where $(a_n) \subset \mathbb{C}, a_n \to 0$ as $n \to \infty$, $q > 0$, $(b(n))$ is a nonnegative sequence such that $b(n) \nearrow$ and $b(n) \to \infty$ as $n \to \infty$. It is clear that $SBVS_2 = GM(1, {}_3\beta(1), 1)$.

In the further part of our paper we will consider the following series:

$$\sum_{n=1}^{\infty} b_n \sin(cnx), \tag{1.2}$$

$$\sum_{n=1}^{\infty} a_n \cos(cnx), \tag{1.3}$$

$$\sum_{n=1}^{\infty} c_n e^{icnx}, \tag{1.4}$$

where $c > 0$.

In the paper [5] Kubiak and Szal showed the following embedding relations:

**Theorem 5.** *Let* $q > 0, r \in \mathbb{N}$ *and* $0 < p_1 \leq p_2$. *Then*

$$GM(p_1, {}_3\beta(q), r) \subseteq GM(p_2, {}_3\beta(q), r).$$

**Theorem 6.** *Let* $p \geq 1$, $q > 0$, $r_1, r_2 \in \mathbb{N}$, $r_1 \leq r_2$. *If* $r_1 \mid r_2$, *then*

$$GM(p, {}_3\beta(q), r_1) \subseteq GM(p, {}_3\beta(q), r_2).$$

Moreover they proved in [5] the following generalization of Theorem 1:

**Theorem 7.** *Let a sequence* $(b_n) \in GM\,(p, {}_3\beta(q), r)$, *where* $p, q \geq 1$, $r \in \mathbb{N}$ *and* $b(n) \geq n$ *for* $n \in \mathbb{N}$. *If*

$$n^{2-\frac{1}{p}} \left|b_n\right| \to 0 \text{ as } n \to \infty \tag{1.5}$$

*and*

$$\sum_{k=1}^{\infty} b_k \sin\left(\frac{2l\pi}{r}k\right) < \infty, \text{ for } r \geq 3,$$

*for all* $l = 1, ..., [\frac{r}{2}] - 1$ *when* $r$ *is an even number and* $l = 1, ..., [\frac{r}{2}]$ *when* $r$ *is an odd number, then the series* (1.2) *is uniformly convergent.*

**Theorem 8.** *Let a sequence* $(a_n) \in GM\,(p, {}_3\beta(q), r)$, *where* $p, q \geq 1$, $r \in \mathbb{N}$ *and* $b(n) \geq n$ *for* $n \in \mathbb{N}$. *If*

$$n^{2-\frac{1}{p}} \left|a_n\right| \to 0 \text{ as } n \to \infty$$

*and*

$$\sum_{k=1}^{\infty} a_k \cos\left(\frac{2l\pi}{r}k\right) < \infty,$$

*for all* $l = 0, 1, ..., [\frac{r}{2}]$, *then the series* (1.3) *is uniformly convergent.*

**Theorem 9.** *Let a sequence* $(c_n) \in GM\left(p, {}_3\beta(q), r\right)$*, where* $p, q \geq 1$*,* $r \in \mathbb{N}$ *and* $b(n) \geq n$ *for* $n \in \mathbb{N}$*. If*

$$n^{2-\frac{1}{p}}\left|c_n\right| \to 0 \text{ as } n \to \infty$$

*and*

$$\sum_{k=1}^{\infty} c_k e^{\left(\frac{2l\pi}{r}k\right)i} < \infty,$$

*for all* $l = 0, 1, ..., [\frac{r}{2}]$*, then the series (1.4) is uniformly convergent.*

In this paper we will show that Theorems 7, 8, 9 are true under weakened assumptions in case $p > 1$.

## 2. Main Results

We have the following results:

**Theorem 10.** *Let a sequence* $(b_n) \in GM\left(p, {}_3\beta(q), r\right)$*, where* $q \geq 1$*,* $p > 1$ *and* $r \in \mathbb{N}$*. If*

$$n \ln n \left|b_n\right| \to 0 \text{ as } n \to \infty \tag{2.1}$$

*and*

$$\sum_{k=1}^{\infty} b_k \sin\left(\frac{2l\pi}{r}k\right) < \infty, \text{ for } r \geq 3, \tag{2.2}$$

*for all* $l = 1, ..., [\frac{r}{2}] - 1$ *when* $r$ *is an even number and* $l = 1, ..., [\frac{r}{2}]$ *when* $r$ *is an odd number, then the series* (1.2) *is uniformly convergent.*

**Proposition 1.** *There exist an* $x_0 \in \mathbb{R}$ *and a sequence* $(b_n) \in GM(p, {}_3\beta(1), 3)$ *for* $p > 1$ *with the properties* $nb_n \to 0$ *as* $n \to \infty$ *and* $(b_n) \notin GM(1, {}_3\beta(1), 3)$*, for which the series* (1.2) *is divergent in* $x_0$*.*

**Theorem 11.** *Let a sequence* $(a_n) \in GM\left(p, {}_3\beta(q), r\right)$*, where* $q \geq 1$*,* $p > 1$ *and* $r \in \mathbb{N}$*. If*

$$n \ln n \left|a_n\right| \to 0 \text{ as } n \to \infty$$

*and*

$$\sum_{k=1}^{\infty} a_k \cos\left(\frac{2l\pi}{r}k\right) < \infty, \tag{2.3}$$

*for all* $l = 0, 1, ..., [\frac{r}{2}]$*, then the series* (1.3) *is uniformly convergent.*

**Theorem 12.** *Let a sequence* $(c_n) \in GM\left(p, {}_3\beta(q), r\right)$*, where* $q \geq 1$*,* $p > 1$ *and* $r \in \mathbb{N}$*. If*

$$n \ln n\, |c_n| \to 0 \ \text{as}\ n \to \infty$$

*and*

$$\sum_{k=1}^{\infty} c_k e^{\left(\frac{2l\pi}{r}k\right)i} < \infty, \tag{2.4}$$

*for all* $l = 0, 1, ..., [\frac{r}{2}]$*, then the series* (1.4) *is uniformly convergent.*

*Remark 1.* It is clear that if a sequence $(b_n)$ satisfies the condition (2.1) then it fulfills the condition (1.5) with $p > 1$, too. Therefore, from Theorem 10 we get Theorem 7 is case $p > 1$. The same remark applies to Theorems 11, 8 and Theorems 12, 9, respectively.

## 3. Lemma

Denote, for $r \in \mathbb{N}$ and $k = 0, 1, 2...$ by

$$\tilde{D}_{k,r}(x) = \frac{\cos\left(k + \frac{r}{2}\right)x}{2\sin\frac{rx}{2}}, \qquad D_{k,r}(x) = \frac{\sin\left(k + \frac{r}{2}\right)x}{2\sin\frac{rx}{2}}$$

the Dirichlet type kernels.

**Lemma 1** . [11,12] *Let* $r, m, n \in \mathbb{N}, l \in \mathbb{Z}$ *and* $(a_k) \subset \mathbb{C}$*. If* $x \neq \frac{2l\pi}{r}$*, then for* $m \geq n$

$$\begin{aligned}\sum_{k=n}^{m} a_k sin(kx) = &-\sum_{k=n}^{m} \Delta_r a_k \tilde{D}_{k,r}(x) + \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}(x)\\ &+ \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}(x)\end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned}\sum_{k=n}^{m} a_k \cos kx = &\sum_{k=n}^{m} \Delta_r a_k \tilde{D}_{k,r}(x) - \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}(x)\\ &+ \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}(x).\end{aligned} \tag{3.2}$$

**Lemma 2** [5]. *Let $r, m, n \in \mathbb{N}, l \in \mathbb{Z}$ and $a = (a_n) \subset \mathbb{C}$. If $x \neq \frac{2l\pi}{r}$, then for $m \geq n$*

$$\sum_{k=n}^{m} a_k e^{ikx} = \frac{-i}{2\sin\left(\frac{rx}{2}\right)} \left( \sum_{k=n}^{m} \Delta_r a_k e^{-i\left(k+\frac{r}{2}\right)x} - \sum_{k=m+1}^{m+r} a_k e^{-i\left(k-\frac{r}{2}\right)x} + \sum_{k=n}^{n+r-1} a_k e^{-i\left(k-\frac{r}{2}\right)x} \right).$$

**Lemma 3.** *Let $n, N \in \mathbb{N}$. Then for $p \geq 1$*

$$\int_{n+N^{\frac{1}{p}}}^{n+N} \frac{1}{k \ln k} dk \leq \ln p.$$

*Proof.* This inequality is true for $p = 1$. Consider the function

$$f(p) = \left(n + N^{\frac{1}{p}}\right)^p$$

for $p > 0$. We get:

$$f'(p) \geq \left(n + N^{\frac{1}{p}}\right)^{p-1} \frac{1}{p} n \ln N \geq 0 \text{ for all } p > 0.$$

It means that the function is non-decresing with respect to $p$.
Thus:

$$n + N = f(1) \leq f(p) = \left(n + N^{\frac{1}{p}}\right)^p \text{ for } p \geq 1.$$

Hence we get that:

$$\ln(n + N) \leq \ln\left(n + N^{\frac{1}{p}}\right)^p. \tag{3.3}$$

Therefore, integrating by substitution with $\ln k = t$ and using (3.3), we get

$$\int_{n+N^{\frac{1}{p}}}^{n+N} \frac{1}{k \ln k} dk = \int_{\ln(n+N)^{\frac{1}{p}}}^{\ln(n+N)} \frac{1}{t} dt = \ln(\ln(n+N)) - \ln(\ln(n + N^{\frac{1}{p}}))$$
$$= \ln\left(\frac{\ln(n+N)}{\ln\left(n + N^{\frac{1}{p}}\right)}\right) = \ln\left(p \frac{\ln(n+N)}{\ln\left(n + N^{\frac{1}{p}}\right)^p}\right) \leq \ln p$$

and the proof is completed. □

# 4. Proofs of the Main Results

## 4.1. Proof of the Theorem 10

Let $\epsilon > 0$. Then from (2.1) and (2.2) we obtain:

$$n \ln n \, |b_n| < \varepsilon, \tag{4.1}$$

$$\left| \sum_{k=n}^{\infty} b_k \sin\left(k \frac{2l\pi}{r}\right) \right| < \varepsilon, \tag{4.2}$$

and

$$\left|\sum_{k=n}^{n+N} b_k \sin\left(k\frac{2l\pi}{r}\right)\right| < \varepsilon, \tag{4.3}$$

for all $n > N_\epsilon$ and $N \in \mathbb{N}$, where $l = 1, ..., [\frac{r}{2}] - 1$ when $r$ is an even number and $l = 1, ..., [\frac{r}{2}]$ when $r$ is an odd number. Denote by

$$\tau_n(x) = \sum_{k=n}^{\infty} b_k \sin(ckx).$$

We will show that

$$|\tau_n(x)| \ll \varepsilon \tag{4.4}$$

holds for any $n \geq \max\{N_\varepsilon, 2\}$ and $x \in \mathbb{R}$. Since $\tau_n(0) = 0$ and $\tau_n\left(\frac{\pi}{c}\right) = 0$ it suffices to prove (4.4) for $0 < x < \frac{\pi}{c}$.

First, we will show that (4.4) is valid for $x = \frac{2l\pi}{rc}$, where $l$ is an integer number such that $0 < 2l < r$. Using (4.2) we get

$$\left|\tau_n\left(\frac{2l\pi}{rc}\right)\right| < \varepsilon.$$

Now, we prove that (4.4) holds for $\frac{2l\pi}{rc} < x \leq \frac{2l\pi}{rc} + \frac{\pi}{rc}$, where $0 \leq 2l < r$. Let $N^{\frac{1}{p}} := N^{\frac{1}{p}}(x) \geq r$ be a natural number such that

$$\frac{2l\pi}{rc} + \frac{\pi}{c(N+1)^{\frac{1}{p}}} < x \leq \frac{2l\pi}{rc} + \frac{\pi}{cN^{\frac{1}{p}}}. \tag{4.5}$$

Then

$$\begin{aligned}\tau_n(x) &= \sum_{k=n}^{n+N^{\frac{1}{p}}-1} b_k \sin(ckx) + \sum_{k=n+N^{\frac{1}{p}}}^{n+N} b_k \sin(ckx) + \sum_{k=n+N+1}^{\infty} b_k \sin(ckx)\\ &= \tau_n^{(1)}(x) + \tau_n^{(2)}(x) + \tau_n^{(3)}(x).\end{aligned}$$

Applying Lagrange's mean value theorem to the function $f(x) = \sin(ckx)$ on the interval $\left[\frac{2l\pi}{rc}, x\right]$ we obtain that for each $k$ there exists $y_k \in \left(\frac{2l\pi}{rc}, x\right)$ such that

$$\sin(ckx) - \sin\left(k\frac{2l\pi}{r}\right) = ck\cos(cky_k)\left(x - \frac{2l\pi}{rc}\right).$$

Using this we get

$$\begin{aligned}\tau_n^{(1)}(x) &= \sum_{k=n}^{n+N^{\frac{1}{p}}-1} ckb_k \cos(cky_k)\left(x - \frac{2l\pi}{rc}\right) + \sum_{k=n}^{n+N^{\frac{1}{p}}-1} b_k \sin\left(k\frac{2l\pi}{r}\right)\\ &= \tau_n^{(1.1)}(x) + \tau_n^{(1.2)}(x).\end{aligned}$$

From (4.3) we have

$$\left|\tau_n^{(1.2)}(x)\right| < \varepsilon.$$

By (4.5) and (4.1)

$$\left|\tau_n^{(1.1)}(x)\right| \le \left(x - \frac{2l\pi}{rc}\right) \sum_{k=n}^{n+N^{\frac{1}{p}}-1} ck\,|b_k| \le \left(x - \frac{2l\pi}{rc}\right) \sum_{k=n}^{n+N^{\frac{1}{p}}-1} \frac{ck\ln k}{\ln k}\,|b_k|$$
$$< \left(x - \frac{2l\pi}{rc}\right) \sum_{k=n}^{n+N^{\frac{1}{p}}-1} \frac{c\varepsilon}{\ln k} \le \frac{\pi\varepsilon}{\ln 2}.$$

Using Lemma 3 we obtain

$$\left|\tau_n^{(2)}(x)\right| = \left|\sum_{k=n+N^{\frac{1}{p}}}^{n+N} b_k \sin(ckx)\right| \le \sum_{k=n+N^{\frac{1}{p}}}^{n+N} \frac{k\ln k}{k\ln k}\,|b_k| \ll \varepsilon \int_{n+N^{\frac{1}{p}}}^{n+N} \frac{1}{k\ln k} dk$$
$$\le \varepsilon \ln p.$$

If $(b_n) \in GM\left(p, {}_3\beta(q), r\right)$, then using Lemma 1, we get

$$\left|\tau_n{}^{(3)}(x)\right| = \left|\sum_{j=0}^{\infty} \sum_{k=2^j(n+N+1)}^{2^{j+1}(n+N+1)-1} b_k \sin(ckx)\right|$$
$$\le \sum_{j=0}^{\infty} \left|\frac{-1}{2\sin(crx/2)} \left\{ \sum_{k=2^j(n+N+1)}^{2^{j+1}(n+N+1)-1} (b_k - b_{k+r}) \cos\left(k + \frac{r}{2}\right) cx \right.\right.$$
$$\left.\left. + \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} b_k \cos\left(k - \frac{r}{2}\right) cx - \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} b_k \cos\left(k - \frac{r}{2}\right) cx \right\}\right|$$
$$\le \frac{1}{2\left|\sin(crx/2)\right|} \sum_{j=0}^{\infty} \left\{ \sum_{k=2^j(n+N+1)}^{2^{j+1}(n+N+1)-1} |b_k - b_{k+r}| + \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} |b_k| \right.$$
$$\left. + \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} |b_k| \right\}.$$

Further applying the Hölder inequality with $p > 1$, the inequality $\frac{rc}{\pi}x - 2\,l \le \left|\sin\frac{rcx}{2}\right|$ $\left(x \in \left[\frac{2\,l\pi}{rc}, \frac{2\,l\pi}{rc} + \frac{\pi}{rc}\right]\right.$ and $\left.0 \le 2\,l < r\right)$, (4.5) and (4.1), we obtain

$$\left|\tau_n{}^{(3)}(x)\right| \le \frac{1}{\frac{rc}{\pi}x - 2l} \sum_{j=0}^{\infty} \left\{ \left( \sum_{k=2^j(n+N+1)}^{2^{j+1}(n+N+1)-1} |b_k - b_{k+r}|^p \right)^{\frac{1}{p}} \right.$$

$$
\left(\sum_{k=2^j(n+N+1)}^{2^{j+1}(n+N+1)-1} 1\right)^{1-\frac{1}{p}} + \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} |b_k| + \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} |b_k| \Bigg\}
$$

$$
\leq \frac{(N+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \Bigg\{ \frac{C\left(2^j(n+N+1)\right)^{1-\frac{1}{p}}}{2^j(n+N+1)} \sup_{m\geq b(2^j(n+N+1))} m
$$

$$
\left(\frac{1}{m}\sum_{k=m}^{2m-1} |b_k|^q\right)^{\frac{1}{q}} + \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} |b_k| + \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} |b_k| \Bigg\}
$$

$$
= \frac{(N+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \Bigg\{ \frac{C}{\left(2^j(n+N+1)\right)^{\frac{1}{p}}} \sup_{m\geq b(2^j(n+N+1))} m^{1-\frac{1}{q}}
$$

$$
\left(\sum_{k=m}^{2m-1} \left(\frac{k\ln k\,|b_k|}{k\ln k}\right)^q\right)^{\frac{1}{q}} + \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} \frac{k\ln k\,|b_k|}{k\ln k}
$$

$$
+ \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} \frac{k\ln k\,|b_k|}{k\ln k} \Bigg\}
$$

$$
< \frac{\varepsilon(N+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \Bigg\{ \frac{C}{\left(2^j(n+N+1)\right)^{\frac{1}{p}}} \sup_{m\geq b(2^j(n+N+1))} m^{1-\frac{1}{q}} \left(\sum_{k=m}^{2m-1} \left(\frac{1}{k}\right)^q\right)^{\frac{1}{q}}
$$

$$
+ \sum_{k=2^j(n+N+1)}^{2^j(n+N+1)+r-1} \frac{1}{k} + \sum_{k=2^{j+1}(n+N+1)}^{2^{j+1}(n+N+1)+r-1} \frac{1}{k} \Bigg\}
$$

$$
\leq \frac{\varepsilon(N+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \Bigg\{ \frac{C}{\left(2^j(n+N+1)\right)^{\frac{1}{p}}} \sup_{m\geq b(2^j(n+N+1))} \left(m^{1-\frac{1}{q}} m^{-1} m^{\frac{1}{q}}\right)
$$

$$
+ \frac{3}{2} r \left(2^j(n+N+1)\right)^{-\frac{1}{p}} \Bigg\}.
$$

Elementary calculations give:

$$
\left|\tau_n^{(3)}(x)\right| < \frac{\varepsilon(N+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \left\{ \left(2^j(n+N+1)\right)^{-\frac{1}{p}} \left(C + \frac{3}{2}r\right) \right\}
$$

$$
\leq \frac{\varepsilon(N+1)^{\frac{1}{p}}\left(C+\frac{3}{2}r\right)}{r\left((n+N+1)\right)^{\frac{1}{p}}} \sum_{j=0}^{\infty} \left(\frac{1}{2^{\frac{1}{p}}}\right)^j \leq \frac{\varepsilon\left(C+\frac{3}{2}r\right)}{r} \frac{1}{1-2^{-\frac{1}{p}}}.
$$

Finally, we prove that (4.4) is true for $\frac{2\,l\pi}{rc} + \frac{\pi}{rc} \leq x < \frac{2(l+1)\pi}{rc}$, where $0 < 2\,(l+1) \leq r$.

Let $M^{\frac{1}{p}} := M^{\frac{1}{p}}(x) \geq r$ be a natural number such that

$$\frac{2(l+1)\pi}{rc} - \frac{\pi}{cM^{\frac{1}{p}}} \leq x < \frac{2(l+1)\pi}{rc} - \frac{\pi}{c(M+1)^{\frac{1}{p}}}. \tag{4.6}$$

Then

$$\begin{aligned}\tau_n(x) &= \sum_{k=n}^{n+M^{\frac{1}{p}}-1} b_k \sin(ckx) + \sum_{k=n+M^{\frac{1}{p}}}^{n+M} b_k \sin(ckx) + \sum_{k=n+M+1}^{\infty} b_k \sin(ckx)\\ &= \tau_n^{(4)}(x) + \tau_n^{(5)}(x) + \tau_n^{(6)}(x).\end{aligned}$$

Applying Lagrange's mean value theorem to the function $f(x) = \sin(ckx)$ on the interval $\left[x, \frac{2(l+1)\pi}{rc}\right]$ we obtain that for each $k$ there exists $z_k \in \left(x, \frac{2(l+1)\pi}{rc}\right)$ such that

$$\sin\left(k\frac{2(l+1)\pi}{r}\right) - \sin(ckx) = ck\cos(ckz_k)\left(\frac{2(l+1)\pi}{rc} - x\right).$$

Using this we get

$$\begin{aligned}\tau_n^{(4)}(x) &= \sum_{k=n}^{n+M^{\frac{1}{p}}-1} ckb_k \cos(ckz_k)\left(\frac{2(l+1)\pi}{rc} - x\right)\\ &\quad + \sum_{k=n}^{n+M^{\frac{1}{p}}-1} b_k \sin\left(k\frac{2(l+1)\pi}{r}\right) = \tau_n^{(4.1)}(x) + \tau_n^{(4.2)}(x).\end{aligned}$$

From (4.2) we have

$$\left|\tau_n^{(4.2)}(x)\right| < \varepsilon.$$

By (4.6) and (4.1)

$$\begin{aligned}\left|\tau_n^{(4.1)}(x)\right| &\leq \left(\frac{2(l+1)\pi}{rc} - x\right) \sum_{k=n}^{n+M^{\frac{1}{p}}-1} ck|b_k| \leq \frac{\pi}{M^{\frac{1}{p}}} \sum_{k=n}^{n+M^{\frac{1}{p}}-1} \frac{k\ln k}{\ln k}|b_k|\\ &\leq \frac{\pi\varepsilon}{\ln 2}.\end{aligned}$$

Using Lemma 3 we get

$$\begin{aligned}\left|\tau_n^{(5)}(x)\right| &= \left|\sum_{k=n+M^{\frac{1}{p}}}^{n+M} b_k \sin(ckx)\right| \leq \sum_{k=n+M^{\frac{1}{p}}}^{n+M} \frac{k\ln k}{k\ln k}|b_k| \ll \varepsilon \int_{n+M^{\frac{1}{p}}}^{n+M} \frac{1}{k\ln k}dk\\ &\leq \varepsilon \ln p.\end{aligned}$$

If $(b_n) \in GM\left(p, {}_3\beta(q), r\right)$, then by Lemma 1

$$\begin{aligned}
\left|\tau_n{}^{(6)}\left(x\right)\right| &= \left|\sum_{j=0}^{\infty} \sum_{k=2^j(n+M+1)}^{2^{j+1}(n+M+1)-1} b_k \sin(ckx)\right| \\
&\le \sum_{j=0}^{\infty} \left| \frac{-1}{2\sin\left(crx/2\right)} \left\{ \sum_{k=2^j(n+M+1)}^{2^{j+1}(n+M+1)-1} \left(b_k - b_{k+r}\right) \cos\left(\left(k+\frac{r}{2}\right)cx\right) \right.\right. \\
&\quad \left.\left. + \sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1} b_k \cos\left(k-\frac{r}{2}\right)cx - \sum_{k=2^j(n+M+1)}^{2^j(n+M+1)+r-1} b_k \cos\left(\left(k-\frac{r}{2}\right)cx\right) \right\}\right| \\
&\le \frac{1}{2\left|\sin\left(crx/2\right)\right|} \sum_{j=0}^{\infty} \left\{ \sum_{k=2^j(n+M+1)}^{2^{j+1}(n+M+1)-1} \left|b_k - b_{k+r}\right| + \sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1} \left|b_k\right| \right. \\
&\quad \left. + \sum_{k=2^j(n+M+1)}^{2^j(n+M+1)+r-1} \left|b_k\right| \right\}.
\end{aligned}$$

Next, applying the Hölder inequality with $p > 1$, then using Lemma 1, the inequality

$$2\left(l+1\right) - \frac{rc}{\pi}x \le \left|\sin\frac{rcx}{2}\right| \left(x \in \left[\frac{2\,l\pi}{rc} + \frac{\pi}{rc}, \frac{2(l+1)\pi}{rc}\right] \text{ and } 0 < 2\left(l+1\right) \le r\right),$$

(4.6) and (4.1), we get

$$\begin{aligned}
\left|\tau_n^{(6)}\left(x\right)\right| &\le \frac{1}{2(l+1) - \frac{rc}{\pi}x} \sum_{j=0}^{\infty} \left\{ \left(\sum_{k=2^j(n+M+1)}^{2^{j+1}(n+M+1)-1} \left|b_k - b_{k+r}\right|^p\right)^{\frac{1}{p}} \right. \\
&\quad \left. \left(\sum_{k=2^j(n+M+1)}^{2^{j+1}(n+M+1)-1} 1\right)^{1-\frac{1}{p}} + \sum_{k=2^j(n+M+1)}^{2^j(n+M+1)+r-1} \left|b_k\right| + \sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1} \left|b_k\right| \right\} \\
&\le \frac{(M+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \left\{ \frac{C\left(2^j(n+M+1)\right)^{1-\frac{1}{p}}}{2^j(n+M+1)} \sup_{m \ge b(2^j(n+M+1))} m \left(\frac{1}{m}\sum_{k=m}^{2m-1} \left|b_k\right|^q\right)^{\frac{1}{q}} \right. \\
&\quad \left. + \sum_{k=2^j(n+M+1)}^{2^j(n+M+1)+r-1} \left|b_k\right| + \sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1} \left|b_k\right| \right\} = \frac{(M+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \\
&\quad \left\{ \frac{C}{\left(2^j(n+M+1)\right)^{\frac{1}{p}}} \sup_{m \ge b(2^j(n+M+1))} m^{1-\frac{1}{q}} \left(\sum_{k=m}^{2m-1} \left(\frac{k\ln k\left|b_k\right|}{k\ln k}\right)^q\right)^{\frac{1}{q}} \right. \\
&\quad \left. + \sum_{k=2^j(n+M+1)}^{2^j(n+M+1)+r-1} \frac{k\ln k\left|b_k\right|}{k\ln k} + \sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1} \frac{k\ln k\left|b_k\right|}{k\ln k} \right\} \\
&< \frac{\varepsilon(M+1)^{\frac{1}{p}}}{r} \sum_{j=0}^{\infty} \left\{ \frac{C}{\left(2^j(n+M+1)\right)^{\frac{1}{p}}} \sup_{m \ge b(2^j(n+M+1))} m^{1-\frac{1}{q}} \right.
\end{aligned}$$

$$\left(\sum_{k=m}^{2m-1}\left(\frac{1}{k}\right)^{q}\right)^{\frac{1}{q}}+\sum_{k=2^{j}(n+M+1)}^{2^{j}(n+M+1)+r-1}\frac{1}{k}+\sum_{k=2^{j+1}(n+M+1)}^{2^{j+1}(n+M+1)+r-1}\frac{1}{k}\Bigg\}.$$

Elementary calculations give

$$\begin{aligned}\left|\tau_n^{(6)}(x)\right| &< \frac{\varepsilon(M+1)^{\frac{1}{p}}}{r}\sum_{j=0}^{\infty}\Bigg\{\frac{C}{\left(2^{j}(n+M+1)\right)^{\frac{1}{p}}}\sup_{m\geq b(2^{j}(n+M+1))}\\ &\quad \left(m^{1-\frac{1}{q}}m^{-1}m^{\frac{1}{q}}\right)+r\left(2^{j}(n+M+1)\right)^{-\frac{1}{p}}+\frac{1}{2}r\left(2^{j}(n+M+1)\right)^{-\frac{1}{p}}\Bigg\}\\ &\leq \frac{\varepsilon(M+1)^{\frac{1}{p}}}{r}\sum_{j=0}^{\infty}\left\{\left(2^{j}(n+M+1)\right)^{-\frac{1}{p}}\left(C+\frac{3}{2}r\right)\right\}\\ &\leq \frac{\varepsilon(M+1)^{\frac{1}{p}}\left(C+\frac{3}{2}r\right)}{r\left((n+M+1)\right)^{\frac{1}{p}}}\sum_{j=0}^{\infty}\left(\frac{1}{2^{\frac{1}{p}}}\right)^{j}\leq\frac{\varepsilon\left(C+\frac{3}{2}r\right)}{r}\frac{1}{1-2^{-\frac{1}{p}}}.\end{aligned}$$

Joining the obtained estimates the uniform convergence of series (1.2) follows and thus the proof is complete. □

### 4.2. Proof of Proposition 1

Let for $n\in\mathbb{N}$:

$$a_n=\begin{cases}\frac{3}{n\ln(n+1)}, & \text{when } n=1 \pmod 3,\\ \frac{1}{n\ln(n+1)}, & \text{when } n=2 \pmod 3,\\ \frac{1}{n\ln(n+1)}, & \text{when } n=0 \pmod 3 \text{ and } n\neq 0 \pmod 6,\\ \frac{1}{(n-3)\ln(n-2)}+\frac{1}{n^{1+\frac{1}{p}}\ln(n+1)}, & \text{when } n=0 \pmod 6.\end{cases}$$

First, we prove that $(a_n)\in GM(p,{}_3\beta(1),3)$ for $p>1$. Let

$$\begin{aligned}A_n&=\{k\in\mathbb{N}: n\leq k\leq 2n-1 \text{ and } k=1 \pmod 3\},\\ B_n&=\{k\in\mathbb{N}: n\leq k\leq 2n-1 \text{ and } k=2 \pmod 3\},\\ C_n&=\{k\in\mathbb{N}: n\leq k\leq 2n-1 \text{ and } k=0 \pmod 3 \text{ and } k\neq 0 \pmod 6\},\\ D_n&=\{k\in\mathbb{N}: n\leq k\leq 2n-1 \text{ and } k=0 \pmod 6\}.\end{aligned}$$

Using elementary calculations we get

$$\left\{\sum_{k=n}^{2n-1}|a_k-a_{k+3}|^p\right\}^{\frac{1}{p}}=\left\{\sum_{k\in A_n}|a_k-a_{k+3}|^p+\sum_{k\in B_n}|a_k-a_{k+3}|^p\right.$$

$$\left.+\sum_{k\in C_n}|a_k-a_{k+3}|^p+\sum_{k\in D_n}|a_k-a_{k+3}|^p\right\}^{\frac{1}{p}}=\left\{\sum_{k\in A_n}\left|\frac{3}{k\ln(k+1)}\right.\right.$$

$$\left.-\frac{3}{(k+3)\ln(k+4)}\right|^p+\sum_{k\in B_n}\left|\frac{1}{k\ln(k+1)}-\frac{1}{(k+3)\ln(k+4)}\right|^p$$

$$+\sum_{k\in C_n}\left|\frac{1}{k\ln(k+1)}-\frac{1}{k\ln(k+1)}-\frac{1}{(k+3)^{1+\frac{1}{p}}\ln(k+4)}\right|^p$$

$$\left.+\sum_{k\in D_n}\left|\frac{1}{(k-3)\ln(k-2)}+\frac{1}{k^{1+\frac{1}{p}}\ln(k+1)}-\frac{1}{(k+3)\ln(k+4)}\right|^p\right\}^{\frac{1}{p}}$$

$$=\left\{\sum_{k\in A_n}3^p\left|\frac{1}{k\ln(k+1)}-\frac{1}{(k+3)\ln(k+4)}\right|^p+\sum_{k\in B_n}\left|\frac{1}{k\ln(k+1)}\right.\right.$$

$$\left.-\frac{1}{(k+3)\ln(k+4)}\right|^p+\sum_{k\in C_n}\left|\frac{1}{(k+3)^{1+\frac{1}{p}}\ln(k+4)}\right|^p$$

$$\left.+\sum_{k\in D_n}\left(\left|\frac{1}{(k-3)\ln(k-2)}-\frac{1}{(k+3)\ln(k+4)}+\frac{1}{k^{1+\frac{1}{p}}\ln(k+1)}\right|\right)^p\right\}^{\frac{1}{p}}.$$

Moreover

$$\left|\frac{1}{k\ln(k+1)}-\frac{1}{(k+3)\ln(k+4)}\right|=\frac{|(k+3)\ln(k+4)-k\ln(k+1)|}{k(k+3)\ln(k+1)\ln(k+4)}$$

$$\le\frac{\frac{3k}{k+1}+3\ln(k+4)}{k(k+3)\ln(k+1)\ln(k+4)}$$

$$\le\frac{6\ln(k+4)}{k^2\ln(k+1)\ln(k+4)}=\frac{6}{k^2\ln(k+1)}$$

for $k\ge 1$ and

$$\left|\frac{1}{(k-3)\ln(k-2)}-\frac{1}{(k+3)\ln(k+4)}\right|=\frac{|(k+3)\ln(k+4)-(k-3)\ln(k-2)|}{(k-3)(k+3)\ln(k-2)\ln(k+4)}$$

$$\le\frac{(k-3)\,|\ln(k+4)-\ln(k-2)|+6\ln(k+4)}{(k-3)(k+3)\ln(k-2)\ln(k+4)}$$

$$\le\frac{\frac{6(k-3)}{k-2}+6\ln(k+4)}{(k-3)(k+3)\ln(k-2)\ln(k+4)}$$

$$\le\frac{12}{(k-3)(k+3)\ln(k-2)}\le\frac{48}{k^2\ln(k+1)}$$

for $k \geq 6$. Thus

$$\left\{ \sum_{k=n}^{2n-1} |a_k - a_{k+3}|^p \right\}^{\frac{1}{p}} \leq \left\{ \sum_{k \in A_n} 3^p \left( \frac{6}{k^2 \ln(k+1)} \right)^p + \sum_{k \in B_n} \left( \frac{6}{k^2 \ln(k+1)} \right)^p \right.$$
$$\left. + \sum_{k \in C_n} \left( \frac{1}{k^{1+\frac{1}{p}} \ln(k+1)} \right)^p + \sum_{k \in D_n} \left( \frac{48}{k^2 \ln(k+1)} + \frac{1}{k^{1+\frac{1}{p}} \ln(k+1)} \right)^p \right\}^{\frac{1}{p}}$$
$$\leq 49 \left\{ \sum_{k=n}^{2n-1} \left( \frac{1}{k^{1+\frac{1}{p}} \ln(k+1)} \right)^p \right\}^{\frac{1}{p}} \leq 49 \frac{1}{n^{1+\frac{1}{p}} \ln(n+1)} n^{\frac{1}{p}}$$
$$= 49 \frac{1}{n \ln(n+1)} \leq 147 \frac{1}{n} \sum_{k=n}^{2n} |a_k| \leq 147 \frac{1}{n} \sup_{m \geq b(n)} \sum_{k=m}^{2m} |a_k| .$$

Hence $(a_n) \in GM(p, {}_3\beta(1), 3)$. Now, we will show that $(a_n) \notin GM(1, {}_3\beta(1), 3)$. We have

$$\sum_{k=n}^{2n-1} |a_k - a_{k+3}| \geq \sum_{k \in C_n} |a_k - a_{k+3}| = \sum_{k \in C_n} \left| \frac{1}{k \ln(k+1)} - \frac{1}{k \ln(k+1)} \right.$$
$$\left. - \frac{1}{(k+3)^{1+\frac{1}{p}} \ln(k+4)} \right| = \sum_{k \in C_n} \frac{1}{(k+3)^{1+\frac{1}{p}} \ln(k+4)}$$
$$\geq \frac{1}{(n+3)^{1+\frac{1}{p}} \ln(n+4)} \frac{n}{12} \geq \frac{1}{48(n+3)^{\frac{1}{p}} \ln(n+4)}.$$

On the other hand, we get

$$\frac{1}{n} \sup_{m \geq b(n)} \sum_{k=m}^{2m} |a_k| \leq C \frac{1}{n}.$$

Therefore, the inequality

$$\sum_{k=n}^{2n} |a_k - a_{k+3}| \leq C \frac{1}{n} \sup_{m \geq b(n)} \sum_{k=m}^{2m} |a_k| .$$

can not be satisfied if $n \to \infty$.

Now, we will show that the series (1.2) is divergent in $x_0 = \frac{2}{3}\pi$. We have

$$\sum_{k=6}^{6N+5} a_k \sin(kx_0) = \sum_{k=1}^{N}\sum_{l=0}^{5} a_{6k+l} \sin\left((6k+l)\frac{2}{3}\pi\right) = \sum_{k=1}^{N} \left(a_{6k} \sin(4\pi)\right.$$

$$+ a_{6k+1} \sin\left((6k+1)\frac{2}{3}\pi\right) + a_{6k+2} \sin\left((6k+2)\frac{2}{3}\pi\right) + a_{6k+3} \sin\left((6k+3)\frac{2}{3}\pi\right)$$

$$\left.+a_{6k+4} \sin\left((6k+4)\frac{2}{3}\pi\right) + a_{6k+5} \sin\left((6k+5)\frac{2}{3}\pi\right)\right)$$

$$= \sum_{k=1}^{N} \left(a_{6k+1} \sin\left(\frac{2}{3}\pi\right) + a_{6k+2} \left(-\sin\left(\frac{2}{3}\pi\right)\right) + a_{6k+4} \sin\left(\frac{2}{3}\pi\right)\right.$$

$$\left.+a_{6k+5} \left(-\sin\left(\frac{2}{3}\pi\right)\right)\right) = \sin\left(\frac{2}{3}\pi\right) \sum_{k=1}^{N} \left[(a_{6k+1} - a_{6k+2}) + (a_{6k+4} - a_{6k+5})\right]$$

$$= \sin\left(\frac{2}{3}\pi\right) \sum_{k=1}^{N} \left[\left(\frac{3}{(6k+1)\ln(6k+2)} - \frac{1}{(6k+2)\ln(6k+3)}\right)\right.$$

$$\left.+ \left(\frac{3}{(6k+4)\ln(6k+5)} - \frac{1}{(6k+5)\ln(6k+6)}\right)\right]$$

$$\geq \sin\left(\frac{2}{3}\pi\right) \sum_{k=1}^{N} \left[\left(\frac{3}{(6k+2)\ln(6k+2)} - \frac{1}{(6k+2)\ln(6k+2)}\right)\right.$$

$$\left.+ \left(\frac{3}{(6k+5)\ln(6k+5)} - \frac{1}{(6k+5)\ln(6k+5)}\right)\right]$$

$$= \sin\left(\frac{2}{3}\pi\right) \sum_{k=1}^{N} \left(\frac{2}{(6k+2)\ln(6k+2)} + \frac{2}{(6k+5)\ln(6k+5)}\right)$$

$$\geq 4\sin\left(\frac{2}{3}\pi\right) \sum_{k=1}^{N} \frac{1}{(6k+5)\ln(6k+5)} \to \infty \text{ as } N \to \infty.$$

This ends our proof. □

### 4.3. Proof of Theorem 11

The proof is similar to the proof of Theorem 10. The only difference is that we use (2.3) and (3.2) instead of (2.2) and (3.1), respectively. □

### 4.4. Proof of Theorem 12

The proof is similar to the proof of Theorem 10. The only difference is that we use (2.4) and Lemma 2 instead of (3.1) and Lemma 1, respectively. □

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**Declarations**

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