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**Results in Mathematics**

# Properties of $K$-Additive Set-Valued Maps

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**Abstract.** For monoids $X, Y$ and a submonoid $K \subset Y$ we define a $K$-additive set-valued map $F\colon X \to 2^Y$ as a map which is additive "modulo $K$". In the paper fundamental properties of $K$-additive set-valued maps are studied. Among others, we prove that in the class of $K$-additive set-valued maps $K$-lower (or weakly $K$-upper) boundedness on a "large" set implies $K$-continuity on the domain, as well as $K$-continuity implies $K$-homogeneity. We also study an algebraic structure of the $K$-homogeneity set for $K$-additive set-valued maps.

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## 1. Introduction

In the paper [7] the notions of $K$-subadditive set-valued maps (shortly called s.v. maps) and $K$-superadditive s.v. maps have been introduced, which generalize the well known notions of subadditive and superadditive real functions.

**Definition 1.** Let $X, Y$ be commutative monoids and $K \subset Y$ be a submonoid.[1] Denote by $n(Y)$ the family of all nonempty subsets of $Y$. A set-valued map $F\colon X \to n(Y)$ is called *$K$-subadditive*, if

$$F(x) + F(y) \subset F(x + y) + K, \quad x, y \in X, \tag{1}$$

and *$K$-superadditve*, if

$$F(x + y) \subset F(x) + F(y) + K, \quad x, y \in X. \tag{2}$$

[1] A monoid $M$ is a semigroup with a neutral element. A submonoid of a monoid $M$ is a subsemigroup of $M$ with the same neutral element as in $M$.

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Here, we would like to introduce the notion of $K$-additivity for s.v. maps in such a way to generalize the notion of additivity of real functions.

Since additive real functions can be characterized as functions which are simultaneously subadditive and superadditive, the natural definition of $K$-additivity is the following one.

**Definition 2.** Let $X, Y$ be commutative monoids and $K \subset Y$ be a submonoid. A s.v. map $F\colon X \to n(Y)$ is called $K$-*additive*, if it is simultaneously $K$-subadditive and $K$-superadditive.

In the case $K = \{0\}$ the notion of $K$-additivity coincides with the definition of additivity of s.v. maps introduced by Nikodem in [10].

If $K = [0, \infty)$, $Y = \mathbb{R}$, and $F$ is additionally single-valued, $K$-additivity of $F$ means classical additivity of the real function $F$.

The following properties of real additive functions defined on a real normed space $X$ seem to be well known (see e.g. [9, Theorems 5.2.1, 5.4.1, 5.4.2, 9.3.1, 9.3.2, 13.2.1, Lemma 13.2.3.]):

(i) each additive function satisfies Jensen's equation, i.e.

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \quad x, y \in X, \tag{3}$$

(ii) each function satisfying Jensen's equation and condition $f(0) = 0$ is additive,

(iii) each additive function bounded above (or below) on a "large" set (i.e. non-meager with the Baire property or of the positive Lebesgue measure) has to be continuous,

(iv) each continuous additive function is linear,

(v) if $X$ is additionally finite dimensional, each linear functional is continuous,

(vi) the set $H_f := \{t \in \mathbb{R}\colon\ f(tx) = tf(x) \text{ for all } x \in X\}$ is a field (called *the homogeneity field of* $f$),

(vii) for every field $L \subset \mathbb{R}$ there is an additive function $f\colon X \to \mathbb{R}$ such that $H_f = L$.

The aim of the paper is to show properties of $K$-additive s.v. maps which are in some sense analogous to those mentioned above, and even are far-reaching generalizations of them.

At the beginning of the paper (in the Sect. 2) we show some examples and basic properties of $K$-additive s.v. maps. Next, in the Sect. 3, we prove that every $K$-additive s.v. map is $K$-Jensen. Moreover, we check that the converse implication generally does not hold, however under some additional assumptions we can get $K$-additivity of a $K$-Jensen s.v. map. In the Sect. 4 we prove that in the class of $K$-additive s.v. maps weak $K$-upper boundedness as well as $K$-lower boundedness on a "large" set imply $K$-continuity on the whole domain and, moreover, $K$-continuity implies $K$-homogeneity. Finally, we show that under some additional assumptions $K$-homogeneity implies $K$-continuity of a $K$-additive s.v. map. At the end of the paper, in the Sect. 5, we study an algebraic structure of the $K$-homogeneity set of a $K$-additive s.v. map.

All necessary notions such as: $K$-Jensen s.v. map, $K$-upper/$K$-lower boundedness, $K$-continuity and $K$-homogeneity, we explain in relevant sections for the convenience of the reader.

## 2. Basic Properties of $K$-Additive s.v. Maps

Lets start with some examples and basic properties of $K$-additive s.v. maps.

*Example 1.* Let $X$ be a submonoid of $\big([0,\infty),+\big)$, $Y$ be a real vector space and $A$ be a nonempty convex subset of $Y$. Then

$$F_A(x) := xA, \quad x \in X,$$

is $\{0\}$-additive, because (e.g. in view of [10, Lemma 1.1])

$$F(x+y) = (x+y)A = xA + yA = F(x) + F(y), \quad x, y \in X.$$

*Example 2.* Let $X$ be a commutative monoid, $Y$ be a real vector space and $K \subset Y$ be a convex cone (i.e. $K + K \subset K$ and $tK \subset K$ for $t \geq 0$). Fix $t \geq 0$ and define

$$(tF)(x) := tF(x), \quad x \in X.$$

Since $tA + tB = t(A+B)$ for $A, B \subset Y$ (see e.g. [10, Lemma 1.1]), if $F$ is $K$-additive, then $tF$ is also $K$-additive.

**Lemma 1.** *Let $X, Y$ be commutative monoids and $K \subset Y$ be a submonoid. If $F, G\colon X \to n(Y)$ are $K$-additive, then*

$$(F+G)(x) := F(x) + G(x), \qquad x \in X,$$

*is also $K$-additive. In particular, for every $A \in n(Y)$ satisfying $0 \in A \subset K$,*

$$(F+A)(x) := F(x) + A, \qquad x \in X,$$

*is $K$-additive, too.*

The proof of the above lemma is obvious.

**Lemma 2.** *Let $X, Y, Z$ be commutative monoids and $K \subset Y$, $L \subset Z$ be submonoids. If $F\colon X \to n(Y)$ is $K$-additive and $G\colon X \to n(Z)$ is $L$-additive, then*

$$(F \times G)(x) := F(x) \times G(x), \quad x \in X,$$

*is $K \times L$-additive.*

*Proof.* For every $x, y \in X$ we get

$$\begin{aligned}\big(F(x)\times G(x)\big)+\big(F(y)\times G(y)\big) &= \big(F(x)+F(y)\big)\times\big(G(x)+G(y)\big)\\ &\subset \big(F(x+y)+K\big)\times\big(G(x+y)+L\big)\\ &= \big(F(x+y)\times G(x+y)\big)+(K\times L),\\ F(x+y)\times G(x+y) &\subset \big(F(x)+F(y)+K\big)\times\big(G(x)+G(y)+L\big)\\ &= \big(F(x)\times G(x)\big)+\big(F(y)\times G(y)\big)+(K\times L),\end{aligned}$$

which ends the proof. □

**Lemma 3.** *Let $X$ be a commutative monoid, $Y$ be a real topological vector space and $K$ be a submonoid of $(Y,+)$. If $F\colon X\to n(Y)$ is $K$-additive and sets $F(x)$ are relatively compact for $x\in X$, then*

$$(\operatorname{cl} F)(x) := \operatorname{cl} F(x), \quad x\in X,$$

*is* $\operatorname{cl} K$*-additive.*

*Proof.* Assume that $F$ is $K$-additive. Since $\operatorname{cl}(A+B)=\operatorname{cl} A+\operatorname{cl} B$ for $A,B\subset Y$ such that the set $\operatorname{cl} A+\operatorname{cl} B$ is closed (see [10, Lemma 1.9]), for every $x,y\in X$ we get

$$\operatorname{cl} F(x)+\operatorname{cl} F(y)=\operatorname{cl}\big(F(x)+F(y)\big)\subset\operatorname{cl}\big(F(x+y)+K\big)=\operatorname{cl} F(x+y)+\operatorname{cl} K,$$

and

$$\begin{aligned}\operatorname{cl} F(x+y) &\subset \operatorname{cl}\big(F(x)+F(y)+K\big)=\operatorname{cl}\big(F(x)+F(y)\big)+\operatorname{cl} K\\ &= \operatorname{cl} F(x)+\operatorname{cl} F(y)+\operatorname{cl} K,\end{aligned}$$

which proves $\operatorname{cl} K$-additivity of $\operatorname{cl} F$. □

**Lemma 4.** *Let $X$ be a commutative monoid, $Y$ be a real topological vector space and $K$ be a convex cone in $Y$. If $F\colon X\to n(Y)$ is $K$-additive and $F(x)$ are convex sets with non-empty interiors for $x\in X$, then*

$$(\operatorname{int} F)(x) := \operatorname{int} F(x), \quad x\in X,$$

*is also $K$-additive.*

*Proof.* Assume that $F$ is $K$-additive. Since $\operatorname{int}(A+B)=\operatorname{int} A+B$ and $\operatorname{int}(A+C)=\operatorname{int} A+\operatorname{int} C$ for convex sets $A,B,C\subset Y$ such that $\operatorname{int} A\neq\emptyset$ and $\operatorname{int} C\neq\emptyset$ (see [10, Lemma 1.11]), for every $x,y\in X$ we get

$$\begin{aligned}\operatorname{int} F(x)+\operatorname{int} F(y) &= \operatorname{int}\big(F(x)+F(y)\big)\subset\operatorname{int}\big(F(x+y)+K\big)=\operatorname{int} F(x+y)+K,\\ \operatorname{int} F(x+y) &\subset \operatorname{int}\big(F(x)+F(y)+K\big)=\operatorname{int}\big(F(x)+F(y)\big)+K\\ &= \operatorname{int} F(x)+\operatorname{int} F(y)+K,\end{aligned}$$

which proves $K$-additivity of $\operatorname{int} F$. □

**Lemma 5.** *Let $X$ be a commutative monoid, $Y$ be a real vector space and $K \subset Y$ be a convex cone satisfying $K \cap (-K) = \{0\}$. Assume that $z_0 \in K \setminus \{0\}$ and $F\colon X \to n(Y)$ is the s.v. map given by*

$$F(x) = [m(x), M(x)]z_0, \quad x \in X,$$

*with $m, M\colon X \to \mathbb{R}$ satisfying $m(x) \leq M(x)$ for $x \in X$. Then $F$ is $K$-additive if and only if $m$ is additive.*

*Proof.* First assume that $m$ is additive. Then

$$\begin{aligned} F(x+y) &= [m(x+y), M(x+y)]z_0 \subset [m(x)+m(y), \infty)z_0 \\ &= [m(x), M(x)]z_0 + [m(y), M(y)]z_0 + [0,\infty)z_0 \subset F(x) + F(y) + K \end{aligned}$$

and

$$\begin{aligned} F(x) + F(y) &= [m(x)+m(y), M(x)+M(y)]z_0 \subset [m(x+y), \infty)z_0 \\ &\subset [m(x+y), M(x+y)]z_0 + [0,\infty)z_0 \subset F(x+y) + K \end{aligned}$$

for every $x, y \in X$, which means that $F$ is $K$-additive.

Now, assume that $F$ is $K$-additive. Then, for every $x, y \in X$,

$$m(x+y)z_0 \in F(x+y) \subset F(x) + F(y) + K = [m(x)+m(y), M(x)+M(y)]z_0 + K$$

and

$$\big(m(x)+m(y)\big)z_0 \in F(x) + F(y) \subset F(x+y) + K = [m(x+y), M(x+y)]z_0 + K.$$

Hence, for $x, y \in X$,

$$\big(m(x+y) - \alpha\big)z_0 \in K, \qquad \big(m(x)+m(y) - \beta\big)z_0 \in K,$$

with some $\alpha \in [m(x)+m(y), M(x)+M(y)]$ and $\beta \in [m(x+y), M(x+y)]$. Since $z_0 \in K \setminus \{0\}$ and $K \cap (-K) = \{0\}$,

$$m(x+y) \geq \alpha \geq m(x) + m(y), \qquad m(x) + m(y) \geq \beta \geq m(x+y)$$

for $x, y \in X$, which proves additivity of $m$. $\square$

From the above lemma we can easy derive the following useful corollary.

**Corollary 6.** *Let $X$ be a commutative monoid, $K = [0, \infty)$ and*

$$F(x) = [m(x), M(x)], \qquad x \in X,$$

*where $m, M\colon X \to \mathbb{R}$ satisfy $m(x) \leq M(x)$ for $x \in X$. Then $K$-additivity of $F$ is equivalent to additivity of $m$.*

Let us recall that a subset $C$ of a uniquely 2-divisible commutative monoid $Y$[2] is called mid-convex, if $\frac{1}{2}C + \frac{1}{2}C \subset C$. It is well known (see e.g. [9, Lemma

[2] The monoid is called uniquely 2-divisible, if for every $y \in Y$ there is a unique $z \in Y$ such that $z + z = y$.

5.1.1]) that mid-convexity is equivalent to $\mathbb{D}$-convexity, i.e. $dC + (1-d)C \subset C$ for any $d \in \mathbb{D} \cap [0,1]$, where $\mathbb{D}$ is the set of dyadic numbers,

$$\mathbb{D} = \left\{ \frac{k}{2^n} : \ k \in \mathbb{Z}, \ n \in \mathbb{N} \cup \{0\} \right\}.$$

Denote by $\mathbb{D}(A)$ $\mathbb{D}$-convex hull of a subset $A$ of a uniquely 2-divisible commutative monoid $Y$ (i.e. $\mathbb{D}(A)$ is the smallest $\mathbb{D}$-convex set containing $A$). Since $\mathbb{D}(A+B) = \mathbb{D}(A) + \mathbb{D}(B)$ for any $A, B \subset Y$, hence we can obtain the following lemma.

**Lemma 7.** *Let $X$ be a commutative monoid, $Y$ be a uniquely 2-divisible commutative monoid and $K \subset Y$ be a uniquely 2-divisible submonoid. If $F\colon X \to n(Y)$ is a $K$-additive s.v. map, then*

$$\mathbb{D}F(x) := \mathbb{D}(F(x)), \quad x \in X,$$

*is $K$-additive.*

*Proof.* By $K$-additivity, for $x, y \in X$ we obtain

$$\mathbb{D}(F(x+y)) \subset \mathbb{D}(F(x) + F(y) + K) = \mathbb{D}(F(x)) + \mathbb{D}(F(y)) + \mathbb{D}(K)$$

and

$$\mathbb{D}(F(x)) + \mathbb{D}(F(y)) = \mathbb{D}(F(x) + F(y)) \subset \mathbb{D}(F(x+y) + K) = \mathbb{D}(F(x+y)) + \mathbb{D}(K).$$

But $K$ is a uniquely 2-divisible submonoid, so it is mid-convex and hence $\mathbb{D}$-convex. Consequently, $\mathbb{D}(K) = K$, which ends the proof. $\square$

Since $\operatorname{conv}(A+B) = \operatorname{conv} A + \operatorname{conv} B$ for any subsets $A, B$ of a real vector space $Y$, in the same way as Lemma 7 we can prove the next lemma.

**Lemma 8.** *Let $X$ be a commutative monoid, $Y$ be a real vector space and $K \subset Y$ be a convex cone. If $F\colon X \to n(Y)$ is a $K$-additive s.v. map, then*

$$\operatorname{conv} F(x) := \operatorname{conv}(F(x)), \quad x \in X,$$

*is $K$-additive.*

At the end of the section, let us introduce a relation $=_K$ in the family $n(Y)$ of all nonempty subsets of a monoid $Y$ with a given submonoid $K \subset Y$:

$$A =_K B \iff (A \subset B + K \ \ \wedge \ \ B \subset A + K)$$

for every $A, B \in n(Y)$.

First let us observe that

$$A =_K B \iff A + K = B + K$$

for $A, B \in n(Y)$. Indeed, if $A =_K B$, then $A + K \subset B + K + K \subset B + K$ and, analogously, $B + K \subset A + K + K \subset A + K$, which means that $A + K = B + K$. On the other hand, if $A + K = B + K$, then $A \subset A + K = B + K$ and $B \subset B + K = A + K$, so $A =_K B$.

**Lemma 9.** *If $Y$ is a commutative monoid and $K \subset Y$ is a submonoid, then $=_K$ is an equivalence relation in $n(Y)$ and for every $A, B, C, D \in n(Y)$ the following properties hold:*

(i) *if $A =_K B$ and $C =_K D$, then $A + C =_K B + D$,*
(ii) *if $0 \in C \subset K$ and $A =_K B + C$, then $A =_K B$,*
(iii) *if $A =_K B$, then $\frac{1}{2}A =_K \frac{1}{2}B$, provided $Y, K$ are uniquely 2-divisible,*
(iv) *if $A =_K B$, then $tA =_K tB$ for every $t > 0$, provided $Y$ is a real vector space and $K$ is a convex cone in $Y$.*

*Proof.* Let $A, B, C, D \in n(Y)$. Reflexivity and symmetry of the relation $=_K$ is obvious. We show that this relation is transitive.

If $A =_K B$ and $B =_K C$, then

$$A \subset B + K, \quad B \subset A + K, \quad B \subset C + K, \quad C \subset B + K,$$

and hence

$$\begin{gathered} A \subset B + K \subset C + K + K \subset C + K, \\ C \subset B + K \subset A + K + K \subset A + K, \end{gathered}$$

which means that $A =_K C$. Consequently, $=_K$ is an equivalence relation.

(i) If $A =_K B$ and $C =_K D$, then

$$A \subset B + K, \quad B \subset A + K, \quad C \subset D + K, \quad D \subset C + K,$$

and hence

$$\begin{gathered} A + C \subset B + D + K + K \subset B + D + K, \\ B + D \subset A + C + K + K \subset A + C + K, \end{gathered}$$

which means that $A + C =_K B + D$.

(ii) Let $0 \in C \subset K$ and $A =_K B + C$. Then

$$\begin{gathered} A \subset B + C + K \subset B + K + K \subset B + K, \\ B \subset B + C \subset A + K, \end{gathered}$$

which means that $A =_K B$.

(iii) Now, assume that $Y$ and $K$ are uniquely 2-divisible. If $A =_K B$, then

$$A \subset B + K, \quad B \subset A + K, \tag{4}$$

and hence

$$\frac{1}{2}A \subset \frac{1}{2}B + \frac{1}{2}K \subset \frac{1}{2}B + K, \quad \frac{1}{2}B \subset \frac{1}{2}A + \frac{1}{2}K \subset \frac{1}{2}A + K,$$

so $\frac{1}{2}A =_K \frac{1}{2}B$.

(iv) Finally, assume that $Y$ is a real vector space and $K$ is a convex cone in $Y$. If $A =_K B$ and $t > 0$, then (4) holds, and hence

$$tA \subset tB + tK \subset tB + K, \quad tB \subset tA + tK \subset tA + K,$$

so $tA =_K tB$. $\square$

Now, for commutative monoids $X, Y$ and a submonoid $K \subset Y$, we can easy write that a s.v. map $F\colon X \to n(Y)$ is $K$-additive, if

$$F(x+y) =_K F(x) + F(y), \quad x, y \in X.$$

We will use this clear notation during the whole paper.

## 3. Connection Between $K$-Additivity and $K$-Jensen s.v. Maps

The next definition generalizes the notion of a Jensen s.v. map which has been introduced in [10].

**Definition 3.** Let $X, Y$ be uniquely 2-divisible commutative monoids and $K \subset Y$ be a submonoid. A s.v. map $F\colon X \to n(Y)$ is called $K$-*Jensen* if it is simultaneously $K$-midconvex and $K$-midconcave, i.e.

$$F\left(\frac{x+y}{2}\right) =_K \frac{1}{2}\big(F(x) + F(y)\big), \quad x, y, \in X.$$

*Example 3.* If $X$ is a uniquely 2-divisible commutative monoid, $K = [0, \infty)$ and $F(x) = [m(x), M(x)]$, where $m, M\colon X \to \mathbb{R}$ satisfy $m(x) \leq M(x)$ for $x \in X$, then $F$ is $K$-Jensen if and only if $m$ satisfies Jensen's equation (3).

**Lemma 10.** *Let $X, Y$ be uniquely 2-divisible commutative monoids and $K \subset Y$ be a uniquely 2-divisible submonoid. If $F\colon X \to n(Y)$ is a $K$-additive mid-convex-valued map, then it is $K$-Jensen.*

*Proof.* For every $x, y \in X$, by $K$-additivity of $F$, mid-convexity of $F\left(\frac{x+y}{2}\right)$ and transitivity of $=_K$, we get

$$2F\left(\frac{x+y}{2}\right) = F\left(\frac{x+y}{2}\right) + F\left(\frac{x+y}{2}\right) =_K F(x+y) =_K F(x) + F(y).$$

To end the proof it is enough to apply Lemma 9 (iii). □

It is clear that there are s.v. maps which are $K$-Jensen but not $K$-additive; it is enough to choose $K = [0, \infty)$ and $F\colon \mathbb{R} \to n(\mathbb{R})$ given by $F(x) = [x+1, x+2]$ for $x \in \mathbb{R}$.

However, under some additional assumptions, $K$-Jensen s.v. maps have to be $K$-additive.

**Theorem 11.** *Let $X, Y$ be uniquely 2-divisible commutative monoids and $K \subset Y$ be a submonoid. If $F\colon X \to n(Y)$ is a $K$-Jensen mid-convex-valued map such that $0 \in F(0) \subset K$, then it is $K$-additive.*

*Proof.* According to Lemma 9 (i),

$$\begin{aligned} F(x) + F(y) &= F\left(\frac{2x+0}{2}\right) + F\left(\frac{2y+0}{2}\right) \\ &=_K \frac{1}{2}\big(F(2x) + F(0)\big) + \frac{1}{2}\big(F(2y) + F(0)\big) = \frac{F(2x) + F(2y)}{2} + F(0) \\ &=_K F\left(\frac{2x+2y}{2}\right) + F(0) = F(x+y) + F(0) \end{aligned}$$

for any $x, y \in X$. Hence, since $0 \in F(0) \subset K$, by Lemma 9 (ii) we get $F(x + y) =_K F(x) + F(y)$ for every $x, y \in X$. $\square$

The converse theorem to Theorem 11 holds under some additional assumptions on $Y, K$ and $F$.

**Theorem 12.** *Let $X$ be a uniquely 2-divisible commutative monoid, $Y$ be a real vector metric space and $K$ be a closed convex cone in $Y$ such that $K \cap (-K) = \{0\}$. Let $F\colon X \to n(Y)$ be a s.v. map such that $F(x)$ are compact convex sets for $x \in X$. Then $F$ is $K$-additive if and only if $F$ is $K$-Jensen and $0 \in F(0) \subset K$.*

*Proof.* By Theorem 11 and Lemma 10 it is enough to show that if $F$ is $K$-additive, then $0 \in F(0) \subset K$.

In view of $K$-additivity,

$$2F(0) = F(0) + F(0) =_K F(0),$$

and hence, according to Lemma 9 (iii),

$$F(0) =_K \frac{1}{2}F(0),$$
$$F(0) =_K \frac{1}{2}F(0) =_K \frac{1}{4}F(0),$$

and, using induction, we get

$$F(0) =_K \frac{1}{2^n}F(0) \quad \text{for every } n \in \mathbb{N}.$$

In the first step we show that $F(0) \subset K$. So, take any $y \in F(0)$. Since

$$F(0) \subset \frac{1}{2^n}F(0) + K \quad \text{for } n \in \mathbb{N},$$

we can find sequences $(y_n)_{n\in\mathbb{N}} \subset F(0)$ and $(k_n)_{n\in\mathbb{N}} \subset K$ such that $y = \frac{y_n}{2^n} + k_n$. But $F(0)$ is compact, so there is a convergent subsequence $(y_{s_n})_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$. Thus

$$k_{s_n} = y - \frac{y_{s_n}}{2^{s_n}} \to y$$

and whence $y \in \operatorname{cl} K = K$.

Next, we prove that $0 \in F(0)$. Since

$$\frac{1}{2^n}F(0) \subset F(0) + K \quad \text{for } n \in \mathbb{N},$$

for fixed $y \in F(0)$ the sequence $\left\{\frac{y}{2^n}\right\}_{n\in\mathbb{N}}$ is contained in $F(0)+K$ and converges to 0. Hence $0 \in \operatorname{cl}(F(0) + K) \subset F(0) + K$. It means that there is some $y_0 \in F(0) \subset K$ such that $-y_0 \in K$. Hence $y_0 \in K \cap (-K) = \{0\}$, which ends the proof. $\square$

At the end of the section let us mention another well known property (see [9, Theorem 13.2.1]) that each real function satisfying Jensen's equality is a translation of an additive function by a constant. A similar result holds also for $\{0\}$–Jensen s.v. maps (see [10, Theorem 5.6]). Unfortunately, we are not able to answer the following question.

**Problem 1.** Let $X, Y, K$ be as in Theorem 12 and $F\colon X \to n(Y)$ be a convex-valued $K$-Jensen map. Are there a $K$-additive s.v. map $A\colon X \to n(Y)$ and a set $B \subset Y$ such that $F(x) =_K A(x) + B$ for $x \in X$?

## 4. $K$-Continuity of $K$-Additive s.v. Maps

From now on we will use the following notations for families of subsets of a real vector space $Y$:

- $\mathcal{B}(Y)$ – the family of all nonempty bounded subsets of $Y$,
- $\mathcal{BC}(Y)$ – the family of all nonempty bounded convex subsets of $Y$,
- $\mathcal{CC}(Y)$ – the family of all nonempty compact convex subsets of $Y$.

First, let us recall definitions of $K$-boundedness and $K$-continuity of s.v. maps from the paper [10].

**Definition 4.** Let $X, Y$ be real topological vector spaces and $K$ be a convex cone in $Y$. A s.v. map $F\colon X \to n(Y)$ is called:

- $K$-*upper bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
$$F(x) \subset B - K \quad \text{for all} \ \ x \in A;$$
- *weakly* $K$-*upper bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that
$$F(x) \cap (B - K) \neq \emptyset \quad \text{for all} \ \ x \in A;$$
- *[weakly]* $K$-*lower bounded* on a set $A \subset X$, if it is [weakly] $(-K)$-upper bounded on this set,
- $K$-*continuous at* $x_0 \in X$, if for every neighborhood $W \subset Y$ of $0$ there is a neighborhood $U$ of $x_0$ such that
$$F(x) \subset F(x_0) + W + K \quad \text{and} \quad F(x_0) \subset F(x) + W + K$$
for all $x \in U$.

*Remark 1.* Clearly, in the case when $K = \{0\}$, [weak] $K$-upper boundedness and [weak] $K$-lower boundedness are equivalent assumptions and $K$-continuity means continuity with respect to the Hausdorff topology on $n(Y)$.

*Remark 2.* For a real topological vector space $X$ and $K = [0, \infty)$ define
$$F(x) = [m(x), M(x)], \quad x \in X,$$
where $m, M\colon X \to \mathbb{R}$ satisfy $m(x) \leq M(x)$ for $x \in X$. It is very easy to show that:

(a) $F$ is $K$-lower (weakly $K$-upper) bounded on a set if and only if $m$ is bounded below (above, resp.) on this set,
(b) $F$ is $K$-continuous at a point if and only if $m$ is continuous at this point.

Let us also recall the notion of "smalness" introduced in [1,2] (see also the notion of a shift-compact set in [3]).

**Definition 5.** The set $A$ in a topological group $X$ is called *null-finite*, if there is a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to 0 in $X$ such that the set $\{n \in \mathbb{N} \colon x+x_n \in A\}$ is finite for every $x \in X$.

If $X$ is a complete metric group, then every Borel null-finite set $B \subset X$ is "small" in topological and measured senses, which means that it is:

- Haar-null, i.e. there exists a probability $\sigma$-additive Borel measure $\nu$ on $X$ such that $\nu(B+x) = 0$ for each $x \in X$ (see [4] and [2, Theorem 6.1]); consequently, $B$ has Haar measure zero provided $X$ is additionally locally compact,
- Haar-meager, and consequently meager, i.e. there exists a continuous function $f \colon 2^\omega \to X$ such that $f^{-1}(B+x)$ is meager for each $x \in X$ (see [5], [2, Theorem 5.1], [1, Proposition 5.1]).

In [6, Theorems 3.1 and 3.2] the authors proved that for real vector metric spaces $X, Y$ each s.v. map $F \colon X \to \mathcal{BC}(Y)$ which is $K$-midconvex and weakly $K$-upper bounded on a non-null finite set, or $K$-midconcave and $K$-lower bounded on a non-null finite set, has to be $K$-continuous. Hence, directly from Lemma 10, we can obtain the following important result, which generalizes [6, Theorem 3.4] and also [10, Theorem 5.1].

**Theorem 13.** *Let $X, Y$ be real vector metric spaces. Assume that $K$ is a convex cone in $Y$. If a s.v. map $F \colon X \to \mathcal{BC}(Y)$ is $K$-additive and satisfies one of the following conditions:*

- *$F$ is weakly $K$-upper bounded on a non-null finite set,*
- *$F$ is $K$-lower bounded on a non-null-finite set,*

*then $F$ is $K$-continuous on $X$.*

In the above theorem the assumption that $F$ is $K$-lower bounded on a non-null-finite set can not be replaced by a weaker one, i.e. by weak $K$-lower boundedness on such a set.

*Example 4.* Let $a \colon \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function, $K = [0, \infty)$ and $F \colon \mathbb{R} \to \mathcal{BC}(\mathbb{R})$ be given by

$$F(x) = [a(x), \max\{1, a(x)+1\}], \quad x \in \mathbb{R}.$$

In view of Corollary 6 and Remark 2b) such a map is $K$-additive, but is not $K$-continuous at any point of $\mathbb{R}$. Moreover, $F(x) \cap (\{0\} + K) \neq \emptyset$ for $x \in \mathbb{R}$, so $F$ is weakly $K$-lower bounded on the whole domain.

It is known that $K$-continuity at a point of a s.v. map implies weak $K$-upper boundedness, as well as $K$-lower boundedness, on a neighbourhood of this point (see e.g. [8, Section 3]). Hence, by Theorem 13, we can easy derive the following corollary.

**Corollary 14.** *Let $X, Y$ be real vector metric spaces and $K \subset Y$ be a convex cone. If a s.v. map $F\colon X \to \mathcal{BC}(Y)$ is $K$-additive and $K$-continuous at a point, then $F$ is $K$-continuous on $X$.*

Knowing that real continuous additive functions have to be linear, we would like to show some kind of $K$-homogeneity of $K$-continuous $K$-additive s.v. maps.

**Theorem 15.** *Let $X, Y$ be real vector metric spaces and $K$ be a closed convex cone in $Y$ such that $K \cap (-K) = \{0\}$. If $F\colon X \to \mathcal{CC}(Y)$ is a $K$-continuous and $K$-additive s.v. map, then it is $K$-homogeneous, i.e.*

$$F(tx) =_K tF(x), \quad x \in X,\ t \geq 0. \tag{5}$$

*Moreover, if $F$ is not single-valued,*

$$\{t \in \mathbb{R}\colon\ F(tx) =_K tF(x) \ \textit{for}\, x \in X\} = [0, \infty).$$

*Proof.* According to Theorem 12, $0 \in F(0) \subset K$ and $F$ is $K$-midconvex and $K$-midconcave. Thus, in view of [10, Theorems 3.1 and 4.1], $F$ is $K$-convex and $K$-concave, i.e.

$$tF(x) + (1-t)F(y) =_K F(tx + (1-t)y), \quad x, y \in X,\ t \in [0,1].$$

Hence, for $y = 0$,

$$tF(x) + (1-t)F(0) =_K F(tx), \quad x \in X,\ t \in [0,1].$$

Clearly, since $0 \in F(0) \subset K$, we get $0 \in (1-t)F(0) \subset (1-t)K \subset K$, and thus, in view of Lemma 9 (ii),

$$tF(x) =_K F(tx), \quad x \in X,\ t \in [0,1].$$

Now, we can complete the proof of (5) by induction. Fix $n \in \mathbb{N}$ and assume that $F(tx) =_K tF(x)$ for every $t \in [n, n+1]$ and $x \in X$. Take any $t \in [n+1, n+2]$. Then $t - 1 \in [n, n+1]$, so, by $K$-additivity,

$$F(tx) =_K F((t-1)x) + F(x) =_K (t-1)F(x) + F(x) = tF(x)$$

for every $x \in X$.

Finally, assume that $F$ is not single-valued. For the proof by contradiction suppose that for some $t > 0$

$$F(-tx) =_K -tF(x), \quad x \in X.$$

Then, by (5), $F(tx) =_K tF(x)$ and, in view of Theorem 12 and $K$-additivity of $F$,

$$\begin{aligned} t\big(F(x) - F(x)\big) &= tF(x) + (-t)F(x) \subset \big(F(tx) + K\big) + \big(F(-tx) + K\big) \\ &\subset F(tx - tx) + K = F(0) + K \subset K \end{aligned}$$

for $x \in X$. Hence

$$-\big(F(x) - F(x)\big) = F(x) - F(x) \subset \frac{1}{t}K \subset K,$$

and, consequently, $F(x) - F(x) \subset K \cap (-K) = \{0\}$ for $x \in X$, which means that all sets $F(x)$ are singletons. This contradiction ends the proof. $\square$

Theorem 15 with $K = \{0\}$ was proved by Nikodem (see [10, Theorem 5.3]).

Now, the question is what about the converse result? More precisely, is it true that every $K$-additive and $K$-homogeneous s.v map has to be $K$-continuous?

In the next example we show that generally it is not true.

*Example 5.* Let $K = [0, \infty)$, $X$ be an infinite dimensional real normed space, $m, M \colon X \to \mathbb{R}$, $m(x) \le M(x)$ for $x \in X$. If $m$ is a discontinuous linear functional, then $F(x) = [m(x), M(x)]$, $x \in X$, is a $K$-additive and $K$-homogeneous s.v. map, which is not $K$-continuous.

However, under some natural additional assumptions, we can obtain the converse result to Theorem 15.

**Theorem 16.** *Let $X$ be a finite dimensional real normed space, $Y$ be a real normed space and $K$ be a convex cone in $Y$. If $F \colon X \to \mathcal{CC}(Y)$ is a $K$-additive and $K$-homogeneous s.v. map, it is $K$-continuous on $X$.*

*Proof.* Let $K^*$ be the set of all continuous linear functionals on $Y$ which are non-negative on $K$. For every $y^* \in K^*$ define

$$f_{y^*}(x) = \inf y^*(F(x)), \quad x \in X.$$

First, observe that for every $y^* \in K^*$ and $A, B \in \mathcal{CC}(Y)$,

$$\text{if } A =_K B, \ \text{ then } \inf y^*(A) = \inf y^*(B). \tag{6}$$

Indeed, let $A =_K B$. Then

$$y^*(A) \subset y^*(B + K) \subset y^*(B) + [0, \infty),$$

and hence $\inf y^*(A) \ge \inf y^*(B)$. In view of symmetry of $=_K$, we get $\inf y^*(B) \ge \inf y^*(A)$. Thus condition (6) holds.

Now, fix $y^* \in K^*$. We prove that $f_{y^*}$ is a linear functional. Since $F$ is $K$-additive, $F(x_1 + x_2) =_K F(x_1) + F(x_2)$ for $x_1, x_2 \in X$, thus, by (6),

$$\begin{aligned} f_{y^*}(x_1) + f_{y^*}(x_2) &= \inf y^*(F(x_1)) + \inf y^*(F(x_2)) = \inf \big(y^*(F(x_1)) + y^*(F(x_2))\big) \\ &= \inf y^*(F(x_1) + F(x_2)) = \inf y^*(F(x_1 + x_2)) = f_{y^*}(x_1 + x_2) \end{aligned}$$

for all $x_1, x_2 \in X$, which proves additivity of $f_{y^*}$. Moreover, by $K$-homogeneity of $F$, $F(tx) =_K tF(x)$ for $t \ge 0$ and $x \in X$. Hence, in view of (6),

$$f_{y^*}(tx) = \inf y^*(F(tx)) = \inf y^*(tF(x)) = \inf t y^*(F(x)) = t f_{y^*}(x)$$

for $t \ge 0$ and $x \in X$. Since every additive real function is odd, $f_{y^*}$ is a linear functional.

In this way we obtained that every functional $f_{y^*}: X \to \mathbb{R}$ is linear, so it is continuous because $X$ is finite dimensional. But in view of [8, Theorem 4] $K$-subadditive s.v. map for which $f_{y^*}$ is continuous for every $y^* \in K^*$ has to be weakly $K$-upper bounded on an open set. Thus, according to Theorem 13, $F$ is $K$-continuous, which ends the proof. $\square$

## 5. Algebraic Structure of a $K$-Homogeneity Set

As we mentioned in the introduction, for every additive function $f: X \to \mathbb{R}$ defined on a real vector space $X$ the set

$$H_f := \{t \in \mathbb{R}: \ f(tx) = tf(x) \text{ for all } x \in X\}$$

is a field, called the *homogeneity field of* $f$. Moreover, the following result holds.

**Theorem 17** [9, Theorem 5.4.2]. *Let $X$ be a real vector space. For every field $L \subset \mathbb{R}$ there is an additive function $f: X \to \mathbb{R}$ such that $H_f = L$.*

Now, for real vector spaces $X, Y$, a s.v. map $F: X \to n(Y)$ and a closed convex cone $K \subset Y$ such that $K \cap (-K) = \{0\}$, we would like to define in the same way the $K$-*homogeneity set of* $F$,

$$H_{F,K} := \{t \in \mathbb{R}: F(tx) =_K tF(x) \text{ for all } x \in X\}.$$

First, let us prove basic properties of $H_{F,K}$.

**Theorem 18.** *Let $X$ be a real vector space, $Y$ be a real vector metric space and $K \subset Y$ be a closed convex cone such that $K \cap (-K) = \{0\}$. If a s.v. map $F: X \to \mathcal{CC}(Y)$ is $K$-additive and not single-valued, then the following conditions hold:*

(i) $\{0, 1\} \subset H_{F,K} \subset [0, \infty)$,
(ii) $s + t \in H_{F,K}$ *for* $s, t \in H_{F,K}$,
(iii) $\frac{s}{t} \in H_{F,K}$ *for* $s, t \in H_{F,K}$ *with* $t \neq 0$;

*i.e. $H_{F,K}$ is a submonoid of $\big([0, \infty), +\big)$ and $H_{F,K}\backslash\{0\}$ is a subgroup of $\big((0, \infty), \cdot\big)$.*

*Proof.* (i) Clearly, $1 \in H_{F,K}$ and, in view of Theorem 12, $0 \in F(0) \subset K$, so $0 \in H_{F,K}$. We have to prove yet that $H_{F,K} \subset [0, \infty)$.

For the proof by contradiction suppose that there is $t > 0$ such that $-t \in H_{F,K}$. Fix arbitrary $x \in X$. Notice that in view of $K$-additivity of $F$

$$F(x) + F(-x) \subset F(0) + K \subset K,$$

which easy implies

$$F(x) \subset -F(-x) + K. \tag{7}$$

Moreover, since $-t \in H_{F,K}$,

$$-tF(-x) \subset F(-t(-x)) + K = F(tx) + K \tag{8}$$

and

$$F(tx) - tF(x) \subset F(tx) + F(-tx) + K \subset F(0) + K \subset K. \tag{9}$$

Hence, according to (7)–(9),

$$\begin{aligned} tF(x) - tF(x) &\subset t\bigl(-F(-x) + K\bigr) - tF(x) \subset -tF(-x) - tF(x) + K \\ &\subset \bigl(F(tx) + K\bigr) - tF(x) + K \subset F(tx) - tF(x) + K \subset K. \end{aligned}$$

Thus

$$-\bigl(F(x) - F(x)\bigr) = F(x) - F(x) \subset \frac{1}{t}K \subset K,$$

which means that $F(x) - F(x) \subset K \cap (-K) = \{0\}$. But this implies that $F(x)$ is a singleton for each $x \in X$, which contradicts the assumption.

(ii) Let $s, t \in H_{F,K}$ and $x \in X$. Then

$$F(tx) =_K tF(x), \quad F(sx) =_K sF(x),$$

and hence, by $K$-additivity of $F$,

$$F((s+t)x) = F(sx + tx) =_K F(sx) + F(tx) =_K sF(x) + tF(x) = (s+t)F(x),$$

which means that $s + t \in H_{F,K}$.

(iii) Now, assume that $s, t \in H_{F,K}$, $t \neq 0$ and $x \in X$. Then

$$F\left(\frac{s}{t}x\right) =_K sF\left(\frac{1}{t}x\right).$$

Hence, according to Lemma 9 (iv),

$$tF\left(\frac{s}{t}x\right) =_K tsF\left(\frac{1}{t}x\right) =_K sF\left(t\frac{1}{t}x\right) = sF(x),$$

and, consequently,

$$F\left(\frac{s}{t}x\right) =_K \frac{s}{t}F(x),$$

which means that $\frac{s}{t} \in H_{F,K}$. □

For a s.v. map $F$ in a special form we can obtain further important properties of $H_{F,K}$.

**Corollary 19.** *Let $X$ be a real vector space, $Y$ be a real vector metric space and $K \subset Y$ be a closed convex cone such that $K \cap (-K) = \{0\}$. Assume that $z_0 \in K \setminus \{0\}$ and*

$$F(x) = [m(x), M(x)]z_0, \quad x \in X,$$

*where $m, M\colon X \to \mathbb{R}$ satisfy $m(x) < M(x)$ for $x \in X$. If $F$ is $K$-additive, then $m$ is additive and the following conditions hold:*

(i) $\{0, 1\} \subset H_{F,K} \subset [0, \infty)$,
(ii) $s + t \in H_{F,K}$ *for* $s, t \in H_{F,K}$,
(iii) $\frac{s}{t} \in H_{F,K}$ *for* $s, t \in H_{F,K}$ *with* $t \neq 0$,
(iv) $H_{F,K} = H_m \cap [0, \infty)$, *where $H_m$ is the homogeneity field of $m$,*

(v) $s-t \in H_{F,K}$ *for* $s,t \in H_{F,K}$ *such that* $s-t \geq 0$.

*Proof.* Since $F$ is $K$-additive, in view of Lemma 5, $m$ is additive. According to Theorem 18, it is enough to show (iv) and (v).

(iv) Fix $t \in H_{F,K}$. By Theorem 18 (i), $t \geq 0$ and, for every $x \in X$, we get

$$[m(tx), M(tx)]z_0 = F(tx) =_K tF(x) = [tm(x), tM(x)]z_0.$$

Hence

$$\big(m(tx) - \alpha\big)z_0 \in K, \quad \big(tm(x) - \beta\big)z_0 \in K$$

for some $\alpha \in [tm(x), tM(x)]$ and $\beta \in [m(tx), M(tx)]$. Since $z_0 \in K \setminus \{0\}$ and $K \cap (-K) = \{0\}$,

$$m(tx) \geq \alpha \geq tm(x), \quad tm(x) \geq \beta \geq m(tx),$$

which proves that $t \in H_m$.

On the other hand, if $t \in H_m \cap (0, \infty)$, then

$$\begin{aligned} F(tx) &= [m(tx), M(tx)]z_0 \subset [m(tx), \infty)z_0 = t[m(x), \infty)z_0 \\ &= t[m(x), M(x)]z_0 + [0, \infty)z_0 \subset tF(x) + K \end{aligned}$$

and

$$\begin{aligned} tF(x) &= t[m(x), M(x)]z_0 = [m(tx), tM(x)]z_0 \subset [m(tx), \infty)z_0 \\ &= [m(tx), M(tx)]z_0 + [0, \infty)z_0 \subset F(tx) + K \end{aligned}$$

for every $x \in X$. It means that $t \in H_{F,K}$. Moreover, if $t = 0$, in view of (i) $0 \in H_{F,K}$.

(v) Take $s,t \in H_{F,K}$ such that $s - t \geq 0$. Then, according to (iv), $s,t$ belongs to the field $H_m$ and hence $s - t \in H_m \cap [0, \infty) = H_{F,K}$, which ends the proof. $\square$

**Problem 2.** Is it true that under assumptions of Theorem 18 condition (v) of Corollary 19 holds?

We can give the positive answer to the above problem only in the case when $F$ is additionally $K$-continuous and not single-valued, because then, in view of Theorem 15, $H_{F,K} = [0, \infty)$.

Finally, let us prove a result which seems to be (in some sense) analogous to Theorem 17.

**Theorem 20.** *Let* $S \subset \mathbb{R}$ *be a set satisfying the following conditions:*

(i) $\{0, 1\} \subset S \subset [0, \infty)$,

(ii) $s + t \in S$ *for* $s,t \in S$,

(iii) $\frac{s}{t} \in S$ *for* $s,t \in S$ *with* $t \neq 0$,

(iv) $s - t \in S$ *for* $s,t \in S$ *with* $s - t \geq 0$.

*Assume that* $X$ *is a real vector space,* $Y$ *is a real vector metric space and* $K \subset Y$ *is a closed convex cone such that* $K \cap (-K) = \{0\}$. *Then there exists a s.v. map* $F\colon X \to \mathcal{CC}(Y)$ *such that* $S = H_{F,K}$.

*Proof.* First we prove that $L := S \cup (-S)$ is a subfield of $\mathbb{R}$, i.e.

$$s - t \in L, \quad s, t \in L, \tag{10}$$

$$\frac{s}{t} \in L, \quad s, t \in L,\ t \neq 0. \tag{11}$$

Let $s, t \in L$. If $s \in S$ and $t \in -S$, then $-t \in S$ and $s - t \in S \subset L$ by (ii). If $s, t \in S$ or $s, t \in -S$, then either $s - t \geq 0$ or $s - t \leq 0$, hence, according to (iv), $s - t \in S \cup (-S) = L$. Thus (10) holds.

Now, let $s, t \in L$ with $t \neq 0$. If $s \in S$ and $t \in -S$, then $-t \in S$ and $\frac{s}{t} = -(\frac{s}{-t}) \in -S \subset L$. If $s, t \in S$ or $s, t \in -S$, then, by (iii), $\frac{s}{t} = \frac{-s}{-t} \in S \subset L$. Hence (11) holds.

Knowing that $L$ is a field, according to Theorem 17 we can find an additive function $f\colon X \to \mathbb{R}$ such that $H_f = L$.

Fix $z_0 \in K \setminus \{0\}$ and define $F\colon X \to \mathcal{CC}(Y)$ by

$$F(x) = [f(x), f(x) + 1]z_0, \quad x \in X.$$

Since $f$ is additive, in view of Lemma 5, $F$ is $K$-additive, and hence, according to Corollary 19,

$$H_{F,K} = H_f \cap [0, \infty) = L \cap [0, \infty) = S,$$

which ends the proof □

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