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Results in Mathematics

Compairing Categories of Lubin–Tate (φ_L, Γ_L)-Modules

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Abstract. In the Lubin–Tate setting we compare different categories of (φ_L, Γ)-modules over various perfect or imperfect coefficient rings. Moreover, we study their associated Herr-complexes. Finally, we show that a Lubin Tate extension gives rise to a weakly decompleting, but not decompleting tower in the sense of Kedlaya and Liu.

Contents

1. Introduction

Since its invention by Fontaine in [17] the concept of (φ, Γ)-modules (for the p-cyclotomic extension) has become a powerful tool in the study of p-adic Galois representations of local fields. In particular, it could be fruitfully applied in Iwasawa theory [2–4,33,34,36–38,48] and in the p-adic local Langlands programme [12]. A good introduction to the subject regarding the state of the art around 2010 can be found in [9,18].

Afterwards a couple of generalisations have been developed. Firstly, Berger and Colmez [6] as well as Kedlaya, Pottharst and Xiao [25] extended the theory to (arithmetic) families of (φ, Γ)-modules, in which representations of the absolute Galois group of a local field on modules over affinoid algebras over $\mathbb{Q}_p$ instead of finite dimensional vector spaces are studied. Secondly, parallel to and influenced by Scholze's point of view of perfectoid spaces as well as the upcoming of the Fargues-Fontaine curve [16] Kedlaya and Liu developed a (geometric) relative p-adic Hodge theory [26,27], in which the Galois group of a local field is replaced by the étale fundamental group of affinoid spaces over $\mathbb{Q}_p$ thereby extending an earlier approach by Andreatta and Brinon. In particular, Kedlaya and Liu have introduced systematically (φ, Γ)-moduels over *perfect* coefficient rings, i.e., for which the Frobenius endomorphism is surjective, and they have studied their decent to *imperfect* coefficient rings, which is needed for Iwasawa theoretic applications and which generalized the work of Cherbonnier and Colmez [11].

Recently there has been a growing interest and activity in introducing and studying (φ_L, Γ_L)-modules for Lubin–Tate extensions of a finite extension L of $\mathbb{Q}_p$, motivated again by requirements from or potential applications to the p-adic local Langlands programme [8,13,19] or Iwasawa theory [7,35,40, 43,44]. The textbook [42] contains a very detailed and thorough approach to the analogue of Fontaine's original equivalence of categories between Galois representations and étale (φ, Γ)-modules to the case of Lubin–Tate extensions as had been proposed, but only sketched in [28], see Theorem 4.1. In this setting it has been shown in [30,31] that - as in the cyclotomic case due to Herr [20] - the Galois cohomology of a L-representation V of the absolute Galois group G_L of L can again be obtained as cohomology of a generalized Herr complex for the (φ_L, Γ_L)-module attached to V, see Theorem 7.1.

The purpose of this article is to spell out in the Lubin–Tate case concretely the various categories of (classical) (φ_L, Γ_L)-modules over perfect and imperfect coefficient rings (analogously to those considered in [26,27] who do not cover the Lubin–Tate situation) such as $\mathbf{A}_L, \mathbf{A}_L^\dagger, \tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L^\dagger, \mathbf{B}_L, \mathbf{B}_L^\dagger, \mathbf{B}_L, \tilde{\mathbf{B}}_L^\dagger$, $\mathcal{R}_L, \tilde{\mathcal{R}}_L$ to be defined in the course of the main text and to compare them among each other. Moreover, we investigate for which versions the generalized Herr complex calculates again the Galois cohomology of a given representation. The results are summarized in diagrams (6) and (32). Finally, we study in the last section how Lubin–Tate extensions fit into Kedlaya's and Liu's concept of *(weakly) decompleting* towers. We show that for $L \neq \mathbb{Q}_p$ they are weakly decompleting, but not decompleting.

See [46] for some results regarding arithmetic families of (φ_L, Γ_L)-modules in the Lubin–Tate setting.

2. Notation

Let $\mathbb{Q}_p \subseteq L \subset \mathbb{C}_p$ be a field of finite degree d over $\mathbb{Q}_p$, o_L the ring of integers of L, $\pi_L \in o_L$ a fixed prime element, $k_L = o_L/\pi_L o_L$ the residue field, $q := |k_L|$ and e the absolute ramification index of L. We always use the absolute value $|\ |$ on $\mathbb{C}_p$ which is normalized by $|\pi_L| = q^{-1}$. We **warn** the reader, though, that we will use the references [19] and [32] in which the absolute value is normalized differently from this paper by $|p| = p^{-1}$. Our absolute value is the dth power of the one in these references. The transcription of certain formulas to our convention will usually be done silently.

We fix a Lubin–Tate formal o_L-module $LT = LT_{\pi_L}$ over o_L corresponding to the prime element π_L. We always identify LT with the open unit disk around zero, which gives us a global coordinate Z on LT. The o_L-action then is given by formal power series $[a](Z) \in o_L[[Z]]$. For simplicity the formal group law will be denoted by $+_{LT}$.

Let T_π be the Tate module of LT. Then T_π is a free o_L-module of rank one, say with generator η, and the action of $G_L := \mathrm{Gal}(\overline{L}/L)$ on T_π is given by a continuous character $\chi_{LT} : G_L \longrightarrow o_L^\times$.

For $n \geq 0$ we let L_n/L denote the extension (in $\mathbb{C}_p$) generated by the π_L^n-torsion points of LT, and we put $L_\infty := \bigcup_n L_n$. The extension L_∞/L is Galois. We let $\Gamma_L := \mathrm{Gal}(L_\infty/L)$ and $H_L := \mathrm{Gal}(\overline{L}/L_\infty)$. The Lubin–Tate character χ_{LT} induces an isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$.

Henceforth we use the same notation as in [43]. In particular, the ring endomorphisms induced by sending Z to $[\pi_L](Z)$ are called φ_L where applicable; e.g. for the ring $\mathscr{A}_L$ defined to be the π_L-adic completion of $o_L[[Z]][Z^{-1}]$ or $\mathscr{B}_L := \mathscr{A}_L[\pi_L^{-1}]$ which denotes the field of fractions of $\mathscr{A}_L$. Recall that we also have introduced the unique additive endomorphism ψ_L of $\mathscr{B}_L$ (and then $\mathscr{A}_L$) which satisfies

$$\varphi_L \circ \psi_L = \pi_L^{-1} \cdot trace_{\mathscr{B}_L/\varphi_L(\mathscr{B}_L)} \ .$$

Moreover, projection formula

$$\psi_L(\varphi_L(f_1)f_2) = f_1\psi_L(f_2) \qquad \text{for any } f_i \in \mathscr{B}_L$$

as well as the formula

$$\psi_L \circ \varphi_L = \frac{q}{\pi_L} \cdot \mathrm{id}$$

hold. An étale (φ_L, Γ_L)-module M comes with a Frobenius operator φ_M and an induced operator denoted by ψ_M.

Let $\widetilde{\mathbf{E}}^+ := \varprojlim o_{\mathbb{C}_p}/p o_{\mathbb{C}_p}$ with the transition maps being given by the Frobenius $\varphi(a) = a^p$. We may also identify $\widetilde{\mathbf{E}}^+$ with $\varprojlim o_{\mathbb{C}_p}/\pi_L o_{\mathbb{C}_p}$ with the transition maps being given by the q-Frobenius $\varphi_q(a) = a^q$. Recall that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring with residue field $\overline{\mathbb{F}_p}$ and its field of fractions

$\widetilde{\mathbf{E}} = \varprojlim \mathbb{C}_p$ being algebraically closed of characteristic p. Let $\mathfrak{m}_{\widetilde{\mathbf{E}}}$ denote the maximal ideal in $\widetilde{\mathbf{E}}^+$.

The q-Frobenius φ_q first extends by functoriality to the rings of the Witt vectors $W(\widetilde{\mathbf{E}})$ and then o_L-linearly to $W(\widetilde{\mathbf{E}})_L := W(\widetilde{\mathbf{E}}) \otimes_{o_{L_0}} o_L$, where L_0 is the maximal unramified subextension of L. The Galois group G_L obviously acts on $\widetilde{\mathbf{E}}$ and $W(\widetilde{\mathbf{E}})_L$ by automorphisms commuting with φ_q. This G_L-action is continuous for the weak topology on $W(\widetilde{\mathbf{E}})_L$ (cf. [42, Lem. 1.5.3]).

By sending the variable Z to $\omega_{LT} \in W(\widetilde{\mathbf{E}})_L$ (see directly after [43, Lem. 4.1]) we obtain an G_L-equivariant, Frobenius compatible embedding of rings

$$\mathscr{A}_L \longrightarrow W(\widetilde{\mathbf{E}})_L$$

the image of which we call $\mathbf{A}_L$. The latter ring is a complete discrete valuation ring with prime element π_L and residue field the image $\mathbf{E}_L$ of $k_L((Z)) \hookrightarrow \widetilde{\mathbf{E}}$ sending Z to $\omega := \omega_{LT} \mod \pi_L$. We form the maximal integral unramified extension (= strict Henselization) $\mathbf{A}_L^{nr}$ of $\mathbf{A}_L$ inside $W(\widetilde{\mathbf{E}})_L$. Its p-adic completion $\mathbf{A}$ still is contained in $W(\widetilde{\mathbf{E}})_L$. Note that $\mathbf{A}$ is a complete discrete valuation ring with prime element π_L and residue field the separable algebraic closure $\mathbf{E}_L^{sep}$ of $\mathbf{E}_L$ in $\widetilde{\mathbf{E}}$. By the functoriality properties of strict Henselizations the q-Frobenius φ_q preserves $\mathbf{A}$. According to [28, Lemma 1.4] the G_L-action on $W(\widetilde{\mathbf{E}})_L$ respects $\mathbf{A}$ and induces an isomorphism $H_L = \ker(\chi_{LT}) \xrightarrow{\cong} \mathrm{Aut}^{cont}(\mathbf{A}/\mathbf{A}_L)$.

Sometimes we omit the index q, L, or M from the Frobenius operator.

Finally, for a valued field K we denote as usual by $\hat{K}$ its completion.

3. An Analogue of Tate's Result

Let $\mathbb{C}_p^\flat$ together with its absolute value $|\cdot|_\flat$ be the tilt of $\mathbb{C}_p$. The aim of this section is to prove an analogue of Tate's classical result [47, Prop. 10] for $\mathbb{C}_p^\flat$ instead of $\mathbb{C}_p$ itself and in the Lubin Tate situation instead of the cyclotomic one. In the following we always consider *continuous* group cohomology.

Proposition 3.1. *$H^n(H, \mathbb{C}_p^\flat) = 0$ for all $n \geq 1$ and $H \subseteq H_L$ any closed subgroup.*

Since the proof is formally very similar to that of loc. cit. or [9, Prop. 14.3.2.] we only sketch the main ingredients. To this aim we fix H and write sometimes W for $\mathbb{C}_p^\flat$ as well as $W_{\geq m} := \{x \in W \| x|_\flat \leq \frac{1}{p^m}\}$.

Lemma 3.2. *The Tate-Sen axiom* **(TS1)** *is satisfied for $\mathbb{C}_p^\flat$ with regard to H, i.e., there exists a real constant $c > 1$ such that for all open subgroups $H_1 \subseteq H_2$ in H there exists $\alpha \in (\mathbb{C}_p^\flat)^{H_1}$ with $|\alpha|_\flat < c$ and $Tr_{H_2|H_1}(\alpha) := \sum_{\tau \in H_2|H_1} \tau(\alpha) = 1$. Moreover, for any sequence $(H_m)_m$ of open subgroups $H_{m+1} \subseteq H_m$ of H there exists a trace compatible system $(y_{H_m})_m$ of elements $y_{H_m} \in (\mathbb{C}_p^\flat)^{H_m}$ with $|y_{H_m}|_\flat < c$ and $Tr_{H|H_m}(y_{H_m}) = 1$.*

Proof. Note that for a perfect field K (like $(\mathbb{C}_p^\flat)^H$) of characteristic p complete for a multiplicative norm with maximal ideal $\mathfrak{m}_K$ and a finite extension F one has $\mathrm{Tr}_{F/K}(\mathfrak{m}_F) = \mathfrak{m}_K$ by [23, Thm. 1.6.4]. Fix some $x \in (\mathbb{C}_p^\flat)^H$ with $0 < |x|_\flat < 1$ and set $c := |x|_\flat^{-1} > 1$. Then we find $\tilde{\alpha}$ in the maximal ideal of $(\mathbb{C}_p^\flat)^{H_1}$ with $Tr_{H|H_1}(\tilde{\alpha}) = x$ and $\alpha := (Tr_{H_2|H_1}(\tilde{\alpha}))^{-1}\tilde{\alpha}$ satisfies the requirement as $|Tr_{H_2|H_1}(\tilde{\alpha})|_\flat^{-1} \leq |x|_\flat^{-1} = c$.

For the second claim we successively choose elements $\tilde{\alpha}_m$ in the maximal ideal of $(\mathbb{C}_p^\flat)^{H_m}$ such that $Tr_{H|H_1}(\tilde{\alpha}_1) = x$ and $Tr_{H_{m+1}|H_m}(\tilde{\alpha}_{m+1}) = \tilde{\alpha}_m$ for all $m \geq 1$. Renormalization $\alpha_m := x^{-1}\tilde{\alpha}_m$ gives the desired system. □

Remark 3.3. Since H is also a closed subgroup of the absolute Galois group G_L of L it possesses a countable fundamental system $(H_m)_m$ of open neighbourhoods of the identity, as for any $n > 0$ the local field L of characteristic 0 has only finitely many extensions of degree smaller than n.

Proof. The latter statement reduces easily to finite Galois extensions L' of L, which are known to be solvable, i.e. L' has a series of at most n intermediate fields $L \subseteq L_1 \subseteq \cdots \subseteq L_n = L'$ such that each subextension is abelian. Now its known by class field theory that each local field in characteristic 0 only has finitely many abelian extensions of a given degree. □

We write $\mathcal{C}^n(G, V)$ for the abelian group of continuous n-cochains of a profinite group G with values in a topological abelian group V carrying a continuous G-action and ∂ for the usual differentials. In particular, we endow $\mathcal{C}^n(H, W)$ with the maximum norm $\|-\|$ and consider the subspace $\mathcal{C}^n(H, W)^\delta := \bigcup_{H' \trianglelefteq H \text{ open}} \mathcal{C}^n(H/H', W) \subseteq \mathcal{C}^n(H, W)$ of those cochains with are even continuous with respect to the discrete topology of W.

Lemma 3.4. *(i) The completion of $\mathcal{C}^n(H, W)^\delta$ with respect to the maximum norm equals $\mathcal{C}^n(H, W)$.*

(ii) There exist $(\mathbb{C}_p^\flat)^H$-linear continuous maps

$$\sigma^n : \mathcal{C}^n(H, W) \to \mathcal{C}^{n-1}(H, W)$$

satisfying $\|f - \partial\sigma^n f\| \leq c\|\partial f\|$.

Proof. Since the space $\mathcal{C}^n(H, W)$ is already complete we only have to show that an arbitrary cochain f in it can be approximated by a Cauchy sequence f_m in $\mathcal{C}^n(H, W)^\delta$. To this end we observe that, given any m, the induced cochain $H^n \xrightarrow{f} W \xrightarrow{pr_m} W/W_{\geq m}$ comes, for some open normal subgroup H_m, from a cochain in $\mathcal{C}^n(H/H_m, W/W_{\geq m})$, which in turn gives rise to $f_m \in \mathcal{C}^n(H, W)^\delta$ when composing with any set theoretical section $W/W_{\geq m} \xrightarrow{s_m} W$ of the canonical projection $W \xrightarrow{pr_m} W/W_{\geq m}$. Note that s_m is automatically continuous, since $W/W_{\geq m}$ is discrete. By construction we have $\|f - f_m\| \leq \frac{1}{p^m}$ and $(f_m)_m$ obviously is a Cauchy sequence. This shows (i).

For (ii) recall from Lemma 3.2 together with Remark 3.3 the existence of a trace compatible system $(y_{H'})_{H'}$ of elements $y_{H'} \in (\mathbb{C}_p^\flat)^{H'}$ with $|y_{H'}|_\flat < c$

and $Tr_{H|H'}(y_{H'}) = 1$, where H' runs over the open normal subgroups of H. Now we first define $(\mathbb{C}_p^\flat)^H$-linear maps

$$\sigma^n : \mathcal{C}^n(H,W)^\delta \to \mathcal{C}^{n-1}(H,W)$$

satisfying $\|f - \partial\sigma^n f\| \leq c\|\partial f\|$ and $\|\sigma^n f\| \leq c\|f\|$ by setting for $f \in \mathcal{C}^n(H/H', W)$

$$\sigma^n(f) := y_{H'} \cup f$$

(by considering $y_{H'}$ as a -1-cochain), i.e.,

$$\sigma^n(f)(h_1,\ldots,h_{n-1}) = (-1)^n \sum_{\tau\in H/H'} (h_1\ldots h_{n-1}\tau)(y_{H'})f(h_1,\ldots,h_{n-1},\tau).$$

The inequality $\|y_{H'} \cup f\| \leq c\|f\|$ follows immediately from this description, see the proof of [9, Lem. 14.3.1.]. Upon noting that $\partial y_{H'} = Tr_{H|H'}(y_{H'}) = 1$, the Leibniz rule for the differential ∂ with respect to the cup-product then implies that

$$f - \partial(y_{H'} \cup f) = y_{H'} \cup \partial f,$$

hence

$$\|f - \partial(y_{H'} \cup f)\| \leq c\|\partial f\|$$

by the previous inequality, see again (loc. cit.). In order to check that this map σ^n is well defined we assume that f arises also from a cochain in $\mathcal{C}^n(H/H'', W)$. Since we may make the comparison within $\mathcal{C}^n(H/(H'\cap H''), W)$ we can assume without loss of generality that $H'' \subseteq H'$. Then

$$\begin{aligned}
&(y_{H''} \cup f)(h_1,\ldots,h_{n-1})\\
&\quad= (-1)^n \sum_{\tau\in H/H''} (h_1\ldots h_{n-1}\tau)(y_{H''})f(h_1,\ldots,h_{n-1},\tau)\\
&\quad= (-1)^n \sum_{\tau\in H/H'} \left(h_1\ldots h_{n-1} \sum_{\tau'\in H'/H''} \tau'\right)(y_{H''})f(h_1,\ldots,h_{n-1},\tau)\\
&\quad= (-1)^n \sum_{\tau\in H/H'} (h_1\ldots h_{n-1})\,(\sum_{\tau'\in H'/H''} \tau'(y_{H''}))f(h_1,\ldots,h_{n-1},\tau)\\
&\quad= (-1)^n \sum_{\tau\in H/H'} (h_1\ldots h_{n-1})\,(y_{H'})f(h_1,\ldots,h_{n-1},\tau)\\
&\quad= (y_{H'} \cup f)(h_1,\ldots,h_{n-1})
\end{aligned}$$

using the trace compatibility in the fourth equality. Finally the inequality $\|\sigma^n f\| \leq c\|f\|$ implies that σ^n is continuous on $\mathcal{C}^n(H,W)^\delta$ and therefore extends continuously to its completion $\mathcal{C}^n(H,W)$. □

The proof of Prop. 3.1 is now an immediate consequence of Lemma 3.4(ii).

4. The Functors D, $\tilde{D}$ and $\tilde{D}^\dagger$

Let $\mathrm{Rep}_{o_L}(G_L)$, $\mathrm{Rep}_{o_L,f}(G_L)$ and $\mathrm{Rep}_L(G_L)$ denote the category of finitely generated o_L-modules, finitely generated free o_L-modules and finite dimensional L-vector spaces, respectively, equipped with a continuous linear G_L-action. The following result is established in [28, Thm. 1.6] (see also [42, Thm. 3.3.10]) and [43, Prop. 4.4 (ii)].

Theorem 4.1. *The functors*

$$T \longmapsto D(T) := (\mathbf{A} \otimes_{o_L} T)^{H_L} \qquad \textit{and} \qquad M \longmapsto (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q \otimes \varphi_M = 1}$$

are exact quasi-inverse equivalences of categories between $\mathrm{Rep}_{o_L}(G_L)$ *and the category* $\mathfrak{M}^{et}(\mathbf{A}_L)$ *of finitely generated étale* (φ_L, Γ_L)*-modules over* $\mathbf{A}_L$*. Moreover, for any* T *in* $\mathrm{Rep}_{o_L}(G_L)$ *the natural map*

$$\mathbf{A} \otimes_{\mathbf{A}_L} D(T) \xrightarrow{\cong} \mathbf{A} \otimes_{o_L} T \tag{1}$$

is an isomorphism (compatible with the G_L*-action and the Frobenius on both sides).*

In the following we would like to establish a version of the above for $\tilde{\mathbf{A}}$ and prove similar properties for it. In the classical situation such versions have been studied by Kedlaya et al. using the unramified rings of Witt vectors $W(R)$. In our Lubin–Tate situation we have to work with ramified Witt vectors $W(R)_L$. Many results and their proofs transfer almost literally from the classical setting. Often we will try to at least sketch the proofs for the convenience of the reader, but when we just quote results from the classical situation, e.g. from [26], this usually means that the transfer is purely formal.

We start defining $\tilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$ and

$$\tilde{\mathbf{A}}^\dagger := \left\{ x = \sum_{n \geq 0} \pi_L^n [x_n] \in \tilde{\mathbf{A}} : |\pi_L^n| |x_n|_\flat^r \xrightarrow{n \to \infty} 0 \text{ for some } r > 0 \right\}$$

as well as $\tilde{D}(T) := (\tilde{\mathbf{A}} \otimes_{o_L} T)^{H_L}$ and $\tilde{D}^\dagger(T) := (\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T)^{H_L}$.

More generally, let K be any perfectoid field containing L and let $K^\flat$ denote its tilt. For $r > 0$ let $W^r(K^\flat)_L$ be the set of $x = \sum_{n=0}^\infty \pi_L^n [x_n] \in W(K^\flat)_L$ such that $|\pi_L|^n |x_n|_\flat^r$ tends to zero as n goes to ∞. This is a subring by [26, Prop. 5.1.2] on which the function

$$|x|_r := \sup_n \{ |\pi_L^n| |x_n|_\flat^r \} = \sup_n \{ q^{-n} |x_n|_\flat^r \}$$

is a complete multiplicative norm; it extends multiplicatively to $W^r(K^\flat)_L[\frac{1}{\pi_L}]$. Furthermore, $W^\dagger(K^\flat)_L := \bigcup_{r>0} W^r(K^\flat)_L$ is a henselian discrete valuation ring by [21, Lem. 2.1.12], whose π_L-adic completion equals $W(K^\flat)_L$ since they coincide modulo π_L^n. Then $\tilde{\mathbf{A}}^\dagger = W^\dagger(\mathbb{C}_p^\flat)_L$, and we write $\tilde{\mathbf{A}}_L$ and $\tilde{\mathbf{A}}_L^\dagger$ for

$W(\hat{L}_\infty^\flat)_L$ and $W^\dagger(\hat{L}_\infty^\flat)_L$, respectively. We set $\tilde{\mathbf{B}}_L = \tilde{\mathbf{A}}_L[\frac{1}{\pi_L}]$, $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[\frac{1}{\pi_L}]$, $\tilde{\mathbf{B}}_L^\dagger = \tilde{\mathbf{A}}_L^\dagger[\frac{1}{\pi_L}]$ and $\tilde{\mathbf{B}}^\dagger = \tilde{\mathbf{A}}^\dagger[\frac{1}{\pi_L}]$ for the corresponding fields of fractions.

Remark 4.2. By the Ax-Tate-Sen theorem [1] and since $\mathbb{C}_p^\flat$ is the completion of an algebraic closure $\overline{\hat{L}_\infty^\flat}$ he have that $(\mathbb{C}_p^\flat)^H = ((\overline{\hat{L}_\infty^\flat})^H)^\wedge$ for any closed subgroup $H \subseteq H_L$, in particular $(\mathbb{C}_p^\flat)^{H_L} = \hat{L}_\infty^\flat$. As completion of an algebraic extension of the perfect field $\hat{L}_\infty^\flat$ the field $(\mathbb{C}_p^\flat)^H$ is perfect, too. Moreover, we have $\tilde{\mathbf{A}}^{H_L} = \tilde{\mathbf{A}}_L$, $(\tilde{\mathbf{A}}^\dagger)^{H_L} = \tilde{\mathbf{A}}_L^\dagger$ and analogously for the rings $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}}^\dagger$. It also follows that $\tilde{\mathbf{A}}$ is the π_L-adic completion of a maximal unramified extension of $\tilde{\mathbf{A}}_L$.

Lemma 4.3. *The rings $\mathbf{A}_L$ and $\mathbf{A}$ embed into $\tilde{\mathbf{A}}_L$ and $\tilde{\mathbf{A}}$, respectively.*

Proof. The embedding $\mathbf{A}_L \hookrightarrow \tilde{\mathbf{A}}_L$ is explained in [42, p. 94]. Moreover, $\mathbf{A}$ is the π_L-adic completion of the maximal unramified extension of $\mathbf{A}_L$ inside $\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L$ (cf. [42, §3.1]). □

On $\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L$ the weak topology is defined to be the product topology of the valuation topologies on the components $\mathbb{C}_p^\flat$. The induced topology on any subring R of it is also called weak topology of R. If M is a finitely generated R-module, then we call the *canonical* topology of M *(with respect to the weak topology of* R) the quotient topology with respect to any surjection $R^n \twoheadrightarrow M$ where the free module carries the product topology; this is independent of any choices. We recall that a (φ_L, Γ_L)-module M over $R \in \{\mathbf{A}_L, \tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L^\dagger\}$ is a finitely generated R-module M together with

- a Γ_L-action on M by semilinear automorphisms which is continuous for the weak topology and
- a φ_L-linear endomorphism φ_M of M which commutes with the Γ_L-action.

We let $\mathfrak{M}(R)$ denote the category of (φ_L, Γ_L)-modules M over R. Such a module M is called étale if the linearized map

$$\varphi_M^{lin} : R \otimes_{R,\varphi_L} M \xrightarrow{\cong} M$$
$$f \otimes m \longmapsto f\varphi_M(m)$$

is bijective. We let $\mathfrak{M}^{\acute{e}t}(R)$ denote the full subcategory of étale (φ_L, Γ_L)-modules over R.

Definition 4.4. For $* = \mathbf{B}_L, \tilde{\mathbf{B}}_L, \tilde{\mathbf{B}}_L^\dagger$ we write $\mathfrak{M}^{\acute{e}t}(*) := \mathfrak{M}^{\acute{e}t}(*') \otimes_{o_L} L$ with $*' = \mathbf{A}_L, \tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L^\dagger$, respectively, and call the objects étale (φ_L, Γ_L)-modules over $*$.

Lemma 4.5. *Let G be a profinite group and $R \to S$ be a topological monomorphism of topological o_L-algebras, for which there exists a system of open neighbourhoods of* 0 *consisting of o_L-submodules. Consider a finitely generated R-module M, for which the canonical map $M \to S \otimes_R M$ is injective (e.g. if S is* *faithfully flat over R or M is free), and endow it with the canonical topology with respect to R. Assume that G acts continuously, o_L-linearly and compatible on R and S as well as continuously and R-semilinearly on M. Then the diagonal G-action on $S \otimes_R M$ is continuous with regard to the canonical topology with respect to S.*

Proof. Imitate the proof of [42, Lem. 3.1.11]. □

Proposition 4.6. *The canonical map*

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T) \xrightarrow{\cong} \tilde{D}(T) \tag{2}$$

is an isomorphism and the functor $\tilde{D}(-) : \mathrm{Rep}_{o_L}(G_L) \to \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L)$ is exact. Moreover, we have a comparison isomorphism

$$\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T) \xrightarrow{\cong} \tilde{\mathbf{A}} \otimes_{o_L} T. \tag{3}$$

Proof. The isomorphism (2) implies formally the isomorphism (3) after base change of the comparison isomorphism (1). Secondly, the isomorphism (2), resp. (3), implies easily that $\tilde{D}(T)$ is finitely generated, resp. étale. Thirdly, since the ring extension $\tilde{\mathbf{A}}_L/\mathbf{A}_L$ is faithfully flat as local extension of (discrete) valuation rings, the exactness of $\tilde{D}$ follows from that of D. Moreover, the isomorphism (2) implies by Lemma 4.5 that Γ_L acts continuously on $\tilde{D}(T)$, i.e., the functor $\tilde{D}$ is well-defined. Thus we only have to prove that

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^{H_L} \xrightarrow{\cong} (\tilde{\mathbf{A}} \otimes_{o_L} T)^{H_L}$$

is an isomorphism. To this aim let us *assume first that T is finite.* Then we find an open normal subgroup $H \trianglelefteq H_L$ which acts trivially on T. Application of the subsequent Lemma 4.7 to $M = (\mathbf{A} \otimes_{o_L} T)^H$ and $G = H_L/H$ interprets the left hand side as $\left(\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^H\right)^{H_L/H}$ while the right hand side equals $\left((\tilde{\mathbf{A}} \otimes_{o_L} T)^H\right)^{H_L/H}$. Hence it suffices to establish the isomorphism

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} (\mathbf{A} \otimes_{o_L} T)^H \xrightarrow{\cong} (\tilde{\mathbf{A}} \otimes_{o_L} T)^H.$$

By Lemma 4.8 below this is reduced to showing that the canonical map

$$\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \otimes_{o_L} T \xrightarrow{\cong} \tilde{\mathbf{A}}^H \otimes_{o_L} T$$

is an isomorphism, which follows from Lemma 4.9 below. *Finally let T be arbitrary.* Then we have isomorphisms

$$\begin{aligned}\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T) &\cong \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \varprojlim_n D(T/\pi_L^n T)\\ &\cong \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} \varprojlim_n D(T)/\pi_L^n D(T)\\ &\cong \varprojlim_n \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T)/\pi_L^n D(T)\\ &\cong \varprojlim_n \tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L} D(T/\pi_L^n T)\\ &\cong \varprojlim_n \tilde{D}(T/\pi_L^n T)\\ &\cong \tilde{D}(T),\end{aligned}$$

where we use for the second and fourth equation exactness of D, for the second last one the case of finite T and for the first, third and last equation the elementary divisor theory for the discrete valuation rings o_L, $\mathbf{A}_L$ and $\tilde{\mathbf{A}}_L$, respectively. □

Lemma 4.7. *Let $A \to B$ be a flat extension of rings and M an A-module with an A-linear action by a finite group G. Then $B \otimes_A M$ carries a B-linear G-action and we have*

$$(B \otimes_A M)^G = B \otimes_A M^G.$$

Proof. Apply the exact functor $B \otimes_A -$ to the exact sequence

$$0 \longrightarrow M^G \longrightarrow M \xrightarrow{(g-1)_{g\in G}} \bigoplus\nolimits_{g\in G} M,$$

which gives the desired description of $(B \otimes_A M)^G$. □

Lemma 4.8. *Let A be $\mathbf{A}$, $\mathbf{A}_L^{nr}$, $\tilde{\mathbf{A}}^\dagger$ or $\tilde{\mathbf{A}}$ and T be a finitely generated o_L-module with trivial action by an open subgroup $H \subseteq H_L$. Then $(A \otimes_{o_L} T)^H = A^H \otimes_{o_L} T$. Moreover, $\mathbf{A}^H$ and $\tilde{\mathbf{A}}^H$ are free $\mathbf{A}_L$- and $\tilde{\mathbf{A}}_L$-modules of finite rank, respectively.*

Proof. Since $T \cong \bigoplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ with $n_i \in \mathbb{N} \cup \{\infty\}$ we may assume that $T = o_L/\pi_L^n o_L$ for some $n \in \mathbb{N} \cup \{\infty\}$. We then we have to show that

$$(A/\pi_L^n A)^H = A^H/\pi_L^n A^H \tag{4}$$

For $n = \infty$ there is nothing to prove.

The case $n = 1$: First of all we have $\mathbf{A}/\pi_L\mathbf{A} = \mathbf{A}_L^{nr}/\pi_L\mathbf{A}_L^{nr} = \mathbf{E}_L^{sep}$. On the other hand, by the Galois correspondence between unramified extensions and their residue extensions, we have that $(\mathbf{E}_L^{sep})^H$ is the residue field of $(\mathbf{A}_L^{nr})^H$. Hence the case $n = 1$ holds true for $A = \mathbf{A}_L^{nr}$. After having finished all cases for $A = \mathbf{A}_L^{nr}$ we will see at the end of the proof that $(\mathbf{A}_L^{nr})^H = \mathbf{A}^H$. Therefore the case $n = 1$ for $A = \mathbf{A}$ will be settled, too.

For $A = \tilde{\mathbf{A}}$ we only need to observe that $\tilde{\mathbf{A}}/\pi_L\tilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)_L/\pi_L W(\mathbb{C}_p^\flat)_L = \mathbb{C}_p^\flat$ and that $(\mathbb{C}_p^\flat)^H$ is the residue field of $(W(\mathbb{C}_p^\flat)_L)^H = W((\mathbb{C}_p^\flat)^H)_L$.

For $A = \tilde{\mathbf{A}}^\dagger$ we argue by the following commutative diagram

$$\begin{array}{ccccc} (\mathbf{C}_p^\flat)^H & \xrightarrow{\cong} & W^\dagger((\mathbf{C}_p^\flat)^H)_L/\pi_L W^\dagger((\mathbf{C}_p^\flat)^H)_L & \xrightarrow{\cong} & (\tilde{\mathbf{A}}^\dagger)^H/\pi_L(\tilde{\mathbf{A}}^\dagger)^H \\ & \searrow^{\cong} & & & \downarrow \\ & & \tilde{\mathbf{A}}^H/\pi_L\tilde{\mathbf{A}}^H \xrightarrow{\cong} (\tilde{\mathbf{A}}/\pi_L\tilde{\mathbf{A}})^H & \xrightarrow{\cong} & (\tilde{\mathbf{A}}^\dagger/\pi_L\tilde{\mathbf{A}}^\dagger)^H. \end{array}$$

The case $1 < n < \infty$: This follows by induction using the commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^H/\pi_L^n A^H & \xrightarrow{\pi_L\cdot} & A^H/\pi_L^{n+1}A^H & \longrightarrow & A^H/\pi_L A^H & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & (A/\pi_L^n A)^H & \xrightarrow{\pi_L\cdot} & (A/\pi_L^{n+1}A)^H & \longrightarrow & (A/\pi_L A)^H, & & \end{array}$$

in which the outer vertical arrows are isomorphism by the case $n = 1$ and the induction hypothesis.

Finally we can check, using the above equality (4) for $A = \mathbf{A}_L^{nr}$ in the third equation:

$$\begin{aligned} \mathbf{A}^H &= \left(\varprojlim_n \mathbf{A}_L^{nr}/\pi_L^n\mathbf{A}_L^{nr}\right)^H \\ &= \varprojlim_n (\mathbf{A}_L^{nr}/\pi_L^n\mathbf{A}_L^{nr})^H \\ &= \varprojlim_n \left(\mathbf{A}_L^{nr}\right)^H/\pi_L^n(\mathbf{A}_L^{nr}\right)^H \\ &= (\mathbf{A}_L^{nr})^H. \end{aligned}$$

Note that $(\mathbf{A}_L^{nr})^H$ is a finite unramified extension of $\mathbf{A}_L$ and therefore is π_L-adically complete. We also see that $\mathbf{A}^H$ is a free $\mathbf{A}_L$-module of finite rank. Similarly, $W(\mathbb{C}_p^\flat)_L^H \cong (W(\hat{L}_\infty^\flat)_L^{nr})^H$ is a free $W(\hat{L}_\infty^\flat)_L$-module of finite rank. □

Lemma 4.9. *For any open subgroup H of H_L the canonical maps*

$$W(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \xrightarrow{\cong} W((\mathbb{C}_p^\flat)^H)_L,$$
$$W(\hat{L}_\infty^\flat)_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H \xrightarrow{\cong} W((\mathbb{C}_p^\flat)^H)_L$$

are isomorphisms.

Proof. We begin with the first isomorphism. Since $\mathbf{A}^H$ is finitely generated free over $\mathbf{A}_L$ by Lemma 4.8, we have

$$W(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \cong \left(\varprojlim_n W_n(\hat{L}_\infty^\flat)_L\right) \otimes_{\mathbf{A}_L} \mathbf{A}^H \cong \varprojlim_n \left(W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H\right).$$

It therefore suffices to show the corresponding assertion for Witt vectors of finite length:

$$W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H/\pi_L^n\mathbf{A}^H = W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H \xrightarrow{\cong} W_n((\mathbb{C}_p^\flat)^H)_L.$$

To this aim we first consider the case $n = 1$. From (4) we know that $\mathbf{A}^H/\pi_L^n\mathbf{A}^H = (\mathbf{E}_L^{sep})^H$. Hence we need to check that

$$\hat{L}_\infty^\flat \otimes_{\mathbf{E}_L} (\mathbf{E}_L^{sep})^H \xrightarrow{\cong} (\mathbb{C}_p^\flat)^H$$

is an isomorphism. Since the perfect hull $\mathbf{E}_L^{perf}$ of $\mathbf{E}_L$ (being purely inseparable and normal) and $(\mathbf{E}_L^{sep})^H$ (being separable) are linear disjoint extensions of $\mathbf{E}_L$ their tensor product is equal to the composite of fields $\mathbf{E}_L^{perf}(\mathbf{E}_L^{sep})^H$ (cf. [10, Thm. 5.5, p. 188]), which moreover has to have degree $[H_L : H]$ over $\mathbf{E}_L^{perf}$. Since the completion of the tensor product is $\hat{L}_\infty^\flat \otimes_{\mathbf{E}_L} (\mathbf{E}_L^{sep})^H$, we see that the completion of the field $\mathbf{E}_L^{perf}(\mathbf{E}_L^{sep})^H$ is the composite of fields $\hat{L}_\infty^\flat(\mathbf{E}_L^{sep})^H$, which has degree $[H_L : H]$ over $\hat{L}_\infty^\flat$. But $\hat{L}_\infty^\flat(\mathbf{E}_L^{sep})^H \subseteq (\mathbb{C}_p^\flat)^H$. By the Ax-Tate-Sen theorem $(\mathbb{C}_p^\flat)^H$ has also degree $[H_L : H]$ over $\hat{L}_\infty^\flat$. Hence the two fields coincide, which establishes the case $n = 1$.

The commutative diagram

$$\begin{array}{ccc}
\hat{L}_\infty^\flat \otimes_{\mathbf{A}_L} \mathbf{A}^H & \xrightarrow{\cong} & (\mathbb{C}_p^\flat)^H \\
{\scriptstyle \varphi_q^m \otimes \mathrm{id}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \varphi_q^m} \\
\hat{L}_\infty^\flat \otimes_{\varphi_q^m, \mathbf{A}_L} \mathbf{A}^H & \xrightarrow{\mathrm{id}\,\varphi_q^m} & (\mathbb{C}_p^\flat)^H
\end{array}$$

shows that also the lower map is an isomorphism. Using that Verschiebung V on $W_n((\mathbb{C}_p^\flat)^H)_L$ and $W_n(\hat{L}_\infty^\flat)_L$ is additive and satisfies the projection formula $V^m(x){\cdot}y = V^m(x{\cdot}\varphi_q^m(y))$ we see that we obtain a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{L}_\infty^\flat \otimes_{\varphi_q^n, \mathbf{A}_L} \mathbf{A}^H & \xrightarrow{V^n \otimes \mathrm{id}} & W_{n+1}(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H & \longrightarrow & W_n(\hat{L}_\infty^\flat)_L \otimes_{\mathbf{A}_L} \mathbf{A}^H & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}\,\varphi_q^n}\downarrow & & {\scriptstyle can}\downarrow & & {\scriptstyle \cong}\downarrow & & \\
0 & \longrightarrow & (\mathbb{C}_p^\flat)^H & \xrightarrow{V^n} & W_{n+1}((\mathbb{C}_p^\flat)^H)_L & \longrightarrow & W_n((\mathbb{C}_p^\flat)^H)_L, & &
\end{array}$$

from which the claim follows by induction because the outer vertical maps are isomorphisms by the above and the induction hypothesis. Here the first non-trivial horizontal morphisms map onto the highest Witt vector component.

The second isomorphism is established as follows: We choose a subgroup $N \subseteq H \subseteq H_L$ which is open normal in H_L and obtain the extensions of henselian discrete valuation rings

$$\tilde{\mathbf{A}}_L^\dagger \subseteq (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)_L \subseteq (\tilde{\mathbf{A}}^\dagger)^N = W^\dagger((\mathbb{C}_p^\flat)^N)_L.$$

The corresponding extensions of their field of fractions

$$\tilde{\mathbf{B}}_L^\dagger \subseteq E := (\tilde{\mathbf{A}}^\dagger)^H[\tfrac{1}{\pi_L}] \subseteq F := (\tilde{\mathbf{A}}^\dagger)^N[\tfrac{1}{\pi_L}]$$

satisfy $F^{H/N} = E$ and $F^{H_L/N} = \tilde{\mathbf{B}}_L^\dagger$. Hence F/E and $F/\tilde{\mathbf{B}}_L^\dagger$ are Galois extensions of degree $[H : N]$ and $[H_L : N]$, respectively. It follows that $E/\tilde{\mathbf{B}}_L^\dagger$ is a finite extension of degree $[H_L : H]$. The henselian condition then implies[1] that $(\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)_L$ is free of rank $[H_L : H]$ over $\tilde{\mathbf{A}}_L^\dagger = W^\dagger(\hat{L}_\infty^\flat)_L$. The π_L-adic completion $\widehat{(-)}$ of the two rings therefore can be obtained by the tensor product with $\tilde{\mathbf{A}}_L = W(\hat{L}_\infty^\flat)_L$. This gives the wanted

$$W(\hat{L}_\infty^\flat)_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger(\hat{L}_\infty^\flat)\widehat{}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H = W^\dagger((\mathbb{C}_p^\flat)^H)\widehat{}_L = W((\mathbb{C}_p^\flat)^H)_L.$$

□

Proposition 4.10. *The sequences*

$$0 \to o_L \to \mathbf{A} \xrightarrow{\varphi_q - 1} \mathbf{A} \to 0, \tag{5}$$

$$0 \to o_L \to \tilde{\mathbf{A}} \xrightarrow{\varphi_q - 1} \tilde{\mathbf{A}} \to 0, \tag{6}$$

$$0 \to o_L \to \tilde{\mathbf{A}}^\dagger \xrightarrow{\varphi_q - 1} \tilde{\mathbf{A}}^\dagger \to 0. \tag{7}$$

are exact.

Proof. The first sequence is [43, (26), Rem. 5.1]. For the second sequence one proves by induction the statement for finite length Witt vectors using that the Artin-Schreier equation has a solution in $\mathbb{C}_p^\flat$. Taking projective limits then gives the claim. For the third sequence only the surjectivity has to be shown. This can be achieved by the same calculation as in the proof of [27, Lem. 4.5.3] with $R = \mathbb{C}_p^\flat$.[2] □

Lemma 4.11. *For any finite T in $Rep_{o_L}(G_L)$ the map $\tilde{\mathbf{A}} \otimes_{o_L} T \xrightarrow{\varphi_q \otimes \mathrm{id} - 1} \tilde{\mathbf{A}} \otimes_{o_L} T$ has a continuous set theoretical section.*

Proof. Since $T \cong \bigoplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ for some natural numbers r, n_i we may assume that $T = o_L/\pi_L^n o_L$ for some n and then we have to show that the surjective map $W_n(\mathbb{C}_p^\flat)_L \xrightarrow{\varphi_q - \mathrm{id}} W_n(\mathbb{C}_p^\flat)_L$ has a continuous set theoretical section. Thus me may neglect the additive structure and identify source and target with $X = (\mathbb{C}_p^\flat)^n$. In order to determine the components of the map $\varphi_q - \mathrm{id} =: f = (f_0, \ldots, f_{n-1}) : X \to X$ with respect to these coordinates we recall that the addition in Witt rings is given by polynomials

$$S_j(X_0, \ldots X_j, Y_0, \ldots, Y_j) = X_j + Y_j + \text{ terms in } X_0, \ldots, X_{j-1}, Y_0, \ldots, Y_{j-1}$$

while the additive inverse is given by

$$I_j(X_0, \ldots, X_j) = -X_j + \text{ terms in } X_0, \ldots, X_{j-1}.$$

[1] See Neukirch, Algebraische Zahlentheorie, proof of Satz II.6.8.

[2] For the other see [27, Lem. 4.5.3] : There the exactness of corresponding sequences for sheaves on the proétale site $Spa(L, o_L)_{pro\acute{e}t}$ is shown, which in turn implies exactness for the corresponding sequences of stalks at the geometric point $Spa(\mathbb{C}_p, o_{\mathbb{C}_p})$. Note that taking stalks at this point is the same as taking sections over it.

Indeed, the polynomials I_j are defined by the property that $\Phi_j(I_0,\dots,I_j) = -\Phi_j(X_0,\dots,X_j)$ where the Witt polynomials have the form $\Phi_j(X_0,\dots,X_j) = X_0^{q^j} + \pi_L X_1^{q^{j-1}} + \cdots + \pi_L^j X_j$. Modulo $(X_0,\dots,X_{j-1})$ we derive that $\pi_L^j I_j(X_0,\dots,X_j) \equiv -\pi_L^j X_j$ and the claim follows. Since φ_q acts componentwise rising the entries to their qth power, we conclude that

$$f_j = S_j(X_0^q,\dots,X_j^q, I_0(X_0),\dots,I_j(X_0,\dots,X_j)).$$

Hence the Jacobi matrix of f at a point $x \in X$ looks like

$$D_x(f) = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ * & & -1 \end{pmatrix},$$

i.e., is invertible in every point. As a polynomial map f is locally analytic. It therefore follows from the inverse function theorem [41, Prop. 6.4] that f restricts to a homeomorphism $f|U_0 : U_0 \xrightarrow{\cong} U_1$ of open neighbourhoods of x and $f(x)$, respectively. By the surjectivity of f every $x \in X$ has an open neighbourhood U_x and a continuous map $s_x : U_x \to X$ with $f \circ s_x = \mathrm{id}_{|U_x}$. But X is strictly paracompact by Remark 8.6 (i) in (loc. cit.), i.e., the covering $(U_x)_x$ has a disjoint refinement. There the restrictions of the s_x glue to a continuous section of f. □

Corollary 4.12. *For T in $Rep_{o_L}(G_L)$, the nth cohomology groups of the complexes concentrated in degrees* 0 *and* 1

$$0 \longrightarrow \tilde{D}(T) \xrightarrow{\varphi-1} \tilde{D}(T) \longrightarrow 0 \quad \textit{and} \tag{8}$$

$$0 \longrightarrow D(T) \xrightarrow{\varphi-1} D(T) \longrightarrow 0 \tag{9}$$

are isomorphic to $H^n(H_L, T)$ for any $n \geq 0$.

Proof. Assume first that T is finite. For (9) see [43, Lem. 5.2]. For (8) we use Lemma 4.11, which says that the right hand map in the exact sequence

$$0 \longrightarrow T \longrightarrow \tilde{\mathbf{A}} \otimes_{o_L} T \xrightarrow{\varphi_q \otimes \mathrm{id} -1} \tilde{\mathbf{A}} \otimes_{o_L} T \longrightarrow 0$$

has a continuous set theoretical section and thus gives rise to the long exact sequence of continuous cohomology groups

$$0 \to H^0(H_L,T) \to \tilde{D}(T) \xrightarrow{\varphi-1} \tilde{D}(T) \to H^1(H_L,T) \to H^1(H_L, \tilde{\mathbf{A}} \otimes_{o_L} T) \to \dots \tag{10}$$

Using the comparison isomorphism (3) and the subsequent Prop. 4.13 we see that all terms from the fifth on vanish.

For the general case (for $\tilde{D}(T)$ as well as $D(T)$) we take inverse limits in the exact sequences for the $(T/\pi_L^m T)$ and observe that $H^n(H_L,T) \cong \varprojlim_m H^n(H_L, T/\pi_L^m T)$. This follows for $n \neq 2$ from [39, Cor. 2.7.6]. For $n = 2$

we use [39, Thm. 2.7.5] and have to show that the projective system $(H^1(H_L, T/\pi_L^m T))_m$ is Mittag-Leffler. Since it is a quotient of the projective system $(D(T/\pi_L^m T))_m$, it suffices for this to check that the latter system is Mittag-Leffler. But due to the exactness of the functor D this latter system is equal to the projective system of artinian $\mathbf{A}_L$-modules $(D(T)/\pi_L^m D(T))_m$ and hence is Mittag-Leffler. We conclude by observing that taking inverse limits of the system of sequences (10) remains exact. The reasoning being the same for $\tilde{D}(T)$ and $D(T)$ we consider only the former. Indeed, we split the 4-term exact sequences into two short exact sequences of projective systems

$$0 \to H^0(H_L, V/\pi_L^m T) \to \tilde{D}(T/\pi_L^m T) \to (\varphi - 1)\tilde{D}(T/\pi_L^m T) \to 0$$

and

$$0 \to (\varphi - 1)\tilde{D}(T/\pi_L^m T) \to \tilde{D}(T/\pi_L^m T) \to H^1(H_L, T/\pi_L^m T) \to 0.$$

Passing to the projective limits remains exact provided the left most projective systems have vanishing $\varprojlim^1$. For the system $H^0(H_L, T/\pi_L^m T)$ this is the case since it is Mittag-Leffler. The system $(\varphi - 1)\tilde{D}(T/\pi_L^m T)$ even has surjective transition maps since the system $\tilde{D}(T/\pi_L^m T)$ has this property by the exactness of the functor $\tilde{D}$ (cf. Prop. 4.6). □

Proposition 4.13. *$H^n(H, \tilde{\mathbf{A}}/\pi_L^m \tilde{\mathbf{A}}) = 0$ for all $n, m \geq 1$ and $H \subseteq H_L$ any closed subgroup.*

Proof. For $j < i$ the canonical projection $W_i(\mathbb{C}_p^\flat) \cong \tilde{\mathbf{A}}/\pi_L^i \tilde{\mathbf{A}} \twoheadrightarrow \tilde{\mathbf{A}}/\pi_L^j \tilde{\mathbf{A}} \cong W_j(\mathbb{C}_p^\flat)$ corresponds to the projection $(\mathbb{C}_p^\flat)^i \twoheadrightarrow (\mathbb{C}_p^\flat)^j$ and hence have set theoretical continuous sections. Using the associated long exact cohomology sequence (after adding the kernel) allows to reduce the statement to Prop. 3.1. □

For any commutative ring R with endomorphism φ we write $\Phi(R)$ for the category of φ-modules consisting of R-modules equipped with a semi-linear φ-action. We write $\Phi^{\acute{e}t}(R)$ for the subcategory of étale φ-modules, i.e., such that M is finitely generated over R and φ induces an R-linear isomorphism $\varphi^* M \xrightarrow{\cong} M$. Finally, we denote by $\Phi_f^{\acute{e}t}(R)$ the subcategory consisting of finitely generated free R-modules.

For $M_1, M_2 \in \Phi(R)$ with M_1 being étale the R-module $\mathrm{Hom}_R(M_1, M_2)$ has a natural structure as a φ-module satisfying

$$\varphi_{\mathrm{Hom}_R(M_1,M_2)}(\alpha)(\varphi_{M_1}(m)) = \varphi_{M_2}(\alpha(m)) \ , \tag{11}$$

hence in particular

$$\mathrm{Hom}_R(M_1, M_2)^{\varphi = \mathrm{id}} = \mathrm{Hom}_{\Phi(R)}(M_1, M_2). \tag{12}$$

Note that with M_1, M_2 also $\mathrm{Hom}_R(M_1, M_2)$ is étale.

Remark 4.14. We recall from [26, §1.5] that the cohomology groups $H^i_\varphi(M)$ of the complex $M \xrightarrow{\varphi - 1} M$ can be identified with the Yoneda extension groups

$\mathrm{Ext}^i_{\Phi(R)}(R, M)$. Indeed, if $S := R[X; \varphi]$ denotes the twisted polynomial ring satisfying $Xr = \varphi(r)X$ for all $r \in R$, then we can identify $\Phi(R)$ with the category S-Mod of (left) S-modules by letting X act via φ_M on X. Using the free resolution

$$0 \longrightarrow S \xrightarrow{\cdot(X-1)} S \longrightarrow R \longrightarrow 0$$

the result follows.

Remark 4.15. Note that $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}_L$ is a faithfully flat ring extension as both rings are discrete valuation rings and the bigger one is the completion of the previous one.

Proposition 4.16. *Base extension induces*

(i) an equivalence of categories

$$\Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger) \leftrightarrow \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$$

(ii) and an isomorphism of Yoneda extension groups

$$\mathrm{Ext}^1_{\Phi(\tilde{\mathbf{A}}_L^\dagger)}(\tilde{\mathbf{A}}_L^\dagger, M) \cong \mathrm{Ext}^1_{\Phi(\tilde{\mathbf{A}}_L)}(\tilde{\mathbf{A}}_L, \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)$$

for all $M \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$.

Proof. For the first item we imitate the proof of [26, Thm. 8.5.3], see also [23, Lem. 2.4.2,Thm. 2.4.5]: First we will show that for every $M \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ it holds that $(\tilde{\mathbf{A}}_L \otimes M)^{\varphi=\mathrm{id}} \subseteq M^{\varphi=\mathrm{id}}$ and hence equality. Applied to $M := \mathrm{Hom}_{\tilde{\mathbf{A}}_L^\dagger}(M_1, M_2)$ this implies that the base change is fully faithful by the equation (12). We observe that the analogue of [26, Lem. 3.2.6] holds in our setting and that S in loc. cit. can be chosen to be a finite separable field extension of the perfect field $R = \hat{L}_\infty^\flat$. Thus we may choose S in the analogue of [26, Prop. 7.3.6] (with $a = 1$, $c = 0$ and M_0 being our M) as completion of a (possibly infinite) separable field extension of R. This means in our situation that there exists a closed subgroup $H \subseteq H_L$ such that $(\tilde{\mathbf{A}}^\dagger)^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \bigoplus (\tilde{\mathbf{A}}^\dagger)^H e_i$ for a basis e_i invariant under φ. Now let $v = \sum x_i e_i$ be an arbitrary element in

$$\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M \subseteq \tilde{\mathbf{A}}^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \tilde{\mathbf{A}}^H \otimes_{(\tilde{\mathbf{A}}^\dagger)^H} (\tilde{\mathbf{A}}^\dagger)^H \otimes_{\tilde{\mathbf{A}}_L^\dagger} M = \bigoplus \tilde{\mathbf{A}}^H e_i$$

with $x_i \in \tilde{\mathbf{A}}^H$ and such that $\varphi(v) = v$. The latter condition implies that $x_i \in \tilde{\mathbf{A}}^{H, \varphi_q = \mathrm{id}} = o_L$, i.e., v belongs to $(M \otimes_{\tilde{\mathbf{A}}_L^\dagger} (\tilde{\mathbf{A}}^\dagger)^H) \cap (M \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{\mathbf{A}}_L) = M$, because M is free and one has $\tilde{\mathbf{A}}_L \cap (\tilde{\mathbf{A}}^\dagger)^H = (\tilde{\mathbf{A}}^\dagger)^{H_L} = \tilde{\mathbf{A}}_L^\dagger$. To show essential surjectivity one proceeds literally as in the proof of [26, Thm. 8.5.3] adapted to ramified Witt vectors.

For the second statement choose a quasi-inverse functor $F : \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L) \to \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L^\dagger)$ with $F(\tilde{\mathbf{A}}_L) = \tilde{\mathbf{A}}_L^\dagger$. Given an extension $0 \longrightarrow M \longrightarrow E \longrightarrow \tilde{\mathbf{A}}_L \longrightarrow 0$ over $\Phi(\tilde{\mathbf{A}}_L)$ with $M \in \Phi_f^{\acute{e}t}(\tilde{\mathbf{A}}_L)$ first observe that $E \in \Phi_f^{ét}(\tilde{\mathbf{A}}_L)$, too. Indeed, $\tilde{\mathbf{A}}_L \xrightarrow{\varphi_q} \tilde{\mathbf{A}}_L$ is a flat ring extension, whence $\varphi^* E \to E$ is an isomorphism, if the corresponding outer maps are. The analogous statement holds over $\tilde{\mathbf{A}}_L^\dagger$. Therefore the sequence $0 \longrightarrow F(M) \longrightarrow F(E) \longrightarrow \tilde{\mathbf{A}}_L^\dagger \longrightarrow 0$ is exact by Remark 4.15, because its base extension - being isomorphic to the original extension - is, by assumption. □

We denote by $\mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ and $\mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L)$ the full subcategories of $\mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ and $\mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L)$, respectively, consisting of finitely generated free modules over the base ring.

Remark 4.17. Let M be in $\mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L)$ and endow $N := \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ with the canonical topology with respect to the weak topology of $\tilde{\mathbf{A}}_L$. Then the induced subspace topology of $M \subseteq N$ coincides with the canonical topology with respect to the weak topology of $\tilde{\mathbf{A}}_L^\dagger$. Indeed for free modules this is obvious while for torsion modules this can be reduced by the elementary divisor theory to the case $M = \tilde{\mathbf{A}}_L^\dagger/\pi_L^n\tilde{\mathbf{A}}_L^\dagger \cong \tilde{\mathbf{A}}_L/\pi_L^n\tilde{\mathbf{A}}_L$. But the latter spaces are direct product factors of $\tilde{\mathbf{A}}_L^\dagger$ and $\tilde{\mathbf{A}}_L$, respectively, as topological spaces, from wich the claim easily follows.

Proposition 4.18. *For $T \in \mathrm{Rep}_{o_L}(G_L)$ and $V \in \mathrm{Rep}_L(G_L)$ we have natural isomorphisms*

$$\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \cong \tilde{D}(T) \text{ and} \tag{13}$$

$$\tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{D}(V), \tag{14}$$

as well as

$$\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \cong \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \text{ and} \tag{15}$$

$$\tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{\mathbf{B}}^\dagger \otimes_L V, \tag{16}$$

respectively. In particular, the functor $\tilde{D}^\dagger(-) : \mathrm{Rep}_{o_L}(G_L) \to \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ is exact.

Moreover, base extension induces equivalences of categories

$$\mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger) \leftrightarrow \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L),$$

and hence also an equivalence of categories

$$\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L^\dagger) \leftrightarrow \mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L).$$

Proof. Note that the base change functor is well-defined - regarding the continuity of the Γ_L-action - by Lemma 4.5 and Remark 4.15 while $\tilde{D}^\dagger$ is well-defined by Remark 4.17, once (13) will have been shown. We first show the equivalence of categories for free modules: By Prop. 4.16 we already have, for $M_1, M_2 \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$, an isomorphism

$$\mathrm{Hom}_{\Phi(\tilde{\mathbf{A}}_L^\dagger)}(M_1, M_2) \cong \mathrm{Hom}_{\Phi(\tilde{\mathbf{A}}_L)}(\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M_1, \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M_2).$$

Taking Γ_L-invariants gives that the base change functor in question is fully faithful.

In order to show that this base change functor is also essentially surjective, consider an arbitrary $N \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L)$. Again by 4.16 we know that there is a free étale φ-module M over $\tilde{\mathbf{A}}_L^\dagger$ whose base change is isomorphic to N. By the fully faithfulness the Γ_L-action descends to M[3]. Since the weak topology of M is compatible with that of N by Remark 4.17, this action is again continuous.

To prepare for the proof of the isomorphism (13) we first observe the following fact. The isomorphism (3) implies that T and $\tilde{D}(T)$ have the same elementary divisors, i.e.: If $T \cong \oplus_{i=1}^r o_L/\pi_L^{n_i} o_L$ as o_L-module (with $n_i \in \mathbb{N} \cup \{\infty\}$) then $\tilde{D}(T) \cong \oplus_{i=1}^r \tilde{\mathbf{A}}_L/\pi_L^{n_i}\tilde{\mathbf{A}}_L$ as $\tilde{\mathbf{A}}_L$-module.

We shall prove (13) in several steps: First assume that T is *finite*. Then T is annihilated by some π_L^n. We have $\tilde{D}^\dagger(T) = \tilde{D}(T)$ and $\tilde{\mathbf{A}}_L^\dagger/\pi_L^n\tilde{\mathbf{A}}_L^\dagger = \tilde{\mathbf{A}}_L/\pi_L^n\tilde{\mathbf{A}}_L$ so that there is nothing to prove. Secondly we suppose that T is *free and* that $\tilde{D}^\dagger(T)$ is free over $\tilde{\mathbf{A}}_L^\dagger$ of the same rank $r := \mathrm{rk}_{o_L} T$. On the other hand, as the functor $\tilde{D}^\dagger$ is always left exact, we obtain the injective maps

$$\tilde{D}^\dagger(T)/\pi_L^n\tilde{D}^\dagger(T) \to \tilde{D}^\dagger(T/\pi_L^nT) = \tilde{D}(T/\pi_L^nT).$$

for any $n \geq 1$. We observe that both sides are isomorphic to $(\tilde{\mathbf{A}}_L^\dagger/\pi_L^n\tilde{\mathbf{A}}_L^\dagger)^r = (\tilde{\mathbf{A}}_L/\pi_L^n\tilde{\mathbf{A}}_L)^r$. Hence the above injective maps are bijections. We deduce that

$$\begin{aligned}\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L^\dagger} \tilde{D}^\dagger(T) &\cong \varprojlim_n \tilde{D}^\dagger(T)/\pi_L^n\tilde{D}^\dagger(T)\\ &\cong \varprojlim_n \tilde{D}(T/\pi_L^nT)\\ &\cong \varprojlim_n \tilde{D}(T)/\pi_L^n\tilde{D}(T)\\ &\cong \tilde{D}(T)\end{aligned}$$

using that the above tensor product means π_L-adic completion for finitely generated $\tilde{\mathbf{A}}_L^\dagger$-modules.

Thirdly let $T \in \mathrm{Rep}_{o_L,f}(G_L)$ be arbitrary and $M \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ such that $\tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} M \cong \tilde{D}(T)$ according the equivalence of categories. Without loss of generality we may treat this isomorphism as an equality. Similarly as in the proof of Prop. 4.16 and with the same notation one shows that $(\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1} = \bigoplus_{i=1}^r o_L e_i$ for some appropriate φ-invariant basis $e_1, \ldots, e_r$

[3] As $\gamma \in \Gamma_L$ acts semilinearly, one formally has to replace $N \xrightarrow{\gamma} N$ by the linearized isomorphism $\tilde{\mathbf{A}}_L \otimes_{\gamma,\tilde{\mathbf{A}}_L} N \xrightarrow{\gamma^{lin}} N$. Upon checking that the source is again a étale φ-module with model $\tilde{\mathbf{A}}_L^\dagger \otimes_{\gamma,\tilde{\mathbf{A}}_L^\dagger} M$ one sees by the fully faithfulness on φ-modules that the linearized isomorphism descends and induces the desired semi-linear action.

of $\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$. Note that $r = \mathrm{rk}_{o_L} T$. Using (3), it follows that

$$\begin{aligned} T &= (\tilde{\mathbf{A}} \otimes_{o_L} T)^{\varphi=1} \cong (\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T))^{\varphi=1} = (\tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1} \\ &= \bigoplus_{i=1}^r \tilde{\mathbf{A}}^{\varphi_q=1} e_i = \bigoplus_{i=1}^r o_L e_i = (\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M)^{\varphi=1}. \end{aligned}$$

It shows that the comparison isomorphism (3) restricts to an injective map $T \hookrightarrow \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$, which extends to a homomorphism $\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \xrightarrow{\alpha} \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ of free $\tilde{\mathbf{A}}^\dagger$-modules of the same rank r. Further base extension by $\tilde{\mathbf{A}}$ gives back the isomorphism (3). Since $\tilde{\mathbf{A}}$ is faithfully flat over $\tilde{\mathbf{A}}^\dagger$ the map α was an isomorphism already. By passing to H_L-invariants we obtain an isomorphism $\tilde{D}^\dagger(T) \cong M$ and see that $\tilde{D}^\dagger(T)$ is free of the same rank as T. Hence the second case applies and gives (13) for free T and (14). Finally, let T be just finitely generated over o_L. Write $0 \to T_{\mathrm{fin}} \to T \to T_{\mathrm{free}} \to 0$ with finite T_{fin} and free T_{free}. We then have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T_{\mathrm{fin}}) & \to & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) & \to & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T_{\mathrm{free}}) & \to & \tilde{\mathbf{A}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} H^1(H_L, \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}}) \\ & \cong\downarrow & & \downarrow & & \cong\downarrow & & \\ 0 \to & \tilde{D}(T_{\mathrm{fin}}) & \to & \tilde{D}(T) & \to & \tilde{D}(T_{\mathrm{free}}) & \to & 0, \end{array}$$

in which we use the first and third step for the vertical isomorphisms. In order to show that the middle perpendicular arrow is an isomorphism it suffices to prove that $H^1(H_L, \tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}}) = 0$. But since T_{fin} is annihilated by some π_L^n we have

$$\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T_{\mathrm{fin}} \cong \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}} \otimes_{o_L} T_{\mathrm{fin}} \cong \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}} \otimes_{\tilde{\mathbf{A}}_L} \tilde{D}(T_{\mathrm{fin}}),$$

the last isomorphism by (3). Thus it suffices to prove the vanishing of $H^1(H_L, \tilde{\mathbf{A}}/\pi_L^n \tilde{\mathbf{A}})$, which is established in Prop. 4.13 and finishes the proof of the isomorphism (13).

Note that this base change isomorphism implies the exactness of $\tilde{D}^\dagger$ as $\tilde{D}$ is exact by Prop. 4.6 and using that the base extension is faithfully flat by Remark 4.15.

For free T the statement (15) (and hence (16)) is already implicit in the above arguments while for finite T the statement coincides with (3). The general case follows from the previous ones by exactness of $\tilde{D}^\dagger$ and the five lemma as above.

□

Corollary 4.19. *For a T in $\mathrm{Rep}_{o_L,f}(G_L)$ and V in $\mathrm{Rep}_L(G_L)$, the nth cohomology group, for any $n \geq 0$, of the complexes concentrated in degrees 0 and 1*

$$0 \to \tilde{D}^\dagger(T) \xrightarrow{\varphi-1} \tilde{D}^\dagger(T) \to 0 \quad \textit{and} \tag{17}$$

$$0 \to \tilde{D}^\dagger(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger(V) \to 0 \quad \textit{and} \tag{18}$$

is isomorphic to $H^n(H_L, T)$ *and* $H^n(H_L, V)$*, respectively.*

Proof. The integral result reduces, by (13), Remark 4.14, and Prop. 4.16, to Corollary 4.12. Since inverting π_L is exact and commutes with taking cohomology [39, Prop. 2.7.11], the second statement follows. □

Set $\mathbf{A}^\dagger := \tilde{\mathbf{A}}^\dagger \cap \mathbf{A}$ and $\mathbf{B}^\dagger := \mathbf{A}^\dagger[\frac{1}{\pi_L}]$ as well as $\mathbf{A}_L^\dagger := (\mathbf{A}^\dagger)^{H_L}$. Note that $\mathbf{B}_L^\dagger := (\mathbf{B}^\dagger)^{H_L} \subseteq \mathbf{B}^\dagger \subseteq \tilde{\mathbf{B}}^\dagger$. For $V \in \mathrm{Rep}_L(G_L)$ we define $D^\dagger(V) := (\mathbf{B}^\dagger \otimes_L V)^{H_L}$. The categories $\mathfrak{M}^{\acute{e}t}(\mathbf{A}_L^\dagger)$ and $\mathfrak{M}^{\acute{e}t}(\mathbf{B}_L^\dagger)$ are defined analogously as in Definition 4.4.

We now introduce the Robba ring $\mathcal{R} = \mathcal{R}_K = \mathcal{R}_K(\mathbf{B})$ of the open unit disk $\mathbf{B}_{/K}$, where $L \subseteq K \subseteq \mathbb{C}_p$ denotes a complete intermediate field. The ring of K-valued global holomorphic functions $\mathcal{O}_K(\mathbf{B})$[4] on $\mathbf{B}$ is the Fréchet algebra of all power series in the variable Z with coefficients in K which converge on the open unit disk $\mathbf{B}(\mathbb{C}_p)$. The Fréchet topology on $\mathcal{O}_K(\mathbf{B})$ is given by the family of norms

$$\Big|\sum_{i\geq 0} c_i Z^i\Big|_r := \max_i |c_i| r^i \qquad \text{for } 0 < r < 1 \ .$$

In the commutative integral domain $\mathcal{O}_K(\mathbf{B})$ we have the multiplicative subset $Z^{\mathbb{N}} = \{Z^j : j \in \mathbb{N}\}$, so that we may form the corresponding localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$. Each norm $|\ |_r$ extends to this localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ by setting $|\sum_{i\gg -\infty} c_i Z^i|_r := \max_i |c_i| r^i$.

The Robba ring $\mathcal{R} \supseteq \mathcal{O}_K(\mathbf{B})$ is constructed as follows. For any $s > 0$, resp. any $0 < r \leq s$, in $p^{\mathbb{Q}}$ let $\mathbf{B}_{[0,s]}$, resp. $\mathbf{B}_{[r,s]}$, denote the affinoid disk around 0 of radius s, resp. the affinoid annulus of inner radius r and outer radius s, over K. For $I = [0, s]$ or $[r, s]$ we denote by

$$\mathcal{R}^I := \mathcal{R}_K^I(\mathbf{B}) := \mathcal{O}_K(\mathbf{B}_I)$$

the affinoid K-algebra of $\mathbf{B}_I$. The Fréchet algebra $\mathcal{R}^{[r,1)} := \varprojlim_{r<s<1} \mathcal{R}^{[r,s]}$ is the algebra of (infinite) Laurent series in the variable Z with coefficients in K which converge on the half-open annulus $\mathbf{B}_{[r,1)} := \bigcup_{r<s<1} \mathbf{B}_{[r,s]}$. The Banach algebra $\mathcal{R}^{[0,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})$ with respect to the norm $|\ |_s$. The Banach algebra $\mathcal{R}^{[r,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ with respect to the norm $|\ |_{r,s} := \max(|\ |_r, |\ |_s)$. It follows that the Fréchet algebra $\mathcal{R}^{[r,1)}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ in the locally convex topology defined by the family of norms $(|\ |_{r,s})_{r<s<1}$. Finally, the Robba ring is $\mathcal{R} = \bigcup_{0<r<1} \mathcal{R}^{[r,1)}$.

Remark 4.20. There is also the following more concrete description for $\mathbf{A}_L^\dagger$ in terms of Laurent series in ω_{LT} :

$$\mathbf{A}_L^\dagger = \{F(\omega_{LT}) \in \mathbf{A}_L | F(Z) \text{ converges on } \rho \leq |Z| < 1 \text{ for some } \rho \in (0,1)\} \subseteq \mathbf{A}_L.$$

[4] In the notation from [13, §1.2] this is the ring $\mathcal{R}^+$.

Indeed this follows from the analogue of [11, Lem. II.2.2] upon noting that the latter holds with and without the integrality condition: "$rv_p(a_n) + n \geq 0$ for all $n \in \mathbb{Z}$" (for $r \in \overline{\mathbf{R}} \setminus \mathbf{R}$) in the notation of that article.

In particular we obtain canonical embeddings $\mathbf{A}_L^\dagger \subseteq \mathbf{B}_L^\dagger \hookrightarrow \mathcal{R}_L$ of rings.

Definition 4.21. V in $\mathrm{Rep}_L(G_L)$ is called *overconvergent*, if $\dim_{\mathbf{B}_L^\dagger} D^\dagger(V) = \dim_L V$. We denote by $\mathrm{Rep}_L^\dagger(G_L) \subseteq \mathrm{Rep}_L(G_L)$ the full subcategory of overconvergent representations.

Remark 4.22. We always have $\dim_{\mathbf{B}_L^\dagger} D^\dagger(V) \leq \dim_L V$. If $V \in \mathrm{Rep}_L(G_L)$ is overconvergent then we have the natural isomorphism

$$\mathbf{B}_L \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V) \xrightarrow{\cong} D(V). \tag{19}$$

Proof. Since $\mathbf{B}_L$ and $\mathbf{B}_L^\dagger$ are fields this is immediate from [18, Thm. 2.13]. □

Remark 4.23. In [5, §10] Berger uses the following condition to define overconvergence of V: There exists a $\mathbf{B}_L$-basis $x_1, \ldots, x_n$ of $D(V)$ such that $M := \bigoplus_{i=1}^n \mathbf{B}_L^\dagger x_i$ is a (φ_L, Γ_L)-module over $\mathbf{B}_L^\dagger$. This then implies a natural isomorphism

$$\mathbf{B}_L \otimes_{\mathbf{B}_L^\dagger} M \cong D(V). \tag{20}$$

Lemma 4.24. *V in $\mathrm{Rep}_L(G_L)$ is overconvergent if and only if V satisfies the above condition of Berger. In this case $M = D^\dagger(V)$.*

Proof. If V is overconvergent, we can take a basis within $M := D^\dagger(V)$. Conversely let V satisfy Berger's condition, i.e. we have the isomorphism (20). One easily checks by faithfully flat descent that with $D(V)$ also M is étale. By [19, Prop. 1.5 (a)][5] we obtain the identity $V = \left(\mathbf{B}^\dagger \otimes_{\mathbf{B}_L^\dagger} M\right)^{\varphi=1}$ induced from the comparison isomorphism

$$\mathbf{B} \otimes_L V \cong \mathbf{B} \otimes_{\mathbf{B}_L} D(V) \cong \mathbf{B} \otimes_{\mathbf{B}_L^\dagger} M. \tag{21}$$

We shall prove that $M \subseteq D^\dagger(V) = (\mathbf{B}^\dagger \otimes_L V)^{H_L}$ as then $M = D^\dagger(V)$ by dimension reasons. To this aim we may write a basis $v_1, \ldots, v_n$ of V over L as $v_i = \sum c_{ij} x_j$ with $c_{ij} \in \mathbf{B}^\dagger$. Then (21) implies that the matrix $C = (c_{ij})$ belongs to $M_n(\mathbf{B}^\dagger) \cap GL_n(\mathbf{B}) = GL_n(\mathbf{B}^\dagger)$. Thus M is contained in $\mathbf{B}^\dagger \otimes_L V$ and - as subspace of $D(V)$ - also H_L-invariant, whence the claim. □

Remark 4.25. Note that the imperfect version of Prop. 4.18 is not true: the base change $\mathfrak{M}^{ét}(\mathbf{B}_L^\dagger) \to \mathfrak{M}^{ét}(\mathbf{B}_L)$ is not essentially surjective in general, whence not an equivalence of categories, by [19]. By definition, its essential image consists of *overconvergent* (φ_L, Γ_L)-modules, i.e., whose corresponding Galois representations are *overconvergent.*

[5] Note that there $\bar{D}$ actually belongs to the category of (φ, G_F)-modules over $\tilde{\mathbf{B}}_{\mathbb{Q}_p} \otimes F$ instead of over $\tilde{\mathbf{B}}_{\mathbb{Q}_p}$ in their notation.

Lemma 4.26. *Assume that $V \in \mathrm{Rep}_L(G_L)$ is overconvergent. Then there is natural isomorphism*

$$\tilde{\mathbf{B}}_L^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \cong \tilde{D}^\dagger(V).$$

Proof. By construction we have a natural map $\tilde{\mathbf{B}}_L^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \to \tilde{D}^\dagger(V)$, whose base change to $\tilde{\mathbf{B}}_L$

$$\tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} D^\dagger(V) \to \tilde{\mathbf{B}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \cong \tilde{D}(V)$$

arises also as the base change of the isomorphism (19), whence is an isomorphism itself. Here we have used the (base change of the) isomorphisms (14), (2). By faithfully flatness the original map is an isomorphism, too. □

5. The Perfect Robba Ring

Again let K be any perfectoid field containing L and $r > 0$. For $0 < s \leq r$, let $\tilde{\mathcal{R}}^{[s,r]}(K)$ be the completion of $W^r(K^\flat)_L[\frac{1}{\pi_L}]$ with respect to the norm $\max\{|\ |_s, |\ |_r\}$, and put

$$\tilde{\mathcal{R}}^r(K) = \varprojlim_{s \in (0,r]} \tilde{\mathcal{R}}^{[s,r]}(K)$$

equipped with the Fréchet topology. Let $\tilde{\mathcal{R}}(K) = \varinjlim_{r>0} \tilde{\mathcal{R}}^r(K)$, equipped with the locally convex direct limit topology (LF topology). We set $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\mathbb{C}_p)$ and $\tilde{\mathcal{R}}_L := \tilde{\mathcal{R}}(\hat{L}_\infty)$. For geometric interpretation of these definitions, see [15]. As in [26, Thm. 9.2.15] we have

$$\tilde{\mathcal{R}}^{H_L} = \tilde{\mathcal{R}}_L.$$

Recall from Sect. 2 the embedding $o_L[[Z]] \to W(\tilde{\mathbf{E}})_L$. As we will explain in Sect. 8 the image ω_{LT} of the variable Z already lies in $W(\hat{L}_\infty^\flat)_L$, so that we actually have an embedding $o_L[[Z]] \to W(\hat{L}_\infty^\flat)_L$. Similarly as in [26, Def. 4.3.1] for the cyclotomic situation one shows that the latter embedding extends to a Γ_L- and φ_L-equivariant topological monomorphism $\mathcal{R}_L \to \tilde{\mathcal{R}}_L$, see also [49, Konstruktion 1.3.27] in the Lubin–Tate setting.

Let R be either $\mathcal{R}_L$ or $\tilde{\mathcal{R}}_L$. A (φ_L, Γ_L)-module over R is a finitely generated free R-module M equipped with commuting semilinear actions of φ_M and Γ_L, such that the action is continuous for the LF topology and such that the semi-linear map $\varphi_M : M \to M$ induces an isomorphism $\varphi_M^{lin} : R \otimes_{R,\varphi_R} M \xrightarrow{\cong} M$. Such M is called étale, if there exists an étale (φ_L, Γ_L)-module N over $\mathbf{A}_L^\dagger$ and $\tilde{\mathbf{A}}_L^\dagger$ (see before Definition 4.4), such that $\mathcal{R}_L \otimes_{\mathbf{A}_L^\dagger} N \cong M$ and $\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} N \cong M$, respectively.

By $\mathfrak{M}(R)$ and $\mathfrak{M}^{ét}(R)$ we denote the category of (φ_L, Γ_L)-modules and étale (φ_L, Γ_L)-modules over R, respectively.

We call the topologies on $\tilde{\mathbf{A}}_L^\dagger$ and $\tilde{\mathbf{A}}^\dagger$, which make the inclusions $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}^\dagger \subseteq \tilde{\mathcal{R}}$ topological embeddings, the LF-topologies.

Lemma 5.1. *For $M \in \mathfrak{M}_f^{ét}(\tilde{\mathbf{A}}_L^\dagger)$ the Γ_L-action is also continuous with respect to the canonical topology with respect to the LF-topology of $\tilde{\mathbf{A}}_L^\dagger$.*

Proof. The proof in fact works in the following generality: Suppose that $\tilde{\mathbf{A}}^\dagger$ is equipped with an o_L-linear ring topology which induces the π_L-adic topology on o_L. Consider on $\tilde{\mathbf{A}}_L^\dagger$ the corresponding induced topology. We claim that then the Γ_L-action on M is continuous with respect to the corresponding canonical topology. By Prop. 6.1 we may choose $T \in \mathrm{Rep}_{o_L,f}(G_L)$ such that $M \cong \tilde{D}^\dagger(T)$. Then we have a homeomorphism $\tilde{\mathbf{A}}^\dagger \otimes_{o_L} T \cong \tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$ with respect to the canonical topology by (15) (as any R-module homomorphism of finitely generated modules is continuous with respect to the canonical topology with regard to any topological ring R). Since $o_L \subseteq \tilde{\mathbf{A}}^\dagger$ is a topological embedding with respect to the π_L-adic and the given topology, respectively, Lemma 4.5 implies that G_L is acting continuously on $\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M$, whence Γ_L acts continuously on $M = \left(\tilde{\mathbf{A}}^\dagger \otimes_{\tilde{\mathbf{A}}_L^\dagger} M\right)^{H_L}$ with respect to the induced topology as subspace of the previous module. Since all involved modules are free and hence carry the product topologies and since $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}^\dagger$ is a topological embedding, it is clear that the latter topology of M coincides with its canonical topology. □

We define the functor

$$\tilde{D}_{rig}^\dagger(-) : \mathrm{Rep}_L(G_L) \longrightarrow \mathfrak{M}(\tilde{\mathcal{R}}_L)$$
$$V \longmapsto (\tilde{\mathcal{R}} \otimes_L V)^{H_L},$$

where the fact, that Γ_L acts continuously on the image with respect to the LF-topology can be seen as follows, once we have shown the next lemma. Indeed, (22) implies that for any G_L-stable o_L-lattice T of V we also have an isomorphism $\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{A}}_L^\dagger} \tilde{D}^\dagger(T) \xrightarrow{\cong} \tilde{D}_{rig}^\dagger$. Now again Lemma 4.5 applies to conclude the claim.

Lemma 5.2. *The canonical map*

$$\tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{B}}_L^\dagger} \tilde{D}^\dagger(V) \xrightarrow{\cong} \tilde{D}_{rig}^\dagger(V) \tag{22}$$

is an isomorphism and the functor $\tilde{D}_{rig}^\dagger(-) : \mathrm{Rep}_L(G_L) \to \mathfrak{M}(\tilde{\mathcal{R}}_L)$ is exact. Moreover, we have a comparison isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}_{rig}^\dagger(V) \xrightarrow{\cong} \tilde{\mathcal{R}} \otimes_{o_L} V. \tag{23}$$

Proof. The comparison isomorphism in the proof of (an analogue of) [24, Thm. 2.13] implies the comparison isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}_{rig}^\dagger(V) \cong \tilde{\mathcal{R}} \otimes_{o_L} V$$

together with the identity $V = (\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} \tilde{D}^\dagger_{rig}(V))^{\varphi_L=1}$. On the other hand the comparison isomorphism (16) induces by base change an isomorphism

$$\tilde{\mathcal{R}} \otimes_{\tilde{\mathbf{B}}^\dagger_L} \tilde{D}^\dagger(V) \xrightarrow{\cong} \tilde{\mathcal{R}} \otimes_{o_L} V.$$

Taking H_L-invariants gives the first claim. The exactness of the functor $\tilde{D}^\dagger_{rig}(-)$ follows from the exactness of the functor $\tilde{D}^\dagger(-)$ by Prop. 4.6. □

Let R be $\mathbf{B}_L$, $\mathbf{B}^\dagger_L$, $\mathcal{R}_L$, $\tilde{\mathbf{B}}_L$, $\tilde{\mathbf{B}}^\dagger_L$, $\tilde{\mathcal{R}}_L$ and let correspondingly R^{int} be $\mathbf{A}_L$, $\mathbf{A}^\dagger_L$, $\mathbf{A}^\dagger_L$, $\tilde{\mathbf{A}}_L$, $\tilde{\mathbf{A}}^\dagger_L$, $\tilde{\mathbf{A}}^\dagger_L$. We denote by $\Phi(R)^{\acute{e}t}$ the essential image of the base change functor $R \otimes_{R^{int}} - : \Phi^{\acute{e}t,f}(R^{int}) \to \Phi^{\acute{e}t,f}(R)$ (sic!).

Proposition 5.3. *Base change induces an equivalence of categories*

$$\Phi(\tilde{\mathbf{B}}^\dagger_L)^{\acute{e}t} \leftrightarrow \Phi(\tilde{\mathcal{R}}_L)^{\acute{e}t}$$

and an isomorphism of Yoneda extension groups

$$\mathrm{Ext}^1_{\Phi(\tilde{\mathbf{B}}^\dagger_L)}(\tilde{\mathbf{B}}^\dagger_L, M) \cong \mathrm{Ext}^1_{\Phi(\tilde{\mathcal{R}}_L)}(\tilde{\mathcal{R}}_L, \tilde{\mathcal{R}}_L \otimes_{\tilde{\mathbf{B}}^\dagger_L} M)$$

for all $M \in \Phi(\tilde{\mathbf{B}}^\dagger_L)^{\acute{e}t}$.

Proof. The first claim is an analogue of [26, Thm. 8.5.6]. The second claim follows as in the proof of Prop. (4.16) using the fact that by Lemma 8.6.3 in loc. cit. any extension of étale φ-modules over $\tilde{R}_L$ is again étale. Note that $\tilde{\mathcal{R}}_L/\tilde{\mathbf{B}}^\dagger_L$ is a faithfully flat ring extension, $\tilde{\mathbf{B}}^\dagger_L$ being a field. □

Corollary 5.4. *If* V *belongs to* $Rep_L(G_L)$*, the following complex concentrated in degrees* 0 *and* 1 *is acyclic*

$$0 \longrightarrow \tilde{D}^\dagger_{rig}(V)/\tilde{D}^\dagger(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger_{rig}(V)/\tilde{D}^\dagger(V) \longrightarrow 0. \tag{24}$$

In particular, we have that the nth cohomology groups of the complex concentrated in degrees 0 *and* 1

$$0 \longrightarrow \tilde{D}^\dagger_{rig}(V) \xrightarrow{\varphi-1} \tilde{D}^\dagger_{rig}(V) \longrightarrow 0$$

are isomorphic to $H^n(H_L, V)$ *for* $n \geq 0$.

Proof. Compare with [26, Thm. 8.6.4] and its proof (Note that the authors meant to cite Thm. 8.5.12 (taking c=0, d=1) instead of Thm. 6.2.9 - a reference which just does not exist within that book). Using the interpretation of the H^i_φ as Hom- and Ext^1-groups, respectively, the assertion is immediate from Prop. 5.3. The last statement now follows from Corollary 4.19. □

Proposition 5.5. *Base extension gives rise to an equivalence of categories*

$$\mathfrak{M}^{\acute{e}t}(\mathbf{B}^\dagger_L) \leftrightarrow \mathfrak{M}^{\acute{e}t}(\mathcal{R}_L).$$

Proof. [19, Prop. 1.6]. □

Lemma 5.6. *(i) $\mathbf{B}_L^\dagger \subseteq \mathcal{R}_L$ are Bézout domains and the strong hypothesis in the sense of [22, Hypothesis 1.4.1] holds, i.e., for any $n \times n$ matrix A over $\mathbf{A}_L^\dagger$ the map $(\mathcal{R}_L/\mathbf{B}_L^\dagger)^n \xrightarrow{1-A\varphi_L} (\mathcal{R}_L/\mathbf{B}_L^\dagger)^n$ is bijective.*

Proof. [22, Prop. 1.2.6]. □

Proposition 5.7. *If V belongs to $Rep_L^\dagger(G_L)$, the following complex concentrated in degrees 0 and 1 is acyclic*

$$0 \longrightarrow D^\dagger_{rig}(V)/D^\dagger(V) \xrightarrow{\varphi-1} D^\dagger_{rig}(V)/D^\dagger(V) \longrightarrow 0, \tag{25}$$

where $D^\dagger_{rig}(V) := \mathcal{R}_L \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V)$. In particular, the complexes

$$0 \longrightarrow D^\dagger_{rig}(V) \xrightarrow{\varphi-1} D^\dagger_{rig}(V) \longrightarrow 0 \quad and \quad 0 \longrightarrow D^\dagger(V) \xrightarrow{\varphi-1} D^\dagger(V) \longrightarrow 0$$

(concentrated in degrees 0 and 1) have the same cohomology groups of for $n \geq 0$.

Proof. This follows from the strong hypothesis in Lemma 5.6 as the Frobenius endomorphism on $M \in \mathfrak{M}^{\acute{e}t}(\mathbf{B}_L^\dagger)$ is of the form $A\varphi_L$ by definition. □

Lemma 5.8. *Base change induces fully faithful embeddings $\Phi(\mathbf{A}_L^\dagger)^{\acute{e}t} \subseteq \Phi(\mathbf{A}_L)^{\acute{e}t}$ and $\Phi(\mathbf{B}_L^\dagger)^{\acute{e}t} \subseteq \Phi(\mathbf{B}_L)^{\acute{e}t}$.*

Proof. As in the proof of Prop. 4.16 this reduces to checking that $\left(\mathbf{A}_L \otimes_{\mathbf{A}_L^\dagger} M\right)^{\varphi=\mathrm{id}} \subseteq M$. By that proposition we know that

$$\left(\mathbf{A}_L \otimes_{\mathbf{A}_L^\dagger} M\right)^{\varphi=\mathrm{id}} \subseteq \left(\tilde{\mathbf{A}}_L \otimes_{\mathbf{A}_L^\dagger} M\right)^{\varphi=\mathrm{id}} \subseteq \tilde{\mathbf{A}}_L^\dagger \otimes_{\mathbf{A}_L^\dagger} M.$$

Since $\mathbf{A}_L \cap \tilde{\mathbf{A}}_L^\dagger = \mathbf{A}_L^\dagger$ within $\tilde{\mathbf{A}}_L$ by definition, the claim follows for the integral version, whence also for the other one by tensoring the integral embedding with L over o_L. □

Remark 5.9. Note that $H^0_\dagger(H_L, V) = H^0(H_L, V)$ and $H^1_\dagger(H_L, V) \subseteq H^1(H_L, V)$. For the latter relation use the previous lemma, which implies that an extension which splits after base change already splits itself, together with Corollary 4.12 and Remark 4.14. In general the inclusion for H^1 is strict as follows indirectly from [19]. Indeed, otherwise the complex

$$0 \longrightarrow D(V)/D^\dagger(V) \xrightarrow{\varphi-1} D(V)/D^\dagger(V) \longrightarrow 0, \tag{26}$$

would be always acyclic, which would imply by the same observation as in Prop. 7.2 below together with [44, Thm. 5.2.10(ii)] that $H^1_\dagger(G_L, V) = H^1(G_L, V)$ in contrast to Remark 5.2.13 in (loc. cit.).

6. The Web of Equivalences

We summarize the various equivalences of categories, for which we only sketch proofs or indicate analogue results whose proofs can be transferred to our setting.

Proposition 6.1. *The following categories are equivalent:*

(i) $Rep_{o_L}(G_L)$,
(ii) $\mathfrak{M}^{ét}(\mathbf{A}_L)$,
(iii) $\mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L)$ *and*
(iv) $\mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger)$.

The equivalences from (*ii*) *and* (*iv*) *to* (*iii*) *are induced by base change.*

Proof. This can be proved in the same way as in [23, Thm. 2.3.5], although it seems to be only a sketch. Another way is to check that the very detailed proof for the equivalence between (i) and (ii) in [42] almost literally carries over to a proof for the equivalence between (i) and (iii). Alternatively, this is a consequence of Prop. 8.2 by [27, Thm. 5.4.6]. See also [29]. For the equivalence between (iii) and (iv) consider the 2-commutative diagram

$$\begin{array}{ccc} \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L^\dagger) & \xrightarrow{\text{faithfully flat base change}} & \mathfrak{M}^{ét}(\tilde{\mathbf{A}}_L) \\ & \nwarrow \quad \swarrow\!\!\nearrow & \\ & \mathrm{Rep}_{o_L}(G_L) & \end{array},$$

which is induced by the isomorphism (13) and immediately implies (essential) surjectivity on objects and morphisms while the faithfulness follows from faithfully flat base change. □

Corollary 6.2. *The following categories are equivalent:*

(i) $Rep_L(G_L)$,
(ii) $\mathfrak{M}^{ét}(\mathbf{B}_L)$,
(iii) $\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L)$ *and*
(iv) $\mathfrak{M}^{ét}(\tilde{\mathbf{B}}_L^\dagger)$.

The equivalences from (*ii*) *and* (*iv*) *to* (*iii*) *are induced by base change.*

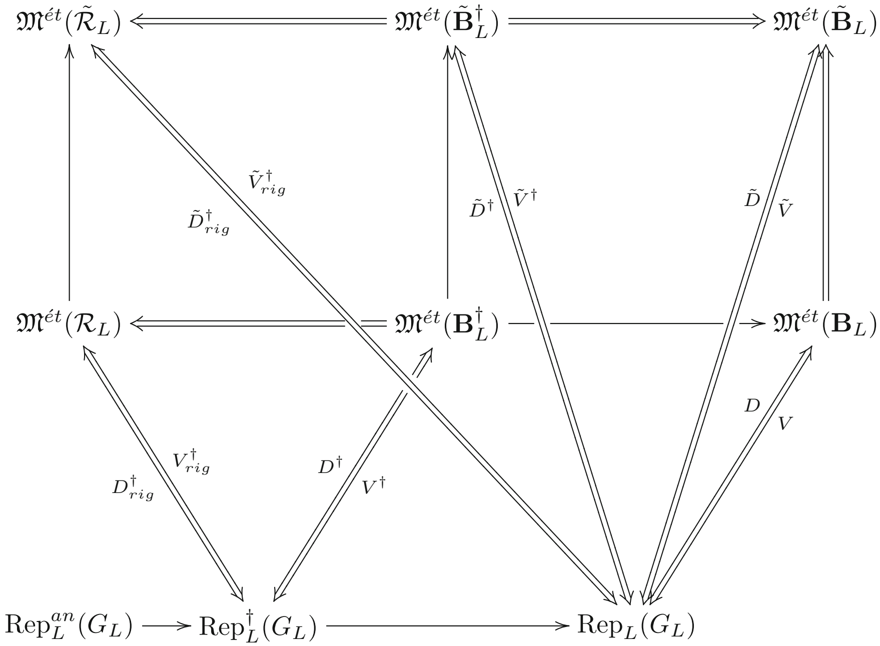
Proof. This follows from Propositions 4.18 and 6.1 by inverting π_L. □

Proposition 6.3. *The categories in Corollary 6.2 are - via base change from (iv) - also equivalent to*

(v) $\mathfrak{M}^{ét}(\tilde{\mathcal{R}}_L)$.

Proof. By definition base change is essentially surjective and it is well-defined - regarding the continuity of the Γ_L-action - by Lemma 5.1 and Lemma 4.5. Since for étale φ_L-modules we know fully faithfulness already, taking Γ_L-invariants gives fully faithfulness for (φ_L, Γ_L)-modules, too.[6] □

Altogether we may visualize the relations between the various categories by the following diagram:



Here all arrows represent functors which are fully faithful, i.e., embeddings of categories. Arrows without label denote base change functors. Under them the functors $D, \tilde{D}, D^\dagger, \tilde{D}^\dagger, D^\dagger_{rig}$, and $\tilde{D}^\dagger_{rig}$ are compatible. The arrows => represent equivalences of categories, while the arrows –> represent embeddings which are not essentially surjective in general. We recall that the quasi-inverse functors are given as follows[7],[8],[9]

$$V(M) = (\mathbf{B} \otimes_{\mathbf{B}_L} M)^{\varphi=1}, \;\; \tilde{V}(M) = (\tilde{\mathbf{B}} \otimes_{\tilde{\mathbf{B}}_L} M)^{\varphi=1}, \;\; V^\dagger(M) = (\mathbf{B}^\dagger \otimes_{\mathbf{B}_L^\dagger} M)^{\varphi=1},$$

$$\tilde{V}^\dagger(M) = (\tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}_L^\dagger} M)^{\varphi=1}, \;\; \tilde{V}^\dagger_{rig}(M) = (\tilde{\mathcal{R}} \otimes_{\tilde{\mathcal{R}}_L} M)^{\varphi=1} \text{ and } V^\dagger_{rig}(M) = (\tilde{\mathcal{R}} \otimes_{\mathcal{R}_L} M)^{\varphi=1}.$$

[6]Regarding φ_L-modules cf. [26, the equivalence between (e) and (f) of Thm. 8.5.6], see also Thm. 8.5.3 in (loc. cit.), the equivalence (d) to (e).

[7]By [19, Prop. 1.5 (a)] the third formula holds while by (c) there is an equivalence of categories.

[8]For the fourth formula compare with the proof of Propositon 4.16 omitting the index L in $\tilde{\mathbf{A}}_L^\dagger$, etc. to conclude that $(\tilde{\mathbf{B}} \otimes_{\mathbf{B}_L^\dagger} M)^{\varphi=1} = (\tilde{\mathbf{B}}^\dagger \otimes_{\mathbf{B}_L^\dagger} M)^{\varphi=1}$.

[9]Since $V^\dagger(M_0) \subseteq V^\dagger_{rig}(M) \subseteq \tilde{V}^\dagger_{rig}(\tilde{\mathcal{R}}_L \otimes_{\mathcal{R}_L} M)$ for some model M_0 over $\mathbf{B}_L^\dagger$ of M we obtain the last formula.

7. Cohomology: Herr Complexes

The aim of this section is to compare the Herr complexes of the various (φ_L, Γ_L)-modules attached to a given Galois representation.

We fix some open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$.

Let M_0 be a complete linearly topologised o_L-module with continuous U-action and with continuous U-equivariant endomorphism f. We define

$$\mathcal{T} := \mathcal{T}_{f,U}(M_0) := \operatorname{cone}\left(\mathcal{C}^\bullet(U, M_0) \xrightarrow{(f)_*-1} \mathcal{C}^\bullet(U, M_0)\right)[-1]$$

the mapping fibre of $\mathcal{C}^\bullet(U, f-1)$. The importance of this generalized Herr-complex is given by the fact that it computes Galois cohomology when applied to $M_0 = D(V)$ and $f = \varphi_{D(V)}$:

Theorem 7.1. *Let V be in $\mathrm{Rep}_L(G_L)$ For $D(V)$ the corresponding (φ_L, Γ_L)-module over $\mathbf{B}_L$ we have canonical isomorphisms*

$$h^* = h^*_{U,V} : H^*(L', V) \xrightarrow{\cong} h^*(\mathcal{T}_{\varphi,U}(D(V))) \tag{27}$$

which are functorial in V and compatible with restriction and corestriction.

Proof. To this aim let T be a G_L-stable lattice of V. In [31, Thm. 5.1.11.], [30, Thm. 5.1.11.] it is shown that the cohomology groups of $\mathcal{T}_{\varphi,U}(D(T))$ are canonically isomorphic to $H^i(L', T)$ for all $i \geq 0$, whence the cohomology groups of $\mathcal{T}_{\varphi,U}(D(T))[\frac{1}{\pi_L}]$ are canonically isomorphic to $H^i(L', V)$ for all $i \geq 0$. □

Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U. We now fix topological generators $\gamma_1, \ldots \gamma_d$ of U' and we set $\Lambda := \Lambda(U')$. By [32, Thm. II.2.2.6] the U'-actions extends to continuous Λ-action and one has $\mathrm{Hom}_{\Lambda,cts}(\Lambda, M_0) = \mathrm{Hom}_\Lambda(\Lambda, M_0)$. Consider the (homological) complexes $K_\bullet(\gamma_i) := [\Lambda \xrightarrow{\gamma_i - 1} \Lambda]$ concentrated in degrees 1 and 0 and define the Koszul complexes

$$K_\bullet := K_\bullet^{U'} := K_\bullet(\gamma) := \bigotimes_{i=1}^{d}{}_{\Lambda} K_\bullet(\gamma_i) \quad \text{and}$$

$$K^\bullet(M_0) := K^\bullet_{U'}(M_0) := \mathrm{Hom}^\bullet_\Lambda(K_\bullet, M_0) \cong \mathrm{Hom}^\bullet_\Lambda(K_\bullet, \Lambda) \otimes_\Lambda M_0 = K^\bullet(\Lambda) \otimes_\Lambda M_0.$$

Following [14, §4.2] and [44, (169)] we obtain a quasi-isomorphism

$$K^\bullet_{U'}(M_0) \xrightarrow{\simeq} \mathcal{C}^\bullet(U', M_0) \tag{28}$$

inducing the quasi-isomorphism

$$K_{f,U'}(M_0) \xrightarrow{\simeq} \mathcal{T}_{f,U'}(M_0), \tag{29}$$

where we denote by $K_{f,U'}(M_0) := \operatorname{cone}\left(K^\bullet(M_0) \xrightarrow{f-\mathrm{id}} K^\bullet(M_0)\right)[-1]$ the mapping fibre of $K^\bullet(f)$. More generally, by [44, Lem. A.0.1] we obtain a canonical quasi-isomorphism

$$K_{f,U'}(M^\Delta) \xrightarrow{\simeq} \mathcal{T}_{f,U}(M), \tag{30}$$

i.e., by Theorem 7.1 we also have canonical isomorphisms

$$h^* = h^*_{U,V} : H^*(L',V) \xrightarrow{\cong} h^*(K_{f,U'}(D(V)^\Delta)). \tag{31}$$

The next proposition extends this result to $\tilde{D}(V), \tilde{D}^\dagger(V)$ and $\tilde{D}^\dagger_{rig}(V)$ instead of $D(V)$.

Proposition 7.2. *If V belongs to $Rep_L(G_L)$, the canonical inclusions of Herr complexes*

$$K^\bullet_{\varphi,U'}(\tilde{D}^\dagger(V)^\Delta) \subseteq K^\bullet_{\varphi,U'}(\tilde{D}^\dagger_{rig}(V)^\Delta),$$
$$K^\bullet_{\varphi,U'}(\tilde{D}^\dagger(V)^\Delta) \subseteq K^\bullet_{\varphi,U'}(\tilde{D}(V)^\Delta) \text{ and}$$
$$K^\bullet_{\varphi,U'}(D(V)^\Delta) \subseteq K^\bullet_{\varphi,U'}(\tilde{D}(V)^\Delta)$$

are quasi-isomorphisms and their cohomology groups are canonically isomorphic to $H^i(L',V)$ for all $i \geq 0$.

Proof. Forming Koszul complexes with regard to U' we obtain the following diagram of (double) complexes with exact columns

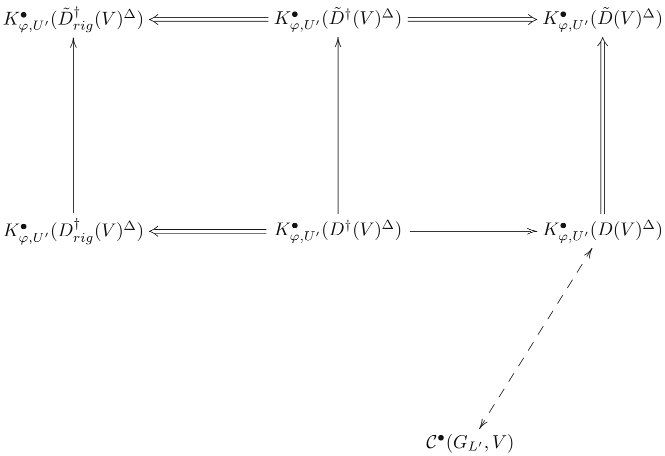
$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K^\bullet(D(V)^\Delta) & \xrightarrow{\varphi-1} & K^\bullet(D(V)^\Delta) \\
\downarrow & & \downarrow \\
K^\bullet(\tilde{D}(V)^\Delta) & \xrightarrow{\varphi-1} & K^\bullet(\tilde{D}(V)^\Delta) \\
\downarrow & & \downarrow \\
K^\bullet((\tilde{D}(V)/D(V))^\Delta) & \xrightarrow[\cong]{\varphi-1} & K^\bullet((\tilde{D}(V)/D(V))^\Delta) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

in which the bottom line is an isomorphism of complexes by 4.12, as the action of Δ commutes with φ. Hence, going over to total complexes gives an exact sequence

$$0 \to K^\bullet_{\varphi,U}(D(V)^\Delta) \to K^\bullet_{\varphi,U}(\tilde{D}(V)^\Delta) \to K^\bullet_{\varphi,U}((\tilde{D}(V)/D(V))^\Delta) \to 0,$$

in which $K^\bullet_{\varphi,U}((\tilde{D}(V)/D(V))^\Delta)$ is acyclic. Thus we have shown the statement regarding the last inclusion. The other two cases are dealt with similarly, now using (24) and 4.19 combined with (8). It follows in particular that all six Koszul complexes in the statement are quasi-isomorphic. Therefore the second part of the assertion follows from (31). □

In accordance with diagram at the end of Sect. 6 we may visualize the relations between the various Herr complexes by the following diagram:



Here all arrows represent injective maps of complexes, among which the arrows => represent quasi-isomorphisms, while the arrows −> need not induce isomorphisms on cohomology, in general. The interrupted arrow − −> means a map in the derived category while < − −> means a quasi-isomorphism in the derived category. By [44, Lem. A.0.1] we have a analogous diagram for $\mathcal{T}_{\varphi,U}(?(V))$ with $? \in \{D, \tilde{D}, D^\dagger, \tilde{D}^\dagger, D^\dagger_{rig}, \tilde{D}^\dagger_{rig}\}$.

Remark 7.3. The image of

$$h^i(\mathcal{T}_{\varphi,U}(D^\dagger_{rig}(V))) \cong h^i(K^\bullet_{\varphi,U'}(D^\dagger_{rig}(V)^\Delta)) \cong h^i(K^\bullet_{\varphi,U'}(D^\dagger(V)^\Delta)) \cong h^i(\mathcal{T}_{\varphi,U}(D^\dagger(V)))$$

in $H^i(L', V)$ is independent of the composite (= path) in above diagram.

8. Weakly Decompleting Towers

Kedlaya and Liu's developed in [27, §5] the concept of perfectoid towers and studied their properties in an axiomatic way. The aim of this section is to show that the Lubin–Tate extensions considered in this article form a *weakly decompleting*, but not a *decompleting* tower, properties which we will recall or refer to in the course of this section. Moreover, we have to show that the *axiomatic* period rings coincide with those introduced earlier.

In the sense of Def. 5.1.1 in (loc. cit.) the sequence $\Psi = (\Psi_n : (L_n, o_{L_n}) \to (L_{n+1}, o_{L_{n+1}}))_{n=0}^{\infty}$ forms a finite étale tower over (L, o_L) or $X := \mathrm{Spa}(L, o_L)$, which is perfectoid as $\hat{L}_\infty$ is by [42, Prop. 1.4.12].[10]

Therefore we can use the perfectoid correspondence [27, Thm. 3.3.8] to associate with $(\hat{L}_\infty, o_{\hat{L}_\infty})$ the pair

$$(\tilde{R}_\Psi, \tilde{R}_\Psi^+) := (\hat{L}_\infty^\flat, o_{\hat{L}_\infty}^\flat).$$

Now we recall the variety of period rings, which Kedlaya and Liu attach to the tower, in our notation, starting with

Perfect period rings:

$$\tilde{\mathbf{A}}_\Psi := \tilde{\mathbf{A}}_L = W(\hat{L}_\infty^\flat)_L,$$

$$\tilde{\mathbf{A}}_\Psi^+ := W(o_{\hat{L}_\infty}^\flat)_L \subseteq \tilde{\mathbf{A}}_\Psi^{\dagger,r} := \tilde{\mathbf{A}}_L^{\dagger,r} = \left\{ x = \sum_{i\geq 0} \pi_L^i [x_i] \in W(\hat{L}_\infty^\flat)_L \mid |\pi_L^i| |x_i|_\flat^r \xrightarrow{i\to\infty} 0 \right\},$$

$$\tilde{\mathbf{A}}_\Psi^\dagger := \bigcup_{r>0} \tilde{\mathbf{A}}_\Psi^{\dagger,r} = \tilde{\mathbf{A}}_L^\dagger$$

Imperfect period rings:

To introduce these we first recall the map $\Theta : W(o_{\mathbb{C}_p}^\flat)_L \to o_{\mathbb{C}_p}$, $\sum_{i\geq 0} \pi_L^i [x_i] \mapsto \sum \pi_L^i x_i^\sharp$, which extends to a map $\Theta : \tilde{\mathbf{A}}_\Psi^{\dagger,s} \to \mathbb{C}_p$ for all $s \geq 1$; for arbitrary $r > 0$ and $n \geq -\log_q r$ the composite $\tilde{\mathbf{A}}_\Psi^{\dagger,r} \xrightarrow{\varphi_L^{-n}} \tilde{\mathbf{A}}_\Psi^{\dagger,1} \xrightarrow{\Theta} \mathbb{C}_p$ is well defined and continuous as it is easy to check. It is a homomorphism of o_L-algebras by [42, Lem. 1.4.18].

Following [27, §5] we set $\mathbf{A}_\Psi^{\dagger,r} := \{x \in \tilde{\mathbf{A}}_\Psi^{\dagger,r} | \Theta(\varphi_q^{-n}(x)) \in L_n \text{ for all } n \geq -\log_q r\}$, $\mathbf{A}_\Psi^\dagger := \bigcup_{r>0} \mathbf{A}_\Psi^{\dagger,r}$, its completion $\mathbf{A}_\Psi := (\mathbf{A}_\Psi^\dagger)^{\wedge \pi_L - adic}$, and residue field $R_\Psi := \mathbf{A}_\Psi/(\pi_L) = (\mathbf{A}_\Psi^\dagger)/(\pi_L) \subseteq \tilde{R}_\Psi$, $R_\Psi^+ := R_\Psi \cap \tilde{R}_\Psi^+$.

Note that $\omega_{LT} = \{[\iota(t)]\} \in \tilde{\mathbf{A}}_\Psi^+ := W(o_{\hat{L}_\infty}^\flat)_L \subseteq \tilde{\mathbf{A}}_\Psi^{\dagger,r}$ for all $r > 0$ (in the notation of [42]). [42, Lem. 2.1.12] shows

$$\Theta(\varphi_q^{-n}(\omega_{LT})) = \Theta(\{[\varphi_q^{-n}(\omega)]\}) = \lim_{i\to\infty} [\pi_L^i]_\varphi(z_{i+n}) = z_n \in L_n,$$

where $t = (z_n)_{n\geq 1}$ is a fixed generator of the Tate module T_π of the formal Lubin–Tate group and $\omega = \iota(t) \in W(o_{\mathbb{C}_p}^\flat)_L$ is the reduction of ω_{LT} modulo π_L satisfying with $\mathbf{E}_L = k((\omega))$. Therefore ω_{LT} belongs to $\mathbf{A}_\Psi^+ := \mathbf{A}_\Psi \cap \tilde{\mathbf{A}}_\Psi^+$. Then it is clear that first $\mathbf{A}_L^+ := o_L[[\omega_{LT}]] \subseteq \tilde{\mathbf{A}}_\Psi^\dagger$ and by the continuity of $\Theta \circ \varphi_L^{-n}$ even $\mathbf{A}_L^+ \subseteq \mathbf{A}_\Psi^\dagger$ holds. Since $\omega_{LT}^{-1} \in \tilde{\mathbf{A}}_\Psi^{\dagger, \frac{q-1}{q}}$ by [45, Lem. 3.10] (in

[10] In the notation of [27]: $E = L$, $\varpi = \pi_L$, $h = r$, $k := o_L/(\pi_L) = \mathbb{F}_q$, i.e. $q = p^r$. $A_{\Psi,n} := L_n$, $A_{\Psi,n}^+ := o_{L_n}$, $X := \mathrm{Spa}(L, o_L)$ with the obvious transition maps which are finite étale.

$(A_\Psi, A_\Psi^+) := \varinjlim_n (A_{\Psi,n}, A_{\Psi,n}^+) = (L_\infty, o_{L_\infty})$

$(\tilde{A}_\Psi, \tilde{A}_\Psi^+) := (A_\Psi, A_\Psi^+)^{\wedge \pi_L - adic} = (\hat{L}_\infty, o_{\hat{L}_\infty})$

analogy with [11, Cor. II.1.5]) and $\Theta\circ\varphi_L^{-n}$ is a ring homomorphism, it follows that $\omega_{LT}^{-1}\in\mathbf{A}_\Psi^{\dagger,\frac{q-1}{q}}$ and $o_L[[\omega_{LT}]][\frac{1}{\omega_{LT}}]\subseteq\mathbf{A}_\Psi^{\dagger}$.

Lemma 8.1. *We have* $R_\Psi^+=\mathbf{E}_L^+$ *and* $R_\Psi=\mathbf{E}_L$.

Proof. From the above it follows that $\mathbf{E}_L\subseteq R_\Psi$, whence $\mathbf{E}_L^{perf}\subseteq R_\Psi^{perf}\subseteq\tilde{R}_\Psi=\hat{L}_\infty^\flat$ the latter being perfect. Since $\widehat{\mathbf{E}_L^{perf}}=\hat{L}_\infty^\flat$ by [42, Prop. 1.4.17] we conclude that

$$\mathrm{R}_\Psi^{\mathrm{perf}}\text{ is dense in }\tilde{R}_\Psi. \tag{32}$$

By [27, Lem. 5.2.2] have the inclusion

$$R_\Psi^+\subseteq\{x\in\tilde{R}_\Psi|x=(\bar{x}_n)\text{ with }\bar{x}_n\in o_{L_n}/(z_1)\text{ for }n>>1\}\overset{(*)}{=}\mathbf{E}_L^+=k[[\omega]]$$

where the equality (*) follows from work of Wintenberger as recalled in [42, Prop. 1.4.29]. Since $\mathbf{E}_L^+\subseteq\tilde{R}_\Psi^+$ by its construction in (loc. cit.), we conclude that $R_\Psi^+=\mathbf{E}_L^+$.

Since each element of R_Ψ is of the form $\frac{a}{\omega^m}$ with $a\in R_\Psi^+$ and $m\geq 0$ by [42, Lem. 1.4.6][11], we conclude that $R_\Psi=\mathbf{E}_L$. □

Thus for each $r>0$ such that $\omega_{LT}^{-1}\in\mathbf{A}_\Psi^{\dagger,r}$, reduction modulo π_L induces a surjection $\mathbf{A}_\Psi^{\dagger,r}\twoheadrightarrow R_\Psi$. Recall that Ψ is called weakly decompleting, if

(i) R_Ψ^{perf} is dense in $\tilde{R}_\Psi$.
(ii) for some $r>0$ we have a strict surjection $\mathbf{A}_\Psi^{\dagger,r}\twoheadrightarrow R_\Psi$ induced by the reduction modulo π_L for the norms $|-|_r$ defined by $|x|_r:=\sup_i\{|\pi_L^i||x_i|_\flat^r\}$ for $x=\sum_{i\geq 0}\pi_L^i[x_i]$, and $|-|_\flat^r$.

We recall from [16, Prop. 1.4.3.] or [26, Prop. 5.1.2 (a)] that $|-|_r$ is multiplicative.

Proposition 8.2. *The above tower* Ψ *is weakly decompleting.*

Proof. Since (32) gives (i), only *(ii)* is missing: Since ω_{LT} has $[\omega]$ in degree zero of its Teichmüller series, we may and do choose $r>0$ such that $|\omega_{LT}-[\omega]|_r<|\omega|_\flat^r$. Then $|\omega_{LT}|_r=\max\{|\omega_{LT}-[\omega]|_r,|\omega|_\flat^r\}=|\omega|_\flat^r$. Consider the quotient norm $\|b\|^{(r)}=\inf_{a\in\mathbf{A}_\Psi^{\dagger,r},a\equiv b\mod\pi_L}|a|_r$.

Now let $b=\sum_{n\geq n_0}a_n\omega^n\in R_\Psi=k((\omega))$ with $a_{n_0}\neq 0$. Lift each $a_n\neq 0$ to $\breve{a}_n\in o_L^\times$ and set $\breve{a}_n=0$ otherwise. Then, for the lift $x:=\sum_{n\geq n_0}\breve{a}_n\omega_{LT}^n$ of b we have by the multiplicativity of $|-|_r$ that

$$\|b\|^{(r)}\leq|x|_r=(|\omega_{LT}|_r)^{n_0}=(|\omega|_\flat^r)^{n_0}=|b|_\flat^r.$$

Since, the other inequality $|b|_\flat^r\leq\|b\|^{(r)}$ giving by continuity is clear, the claim follows. □

[11] For $\alpha\in R_\Psi$ there exist $m\geq 0$ such that $|\omega^m\alpha|_\flat\leq 1$, i.e., $\omega^m\alpha\in R_\Psi^+$.

Proposition 8.3. $\mathbf{A}_L = \mathbf{A}_\Psi$.

Proof. Both rings have the same reduction modulo π_L. And using that the latter element is not a zero-divisor in any of these rings we conclude inductively, that $\mathbf{A}_L/\pi_L^n\mathbf{A}_L = \mathbf{A}_\Psi/\pi_L^n\mathbf{A}_\Psi$ for all n. Taking projective limits gives the result. □

Proposition 8.4. $\mathbf{A}_L^\dagger = \mathbf{A}_\Psi^\dagger$.

Proof. By [27, Lem. 5.2.10] we have that $\mathbf{A}_\Psi^\dagger = \tilde{\mathbf{A}}_L^\dagger \cap \mathcal{R}_L$. On the other hand $\mathbf{A}_L^\dagger = (\tilde{\mathbf{A}}^\dagger \cap \mathbf{A})^{H_L} = \tilde{\mathbf{A}}_L^\dagger \cap \mathbf{A}$ is contained in $\mathcal{R}_L$ by Remark 4.20, whence $\mathbf{A}_L^\dagger \subseteq \mathbf{A}_\Psi^\dagger$ while the inclusion $\mathbf{A}_\Psi^\dagger \subseteq \tilde{\mathbf{A}}^\dagger \cap \mathbf{A}_L = \mathbf{A}_L^\dagger$ follows from Prop. 8.3. □

In Definition 5.6.1 in (loc. cit.) they define the property *decompleting* for a tower Ψ, which we are not going to recall here as it is rather technical. The cyclotomic tower over $\mathbb{Q}_p$ is of this kind for instance. If our Ψ would be decompleting, the machinery of (loc. cit.), in particular Theorems 5.7.3/4, adapted to the Lubin–Tate setting would imply that all the categories at the end of Sect. 6 are equivalent, which contradicts Remark 4.25.

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