Results Math (2023) 78:210

1422-6383/23/060001-29
*published online* August 19, 2023
https://doi.org/10.1007/s00025-023-01992-6

**Results in Mathematics**

# Topological Properties of Weighted Composition Operators in Sequence Spaces

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**Abstract.** For fixed sequences $u = (u_i)_{i\in\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}}$, we consider the weighted composition operator $W_{u,\varphi}$ with symbols $u, \varphi$ defined by $x = (x_i)_{i\in\mathbb{N}} \mapsto u(x \circ \varphi) = (u_i x_{\varphi_i})_{i\in\mathbb{N}}$. We characterize the continuity and the compactness of the operator $W_{u,\varphi}$ when it acts on the weighted Banach spaces $l^p(v)$, $1 \leq p \leq \infty$, and $c_0(v)$, with $v = (v_i)_{i\in\mathbb{N}}$ a weight sequence on $\mathbb{N}$. We extend these results to the case in which the operator $W_{u,\varphi}$ acts on sequence (LF)-spaces of type $l_p(\mathcal{V})$ and on sequence (PLB)-spaces of type $a_p(\mathcal{V})$, with $p \in [1,\infty] \cup \{0\}$ and $\mathcal{V}$ a system of weights on $\mathbb{N}$. We also characterize other topological properties of $W_{u,\varphi}$ acting on $l_p(\mathcal{V})$ and on $a_p(\mathcal{V})$, such as boundedness, reflexivity and to being Montel.

**Mathematics Subject Classification.** Primary 47B33, 47B37, 46A13, 46A45; Secondary 46M40, 47B07.

**Keywords.** Sequence (LF)-space, sequence (PLB)-space, weighted composition operator, bounded operator, compact operator, montel operator, reflexive operator.

## 1. Introduction

Shift operators are of interest because many classical operators can be viewed as such operators and also because they have been through the years a favourite testing ground for operator-theorists. The basic model of all shifts is the *(unilateral) backward shift*

$$B(x_1, x_2, x_3 \ldots) = (x_2, x_3, x_4, \ldots).$$

Shift operators in a Banach space setting where first studied by Rolewicz [34], who showed that for any $c > 1$ the multiple $cB$ of the backward shift $B$ on the sequence space $l^p$, $1 \leq p < \infty$, or $c_0$ is hypercyclic. A generalization of the (unilateral) backward shift is the *(unilateral) weighted backward shift*

$$B_w(x_1, x_2, x_3 \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots),$$

where $w = (w_n)_{n\in\mathbb{N}}$ is a sequence of non-zero scalars, called a *weight sequence*. For instance, the differentiation operator $D$ on the space $H(\mathbb{C})$ of entire functions can be regarded as a particular weighted backward shift operator. Indeed, we have

$$De_n = ne_{n-1},\ n \in \mathbb{N}_0,$$

where $(e_n)_{n\in\mathbb{N}_0}$ denotes the canonical basis on $H(\mathbb{C})$ given by $e_n(z) := z^n$, for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$, and where $e_{-1} = 0$.

The study of (weighted) shift operators on various spaces of sequences or functions has made possible to obtain an astounding number of deep results in the setting of the dynamic properties of linear operators such as topological transitivity, hypercyclicity and linear chaos (see, e.g., [10,11,17,20,21,28,29,35,36,40] and the references therein).

In the last years, the attention of the researchers has been attracted by the study of power boundedness, topologizability and mean ergodicity for linear operators acting on Banach spaces as well as on locally convex Hausdorff spaces such as Fréchet spaces or (LB)-spaces (see, e.g., [1–3,14,32,33]). Particular interest has been devoted to the study of these properties for weighted composition operators in function spaces or in sequence spaces (see [5,7–9,15,16,23–26,37] and the references therein). A weighted composition operator $W_{u,\varphi}$, with $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, when it acts on a sequence space $X \subseteq \mathbb{K}^N$ is of this type

$$x = (x_i)_{i\in\mathbb{N}} \in X \mapsto W_{u,\varphi}(x) := (w_i x_{\varphi_i})_{i\in\mathbb{N}}.$$

(the operator $W_{u,\varphi}$ is also called weighted pseudo shift if $\varphi\colon \mathbb{N} \to \mathbb{N}$ is injective). Weighted composition operators acting on sequence spaces are then a generalization of weighted shift operators. Observe that if $\varphi_i = i$ for every $i \in \mathbb{N}$, then $W_{u,\varphi}$ becomes the multiplication operator defined by $M_u(x) := ux = (u_i x_i)_{i\in\mathbb{N}}$, for $x \in X$. For $u_i = 1$ for every $i \in \mathbb{N}$, the operator $W_{u,\varphi}$ becomes the composition operator defined by $C_\varphi(x) := x \circ \varphi = (x_{\varphi_i})_{i\in\mathbb{N}}$, for $x \in X$.

As some properties like being a compact operator or a Montel operator are closely related to the (uniform) mean ergodicity (see, e.g., [4] and the rerences therein), the aim of this paper is to investigate and to characterize the continuity, the boundedness, the compactness and the property of being Montel or reflexive for weighted composition operators $W_{u,\varphi}$ in sequence (LF)-spaces of type $l_p(\mathcal{V})$ and in sequence (PLB)-spaces of type $a_p(\mathcal{V})$, with $p \in [1,\infty] \cup \{0\}$ and $\mathcal{V}$ a system of weights on $\mathbb{N}$. Köthe echelon spaces and Köthe co-echelon spaces are spaces of type $a_p(\mathcal{V})$ and $l_p(\mathcal{V})$, respectively, for particular systems $\mathcal{V}$ of weights.

The paper is organized as follows. In Sect. 2, we recall general definitions and results on (LF)- and (PLB)-spaces and on operators acting on these spaces. Section 3 is devoted to the study of the topological properties of weigthed composition operators. In particular, in Sect. 3.1 we introduce the sequence (LF)- and (PLB)-spaces and give their relevant properties. In Sect. 3.2 we establish when the weighted composition operator $W_{u,\varphi}$ acts continuously or compactly on the sequence Banach spaces $l^p(v)$, for $p \in [1,\infty] \cup \{0\}$ (Theorems 3.4 and 3.12). The continuity, the boundedness and the compactness of the operator $W_{u,\varphi}$ in the sequence spaces $l_p(\mathcal{V})$ and $a_p(\mathcal{V})$, for $p \in [1,\infty] \cup \{0\}$, are characterized in Sect. 3.3 (Theorems 3.14, 3.15 and 3.16 for the (LF)-case; Theorems 3.17, 3.18 and 3.19 for the (PLB)-case). Sects. 3.5 and 3.6 are devoted to establish when the weighted composition operator $W_{u,\varphi}$ acting either on Köthe echelon spaces or on sequence (LF)-spaces of type $l_p(\mathcal{V})$, with $p \in [1,\infty] \cup \{0\}$, is Montel or reflexive, respectively.

## 2. Definitions and General Results on (LF)- and (PLB)-Spaces

Let $E$ and $F$ be two locally convex Hausdorff spaces (briefly, lcHs for locally convex Hausdorff space). We denote by $\mathcal{L}(E,F)$ the space of all continuous linear operators from $E$ into $F$. In particular, $\mathcal{L}_s(E,F)$ ($\mathcal{L}_b(E,F)$, resp.) denotes $\mathcal{L}(E,F)$ endowed with the strong operator topology $\tau_s$ ($\mathcal{L}(E,F)$ endowed with the topology $\tau_b$ of the uniform convergence on bounded subsets of $E$, resp.). In case $F = E$, we simply write $\mathcal{L}(E)$, $\mathcal{L}_s(E)$ and $\mathcal{L}_b(E)$.

Let $E$ and $F$ be two lcHs and $T$ be a linear operator from $E$ into $F$. The operator $T$ is called *bounded* if $T$ maps some 0-neighborhood of $E$ into a bounded subset of $F$. The operator $T$ is called *compact* if $T$ maps some 0-neighborhood of $E$ into a relatively compact subset of $F$. We denote by $\mathcal{K}(E,F)$ the space of all compact linear operators from $E$ into $F$. We observe that if $T$ is a bounded or compact operator from $E$ into $F$, then it is necessarily continuous, i.e., $T \in \mathcal{L}(E,F)$. Moreover, an operator $T \in \mathcal{L}(E,F)$ is called *Montel* (*reflexive*, resp.) if $T$ maps bounded subsets of $E$ into relatively compact (relatively weakly compact, resp.) subsets of $F$. In case $E$ is a bornological lcHs, the assumption on the continuity of $T$ is not necessary because in such a case every linear operator from $E$ into $F$ mapping bounded subsets of $E$ into relatively (weakly) compact subsets of $F$ is continuous. If $F$ is a reflexive lcHs, then every $T \in \mathcal{L}(E,F)$ is reflexive. If $E$ and $F$ are Banach spaces, then a linear operator $T \colon E \to F$ is Montel if, and only if, it is compact. We refer the reader to [27] for more details.

In the following we recall some necessary definitions and collect some results on (LF)- or (PLB)-spaces and on operators acting in such spaces. We first consider the case of (LF)-spaces.

A lcHs $E$ is called an (LF)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of Fréchet spaces with $E_n \hookrightarrow E_{n+1}$ continuously such that $E = \bigcup_{n\in\mathbb{N}} E_n$

and the topology of $E$ coincides with the finest locally convex topology for which each inclusion $E_n \hookrightarrow E$ is continuous. In such a case, we simply write $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$. The sequence $\{E_n\}_{n\in\mathbb{N}}$ is called a *defining inductive spectrum* for $E$. In this paper the (LF)-spaces are Hausdorff by definition. The space $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ is called an (LB)-space if $E_n$ is a Banach space for all $n \in \mathbb{N}$. An (LF)-space $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ is called *regular* if every bounded subset $B$ of E is contained and bounded in $E_n$ for some $n \in \mathbb{N}$. Every complete (LF)-space is always regular.

An (LF)-space can satisfy stronger regularity conditions.

Let $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ be an (LF)-space and $\tau$ ($\tau_n$, resp.) denote the locally convex topology of $E$ (of $E_n$, for $n \in \mathbb{N}$, resp.). The (LF)-space $E$ is said to satisfy the *condition (M)* (($\mathrm{M}_0$), resp.) *of Retakh* if there exists an increasing sequence $\{U_n\}_{n\in\mathbb{N}}$ of subsets of $E$ such that $U_n$ is an absolutely convex 0-neighborhood of $E_n$ for all $n \in \mathbb{N}$ satisfying

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall \mu \geq m\colon \tau_\mu \text{ and } \tau_m \text{ induce the same topology on } U_n,$$

($\forall n \in \mathbb{N}\, \exists m \geq n\ \forall \mu \geq m\colon \sigma(E_\mu, E'_\mu)$ and $\sigma(E_m, E'_m)$ induce the same topology on $U_n$, resp.). An (LF)-space satisfying the condition (M) (($\mathrm{M}_0$), resp.) is called *acyclic* (*weakly acyclic*, resp.). Every acyclic (LF)-space is weakly acyclic and also complete (see [41, Corollary 6.5]).

The (LF)-space $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ is called *compactly retractive* if every compact subset $K$ of $E$ is contained and compact in $E_n$ for some $n \in \mathbb{N}$. The (LF)-space $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ is called *boundedly retractive* if every bounded subet $B$ of $E$ is contained and bounded in some step $E_n$ and the topologies of $E$ and $E_n$ coincide on $B$. While, the (LF)-space $E$ is called *sequentially retractive* if every convergent sequence in $E$ is contained in some step $E_n$ and converges there. We observe that, in view of Grothendieck's factorization theorem [22, p.147], all conditions above do not depend on the defining inductive spectrum of $E$.

Recall the following deep result due to Wengenroth [41].

**Theorem 2.1** ([41, Theorem 6.4]). *For an (LF)-space $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ the following conditions are equivalent:*

(1) *$E$ satisfies condition (M);*
(2) *$E$ is boundedly retractive;*
(3) *$E$ is compactly retractive;*
(3) *$E$ is sequentially retractive.*

The characterization of the continuity of operators between (LF)-spaces is well-known and due to Grothendieck. The characterization of the boundedness or the compactness as well as the property to being Montel or reflexive for operators acting between (LF)-spaces has been given in [31] (see also [16]) as it follows.

**Theorem 2.2** ([31, §2.2]). *Let $E = \mathrm{ind}_{m\in\mathbb{N}} E_m$ and $F = \mathrm{ind}_{n\in\mathbb{N}} F_n$ be two (LF)-spaces. Let $T\colon E \to F$ be a linear operator. The following assertions hold true:*

(1) *Assume that $F$ is regular. The operator $T$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T\colon E_m \to F_n$ is bounded.*
(2) *Assume that $F$ satisfies the condition (M). The operator $T$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T\colon E_m \to F_n$ is compact.*
(3) *Assume that $E$ is regular and $F$ satisfies the condition (M). The operator $T$ is Montel if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $T(E_m) \subset F_n$ and the restriction $T\colon E_m \to F_n$ is Montel.*
(4) *Assume that $E$ is regular and $F$ satisfies the condition ($M_0$). The operator $T$ is reflexive if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $T(E_m) \subset F_n$ and the restriction $T\colon E_m \to F_n$ is reflexive.*

A lcHs $E$ is called a (PLB)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of (LB)-spaces with $E_{n+1} \hookrightarrow E_n$ continuously, for $n \in \mathbb{N}$, such that $E = \bigcap_{n\in\mathbb{N}} E_n$ and the topology of $E$ is the coarsest locally convex topology for which each inclusion $E \hookrightarrow E_n$ is continuous. In such a case, we simply write $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$. Clearly, a (PLB)-space $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ is complete whenever $E_n$ is a complete (LB)-space for an infinite number of indices $n$.

The characterization of the continuity as well as the boundedness and the compactness for operators acting between (PLB)-spaces has been given in [6]. We collect these characterizations in the following result.

**Theorem 2.3** [6, §2]. *Let $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ be a (PLB)-space such that the continuous inclusion $E \hookrightarrow E_n$ has dense range for all $n \in \mathbb{N}$. Let $F = \operatorname{proj}_{k\in\mathbb{N}} F_k$ be a (PLB)-space such that $F_k$ is a complete (LB)-space for all $k \in \mathbb{N}$. Let $T\colon E \to F$ be a linear operator. Then the following assertions hold true:*

(1) *The operator $T$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator $T$ admits a unique linear continuous extention $T_k^n$ from $E_n$ into $F_k$.*
(2) *The operator $T$ is bounded (compact, resp.) if, and only if, there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator $T$ admits a unique linear extension $T_k^n\colon E_n \to F_k$ which is bounded (compact, resp.).*

We recall from [6] the following remarks, which are useful for the next sections.

*Remark 2.4* Let $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ be a (PLB)-space. Let $\tau_n$ denote the locally convex topology of $E_n$, for $n \in \mathbb{N}$.

(1) From the proof of Theorem 2.3(1), it follows that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such the operator $T\colon (E, \tau_n) \to F_k$ is continuous also in the case $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ is a (PLB)-space with no dense inclusion in $E_n$ for any $n \in \mathbb{N}$.
(2) From the proof of Theorem 2.3(2), it follows that there exists $n_0 \in \mathbb{N}$ such that the operator $T\colon (E, \tau_{n_0}) \to F_k$ is bounded (compact, resp.) for

all $k \in \mathbb{N}$ also in the case that $E = \mathrm{proj}_{n\in\mathbb{N}} E_n$ is a (PLB)-space with no dense inclusion in $E_n$ for any $n \in \mathbb{N}$.

# 3. Weighted Composition Operators Between Sequence (LF)- and (PLB)-Spaces

## 3.1. Sequence (LF)- and (PLB)-Spaces

Throughout the paper, $\omega := \mathbb{K}^{\mathbb{N}}$ (where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) denotes the space of all $\mathbb{K}$-valued sequences, endowed with the locally convex (briefly, lc) topology of the coordinate convergence. Therefore, $\omega$ is a reflexive Fréchet space whose topological dual is the space $\omega' = \mathbb{K}^{(\mathbb{N})}$ of all $\mathbb{K}$-valued sequences with only a finite number of not-zero coordinates. In particular, $\omega'$ is an (LB)-space when it is endowed with the strong topology.

For all $n \in \mathbb{N}$, let $V_n = (v_{n,k})_{k\in\mathbb{N}}$ be a countable family of (strictly) positive sequences, called *weights*, on $\mathbb{N}$. We denote by $\mathcal{V}$ the sequence $(V_n)_{n\in\mathbb{N}}$ and we assume that the following two conditions are satisfied:

(1) $v_{n,k}(i) \le v_{n,k+1}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$;
(2) $v_{n,k}(i) \ge v_{n+1,k}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$.

The family $\mathcal{V}$ is called a *system of weights* on $\mathbb{N}$.

Given a system $\mathcal{V}$ of weights on $\mathbb{N}$, for $n, k \in \mathbb{N}$ and $1 \le p \le \infty$, we define as usual

$$l^p(v_{n,k}) := \left\{x = (x_i)_{i\in\mathbb{N}} \in \omega \colon \|x\|_{p,v_{n,k}} := \|(x_i v_{n,k}(i))_{i\in\mathbb{N}}\|_p < \infty\right\},$$

where $\|\cdot\|_p$ denotes the $l^p$ norm. For $p = 0$, we set

$$c_0(v_{n,k}) := \left\{x = (x_i)_{i\in\mathbb{N}} \in \omega \colon \lim_{i\to\infty} v_{n,k}(i) x_i = 0\right\}.$$

Clearly, $(l^p(v_{n,k}), \|\cdot\|_{p,v_{n,k}})$, $1 \le p \le \infty$, are Banach spaces, and $c_0(v_{n,k})$ is a Banach space with the norm of $l^\infty(v_{n,k})$. Since $l^p(v_{n,k+1})$ is continuously included into $l^p(v_{n,k})$, the sequence $\{l^p(v_{n,k})\}_{k\in\mathbb{N}}$ of Banach spaces forms a projective spectrum. Hence, for all $n \in \mathbb{N}$ and $1 \le p \le \infty$, we can consider the *echelon spaces*

$$\lambda_p(V_n) := \bigcap_{k\in\mathbb{N}} l^p(v_{n,k}) \text{ and } \lambda_0(V_n) := \bigcap_{k\in\mathbb{N}} c_0(v_{n,k}).$$

Endowed with the projective topologies $\lambda_p(V_n) = \mathrm{proj}_{k\in\mathbb{N}} l^p(v_{n,k})$ ($\lambda_0(V_n) = \mathrm{proj}_{k\in\mathbb{N}} c_0(v_{n,k})$, resp.), these spaces are Fréchet spaces with the topology defined by the corresponding seminorms. We point out that these spaces are Köthe echelon spaces.

The sequence $\{\lambda_p(V_n)\}_{n\in\mathbb{N}}$ of Fréchet spaces forms an inductive spectrum. Thus, the spaces

$$l_p(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_p(V_n) \quad (1 \le p \le \infty) \text{ and } l_0(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_0(V_n)$$

endowed with the inductive topologies, i.e., $l_p(\mathcal{V}) = \mathrm{ind}_{n\in\mathbb{N}} \lambda_p(V_n)$ ($l_0(\mathcal{V}) = \mathrm{ind}_{n\in\mathbb{N}} \lambda_0(V_n)$, resp.) are (LF)-spaces.

We say that the system $\mathcal{V}$ of weights on $\mathbb{N}$ satisfies the condition $(WQ)$ (or is of type $(WQ)$) if

$$\forall n \in \mathbb{N}\ \exists \mu, m \in \mathbb{N}\ \forall k, N \in \mathbb{N}\ \exists K \in \mathbb{N}, S > 0, \\ \text{s.t. } \forall i \in \mathbb{N}\colon v_{m,k}(i) \leq S(v_{n,\mu}(i) + v_{N,K}(i)).$$

While, we say that the system $\mathcal{V}$ satisfies the condition $(Q)$ (or is of type $(Q)$) if

$$\forall n \in \mathbb{N}\ \exists \mu, m \in \mathbb{N}\ \forall k, N \in \mathbb{N}, R > 0\ \exists K \in \mathbb{N}, S > 0, \text{s.t. } \forall i \in \mathbb{N}\colon \\ v_{m,k}(i) \leq \frac{1}{R} v_{n,\mu}(i) + S v_{N,K}(i).$$

The properties of regularity of the (LF)-space $l_p(\mathcal{V})$ and the conditions (Q) and (WQ) are related, as the following theorem of Vogt [39] shows.

**Theorem 3.1** *Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a system of weights on $\mathbb{N}$. Then the following properties hold true:*

(1) *For $1 < p < \infty$, the following conditions are equivalent:*
  (i) *$\mathcal{V}$ satisfies the condition (WQ);*
  (ii) *$l_p(\mathcal{V})$ is regular;*
  (iii) *$l_p(\mathcal{V})$ is complete;*
  (iv) *$l_p(\mathcal{V})$ is reflexive.*

(2) *For $p = 1, \infty$, the following conditions are equivalent:*
  (i) *$\mathcal{V}$ satisfies the condition (WQ);*
  (ii) *$l_p(\mathcal{V})$ is regular;*
  (iii) *$l_p(\mathcal{V})$ is complete.*

(3) *For $p = 0$, the following conditions are equivalent:*
  (i) *$\mathcal{V}$ satisfies the condition (Q);*
  (ii) *$l_0(\mathcal{V})$ is regular;*
  (iii) *$l_0(\mathcal{V})$ is complete.*

Vogt [39] also characterized when the (LF)-spaces $l_p(\mathcal{V})$ satisfy the condition (M) and ($M_0$), as stated in the following result.

**Theorem 3.2** *Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a system of weights on $\mathbb{N}$.*

*For $p \in [1, \infty] \cup \{0\}$, the (LF)-space $l_p(\mathcal{V})$ satisfies the condition (M) if, and only if, $\mathcal{V}$ satisfies the condition (Q).*

*Moreover, if $p \neq 1, \infty$, the (LF)-space $l_p(\mathcal{V})$ satisfies the condition ($M_0$) if, and only if, $\mathcal{V}$ satisfies the condition (WQ). The (LF)-space $l_1(\mathcal{V})$ satisfies the condition ($M_0$) if, and only if, $\mathcal{V}$ satisfies the condition (Q).*

Given a system of weights $\mathcal{V} = (v_{n,k})_{n,k\in\mathbb{N}}$ on $\mathbb{N}$, we set $V^k = (v_{n,k})_{n\in\mathbb{N}}$, for all $k \in \mathbb{N}$. Then for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, both the sequences $\{l^p(v_{n,k})\}_{n\in\mathbb{N}}$

and $\{c_0(v_{n,k})\}_{n\in\mathbb{N}}$ of Banach spaces form an inductive spectrum. Hence, we can consider the *co-echelon spaces*

$$a_p(V^k) := \bigcup_{n\in\mathbb{N}} l^p(v_{n,k}) \ \ (1 \le p \le \infty) \text{ and } a_0(V^k) := \bigcup_{n\in\mathbb{N}} c_0(v_{n,k}),$$

which are (LB)-spaces when they are endowed with the inductive topologies, i.e., $a_p(V^k) = \mathrm{ind}_{n\in\mathbb{N}}\, l^p(v_{n,k})$ ($a_0(V^k) = \mathrm{ind}_{n\in\mathbb{N}}\, c_0(v_{n,k})$, resp.). We point out that these spaces are Köthe co-echelon spaces.

The sequence $\{a_p(V^k)\}_{k\in\mathbb{N}}$ of (LB)-spaces forms a projective spectrum. Hence, the spaces

$$a_p(\mathcal{V}) := \bigcap_{k\in\mathbb{N}} a_p(V^k) \ \ (1 \le p \le \infty) \text{ and } a_0(\mathcal{V}) := \bigcap_{k\in\mathbb{N}} a_0(V^k),$$

endowed with the projective topologies, i.e., $a_p(\mathcal{V}) = \mathrm{proj}_{k\in\mathbb{N}}\, a_p(V^k)$ and $a_0(\mathcal{V}) = \mathrm{proj}_{k\in\mathbb{N}}\, a_0(V^k)$ are (PLB)-spaces. We observe that by [13, Theorem 2.3 and Corollary 2.8], the co-echelon spaces $a_p(V^k)$ are complete (LB)-spaces for every $1 \le p \le \infty$. Accordingly, $a_p(\mathcal{V}) = \mathrm{proj}_{k\in\mathbb{N}}\, a_p(V^k)$ is a complete (PLB)-space for every $1 \le p \le \infty$. The (LB)-space $a_0(V^k)$, for $k \in \mathbb{N}$, need not be regular. The regularity is ensured by a stronger condition on the system $V^k$ of weights. To see this, we recall that, given a sequence $(v_n)_{n\in\mathbb{N}}$ of decreasing weights on $\mathbb{N}$, the sequence $(v_n)_{n\in\mathbb{N}}$ is called *regularly decreasing* if given $n \in \mathbb{N}$, there exists $m \ge n$ such that, on each subset of $\mathbb{N}$ on which the quotient $\frac{v_m}{v_n}$ is bounded away from zero, also all quotients $\frac{v_k}{v_n}$, $k \ge m$, are bounded away from zero. By [13, Theorem 3.7], the co-echelon space $a_0(V^k)$, for $k \in \mathbb{N}$, is regular if, and only if, it is complete if, and only if, it satisfies condition (M) if, and only if, it is (strongly) boundedly retractive if, and only if, the sequence $V^k = (v_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing. Furthermore, every co-echelon space $a_p(V^k)$, for $k \in \mathbb{N}$ and $1 \le p \le \infty$, satisfies condition (M) if, and only if, it is (strongly) boundedly retractive if, and only if, the sequence $V^k = (v_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing.

Finally, we point out that the spaces introduced above are all continuously included in $\omega$ with dense range.

### 3.2. Weighted Composition Operators or Pseudo Shifts

For fixed $u = (u_i)_{i\in\mathbb{N}} \in \omega, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, we can define the weighted composition operator $W_{u,\varphi}$ acting on $\omega$ with symbols $u, \varphi$ by setting

$$W_{u,\varphi}(x) := u(x \circ \varphi) = (u_i x_{\varphi_i})_{i\in\mathbb{N}}, \quad x = (x_i)_{i\in\mathbb{N}} \in \omega.$$

Observe that this operator is obtained by composition of two well-known operators: the multiplication operator $M_u$ and the composition operator $C_\varphi$. In fact, when $\varphi$ is the identity map on $\mathbb{N}$, $W_{u,\varphi}$ becomes a multiplication operator which is defined pointwise by $M_u(x) := ux = (u_i x_i)_{i\in\mathbb{N}}$. If $u_i = 1$ for all $i \in \mathbb{N}$, then $W_{u,\varphi}$ becomes a composition operator defined as $C_\varphi(x) := x \circ \varphi = (x_{\varphi_i})_{i\in\mathbb{N}}$. Clearly, $W_{u,\varphi} \in \mathcal{L}(\omega)$ for every pair $u, \varphi \in \omega$.

*Remark 3.3* Let $X$ be a non complete lcHs, $Y$ be a complete lcHs and $u \in \omega, \varphi \in \mathbb{N}^{\mathbb{N}}$. If $\overline{X}$ and $Y$ are continuously included in $\omega$ (here, $\overline{X}$ denotes the completion of $X$) and $W_{u,\varphi} \in \mathcal{L}(X,Y)$, then the continuous linear extension $\overline{W}_{u,\varphi}\colon \overline{X} \to Y$ coincides with $W_{u,\varphi}$. Indeed, if we denote by $j_{\overline{X}}\colon \overline{X} \hookrightarrow \omega$ (by $j_Y\colon Y \hookrightarrow \omega$, resp.) the continuous inclusion of $\overline{X}$ (of $Y$, resp.) in $\omega$, then $j_Y \circ W_{u,\varphi} = W_{u,\varphi} \circ (j_{\overline{X}})|_X$. Since $W_{u,\varphi} \in \mathcal{L}(\omega)$, by passing to the completion of $X$, it follows that $j_Y \circ \overline{W}_{u,\varphi} = W_{u,\varphi} \circ j_{\overline{X}}$. Therefore, $\overline{W}_{u,\varphi} = W_{u,\varphi}$.

Throughout the paper, we denote by $e_n$, for $n \in \mathbb{N}$, the element $(\delta_{n,i})_{i\in\mathbb{N}}$ of $\omega$. We observe that the sequence $(e_n)_{n\in\mathbb{N}}$ forms an unconditional basis for $\omega$.

### 3.3. The Operator $W_{u,\varphi}$ Acting on Weighted $l^p$ Sequence Banach Spaces

We start by studying the continuity and the compactness of the weighted composition operator $W_{u,\varphi}$ when it acts from $l^p(v)$ into $l^p(w)$, for $1 \le p \le \infty$ and $v, w$ two weights on $\mathbb{N}$. We first characterize the continuity as follows. For $p = 2$ and $p = \infty$ the result is given in [26, Theorem 2.3] and in [18, Theorem 2.1], respectively.

**Theorem 3.4** *Let $u = (u_i)_{i\in\mathbb{N}} \in \omega, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, let $v, w$ be two weights on $\mathbb{N}$ and $1 \le p < \infty$. The weighted composition operator $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$ if, and only if, there exists $M > 0$ such that*

$$\frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \le M, \quad \forall n \in \mathbb{N}, \tag{3.1}$$

*where the sum is defined equal to $0$ if $\varphi^{-1}(n) = \emptyset$ for some $n \in \mathbb{N}$.*

*Proof* Suppose that $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$. Then there exists $M > 0$ such that for every $x \in l^p(v)$

$$\|W_{u,\varphi}(x)\|_{p,w}^p \le M\|x\|_{p,v}^p. \tag{3.2}$$

Fix $n \in \mathbb{N}$. Obviously, if $\varphi^{-1}(n) = \emptyset$, then (3.1) is clearly satisfied. So, let $\varphi^{-1}(n) \ne \emptyset$. Observe that $W_{u,\varphi}(e_n) = (u_j(e_n)_{\varphi_j})_{j\in\mathbb{N}} = (u_j\delta_{n,\varphi_j})_{j\in\mathbb{N}}$, where

$$\delta_{n,\varphi_j} = \begin{cases} 1 & \text{if } n = \varphi_j, \\ 0 & \text{if } n \ne \varphi_j. \end{cases}$$

Therefore, applying (3.2) with $x = e_n$, we get that

$$\sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p = \|W_{u,\varphi}(e_n)\|_{p,w}^p \le M\|e_n\|_{p,v}^p = v_n^p.$$

Conversely, suppose that (3.1) is satisfied. Then for every $x \in l^p(v)$ we have

$$\begin{aligned} \|W_{u,\varphi}(x)\|_{p,w}^p &= \sum_{j\in\mathbb{N}} |u_j|^p |x_{\varphi_j}|^p w_j^p = \sum_{n\in\mathbb{N}} \sum_{j\in\varphi^{-1}(n)} |u_j|^p |x_n|^p w_j^p \\ &= \sum_{n\in\mathbb{N}} |x_n|^p \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \le M \sum_{n\in\mathbb{N}} |x_n|^p v_n^p = M\|x\|_{p,v}^p. \end{aligned}$$

This means that $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$. $\square$

*Remark 3.5* For $p = \infty$ the operator $W_{u,\varphi}$ belongs to $\mathcal{L}(l^\infty(v), l^\infty(w))$ if, and only if, $\sup_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(e_n)\|_{\infty,w}}{\|e_n\|_{\infty,v}} < \infty$, see [18, Theorem 2.1]. This is equivalent to the existence of $M > 0$ such that

$$\sup_{j\in\varphi^{-1}(n)} |u_j| w_j \le M v_n, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

*Remark 3.6* Let $X$ be a locally compact Hausdorff topological space. A continuous map $\varphi\colon X \to X$ is called *proper* if the preimage of every compact set $K$ in $X$ is also a compact set in $X$. If $\varphi\colon \mathbb{N} \to \mathbb{N}$, then $\varphi$ is proper if, and only if, $\lim_{i\to\infty} \varphi_i = \infty$, as it is easy to show.

If we assume that $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper map, then the operator $W_{u,\varphi}$ belongs to $\mathcal{L}(c_0(v), c_0(w))$ if, and only if, $W_{u,\varphi}$ belongs to $\mathcal{L}(l^\infty(v), l^\infty(w))$. The proof follows as in [6, Proposition 4] with some obvious changes.

*Remark 3.7* The proof of Theorem 3.4 yields for each $1 \le p < \infty$ that

$$\|W_{u,\varphi}\|^p = \inf\left\{M > 0\colon \frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \le M \quad \forall n \in \mathbb{N}\right\}.$$

In a similar way, one shows for $p = \infty$ that

$$\|W_{u,\varphi}\|_\infty = \inf\left\{M > 0\colon \sup_{j\in\varphi^{-1}(n)} |u_j| w_j \le M v_n \quad \forall n \in \mathbb{N}\right\}.$$

Moreover, we observe that:

(1) If $\varphi$ is one-to-one, then $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$ if, and only if, there exists $M > 0$ such that $\frac{|u_{\varphi^{-1}(n)}| w_{\varphi^{-1}(n)}}{v_n} \le M$, for all $n \in \mathbb{N}$. In such a case, we have

$$\|W_{u,\varphi}\| = \inf\left\{M > 0\colon \left|u_{\varphi^{-1}(n)}\right| w_{\varphi^{-1}(n)} \le M v_n \quad \forall n \in \mathbb{N}\right\}.$$

(2) Suppose that there exists $L > 0$ such that $\lambda(\varphi^{-1}(n)) \le L$ for all $n \in \mathbb{N}$, where $\lambda$ denotes the counting measure on $\mathbb{N}$. If there exists $M > 0$ such that $\frac{|u_{\varphi^{-1}(n)}| w_{\varphi^{-1}(n)}}{v_n} \le M$, for all $n \in \mathbb{N}$, then $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$ and $\|W_{u,\varphi}\| \le M L^{1/p}$.

We now deal with the study of the compactness of weighted composition operators $W_{u,\varphi}$ acting on weighted $l^p$ sequence Banach spaces. To obtain a characterization of the compactness of the operator $W_{u,\varphi}$, we need some auxiliary results about the dual operator $W'_{u,\varphi}$ of $W_{u,\varphi}$ which acts from $(l^p(w), \|\cdot\|_{p,w})'$ into $(l^p(v), \|\cdot\|_{p,v})'$. For this, recall that if $v$ is a weight on $\mathbb{N}$, then for $1 \le p < \infty$ the Banach space $\left(l^{p'}\left(\frac{1}{v}\right), \|\cdot\|_{p',\frac{1}{v}}\right)$ $\left(\left(l^1\left(\frac{1}{v}\right), \|\cdot\|_{1,\frac{1}{v}}\right), \text{resp.}\right)$ is the strong dual of the Banach space $(l^p(v), \|\cdot\|_{p,v})$ $((c_0(v), \|\cdot\|_{\infty,v})$, resp.), with $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, the following representation for the dual operator $W'_{u,\varphi}$ is valid.

**Proposition 3.8** *Let $u = (u_i)_{i\in\mathbb{N}} \in \omega, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, let $v, w$ be two weights on $\mathbb{N}$ and $p \in [1,\infty) \cup \{0\}$. If $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$, then $W'_{u,\varphi} \in \mathcal{L}\left(l^{p'}\left(\frac{1}{w}\right), l^{p'}\left(\frac{1}{v}\right)\right)$ and*

$$W'_{u,\varphi}(y) = \left(\sum_{j\in\varphi^{-1}(n)} u_j y_j\right)_{n\in\mathbb{N}}, \quad \forall y \in l^{p'}\left(\frac{1}{w}\right),$$

*where $p'$ is the conjugate exponent of $p$ if $p \in [1,\infty)$, while $p' = 1$ if $p = 0$.*

*Proof* Suppose $1 \leq p < \infty$. Then the assumption implies that $W'_{u,\varphi} \in \mathcal{L}\left(l^{p'}\left(\frac{1}{w}\right), l^{p'}\left(\frac{1}{v}\right)\right)$. Moreover, for every $y \in l^{p'}\left(\frac{1}{w}\right)$ and $x \in l^p(v)$ we have

$$\begin{aligned}\langle x, W'_{u,\varphi} y\rangle &= \langle W_{u,\varphi} x, y\rangle = \sum_{j\in\mathbb{N}} u_j x_{\varphi_j} y_j = \sum_{n\in\mathbb{N}} \sum_{j\in\varphi^{-1}(n)} u_j x_n y_j \\ &= \sum_{n\in\mathbb{N}} x_n \sum_{j\in\varphi^{-1}(n)} u_j y_j = \left\langle x, \left(\sum_{j\in\varphi^{-1}(n)} u_j y_j\right)_{n\in\mathbb{N}}\right\rangle.\end{aligned}$$

It follows that $W'_{u,\varphi}(y) = \left(\sum_{j\in\varphi^{-1}(n)} u_j y_j\right)_{n\in\mathbb{N}}$ for every $y \in l^{p'}\left(\frac{1}{w}\right)$.

For $p = 0$ the proof is analogous. □

To obtain the desired characterization for the compactness of $W_{u,\varphi}$ we will require the following two results.

*Remark 3.9* Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a map. Then for all $J \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n > N$ and $j \in \varphi^{-1}(n)$ we have $j > J$.

Indeed, for a fixed $J \in \mathbb{N}$, set $N := \max_{1\leq j\leq J} \varphi(j)$. Fixed $n > N$ and $j \in \varphi^{-1}(n)$, if $j \leq J$, then

$$N \geq \varphi(j) = n > N;$$

a contradiction.

**Lemma 3.10** *Let $u = (u_i)_{i\in\mathbb{N}} \in \omega, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, let $v, w$ be two weights on $\mathbb{N}$. If $W_{u,\varphi} \in \mathcal{L}(l^1(v), l^1(w))$, then the dual operator $W'_{u,\varphi} \in \mathcal{L}\left(l^\infty\left(\frac{1}{w}\right), l^\infty\left(\frac{1}{v}\right)\right)$ maps $c_0\left(\frac{1}{w}\right)$ into $c_0\left(\frac{1}{v}\right)$ and $W'_{u,\varphi}|_{c_0\left(\frac{1}{w}\right)} \in \mathcal{L}\left(c_0\left(\frac{1}{w}\right), c_0\left(\frac{1}{v}\right)\right)$.*

*Proof* By Proposition 3.8 we have that $W'_{u,\varphi} \in \mathcal{L}\left(l^\infty\left(\frac{1}{w}\right), l^\infty\left(\frac{1}{v}\right)\right)$ is given by

$$W'_{u,\varphi}(y) = \left(\sum_{j\in\varphi^{-1}(n)} u_j y_j\right)_{n\in\mathbb{N}}, \quad \forall y \in l^\infty\left(\frac{1}{w}\right).$$

On the other hand, by Theorem 3.4 the continuity of $W_{u,\varphi}$ implies the existence of $M > 0$ such that

$$\sum_{j\in\varphi^{-1}(n)} |u_j| w_j \leq M v_n, \quad \forall n \in \mathbb{N}.$$

Now, for fixed $y \in c_0\left(\frac{1}{w}\right)$ and $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $\frac{|y_j|}{w_j} \leq \frac{\varepsilon}{M}$ for all $j \geq j_0$. On the other hand, by Remark 3.9 there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $j \in \varphi^{-1}(n)$ we have $j > j_0$. Therefore, we get for all $n > n_0$ that

$$\left|\sum_{j\in\varphi^{-1}(n)} u_j y_j\right| \leq \sum_{j\in\varphi^{-1}(n)} |u_j y_j| = \sum_{j\in\varphi^{-1}(n)} |u_j y_j| \frac{w_j}{w_j}$$
$$\leq \frac{\varepsilon}{M} \sum_{j\in\varphi^{-1}(n)} |u_j| w_j \leq v_n \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, this means that $W'_{u,\varphi}(y) = \left(\sum_{j\in\varphi^{-1}(n)} u_j y_j\right)_{n\in\mathbb{N}}$ belongs to $c_0\left(\frac{1}{v}\right)$. Since $y$ is also arbitrary, it follows that $W'_{u,\varphi}$ maps $c_0\left(\frac{1}{w}\right)$ into $c_0\left(\frac{1}{v}\right)$ and hence, $W'_{u,\varphi}|_{c_0(\frac{1}{w})} \in \mathcal{L}\left(c_0\left(\frac{1}{w}\right), c_0\left(\frac{1}{v}\right)\right)$. □

*Remark 3.11* Let $X, Y$ be two lcHs. We recall that if $T \in \mathcal{L}(X, Y)$, then $T$ is $\sigma(X, X') - \sigma(Y, Y')$ continuous ($w - w$ continuous). Moreover, $T' \in \mathcal{L}(Y', X')$ is also $\sigma(Y', Y) - \sigma(X', X)$ continuous ($w^* - w^*$ continuous) and $\sigma(Y', Y'') - \sigma(X', X'')$ continuous ($w - w$ continuous).

It is well-known that there are no compact composition operator from $l^2$ into itself. On the other hand, Singh and Manhas [38] established necessary and sufficient conditions in order to have a compact composition operator acting on $l^2(v)$, with $v$ a weight on $\mathbb{N}$. An analogous result was given in [26]. In [18] the authors characterized the compactness of weighted composition operators in $l^\infty(v)$. In the next result we extend this characterization in the setting of $l^p(v)$ spaces, for $1 \leq p < \infty$, without any addditional assumptions on the multiplier $u$ and on the map $\varphi$.

**Theorem 3.12** *Let $u = (u_i)_{i\in\mathbb{N}} \in \omega, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, let $v, w$ be two weights on $\mathbb{N}$ and $1 \leq p < \infty$. Then $W_{u,\varphi} \in \mathcal{K}(l^p(v), l^p(w))$ if, and only if, $\left(\frac{1}{v_n^p}\sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p\right)_{n\in\mathbb{N}} \in c_0$.*

*Proof* We distinguish the cases $p = 1$ and $1 < p < \infty$.

*Case:* $1 < p < \infty$. Suppose that the operator $W_{u,\varphi}$ is compact. As $\left\{\frac{e_n}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is a bounded subset of $l^p(v)$, it then follows that the set $\left\{\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is relatively compact in $l^p(w)$. On the other hand, the sequence $\left(\frac{e_n}{\|e_n\|_{p,v}}\right)_{n\in\mathbb{N}}$ weakly converges to 0 in $l^p(v)$, as it is easy to show. Thus, $\left(\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}}\right)_{n\in\mathbb{N}}$ weakly converges to 0 in $l^p(w)$, thereby implying that the set $\left\{\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is relatively weakly compact in $l^p(w)$. Accordingly, the norm topology of $l^p(w)$ and the weak topology $\sigma\left(l^p(w), l^{p'}(\frac{1}{w})\right)$ necessarily coincide on the set $\left\{\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$. Hence, $\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} \to 0$ in $l^p(w)$.

Since

$$\frac{\|W_{u,\varphi}(e_n)\|_{p,w}^p}{\|e_n\|_{p,v}^p} = \frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p, \quad \forall n \in \mathbb{N},$$

the thesis follows.

Now, assume that $\left(\frac{1}{v_n^p}\sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p\right)_{n\in\mathbb{N}} \in c_0$. By Theorem 3.4, this condition implies that $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$ with

$$\|W_{u,\varphi}\|^p = \left\| \left( \frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \right)_{n\in\mathbb{N}} \right\|_\infty =: M.$$

In order to show that $W_{u,\varphi} \in \mathcal{K}(l^p(v), l^p(w))$, we fix a bounded sequence $(x_i)_{i\in\mathbb{N}}$ of $l^p(v)$ and set $L := \sup_{i\in\mathbb{N}} \|x_i\|_{p,v}$. Since $l^p(v)$ is a reflexive Banach space, there exists a subsequence of $(x_i)_{i\in\mathbb{N}}$, denoted again by $(x_i)_{i\in\mathbb{N}}$ for the sake of simplicity, such that $(x_i)_{i\in\mathbb{N}}$ weakly converges in $l^p(v)$ to some $x \in l^p(v)$. Accordingly, $\|x\|_{p,v} \le L$. Since $W_{u,\varphi}\colon l^p(v) \to l^p(w)$ is also weakly continuous, it follows that the sequence $(W_{u,\varphi}(x_i))_{i\in\mathbb{N}}$ weakly converges to $W_{u,\varphi}(x)$.

Now, for a fixed $\varepsilon > 0$, by assumption there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p < \frac{\varepsilon^p}{(3L)^p}$$

for all $n > n_0$. Therefore, we get for all $n > n_0$ and $i \in \mathbb{N}$ that

$$\begin{aligned}
\|W_{u,\varphi}(x_i) - W_{u,\varphi}(x)\|_{p,w}^p &= \sum_{j\in\mathbb{N}} w_j^p |u_j x_{i\varphi_j} - u_j x_{\varphi_j}|^p \\
&= \sum_{n\in\mathbb{N}} \sum_{j\in\varphi^{-1}(n)} w_j^p |u_j|^p |x_{in} - x_n|^p \\
&= \sum_{n\in\mathbb{N}} |x_{in} - x_n|^p \sum_{j\in\varphi^{-1}(n)} w_j^p |u_j|^p \\
&= \sum_{n=1}^{n_0} |x_{in} - x_n|^p \sum_{j\in\varphi^{-1}(n)} w_j^p |u_j|^p \\
&\quad + \sum_{n>n_0} |x_{in} - x_n|^p \sum_{j\in\varphi^{-1}(n)} w_j^p |u_j|^p \\
&\le M \sum_{n=1}^{n_0} v_n^p |x_{in} - x_n|^p \\
&\quad + \frac{\varepsilon^p}{(3L)^p} \sum_{n>n_0} v_n^p |x_{in} - x_n|^p \\
&\le M \sum_{n=1}^{n_0} v_n^p |x_{in} - x_n|^p + \frac{\varepsilon^p (2L)^p}{(3L)^p}.
\end{aligned}$$

Since $x_i \to x$ weakly in $l^p(v)$ and hence, pointwise, there is $i_0 \in \mathbb{N}$ such that

$$M \sum_{n=1}^{n_0} v_n^p |x_{in} - x_n|^p < \frac{\varepsilon^p}{3^p}$$

for all $i \geq i_0$. Summing up, we obtain for all $i \geq i_0$ that

$$\|W_{u,\varphi}x_i - W_{u,\varphi}x\|_{p,w}^p < \frac{\varepsilon^p}{3^p} + \frac{\varepsilon^p 2^p}{3^p} \leq \varepsilon^p.$$

This means that $W_{u,\varphi}(x_i) \to W_{u,\varphi}(x)$ in $l^p(w)$. So, the proof is complete.

*Case:* $p = 1$. Suppose that $W_{u,\varphi} \in \mathcal{K}(l^1(v), l^1(w))$. Then $W_{u,\varphi} \in \mathcal{L}(l^1(v), l^1(w))$ and hence, by Lemma 3.10, the dual operator $W'_{u,\varphi} \in \mathcal{L}\left(l^\infty\left(\frac{1}{w}\right), l^\infty\left(\frac{1}{v}\right)\right)$ maps $c_0\left(\frac{1}{w}\right)$ into $c_0\left(\frac{1}{v}\right)$ and $T := W'_{u,\varphi}|_{c_0(\frac{1}{w})} \in \mathcal{L}\left(c_0\left(\frac{1}{w}\right), c_0\left(\frac{1}{v}\right)\right)$. Therefore, $T' \in \mathcal{L}(l^1(v), l^1(w))$ is also $\sigma\left(l^1(v), c_0(\frac{1}{v})\right) - \sigma\left(l^1(w), c_0(\frac{1}{w})\right)$ continuous, i.e., w*-w* continuous (see Remark 3.11). Moreover, $T' = (W'_{u,\varphi}|_{c_0(\frac{1}{w})})' = W_{u,\varphi}$, as it easily follows by using standard duality arguments. So, since the sequence $\left(\frac{e_n}{\|e_n\|_{1,v}}\right)_{n\in\mathbb{N}} \subset l^1(v)$ weakly* converges to 0 in $l^1(v)$, the sequence $\left(W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right)\right)_{n\in\mathbb{N}}$ necessarily weakly* converges to 0 in $l^1(w)$. This implies that the set $\left\{W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) : n \in \mathbb{N}\right\}$ is relatively weakly* compact in $l^1(w)$. On the other hand, by the facts that $W_{u,\varphi} \in \mathcal{K}(l^1(v), l^1(w))$ and $\left\{\frac{e_n}{\|e_n\|_{1,v}} : n \in \mathbb{N}\right\}$ is a bounded set in $l^1(v)$, we get that $\left\{W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) : n \in \mathbb{N}\right\}$ is a relatively compact subset of $l^1(w)$. So, we get that the norm topology of $l^1(w)$ and the weak* topology $\sigma(l^1(w), c_0(\frac{1}{w}))$ necessarily coincide on the set $\left\{W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) : n \in \mathbb{N}\right\}$. Accordingly, $W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) \to 0$ in $l^1(w)$. This means that

$$\frac{\|W_{u,\varphi}e_n\|_{1,w}}{\|e_n\|_{1,v}} = \frac{1}{v_n} \sum_{j\in\varphi^{-1}(n)} |u_j| w_j \to 0.$$

Now, assume that $\left(\frac{1}{v_n}\sum_{j\in\varphi^{-1}(n)} |u_j| w_j\right)_{n\in\mathbb{N}} \in c_0$. By Theorem 3.4, this condition implies that $W_{u,\varphi} \in \mathcal{L}(l^1(v), l^1(w))$ with

$$\|W_{u,\varphi}\| = \left\| \left( \frac{1}{v_n} \sum_{j\in\varphi^{-1}(n)} |u_j| w_j \right)_{n\in\mathbb{N}} \right\|_\infty =: M.$$

In order to prove that $W_{u,\varphi} \in \mathcal{K}(l^1(v), l^1(w))$, we fix a bounded sequence $(x_i)_{i\in\mathbb{N}}$ of $l^1(v)$ and set $L := \sup_{i\in\mathbb{N}} \|x_i\|_{1,v}$. Since $l^1(v) \hookrightarrow \omega$ continuously and $\omega$ is a Fréchet Montel space, we have that $(x_i)_{i\in\mathbb{N}}$ is a bounded sequence in $\omega$ and hence, it contains a subsequence convergent in $\omega$ to some $x \in \omega$. For the sake of simplicity, we still denote the subsequence by $(x_i)_{i\in\mathbb{N}}$. Since

$$\sum_{j=1}^{n}|x_{ij}|v_j \le L, \quad \forall i,n\in\mathbb{N},$$

by letting $i\to\infty$, it follows that

$$\sum_{j=1}^{n}|x_{j}|v_j \le L, \quad \forall n\in\mathbb{N}.$$

This implies that $\|x\|_{1,v}\le L$ and hence, $x\in l^1(v)$. Now, to get the thesis it suffices to proceed as in the proof of the case $1<p<\infty$ (just to put 1 instead of $p$). $\square$

*Remark 3.13* The operator $W_{u,\varphi}$ belongs to $\mathcal{K}(l^\infty(v),l^\infty(w))$ if, and only if, we have $\lim_{n\to\infty}\frac{\|W_{u,\varphi}(e_n)\|_{\infty,w}}{\|e_n\|_{\infty,v}}=0$ (see [18, Theorem 3.2]). This is equivalent to require that

$$\lim_{n\to\infty}\frac{\sup_{j\in\varphi^{-1}(n)}|u_j|w_j}{v_n}=0.$$

If $p=0$ and $\varphi\colon\mathbb{N}\to\mathbb{N}$ is a proper map, then the operator $W_{u,\varphi}$ belongs to $\mathcal{K}(c_0(v),c_0(w))$ if, and only if, it belongs to $\mathcal{K}(l^\infty(v),l^\infty(w))$.

### 3.4. The Operator $W_{u,\varphi}$ Acting on Sequence (LF)- and (PLB)-Spaces

Thanks to the results in Sect. 2 and in Sect. 3.3, we can characterize the continuity, the boundedness and the compactness of weighted composition operators in sequence (LF)-spaces of type $l_p(\mathcal{V})$ and in sequence (PLB)-spaces of type $a_p(\mathcal{V})$, for $p\in[1,\infty]\cup\{0\}$.

Concerning the weighted composition operator $W_{u,\varphi}$ in sequence (LF)-spaces of type $l_p(\mathcal{V})$ we have the following results. In the first one we characterize the continuity.

**Theorem 3.14** *Let $\mathcal{V},\mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi=(\varphi_i)_{i\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}, u=(u_i)_{i\in\mathbb{N}}\in\omega$. For $1\le p<\infty$, the following properties are equivalent:*

1. *$W_{u,\varphi}\colon l_p(\mathcal{V})\to l_p(\mathcal{W})$ is well-defined;*
2. *$W_{u,\varphi}\colon l_p(\mathcal{V})\to l_p(\mathcal{W})$ is continuous;*
3. *For all $m\in\mathbb{N}$ there exists $n\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ there exist $l\in\mathbb{N}$, $M>0$ for which*

$$\frac{1}{v_{m,l}^p(i)}\sum_{j\in\varphi^{-1}(i)}|u_j|^p w_{n,k}^p(j)\le M, \quad \forall i\in\mathbb{N},$$

*where the sum is defined equal to 0 if $\varphi^{-1}(i)=\emptyset$ for some $i\in\mathbb{N}$.*

*If $p=\infty$, the following properties are equivalent:*

1. *$W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ is well-defined;*
2. *$W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ is continuous;*

(3) *For all* $m \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$ *there exist* $l \in \mathbb{N}$, $M > 0$ *for which*

$$\sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) \leq M v_{m,l}(i), \quad \forall i \in \mathbb{N}.$$

*If* $p = 0$ *and* $\varphi\colon \mathbb{N} \to \mathbb{N}$ *is a proper map, then* $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ *is continuous if, and only if,* $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ *is continuous.*

*Proof* Case $1 \leq p < \infty$. Clearly, (2) implies (1) and (1) implies (2) by the Closed Graph theorem (see, f.i., [27, p.57]). Indeed, if $(x_\alpha)_\alpha \subset l_p(\mathcal{V})$ is a net convergent to $x$ in $l_p(\mathcal{V})$ and $(W_{u,\varphi}(x_\alpha))_\alpha$ is convergent to $y$ in $l_p(\mathcal{W})$, then $(x_\alpha)_\alpha$ and $(W_{u,\varphi}(x_\alpha))_\alpha$ converge in $\omega$ to $x$ and $y$ respectively. Since $W_{u,\varphi} \in \mathcal{L}(\omega)$, it follows that $W_{u,\varphi}(x) = y$. This proves that the graph of $W_{u,\varphi}$ is closed.

(2)$\Leftrightarrow$(3). The weighted composition operator $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous if, and only if, for all $m \in \mathbb{N}$ the operator $W_{u,\varphi}\colon \lambda_p(V_m) \to l_p(\mathcal{W})$ is continuous. By Grothendieck's factorization theorem [22, p.147], it follows that $W_{u,\varphi}$ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $W_{u,\varphi}\colon \lambda_p(V_m) \to \lambda_p(W_n)$ is well-defined and continuous. Since each space $l^p(v_{m,l})$ is dense in $\lambda_p(V_m)$, by Theorem 2.3(1), this holds if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that the operator $W_{u,\varphi}$ admits a unique continuous linear extension $(W_{u,\varphi})^l_k\colon l^p(v_{m,l}) \to l^p(w_{n,k})$. Since $W_{u,\varphi} \in \mathcal{L}(\omega)$ and the spaces $l^p(v_{m,l})$ and $l^p(w_{n,k})$ are continuously included in $\omega$, $(W_{u,\varphi})^l_k = W_{u,\varphi}$ (see Remark 3.3), thereby implying that $W_{u,\varphi}\colon l^p(v_{m,l}) \to l^p(w_{n,k})$ is continuous. The thesis now follows by applying Theorem 3.4.

For $p = \infty$, it suffices to observe that $W_{u,\varphi}\colon \lambda_\infty(V_m) \to \lambda_\infty(W_n)$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$, $M > 0$ such that

$$\|W_{u,\varphi}(x)||_{\infty, w_{n,k}} \leq M \|x\|_{\infty, v_{m,l}}, \quad \forall x \in \lambda_\infty(V_m). \tag{3.4}$$

If we put $x = e_i \in \lambda_\infty(V_m)$ in (3.4) for some fixed $i \in \mathbb{N}$, we get that

$$\sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) = \|W_{u,\varphi}(e_i)||_{\infty, w_{n,k}} \leq M \|e_i\|_{\infty, v_{m,l}} = M v_{m,l}(i).$$

Since $i \in \mathbb{N}$ is arbitrary, we can then conclude that (2)$\Rightarrow$(3) also for $p = \infty$.

Conversely, suppose that (3) holds true. Then by Remark 3.5, we get that $W_{u,\varphi} \in \mathcal{L}(l^\infty(v_{m,l}), l^\infty(w_{n,k}))$ and hence, the thesis follows in view of Theorem 2.3(1).

For $p = 0$, we can argue as in the proof of the case $1 \leq p < \infty$, by obtaining that $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}, M > 0$ for which $\sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) \leq M v_{m,l}(i)$ for all $i \in \mathbb{N}$. So, $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is continuous if, and only if, $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is continuous. □

The proof of the following characterization of the boundedness of $W_{u,\varphi}$ in $l_p(\mathcal{V})$ is an application of Theorems 2.2(1), 2.3(2) and 3.4, and Remark 3.5.

**Theorem 3.15** *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. The following assertions hold true:*

(1) *If $1 \le p < \infty$ and $l_p(\mathcal{W})$ is regular, then $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $M > 0$ for which*

$$\frac{1}{v^p_{m,l}(i)} \sum_{j\in\varphi^{-1}(i)} |u_j|^p w^p_{n,k}(j) \le M, \quad \forall i \in \mathbb{N},$$

*where the sum is defined equal to $0$ if $\varphi^{-1}(i) = \emptyset$ for some $i \in \mathbb{N}$.*

(2) *If $p = \infty$ and $l_\infty(\mathcal{W})$ is regular, then $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $M > 0$ for which*

$$\sup_{j\in\varphi^{-1}(i)} |u_j| w_{n,k}(j) \le M v_{m,l}(i), \quad \forall i \in \mathbb{N}.$$

(3) *If $p = 0$, $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper map and $l_0(\mathcal{W})$ is regular, then $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is bounded if, and only if, $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is bounded.*

*Proof* For $p \in [1, \infty) \cup \{0\}$ the proof follows by arguing as in the proof of Theorem 3.14 in view of Theorems 2.2(1), 2.3(2) and 3.4, and Remark 3.6. In particular, for $p = 0$ one shows that $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is bounded if, and only if, the condition in (2) is satisfied.

For $p = \infty$ we observe that $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $W_{u,\varphi}\colon \lambda_\infty(V_m) \to \lambda_\infty(W_n)$ is bounded. By Theorem 2.3(2) combined with Remark 2.4, the operator $W_{u,\varphi}\colon \lambda_\infty(V_m) \to \lambda_\infty(W_n)$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator $W_{u,\varphi}\colon (\lambda_\infty(V_m), \tau_{m,l}) \to l^\infty(w_{n,k})$ is continuous, where $\tau_{m,l}$ denotes the lc-topology of $l^\infty(v_{m,l})$, i.e., there exists $M > 0$ such that

$$\|W_{u,\varphi}(x)\|_{\infty,w_{n,k}} \le M\|x\|_{\infty,v_{m,l}}, \quad \forall x \in \lambda_\infty(V_m).$$

In view of the inequality above, we can argue as in the proof of Theorem 3.14 to conclude that this is equivalent to require that the conditon (2) is satisfied. □

In the next result we characterize the compactness of weighted composition operators in sequence (LF)-spaces of type $l_p(\mathcal{V})$.

**Theorem 3.16** *Let $\mathcal{V}, \mathcal{W}$ be two system of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. The following assertions hold true:*

(1) *If $1 \le p < \infty$ and $l_p(\mathcal{W})$ satisfies condition (M), then $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$*

$$\lim_{i\to\infty} \frac{1}{v^p_{m,l}(i)} \sum_{j\in\varphi^{-1}(i)} |u_j|^p w^p_{n,k}(j) = 0.$$

(2) *If* $p=\infty$, $l_\infty(\mathcal{W})$ *satisfies condition (M) and the space* $\lambda_\infty(V_m)$ *is dense in* $l^\infty(v_{m,l})$ *for all* $m,l\in\mathbb{N}$, *then* $W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ *is compact if, and only if, there exists* $n\in\mathbb{N}$ *such that for all* $m\in\mathbb{N}$ *there exists* $l\in\mathbb{N}$ *such that for all* $k\in\mathbb{N}$

$$\lim_{i\to\infty}\frac{\sup_{j\in\varphi^{-1}(i)}|u_j|w_{n,k}(j)}{v_{m,l}(i)}=0.$$

(3) *If* $p=0$, $l_0(\mathcal{W})$ *satisfies condition (M) and* $\varphi\colon\mathbb{N}\to\mathbb{N}$ *is a proper map, then* $W_{u,\varphi}\colon l_0(\mathcal{V})\to l_0(\mathcal{W})$ *is compact if, and only if,* $W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ *is compact.*

*Proof* Let $1\le p<\infty$. By Theorem 2.2(2), $W_{u,\varphi}\colon l_p(\mathcal{V})\to l_p(\mathcal{W})$ is compact if, and only if, there exists $n\in\mathbb{N}$ such that for all $m\in\mathbb{N}$ the restriction $W_{u,\varphi}\colon\lambda_p(V_m)\to\lambda_p(W_n)$ is compact. On the other hand, by Theorem 2.3(2), this holds true if, and only if, there exists $l\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ the operator $W_{u,\varphi}$ admits a unique linear extension $(W_{u,\varphi})^l_k\colon l^p(v_{m,l})\to l^p(w_{n,k})$ which is compact. Since $W_{u,\varphi}\in\mathcal{L}(\omega)$ and the space $l^p(v_{m,l})$ and $l^p(w_{n,k})$ are continuously included in $\omega$, necessarily, $(W_{u,\varphi})^l_k=W_{u,\varphi}$ (see Remark 3.3), i.e., $W_{u,\varphi}\in\mathcal{K}(l^p(v_{m,l}),l^p(w_{n,k}))$. Accordingly, the thesis now follows by applying Theorem 3.12.

For $p=0,\infty$ the proof is analogous and so, it is omitted. □

We now turn our attention on the study of weighted composition operators $W_{u,\varphi}$ in sequence (PLB)-spaces of type $a_p(\mathcal{V})$. In view of Theorems 2.3(1) and 3.4, we can characterize the continuity of $W_{u,\varphi}$ as follows.

**Theorem 3.17** *Let* $\mathcal{V},\mathcal{W}$ *be two systems of weights on* $\mathbb{N}$ *and* $\varphi=(\varphi_i)_{i\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$, $u=(u_i)_{i\in\mathbb{N}}\in\omega$. *The following assertions hold true:*

(1) *If* $1\le p<\infty$, *then* $W_{u,\varphi}\colon a_p(\mathcal{V})\to a_p(\mathcal{W})$ *is continuous if, and only if, for all* $k\in\mathbb{N}$ *there exists* $l\in\mathbb{N}$ *such that for all* $m\in\mathbb{N}$ *there exist* $n\in\mathbb{N}$, $M>0$ *for which*

$$\frac{1}{v^p_{m,l}(i)}\sum_{j\in\varphi^{-1}(i)}|u_j|^pw^p_{n,k}(j)\le M,\quad\forall i\in\mathbb{N}.$$

(2) *If* $p=\infty$, *then* $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ *is continuous if, and only if, for all* $k\in\mathbb{N}$ *there exists* $l\in\mathbb{N}$ *such that for all* $m\in\mathbb{N}$ *there exist* $n\in\mathbb{N}$, $M>0$ *for which*

$$\sup_{j\in\varphi^{-1}(i)}|u_j|w_{n,k}(j)\le Mv_{m,l}(i),\quad\forall i\in\mathbb{N}.\tag{3.5}$$

(3) *If* $p=0$, $\varphi\colon\mathbb{N}\to\mathbb{N}$ *is a proper map and the sequence* $W^k=(w_{n,k})_{n\in\mathbb{N}}$ *is regularly decreasing for all* $k\in\mathbb{N}$, *then* $W_{u,\varphi}\colon a_0(\mathcal{V})\to a_0(\mathcal{W})$ *is continuous if, and only if,* $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ *is continuous.*

*Proof* For $1 \le p < \infty$ or $p = 0$, the characterization of the continuity follows by arguing as in the proof of Theorem 3.14 in view of Theorems 2.3(1) and 3.4 and Remarks 3.5 and 3.6. In particular, for $p = 0$ one shows that the operator $W_{u,\varphi}\colon a_0(\mathcal{V}) \to a_0(\mathcal{W})$ is continuous if, and only if, the condition in (2) is satisfied.

For $p = \infty$, the space $a_\infty(\mathcal{V})$ could be not dense in each $a_\infty(V^l)$. So, to show the statement, we proceed as follows.

By Remark 2.4(1), the operator $W_{u,\varphi}\colon a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $W_{u,\varphi}\colon (a_\infty(\mathcal{V}), \tau_l) \to a_\infty(W^k)$ is continuous, where $\tau_l$ denotes the lc-topology of the co-echelon space $a_\infty(V^l)$. Now, for a fixed $m \in \mathbb{N}$, let $B := \{\frac{e_i}{v_{m,l}(i)} \colon i \in \mathbb{N}\} \subset a_\infty(\mathcal{V})$. Then the set $B$ is contained in $l^\infty(v_{m,l})$ and bounded there. Indeed, for all $i \in \mathbb{N}$, we have

$$\left\| \frac{e_i}{v_{m,l}(i)} \right\|_{\infty, v_{m,l}} = 1.$$

Accordingly, $B$ is a bounded subset of $(a_\infty(\mathcal{V}), \tau_l)$. The continuity of $W_{u,\varphi}$ from $(a_\infty(\mathcal{V}), \tau_l)$ into $a_\infty(W^k)$ implies that $W_{u,\varphi}(B)$ is a bounded subset of $a_\infty(W^k)$. Therefore, there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\frac{1}{v_{m,l}(i)} \sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) = \left\| W_{u,\varphi}\left( \frac{e_i}{v_{m,l}(i)} \right) \right\|_{\infty, w_{n,k}} \le C, \quad \forall i \in \mathbb{N},$$

i.e., (3.5) is satisfied.

Conversely, if (3.5) is satisfied, then by Remark 3.5, this implies that the operator $W_{u,\varphi} \in \mathcal{L}(l^\infty(v_{m,l}), l^\infty(w_{n,k}))$ and hence, the thesis follows. □

The characterization of the boundedness of weighted composition operators in sequence (PLB)-spaces of type $a_p(\mathcal{V})$ is contained in the following result. The proof relies on Theorems 2.2(1), 2.3(2) and 3.4, and Remark 3.5.

**Theorem 3.18** *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}, u = (u_i)_{i \in \mathbb{N}} \in \omega$. The following assertions hold true:*

(1) *If $1 \le p < \infty$, then $W_{u,\varphi}\colon a_p(\mathcal{V}) \to a_p(\mathcal{W})$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exist $M > 0$ for which*

$$\frac{1}{v^p_{m,l}(i)} \sum_{j \in \varphi^{-1}(i)} |u_j|^p w^p_{n,k}(j) \le M, \quad \forall i \in \mathbb{N}.$$

(2) *If $p = \infty$, then $W_{u,\varphi}\colon a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exist $M > 0$ for which*

$$\sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) \le M v_{m,l}(i), \quad \forall i \in \mathbb{N}.$$

(3) *If $p=0$, $\varphi\colon \mathbb{N}\to\mathbb{N}$ is a proper map and the sequence $W^k=(w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k\in\mathbb{N}$, then $W_{u,\varphi}\colon a_0(\mathcal{V})\to a_0(\mathcal{W})$ is bounded if, and only if, $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ is bounded.*

*Proof* For $p\in[1,\infty)\cup\{0\}$ the proof follows as in the proof of Theorem 3.14 in view of Theorems 2.2(1), 2.3(2) and 3.4, and Remarks 3.5 and 3.6. In particular, for $p=0$ one shows that the operator $W_{u,\varphi}\colon a_0(\mathcal{V})\to a_0(\mathcal{W})$ is bounded if, and only if, the condition in (2) is satisfied.

Let $p=\infty$. By Theorem 2.3(2) and Remark 2.4, the operator $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ is bounded if, and only if, there exists $l\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ the operator $W_{u,\varphi}\colon (a_\infty(\mathcal{V}),\tau_l)\to a_\infty(W^k)$ is bounded, where $\tau_l$ denotes the lc-topology of the (LB)-space $a_\infty(V^l)$.

Fix $k\in\mathbb{N}$. The fact that $W_{u,\varphi}\colon (a_\infty(\mathcal{V}),\tau_l)\to a_\infty(W^k)$ is bounded implies that there exists a 0-neighborhood $U$ of $a_\infty(V^l)$ such that $W_{u,\varphi}(U\cap a_\infty(\mathcal{V}))$ is a bounded subset of $a_\infty(W^k)$. Since $a_\infty(W^k)$ is regular, there exists $n\in\mathbb{N}$ such that such that $W_{u,\varphi}(U\cap a_\infty(\mathcal{V}))$ is contained in $l^\infty(w_{n,k})$ and bounded there. Now, for a fixed $m\in\mathbb{N}$, set $B:=\{\frac{e_i}{v_{m,l}(i)}: i\in\mathbb{N}\}\subset a_\infty(\mathcal{V})$. Then $B$ is contained in $l^\infty(v_{m,l})$ and bounded there (see the proof of Theorem 3.14). Accordingly, $B$ is a bounded subset of $(a_\infty(\mathcal{V}),\tau_l)$. Thus, there exists $\lambda>0$ such that $B\subseteq\lambda(U\cap a_\infty(\mathcal{V}))$. It follows that $W_{u,\varphi}(B)$ is also a bounded subset of $l^\infty(w_{n,k})$. So, there exists $C>0$ such that

$$\frac{1}{v_{m,l}(i)}\sup_{j\in\varphi^{-1}(i)}|u_j|w_{n,k}(j)=\left\|W_{u,\varphi}\left(\frac{e_i}{v_{m,l}(i)}\right)\right\|_{\infty,w_{n,k}}\le C,\quad \forall i\in\mathbb{N},$$

i.e., the condition in (2) is satisfied. Conversely, if the condition in (2) is satisfied, then by Remark 3.5 the operator $W_{u,\varphi}\in\mathcal{L}(l^\infty(v_{m,l}),l^\infty(w_{n,k}))$. So, the thesis follows. $\square$

In the final result of this section, we characterize the compactness of weighted composition operators in sequence (PLB)-spaces of type $a_p(\mathcal{V})$.

**Theorem 3.19** *Let $\mathcal{V},\mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi=(\varphi_i)_{i\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$, $u=(u_i)_{i\in\mathbb{N}}\in\omega$. The following assertions hold true:*

(1) *If $1\le p<\infty$, then $W_{u,\varphi}\colon a_p(\mathcal{V})\to a_p(\mathcal{W})$ is compact if, and only if, there exists $l\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ there exists $n\in\mathbb{N}$ such that for all $m\in\mathbb{N}$*

$$\lim_{i\to\infty}\frac{1}{v_{m,l}^p(i)}\sum_{j\in\varphi^{-1}(i)}|u_j|^p w_{n,k}^p(j)=0.$$

(2) *If $p=\infty$, $a_\infty(\mathcal{V})$ is dense in $a_\infty(V^l)$ for all $l\in\mathbb{N}$ and the sequence $W^k=(w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k\in\mathbb{N}$, then $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ is compact if, and only if, there exists $l\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ there exists $n\in\mathbb{N}$ such that for all $m\in\mathbb{N}$*

$$\lim_{i\to\infty}\frac{\sup_{j\in\varphi^{-1}(i)}|u_j|w_{n,k}(j)}{v_{m,l}(i)}=0.$$

(3) *If* $p=0$, $\varphi\colon \mathbb{N}\to\mathbb{N}$ *is a proper map and the sequence* $W^k=(w_{n,k})_{n\in\mathbb{N}}$ *is regularly decreasing for all* $k\in\mathbb{N}$, *then* $W_{u,\varphi}\colon a_0(\mathcal{V})\to a_0(\mathcal{W})$ *is compact if, and only if,* $W_{u,\varphi}\colon a_\infty(\mathcal{V})\to a_\infty(\mathcal{W})$ *is compact.*

*Proof* The thesis follows by arguing as in the proof of Theorem 3.16 in view of Theorems 2.2(2), 2.3(2) and 3.12, and of the remark thereafter. □

*Remark 3.20* The results of this subsection extend to the case $p\in[1,\infty)$ the characterization of the continuity, the boundedness and of the compactness of composition operators acting either on sequence (LF)-spaces of type $l_p(\mathcal{V})$ or on sequence (PLB)-spaces of type $a_p(\mathcal{V})$ for $p=0,\infty$ given in [6]. The results above also extend to the case of weighted composition operators the characterization of the continuity, the boundedness and of the compactness of multiplication operators acting on sequence (LF)-spaces of type $l_p(\mathcal{V})$, for $p\in[1,\infty]\cup\{0\}$, given in [31].

### 3.5. Montel Weighted Composition Operators Acting on Köthe Echelon Spaces and on Sequence (LF)-Spaces

In this section we give necessary and sufficient conditions in order that a weighted composition operator acting on Köthe echelon spaces and on sequence (LF)-spaces of type $l_p(\mathcal{V})$ is Montel. To this end, we recall a known result about the relative compactness of subsets of the spaces $l^p(v)$, $1\le p<\infty$, and $c_0(v)$ (see, f.i., [30, Chapter 15]), where $v$ is a weight on $\mathbb{N}$. For $1\le p<\infty$, a subset $K$ of $l^p(v)$ is relatively compact if, and only if, for every $\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that $\sum_{j=j_0+1}^{\infty}|x_j|^p v_j^p<\varepsilon^p$ for every $x\in K$. A subset $K$ of $c_0(v)$ is relatively compact if, and only if, for every $\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that $\sup_{j\ge j_0+1}|x_j|v_j<\varepsilon$ for every $x\in K$.

Let $A=(a_n)_{n\in\mathbb{N}}$ be a *Köthe matrix*, i.e., $a_n\in\omega$ for all $n\in\mathbb{N}$ and $0<a_n(i)\le a_{n+1}(i)$ for all $i,n\in\mathbb{N}$. Denote by

$$\overline{A}:=\lambda_\infty(A)_+=\{x=(x_n)_{n\in\mathbb{N}}\in\omega\colon \|(a_nx_n)_{n\in\mathbb{N}}\|_\infty<\infty \,\text{and}\, x_n>0\,\forall n\in\mathbb{N}\}.$$

The following useful description of the bounded sets in a Köthe echelon space is due to Bierstedt, Meise and Summers [13].

**Proposition 3.21** *Let* $A=(a_n)_{n\in\mathbb{N}}$ *be a Köthe matrix. For* $p\in[1,\infty)\cup\{0\}$, *a subset* $B$ *of* $\lambda_p(A)$ *is bounded if, and only if, there exists* $\overline{a}\in\overline{A}$ *such that*

$$B\subseteq B_{\overline{a}}:=\left\{x\in\omega\colon \left\|\left(\frac{x_i}{\overline{a}(i)}\right)_{i\in\mathbb{N}}\right\|_p\le 1\right\}.$$

In the following result we characterize Montel weighted composition operators acting on Köthe echelon spaces.

**Proposition 3.22** *Let* $A=(a_n)_{n\in\mathbb{N}}$, $B=(b_m)_{m\in\mathbb{N}}$ *be two Köthe matrices and let* $\varphi=(\varphi_i)_{i\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}, u=(u_i)_{i\in\mathbb{N}}\in\omega$, *with* $\varphi$ *increasing. The following assertions hold true:*

(1) *If* $1 \leq p < \infty$ *and* $W_{u,\varphi} \in \mathcal{L}(\lambda_p(A), \lambda_p(B))$*, then* $W_{u,\varphi} \colon \lambda_p(A) \to \lambda_p(B)$ *is Montel if, and only if, for every* $\overline{a} \in \overline{A}$ *and* $m \in \mathbb{N}$ *we have*

$$\lim_{n\to\infty} \overline{a}^p(n) \sum_{j\in\varphi^{-1}(n)} b_m^p(j)|u_j|^p = 0. \tag{3.6}$$

(2) *If* $p = 0$*,* $\varphi \colon \mathbb{N} \to \mathbb{N}$ *is a proper map and* $W_{u,\varphi} \in \mathcal{L}(\lambda_0(A), \lambda_0(B))$*, then* $W_{u,\varphi} \colon \lambda_0(A) \to \lambda_0(B)$ *is Montel if, and only if, for every* $\overline{a} \in \overline{A}$ *and* $m \in \mathbb{N}$ *we have*

$$\lim_{n\to\infty} \overline{a}(n) \sup_{j\in\varphi^{-1}(n)} b_m(j)|u_j| = 0. \tag{3.7}$$

*Proof* We prove the statement only for $1 \leq p < \infty$. The proof for $p = 0$ is analogous and so, it is omitted.

Suppose that $W_{u,\varphi} \colon \lambda_p(A) \to \lambda_p(B)$ is Montel. So, in order to show that the condition is necessary, we fix $\overline{a} \in \overline{A}$ and $m \in \mathbb{N}$, and consider the set $B_{\overline{a}} = \{x \in \omega \colon \|(x_i(\overline{a}(i))^{-1})_{i\in\mathbb{N}}\|_p \leq 1\}$. In view of Proposition 3.21, the set $B_{\overline{a}}$ is bounded in $\lambda_p(A)$. By applying the assumption on $W_{u,\varphi}$, we get that the set $W_{u,\varphi}(B_{\overline{a}})$ is relatively compact in $\lambda_p(B)$ and hence, in $l^p(b_m)$. Since $\{\overline{a}(n)e_n \colon n \in \mathbb{N}\} \subset B_{\overline{a}}$, it follows that the set $\{W_{u,\varphi}(\overline{a}(n)e_n) \colon n \in \mathbb{N}\}$ is relatively compact in $l^p(b_m)$. On the other hand, the sequence $(\overline{a}(n)e_n)_{n\in\mathbb{N}}$ weakly converges to 0 in $\lambda_p(A)$, thereby implying that the sequence $(W_{u,\varphi}(\overline{a}(n)e_n))_{n\in\mathbb{N}}$ weakly converges to 0 in $\lambda_p(B)$ and hence, also in $l^p(b_m)$. But, as the set $\{W_{u,\varphi}(\overline{a}(n)e_n) \colon n \in \mathbb{N}\}$ is relatively compact in $l^p(b_m)$, the norm topology of $l^p(b_m)$ and the weak topology of $l^p(b_m)$ coincide on $\{W_{u,\varphi}(\overline{a}(n)e_n) \colon n \in \mathbb{N}\}$. Accordingly, the sequence $(W_{u,\varphi}(\overline{a}(n)e_n))_{n\in\mathbb{N}}$ converges to 0 in $l^p(b_m)$. This means that (3.6) is satisfied. Indeed, for all $n \in \mathbb{N}$ we have

$$\|W_{u,\varphi}(\overline{a}(n)e_n)\|_{p,b_m}^p = \overline{a}^p(n) \sum_{j\in\varphi^{-1}(n)} b_m^p(j)|u_j|^p.$$

Now, suppose that the condition is fulfilled. In order to prove that $W_{u,\varphi} \colon \lambda_p(A) \to \lambda_p(B)$ is Montel, we fix a bounded set $B$ of $\lambda_p(A)$. In view of Proposition 3.21, there exists $\overline{a} \in \overline{A}$ such that $B \subset B_{\overline{a}}$. To conclude the proof, it then suffices to show that the set $W_{u,\varphi}(B_{\overline{a}})$ is relatively compact in $l^p(b_m)$ for all $m \in \mathbb{N}$, i.e., to show that for every $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that $\sum_{j\geq j_0+1} b_m^p(j)|y_j^p| < \varepsilon^p$ for every $y \in W_{u,\varphi}(B_{\overline{a}})$. Hence, for fixed $m \in \mathbb{N}$ and $\varepsilon > 0$, due to (3.6), we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\overline{a}^p(n) \sum_{j\in\varphi^{-1}(n)} b_m^p(j)|u_j|^p < \varepsilon^p.$$

If $y \in W_{u,\varphi}(B_{\overline{a}})$, then $y = (y_j)_{j\in\mathbb{N}} = (u_j x_{\varphi_j})_{j\in\mathbb{N}}$. Set $j_0 := \min \varphi^{-1}(n_0)$ (if $\varphi^{-1}(n_0) = \emptyset$, we get trivially the thesis). Then, for every $y \in W_{u,\varphi}(B_{\overline{a}})$, we get

$$\sum_{j\geq j_0+1}|y_j|^p b_m^p(j)=\sum_{j\geq j_0+1}|u_j x_{\varphi_j}|^p b_m^p(j)=\sum_{n\geq n_0+1}\sum_{j\in\varphi^{-1}(n)}|u_j x_n|^p b_m^p(j)$$
$$\leq \sum_{n\geq n_0+1}|x_n|^p\sum_{j\in\varphi^{-1}(n)}|u_j|^p b_m^p(j)\leq \sum_{n\geq n_0+1}\frac{|x_n|^p\varepsilon^p}{\overline{a}^p(n)}\leq\varepsilon^p,$$

where we used the fact that $\varphi$ is increasing. □

*Remark 3.23* If $\varphi$ is an increasing self-map on $\mathbb{N}$, i.e., $\varphi(i)\leq\varphi(i+1)$ for all $i\in\mathbb{N}$, the map $\varphi$ could not be proper. For example, let $\varphi(i)=1$ for all $i\in\mathbb{N}$. While, $\varphi$ strictly increasing, i.e., $\varphi(i)<\varphi(i+1)$ for all $i\in\mathbb{N}$, implies that $\lim_{i\to\infty}\varphi(i)=\infty$ and hence, $\varphi$ is a proper map (see Remark 3.6).

The case $p<\infty$ is completely characterized. So, it remains to prove that the same characterization holds for $p=\infty$. In order to do this, we observe what follows.

*Remark 3.24* Let $\mathcal{X}$ be the family of all sequence lcHs X satisfying the following properties:

(a) The inclusion $j\colon X\hookrightarrow\omega$ is continuous with dense range;
(b) The dual operator $j'\colon\omega'_\beta\hookrightarrow X'_\beta$ has dense range.

We point out that, for any $X\in\mathcal{X}$, by (b), the bidual operator $j''\colon X''\hookrightarrow\omega$ is a continuous inclusion.

We observe that, for fixed $X,Y\in\mathcal{X}$ and $\varphi\in\mathbb{N}^{\mathbb{N}}$, if the composition operator $C_\varphi\in\mathcal{L}(X,Y)$ (i.e., $C_\varphi(x)=x\circ\varphi$, for $x\in X$), then the bidual operator $C''_\varphi\in\mathcal{L}(X''_\beta,Y''_\beta)$ is given by $C''_\varphi(x'')=x''\circ\varphi$ for every $x''\in X''$. Indeed, if we denote by $\Phi$ the composition operator by $\varphi$ acting on $\omega$ and by $j_X\colon X\hookrightarrow\omega$ ($j_Y\colon Y\hookrightarrow\omega$, resp.) the inclusion of $X$ ($Y$, resp.) into $\omega$, we have that $j_Y\circ C_\varphi=\Phi\circ j_X$. Passing to the bidual operators, we obtain that $j''_Y\circ C''_\varphi=\Phi''\circ j''_X$. Since $\omega$ is reflexive, $\Phi''$ coincides with $\Phi$ on $\omega$. Since $j''_X$ and $j''_Y$ are inclusion maps, it follows that $C''_\varphi(x'')=x''\circ\varphi$ for every $x''\in X''$.

An analogous result holds for the weighted composition operators, i.e., $W_{u,\varphi}=W''_{u,\varphi}$, if $X,Y\in\mathcal{X}$ and $W_{u,\varphi}\in\mathcal{L}(X,Y)$, for $u,\varphi\in\omega$.

**Proposition 3.25** *Let $A=(a_n)_{n\in\mathbb{N}}$, $B=(b_m)_{m\in\mathbb{N}}$ be two Köthe matrices and let $\varphi=(\varphi_i)_{i\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}},u=(u_i)_{i\in\mathbb{N}}\in\omega$, with $\varphi$ proper and increasing. If the operator $W_{u,\varphi}\in\mathcal{L}(\lambda_\infty(A),\lambda_\infty(B))$, then $W_{u,\varphi}\colon\lambda_\infty(A)\to\lambda_\infty(B)$ is Montel if, and only if, $W_{u,\varphi}\colon\lambda_0(A)\to\lambda_0(B)$ is Montel.*

*Proof* We first observe that $W_{u,\varphi}\colon\lambda_\infty(A)\to\lambda_\infty(B)$ is continuous if, and only if, $W_{u,\varphi}\colon\lambda_0(A)\to\lambda_0(B)$ is continuous (see Theorem 3.14) and that $\lambda_0(A)$ ($\lambda_0(B)$, resp.) is a closed subspace of $\lambda_\infty(A)$ ($\lambda_\infty(B)$, resp.).

If $W_{u,\varphi}\colon\lambda_\infty(A)\to\lambda_\infty(B)$ is Montel and $B$ is a bounded subset of $\lambda_0(A)$ (hence, of $\lambda_\infty(A)$), then the set $W_{u,\varphi}(B)$ is relatively compact in $\lambda_\infty(B)$ and hence, in $\lambda_0(B)$. This means that $W_{u,\varphi}\colon\lambda_0(A)\to\lambda_0(B)$ is Montel.

Now, suppose that $W_{u,\varphi}\colon \lambda_0(A) \to \lambda_0(B)$ is Montel. By [19, Corollary 2.3], the dual operator $W'_{u,\varphi}\colon \lambda_0(A)'_\beta \to \lambda_0(B)'_\beta$ is Montel. Since $\lambda_0(A)'_\beta$ and $\lambda_0(B)'_\beta$ are complete (LB)-spaces (see, f.i., [12, Proposition 10]), we can apply [19, Corollary 2.4] to get that the bidual operator $W''_{u,\varphi}\colon \lambda_0(A)''_\beta \to \lambda_0(B)''_\beta$ is also Montel. Since $\lambda_\infty(A)$ is the strong bidual of $\lambda_0(A)$ ($\lambda_\infty(B)$ is the strong bidual of $\lambda_0(B)$) and $W''_{u,\varphi} = W_{u,\varphi}$ (apply Remark 3.24 with $X = \lambda_0(A)$ and $Y = \lambda_0(B)$), we get the claim. $\square$

In the next result, we characterize when a weighted composition operator acting on sequence (LF)-spaces of type $l_p(\mathcal{V})$ is Montel. The proof is an application of Theorem 2.2(3), and of Propositons 3.22 and 3.25, taking into account that for a system $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ of weights on $\mathbb{N}$ we have that $l_p(\mathcal{V}) = \text{ind}_{\,n\in\mathbb{N}}\lambda_p(V_n)$, for $1 \le p \le \infty$ or $p = 0$, where each $\lambda_p(V_n)$ is a Köthe echelon space.

**Theorem 3.26** *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights on $\mathbb{N}$ and let $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $u = (u_i)_{i\in\mathbb{N}} \in \omega$, with $\varphi$ increasing. Let $p \in [1,\infty] \cup \{0\}$. Suppose that $l_p(\mathcal{V})$ is regular and that $l_p(\mathcal{W})$ satisfies the condition (M). Moreover, suppose that the operator $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous. Then the following assertions hold true:*

(1) *If $1 \le p < \infty$, then $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is Montel if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $\overline{v}_m \in \lambda_\infty(V_m)_+$ and $k \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \overline{v}_m^p(n) \sum_{j\in\varphi^{-1}(n)} w_{n,k}^p(j)|u_j|^p = 0.$$

(2) *If $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper map, then $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is Montel if, and only if, $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is Montel if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $\overline{v}_m \in \lambda_\infty(V_m)_+$ and $k \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \overline{v}_m(n) \sup_{j\in\varphi^{-1}(n)} w_{n,k}(j)|u_j| = 0.$$

Theorem 3.26 extends to the case of weighted composition operators the characterization of when a multiplication operator acting on sequence (LF)-spaces of type $l_p(\mathcal{V})$, for $p \in [1,\infty] \cup \{0\}$ is Montel given in [31, Theorem 4.12].

### 3.6. Reflexive Weighted Composition Operators Acting on Köthe Echelon Spaces and on Sequence (LF)-Spaces

In the following, we give necessary and sufficient conditions in order that a weighted composition operator acting either on Köthe echelon spaces or on sequence (LF)-spaces of type $l_p(\mathcal{V})$ is reflexive.

For $p = 1, 0, \infty$, we will show that a weighted composition operator acting on Köthe echelon spaces is Montel if, and only if, it is reflexive. In order to see this, we observe what follows.

Let $A = (a_n)_{n\in\mathbb{N}}$, $B = (b_m)_{m\in\mathbb{N}}$ be two Köthe matrices and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $u = (u_i)_{i\in\mathbb{N}} \in \omega$. For $p = 1, 0, \infty$, assume that $W_{u,\varphi}\colon \lambda_p(A) \to \lambda_p(B)$ is continuous. Then we have:

(i) If $p = 1$, by Schur's Theorem (see [42]) a bounded subset $B$ of $\lambda_1(A)$ is weakly (relatively) compact if, and only if, it is (relatively) compact. Accordingly, $W_{u,\varphi}\colon \lambda_1(A) \to \lambda_1(B)$ is reflexive if, and only if, it is Montel;

(ii) If $p = 0$ and $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper increasing map, we claim that $W_{u,\varphi}\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive if, and only if, it is Montel. We only have to show that the condition is necessary. So, suppose that $W_{u,\varphi}\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive. By a result of Grothendieck [22] (see also [27, p.204]), this implies that $W_{u,\varphi} = W''_{u,\varphi}$ (see Remark 3.24) maps $\lambda_\infty(A) = (\lambda_0(A)'_\beta)'_\beta$ in $\lambda_0(B)$. If we set $\overline{A} = \lambda_\infty(A)_+$, then we get that $W_{u,\varphi}(\overline{A}) \subset \lambda_0(B)$. But, this is equivalent to say that for every $\overline{a} \in \overline{A}$ and $m \in \mathbb{N}$ the sequence $(\overline{a}(\varphi_n)u_n b_m(n))_{n\in\mathbb{N}}$ vanishes at infinity, that is

$$\lim_{n\to\infty} \overline{a}(n) \sup_{j\in\varphi^{-1}(n)} b_m(j)|u_j| = 0$$

Therefore, the condition (2) (see (3.7)) in Proposition 3.22 is satisfied and hence, the claim is proved.

(iii) If $p = \infty$ and $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper increasing map, then the weighted composition operator $W_{u,\varphi}\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is reflexive if, and only if, it is Montel. Indeed, by [19, Corollary 2.3, 2.4] the fact that $W_{u,\varphi}\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is reflexive implies that $W_{u,\varphi}\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive. So, in view of point (ii) above, we can conclude that $W_{u,\varphi}\colon \lambda_0(A) \to \lambda_0(B)$ is Montel. Thus, by Proposition 3.25 also $W_{u,\varphi}\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is Montel.

Therefore, we can give a first characterization. The proof is an application of Theorem 2.2(3)–(4) and the considerations above, taking into account that for a system $\mathcal{V}$ of weights on $\mathbb{N}$ we have $l_p(\mathcal{V}) = \mathrm{ind}_{\,n\in\mathbb{N}}\lambda_p(V_n)$, for $1 \le p \le \infty$ or $p = 0$, where each $\lambda_p(V_n)$ is a Köthe sequence space.

**Theorem 3.27** *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights on $\mathbb{N}$ and let $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $u = (u_i)_{i\in\mathbb{N}} \in \omega$, with $\varphi$ an increasing sequence. Let $p = 1, 0, \infty$. Suppose that $l_p(\mathcal{V})$ is regular and that $l_p(\mathcal{W})$ satisfies the condition ($M_0$). Furthermore, for $p \neq 1$, suppose also that $\varphi$ is a proper map. If $W_{u,\varphi} \in \mathcal{L}(l_p(\mathcal{V}), l_p(\mathcal{W}))$, then $W_{u,\varphi}$ is reflexive if, and only if, $W_{u,\varphi}$ is Montel.*

We now consider the case $1 < p < \infty$. Since $\lambda_p(A)$ and $\lambda_p(B)$ are reflexive Fréchet spaces ([12, Proposition 9]), the weighted composition operator $W_{u,\varphi}\colon \lambda_p(A) \to \lambda_p(B)$ is clearly reflexive. In this case the following characterization is valid. The proof is an obvious consequence of Theorem 3.1.

**Theorem 3.28** *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights on $\mathbb{N}$ and let $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. Let $1 < p < \infty$. Suppose that $l_p(\mathcal{W})$ is regular. The weighted composition operator $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous if, and only if, it is reflexive.*

Theorems 3.27 and 3.28 extend to the case of weighted composition operators the characterization of reflexive multiplication operators acting on sequence (LF)-spaces of type $l_p(\mathcal{V})$, for $1 \in [1,\infty] \cup \{0\}$, given in [31, Theorem 4.13] and [31, Proposition 4.14], respectively.

**Funding** Open access funding provided by Università del Salento within the CRUI-CARE Agreement.

**Data Availibility Statement** Not applicable.

## Declarations

**Conflict of interest** The author has no relevant financial or non-financial interests to disclose.



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Received: February 6, 2023.
Accepted: July 29, 2023.