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Results in Mathematics

Nilpotent Cone and Bivariant Theory

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Abstract. We exhibit a new proof, relying on bivariant theory, that the nilpotent cone is rationally smooth. Our approach enables us to prove a slightly more general statement.

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1. Introduction

In [2] Borho and MacPherson proved that the nilpotent cone is a rational homology manifold. The proof relies on the celebrated Decomposition Theorem by Beilinson, Bernstein, Deligne and Gabber [1] and on the Springer's theory of Weyl group representations (see [2] and the references therein).

The aim of this paper is to present a new proof, in our opinion conceptually very simple, based on the bivariant theory founded by Fulton and MacPherson in [4]. Actually, our approach enables us to prove a slightly more general statement (see Remark 2.4 below). By *bivariant theory* we intend *the topological bivariant homology theory with coefficients in a Noetherian commutative ring with identity* $\mathbb{A}$ [4, pp. 32, 83 and p. 86, Corollary 7.3.4].

That the nilpotent cone is a rational homology manifold can be seen as an easy consequence of a characterization of homology manifolds we recently proved in [3, Theorem 6.1]: *given a resolution of singularities* $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ *of a quasi-projective variety* $\mathcal{N}$*, then* $\mathcal{N}$ *is a homology manifold if and only if there exists a bivariant class of degree one for* π. A *bivariant class of degree one for*

π is an element $\eta \in H^0(\widetilde{\mathcal{N}} \xrightarrow{\pi} \mathcal{N})$ such that the induced Gysin homomorphism $\eta_0 : H^0(\widetilde{\mathcal{N}}) \to H^0(\mathcal{N})$ sends $1_{\widetilde{\mathcal{N}}}$ to $1_{\mathcal{N}}$.

2. The Main Result

Theorem 2.1. *Let $\pi' : \widetilde{\mathbf{g}} \to \mathbf{g}$ be a projective morphism between complex quasi-projective nonsingular varieties of the same dimension. Assume that π' is generically finite, of degree δ. Let $\mathcal{N} \subset \mathbf{g}$ be a closed irreducible subvariety. Consider the induced fibre square diagram:*

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \hookrightarrow & \widetilde{\mathbf{g}} \\ \downarrow \pi & & \downarrow \pi' \\ \mathcal{N} & \xhookrightarrow{i} & \mathbf{g}, \end{array}$$

where $\widetilde{\mathcal{N}} := \mathcal{N} \times_{\mathbf{g}} \widetilde{\mathbf{g}}$. If $\widetilde{\mathcal{N}}$ is irreducible and nonsingular and π is birational, then $\mathcal{N}$ is an $\mathbb{A}$-homology manifold for every Noetherian commutative ring with identity $\mathbb{A}$ for which δ is a unit.

Proof. Since $\pi' : \widetilde{\mathbf{g}} \to \mathbf{g}$ is a projective morphism between complex quasi-projective nonsingular varieties of the same dimension, it is a local complete intersection morphism of relative codimension 0 [4, p. 130]. Let

$$\theta' \in H^0(\widetilde{\mathbf{g}} \xrightarrow{\pi'} \mathbf{g}) \cong Hom_{D^b_c(\mathbf{g})}(R\pi'_*\mathbb{A}_{\widetilde{\mathbf{g}}}, \mathbb{A}_{\mathbf{g}})$$

be the orientation class of π' [4, p. 131]. Let $\theta'_0 : H^0(\widetilde{\mathbf{g}}) \to H^0(\mathbf{g})$ be the induced Gysin map. It is clear that $\theta'_0(1_{\widetilde{\mathbf{g}}}) = \delta \cdot 1_{\mathbf{g}} \in H^0(\mathbf{g})$, where δ is the degree of π'. Therefore, if we denote by

$$\theta := i^*\theta' \in H^0(\widetilde{\mathcal{N}} \xrightarrow{\pi} \mathcal{N}) \cong Hom_{D^b_c(\mathcal{N})}(R\pi_*\mathbb{A}_{\widetilde{\mathcal{N}}}, \mathbb{A}_{\mathcal{N}})$$

the pull-back of θ', then $\delta^{-1} \cdot \theta$ is a bivariant class of degree one for π [3, 2. Notations, (ii)]. At this point, our claim follows by [3, Theorem 6.1]. *For the Reader's convenience, let us briefly summarize the argument.*

Since $\delta^{-1} \cdot \theta$ is a bivariant class of degree one for π, it follows that $\left(\delta^{-1} \cdot \theta\right) \circ \pi^* = \mathrm{id}_{\mathbb{A}_{\mathcal{N}}}$ in $D^b_c(\mathcal{N})$, i.e. that $\delta^{-1} \cdot \theta$ is a section of the pull-back $\pi^* : \mathbb{A}_{\mathcal{N}} \to R\pi_*\mathbb{A}_{\widetilde{\mathcal{N}}}$ [3, Remark 2.1, (i)]. Hence, $\mathbb{A}_{\mathcal{N}}$ is a direct summand of $Rf_*\mathbb{A}_{\widetilde{\mathcal{N}}}$ in $D^b_c(\mathcal{N})$ [3, Lemma 3.2] and so we have a decomposition

$$Rf_*\mathbb{A}_{\widetilde{\mathcal{N}}} \cong \mathbb{A}_{\mathcal{N}} \oplus \mathcal{K}. \tag{1}$$

Now, set $\nu = \dim \widetilde{\mathcal{N}} = \dim \mathcal{N}$ and let $[\widetilde{\mathcal{N}}] \in H_{2\nu}(\widetilde{\mathcal{N}})$ be the fundamental class of $\widetilde{\mathcal{N}}$. We have:

$$[\widetilde{\mathcal{N}}] \in H_{2\nu}(\widetilde{\mathcal{N}}) \cong H^{-2\nu}(\widetilde{\mathcal{N}} \to pt.) \cong Hom_{D^b_c(\widetilde{\mathcal{N}})}(\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu], D\left(\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu]\right)),$$

where D denotes Verdier dual. Therefore, $[\widetilde{\mathcal{N}}]$ corresponds to a morphism

$$\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu] \to D\left(\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu]\right), \tag{2}$$

whose induced map in hypercohomology is nothing but the duality morphism

$$\mathcal{D}_{\widetilde{\mathcal{N}}} : x \in H^{\bullet}(\widetilde{\mathcal{N}}) \to x \cap [\widetilde{\mathcal{N}}] \in H_{2\nu-\bullet}(\widetilde{\mathcal{N}}). \tag{3}$$

If we assume that $\widetilde{\mathcal{N}}$ is nonsingular (actually it suffices that $\widetilde{\mathcal{N}}$ is an $\mathbb{A}$-homology manifold), the morphisms (2) and (3) are isomorphisms. The first one induces an isomorphism

$$R\pi_*\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu] \to D\left(R\pi_*\mathbb{A}_{\widetilde{\mathcal{N}}}[\nu]\right),$$

which in turn, via decomposition (1), induces two projections

$$\mathbb{A}_{\mathcal{N}}[\nu] \to D\left(\mathbb{A}_{\mathcal{N}}[\nu]\right), \quad \mathcal{K}[\nu] \to D\left(\mathcal{K}[\nu]\right). \tag{4}$$

Making explicit the isomorphism induced in cohomology and homology by (1), one may prove [3, Corollary 5.1] that $\mathcal{D}_{\widetilde{\mathcal{N}}}$ is the direct sum of P_1 and P_2, where

$$P_1 : H^{\bullet}(\mathcal{N}) \to H_{2\nu-\bullet}(\mathcal{N}) \quad \text{and} \quad P_2 : \mathbb{H}(\mathcal{K}[\nu]) \to \mathbb{H}(D\left(\mathcal{K}[\nu]\right))$$

are the maps induced in hypercohomology by the projections (4). It follows that P_1 is an isomorphism, because so is $\mathcal{D}_{\widetilde{\mathcal{N}}}$, and this holds true when restricting to every open subset U of $\mathcal{N}$. For instance (see also [3, Corollary 5.1]), if $\widetilde{U} = \pi^{-1}(U)$, the vanishing of the morphism $\mathbb{H}^{\bullet}(\mathcal{K}_U[\nu]) \to H_{2\nu-\bullet}(U)$ derives from projection formula [4, p. 26, G4, (ii)]:

$$\pi_*([\widetilde{U}] \cap \lambda_* w) = \pi_*(\delta^{-1}\theta^*[U] \cap \lambda_* w) = \delta^{-1}(\theta_*\lambda_* w) \cap [U] = 0, \quad \forall w \in \mathbb{H}^{\bullet}(\mathcal{K}_U[\nu]),$$

where λ_* is the morphism induced in hypercohomology by $\mathcal{K}_U[\nu] \to R\pi_*\mathbb{A}_{\widetilde{U}}[\nu]$.

Therefore, we have $\mathbb{A}_{\mathcal{N}}[\nu] \cong D\left(\mathbb{A}_{\mathcal{N}}[\nu]\right)$, which is equivalent to say that $\mathcal{N}$ is an $\mathbb{A}$-homology manifold. □

Remark 2.2. Observe that, as a scheme, $\widetilde{\mathcal{N}}$ could also be nonreduced, but what matters is that, for the usual topology, it is a nonsingular variety [4, p. 32, 3.1.1].

Corollary 2.3. *The nilpotent cone is a rational homology manifold.*

Proof. Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution of the nilpotent cone $\mathcal{N}$. It extends to a generically finite projective morphism $\pi' : \widetilde{\mathbf{g}} \to \mathbf{g}$, known as the Grothendieck simultaneous resolution, between complex quasi-projective nonsingular varieties of the same dimension [2, p. 49]. Therefore, Theorem 2.1 applies. □

Remark 2.4. If the Grothendieck simultaneous resolution $\pi' : \widetilde{\mathbf{g}} \to \mathbf{g}$ has degree δ, by Theorem 2.1 we deduce that *the nilpotent cone $\mathcal{N}$ is an $\mathbb{A}$-homology manifold for every Noetherian commutative ring with identity $\mathbb{A}$ for which δ is a unit.* For instance, for the variety $\mathcal{N}$ of nilpotent matrices in $\mathrm{GL}(n, \mathbb{C})$, we have $\delta = n!$. Therefore, in this case, $\mathcal{N}$ is also a $\mathbb{Z}_h$-homology manifold for every integer h relatively prime with $n!$ in $\mathbb{Z}$.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

[1] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, 100, Soc. Math. France, Paris, , 5–171 (1982)

[2] Borho, W., MacPherson, R.: Partial Resolutions of Nilpotent Varieties. Astérisque **101–102**, 23–74 (1983)

[3] Di Gennaro, V., Franco, D., Sessa, C.: Bivariant class of degree one. Results Math. **77**(6),(2022). Paper No. 227

[4] Fulton, W., MacPherson, R.: Categorical framework for the study of singular spaces. Mem. Am. Math. Soc. **31**(243) (1981). pp. vi+165

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