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Results in Mathematics

Topological Ordered Rings and Measures

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Abstract. Given a ring endowed with a ring order, we provide sufficient conditions for the order topology induced by the ring order to become a ring topology (analogous results for module orders are consequently derived). Finally, the notions of Radon and regular measures are transported to the scope of module-valued measures through module orders. Classical characterizations of these measures are obtained as well as the hereditariness of regularity for conditional ring-valued measures.

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1. Introduction

Topological rings and modules have been widely studied in the literature. We refer the reader to [2,8,17] for a broad perspective on the topic. In most studies of topological ordered rings, the order topology is typically considered only on totally ordered sets. In this manuscript, we will consider the order topology in partially ordered rings and modules. As an application of our study, we are capable of extending classical concepts from measure theory, such as Radon and regular measures to the scope of measures valued on ordered modules.

Classical Measure Theory (see [12]) deals with positive real-valued measures defined on Boolean algebras of sets. Due to the famous Stone Representation Theorem for Boolean algebras [15], every Boolean algebra is isomorphic (in the category of Boolean algebras) to a Boolean algebra of sets. This way, Classical Measure Theory retained full generality when defining classical measures until the birth of effect algebras [9]. However, these classical measures were always real or complex valued. In the remarkable book [7], real and complex Banach space valued measures defined on a Boolean algebra of sets were deeply studied. After the irruption of effect algebras, which extend the concept of boolean algebras of sets, an extensive study of non-real or complex valued measures has been performed. We refer the reader to [1,3–6,9,13,19] where valued measures on commutative groups, commutative topological groups, Hausdorff locally convex topological vector spaces and normed spaces were considered. Finally, in [10], topological module valued measures were analysed by using the algebraic structure of modules over rings. Classical concepts such as Radon and regularity for measures were studied for positive real-valued measures (see [11,12]) and in [7], the notion of Radon measure was taken to the scope of Banach space valued measures.

Our paper is organized as follows: in Sect. 2 we first recall the basic properties of ordered rings. We introduce the notion of faithful ring order and illustrate this concept by providing some examples in the context of bounded operators on a Hilbert space. We later obtain similar results for ordered modules over ordered rings. We concentrate on topological ordered rings, which are ordered rings where the order topology is a ring topology and we obtain necessary and sufficient conditions to ensure when an ordered ring is an ordered topological ring. In Sect. 3 we extend the classical notions of Radon and regular measures to the scope of module-valued measures. We provide classical characterizations of these measures and the hereditariness of regularity for conditional ring-valued measures in this context. When the order topology in an ordered module is a module topology, we then relate inner and outer regular measures and inner regular measures with Radon measures.

2. Ring and Module Orders and Topologies

Throughout this manuscript and unless otherwise stated, all rings will be associative, unitary and nonzero, all monoid actions considered will be left, and all modules M over rings will be unital ($1m = m$, $\forall m \in M$) and nonzero. Recall that if X is a partially ordered set (in what follows a poset), then $\uparrow x := \{y \in X : y \geq x\}$, $\downarrow x := \{y \in X : y \leq x\}$, $\uparrow_{\times} x := (\uparrow x) \setminus \{x\} = \{y \in X : y > x\}$, $\downarrow^{\times} x := (\downarrow x) \setminus \{x\} = \{y \in X : y < x\}$ for every $x \in X$. The interval notation will be used in the manuscript, that is, $(-\infty, y] := \downarrow y$, $[x, \infty) := \uparrow x$, and $[x, y] = \uparrow x \cap \downarrow y$ (similarly for open intervals). Finally, if X is a topological space and A is a subset of X, then $\mathrm{cl}(A)$ stands for the closure of A.

2.1. Ordered Rings/Modules

Let R be a ring. A partial order $\leq$ on R is called a ring order provided that for all $r, s \in R$ with $r \leq s$, $r + t \leq s + t$ for all $t \in R$, and for all $r, s \in R^+$, $rs \in R^+$, where $R^+ := \uparrow 0 = \{t \in R : t \geq 0\}$, that is, a partial order compatible with the ring operations. Ordered rings are just rings endowed with a ring order. We refer the reader to [14] for the basics on ring orders such as the following proposition.

Proposition 2.1. *Let R be a nonzero ordered ring. For all $r, s, t, u \in R$:*

- *If $r < s$ and $t \leq u$, then $r + t < s + u$.*
- *If $r < s$ and $t > 0$ is not a left zero-divisor (resp. right zero-divisor), then $tr < ts$ (resp. $rt < st$).*
- *If $r < s$, then $s - r > 0$ and $-s < -r$. In particular, $R^- := \downarrow 0 = \{a \in R : a \leq 0\} = -R^+$. Also, $-(\downarrow r) = \uparrow(-r)$ and $-(\uparrow r) = \downarrow(-r)$.*
- *If $0, 1$ are comparable, then $0 < 1$ and $\mathrm{char}(R) = 0$.*
- *If $r, 0$ are comparable, then $r^2 \geq 0$.*
- *R is totally ordered if and only if $R^+ \cup R^- = R$.*

Remark 2.2. Notice that there are other examples of ordered rings for which $0, 1$ are not comparable. Indeed, the trivial partial order $r \leq s \Leftrightarrow r = s$ is clearly a ring order for which $0, 1$ are not comparable (unless $1 = 0$).

Let R be a ring. If $\leq$ is a ring order on R, then the positive cone R^+ satisfies that $0 \in R^+$, $R^+ \cap R^- = \{0\}$, $R^+ + R^+ \subseteq R^+$, and $R^+R^+ \subseteq R^+$. Conversely, if $S \subseteq R$ is a subset of R satisfying the four conditions above, then there exists a unique ring order on R, given by $r \leq s \Leftrightarrow s - r \in S$ for all $r, s \in R$, for which $S = R^+$. A subset $S \subseteq R$ satisfying all four previous conditions will be called a ring ordered set. It is trivial that the intersection of any family of ring ordered subsets is again a ring ordered subset, and the union of any increasing family of ring ordered subsets is again a ring ordered subset. By means of Zorn's Lemma, we will demonstrate the existence of ring orders.

Lemma 2.3. *Let R be a ring. There exists a maximal ring ordered subset $S \subseteq R$. If, in addition, $\mathrm{char}(R) = 0$, then S can be taken in such a way that $1 \in S$, resulting in $0, 1$ being comparable.*

Proof. Let $\mathcal{L} := \{T \subseteq R : T \text{ is a ring ordered subset}\}$. Notice that $\mathcal{L} \neq \varnothing$ since $\{0\} \in \mathcal{L}$. If $\mathcal{C} \subseteq \mathcal{L}$ is a chain of $\mathcal{L}$, then $\bigcup \mathcal{C} \in \mathcal{L}$. Finally, Zorn's Lemma guarantees the existence of maximal elements in $\mathcal{L}$. If $\mathrm{char}(R) = 0$, then we can consider $\mathcal{L}_1 := \{T \in \mathcal{L} : 1 \in T\}$. Notice that $\mathcal{L}_1 \neq \varnothing$ since $\{0\} \cup \{\overbrace{1 + \cdots + 1}^{n} : n \in \mathbb{N}\} \in \mathcal{L}_1$. Also, any chain $\mathcal{C} \subseteq \mathcal{L}_1$ satisfies that $\bigcup \mathcal{C} \in \mathcal{L}_1$. Zorn's Lemma guarantees again the existence of maximal elements in $\mathcal{L}_1$. □

Ring orders for which $0, 1$ are comparable will be called unital ring orders. Similarly, if a ring ordered subset contains 1, then it will be called a unital ring ordered subset. We now introduce the following definition that will be later needed.

Definition 2.4. A ring order $\leq$ on a ring R is called faithful provided that $R^+ \cup R^-$ is a subring of R.

Notice that a ring order is faithful if and only if $r - s \in R^+ \cup R^-$ for all $r, s \in R^+$. In other words, a ring order is faithful if and only if R^+ is totally ordered. A slight modification in the proof of Lemma 2.3 serves to show the existence of maximal faithful ring orders in rings of characteristic 0.

In order to illustrate the concept of a faithful ring order, we will now provide some examples of faithful and non-faithful ring orders in the space of bounded operators defined on a complex Hilbert space H, denoted as $\mathcal{B}(H)$. A bounded operator $T \in \mathcal{B}(H)$ is said to be selfadjoint provided that $T = T'$, that is, $(T(h)|k) = (h|T(k))$ for all $h, k \in \mathcal{B}(H)$. The subset of selfadjoint bounded operators has structure of real vector space. In fact, selfadjoint bounded operators can be characterized as those operators for which $(T(h)|h) \in \mathbb{R}$ for all $h \in H$. This fact induces the following partial order in the real vector subspace of selfadjoint bounded operators: $T \leq S \Leftrightarrow (T(h)|h) \leq (S(h)|h)$ for all $h \in H$. This order is a real vector space order, that is, $T \leq S \Rightarrow T + R \leq S + R$ and $T \leq S \Rightarrow \lambda T \leq \lambda S$ for all selfadjoint bounded operators T, R, S and all $\lambda \geq 0$. However, the previous order does not behave well as a ring order. In fact, the set of selfadjoint bounded operators is not a subring of $\mathcal{B}(H)$ since $(T \circ S)' = S' \circ T' = S \circ T$ for T, S selfadjoint and bounded. Even if T, S are commuting positive selfadjoint bounded operators, it cannot be guaranteed that $T \circ S$ is positive (this fact, however, holds if $\dim(H) < \infty$). We refer the reader to [16] for more background about selfadjoint operators. In the upcoming results, we will unveil a unital ring ordered subset in $\mathcal{B}(H)$ consisting of positive selfadjoint bounded operators.

Remark 2.5. Let H be a complex Hilbert space. Every bounded operator $T \in \mathcal{B}(H)$ satisfies that $|(T(h)|h)| \leq \|T\|(h|h)$ for all $h \in H$, meaning that if T is selfadjoint, then $-\|T\|I \leq T \leq \|T\|I$.

The previous remark establishes that any selfadjoint bounded operator is comparable with a multiple of the identity. However, there are examples of positive selfadjoint bounded operators not comparable with the identity. Recall that the numerical radius of a bounded operator $T \in \mathcal{B}(H)$ is defined as $r(T) := \sup_{h \in \mathsf{S}_H} |(T(h)|h)|$, where S_H stands for the unit sphere of H.

Lemma 2.6. *Let H be a complex Hilbert space. Let $T \in \mathcal{B}(H)$ be selfadjoint. If $r(T) > 1$ and $\ker(T) \neq \{0\}$, then T is not comparable with the identity I.*

Proof. Since $r(T) > 1$, there exists $h \in \mathsf{S}_H$ such that $(T(h)|h) > 1 = (h|h)$. On the other hand, $(T(h)|h) = 0 < (h|h)$ for each $h \in \ker(T) \setminus \{0\}$. □

Example 2.7. Let H be a complex Hilbert space. Fix a proper closed subspace P of H. Take $T = 2\pi_P$, where π_P stands for the orthogonal projection onto P. If $p \in \mathsf{S}_P$, then $(T(p)|p) = (2p|p) = 2 > 1$, hence $r(T) > 1$. Next, if $q \in P^\perp$, then $T(q) = 0$, so $\ker(T) \neq \{0\}$. Finally, T is clearly selfadjoint and positive.

Theorem 2.8. *Let H be a complex Hilbert space. Let $T \in \mathcal{B}(H)$ selfadjoint and positive. Then $S := \mathbb{R}^+[T]$ is a unital ring ordered subset of $\mathcal{B}(H)$. If, in* *addition, T is not comparable with I, then the ring order induced by S in $\mathcal{B}(H)$ is not faithful. However, if $T := I$, then the ring order induced by S in $\mathcal{B}(H)$ is faithful.*

Proof. First off, it is clear that $S + S \subseteq S$, $0, I \in S$, and $SS \subseteq S$. Let us prove that $S \cap -S = \{0\}$. Indeed, let $p, q \in \mathbb{R}^+[x]$ such that $p(T) = -q(T)$. Note that $p(T)$ is selfadjoint and positive, so $-q(T)$ is selfadjoint and positive, meaning that $q(T)$ is selfadjoint and negative, so $q(T) = 0$. Next, let us assume that T is not comparable with I. We will show that the ring order induced by S in $\mathcal{B}(H)$ is not faithful. Indeed, take $2I + T, I + 2T \in \mathbb{R}^+[T]$. Then $(2I + T) - (I + 2T) = I - T$. If $I - T \in \mathbb{R}^+[T]$, then $I - T \geq 0$, so I and T are comparable. If $T - I \in \mathbb{R}^+[T]$, then $T - I \geq 0$, so again I and T are comparable. Finally, suppose that $T := I$. Notice that $S := \mathbb{R}^+[I] = \mathbb{R}^+ I$, which is clearly totally ordered, hence the ring order induced by S in $\mathcal{B}(H)$ is faithful. □

Let M be a module over an ordered ring R. A partial order $\leq$ on M is called a module order provided that for all $m, n \in M$ with $m \leq n$, $m+p \leq n+p$ for all $p \in M$, and for all $m, n \in M$ with $m \leq n$, $rm \leq rn$ for all $r \in R^+$. As usual, we will let $M^+ := \uparrow 0 = \{m \in M : m \geq 0\}$. Ordered modules are just modules, over ordered rings, endowed with a partial order compatible with the module operations. Notice that ordered rings are ordered modules over themselves.

The following are basic properties satisfied by module orders which can be found in [14].

Proposition 2.9. *Let R be an ordered ring and M an ordered R-module. For all $m, n, p, q \in M$ and all $r, s \in R$:*

- *If $m < n$ and $p \leq q$, then $m + p < n + q$.*
- *If $r \leq s$ and $m \geq 0$, then $rm \leq sm$.*
- *If $m < n$, then $m - n > 0$ and $-n < -m$. In particular, $M^- := \downarrow 0 = \{m \in M : m \leq 0\} = -M^+$. Also, $-(\downarrow m) = \uparrow(-m)$, $-(\uparrow m) = \downarrow(-m)$, $\downarrow m = m + M^-$, and $\uparrow m = m + M^+$.*
- *M is totally ordered if and only if $M^+ \cup M^- = M$.*

Let M be an R-module. If $\leq$ is a module order on M, then the positive cone M^+ satisfies that $0 \in M^+$, $M^+ \cap M^- = \{0\}$, $M^+ + M^+ \subseteq M^+$, and $R^+ M^+ \subseteq M^+$. Conversely, if $S \subseteq M$ is a subset of M satisfying the previous four conditions, then there exists a unique module order on M, given by $m \leq n \Leftrightarrow m - n \in S$ for all $m, n \in M$, for which $S = M^+$. Such a set S will be called a module ordered set. The intersection of any family of module ordered subsets is again a module ordered subset, and the union of any increasing family of module ordered subsets is again a module ordered subset. Thus, the collection of all module ordered subsets of a module over an ordered ring is inductive when endowed with the inclusion, so by Zorn's Lemma, following a similar proof as in Lemma 2.3, every module over an ordered ring has a maximal module ordered subset.

We now introduce the concept of faithful module order.

Definition 2.10. A module order in a module M, over a faithful ordered ring R, is called faithful provided that $M^+ \cup M^-$ is a $(R^+ \cup R^-)$-submodule of M.

Note that a module order in a module M, over a faithful ordered ring R, is faithful if and only if $m - n \in M^+ \cup M^-$ for all $m, n \in M^+$, or equivalently, if and only if M^+ is totally ordered. The existence of maximal faithful module orders is guaranteed by the Zorn's Lemma.

2.2. Topological Ordered Rings/Modules

We start this section by introducing the notion of topological ordered ring.

Definition 2.11. An ordered ring R is called a topological ordered ring whenever the order topology on R is a ring topology.

Note that in ordered rings, the ring order does not necessarily have to be compatible with the ring topology in any sense. In fact, the existence of nonclosed ring ordered subsets is guranteed by the following example.

Example 2.12. Let R be a non-Hausdorff topological ring. Consider the trivial ring order in R, that is, the one whose ordered subset is $S := \{0\}$, which trivially induces the trivial order $r \leq s \iff r = s$. Notice that S is not closed because R is not Hausdorff.

Remark that Example 2.12 can actually be exploited to obtain many more examples of nonclosed ring ordered subsets. Indeed, if R is a ring and we endow it with the trivial topology, which is a ring topology, then no ring ordered subset is closed since every ring ordered subset is proper (unless $R = 0$).

Remark 2.13. Let R be a topological ordered ring. For every $r \in R$, $\downarrow r = r + R^-$ and $\uparrow r = r + R^+$. As a consequence, $R^+, R^-, \downarrow r, \uparrow r$ are all homeomorphic and then the following conditions are equivalent:

(1) R^+ is closed.
(2) R^- is closed.
(3) For every $r \in R$, $\downarrow r$ is closed.
(4) There exists $r \in R$ such that $\downarrow r$ is closed.
(5) For every $r \in R$, $\uparrow r$ is closed.
(6) There exists $r \in R$ such that $\uparrow r$ is closed.

We can easily obtain the following result that shows equivalent conditions for R^+ being closed.

Proposition 2.14. *Let R be a topological ordered ring. The following conditions are equivalent:*

(1) *R^+ is closed.*

(2) *For every* $r \in R$ *and every prefilter* B *of* R *such that* $\bigcup_{C \in B} C \subseteq \downarrow r$, *it follows that* $\lim B \subseteq \downarrow r$.
(3) *For every* $r \in R$, *every net* $(s_i)_{i \in I} \subseteq R$ *such that* $s_i \leq r$ *for all* $i \in I$, *and every* $s \in \lim_{i \in I} s_i$, *it follows that* $s \leq r$.
(4) *For every* $r \in R$ *and every prefilter* B *of* R *such that* $\bigcup_{C \in B} C \subseteq \uparrow r$, *it follows that* $\lim B \subseteq \uparrow r$.
(5) *For every* $r \in R$ *and every net* $(s_i)_{i \in I} \subseteq R$ *such that* $s_i \geq r$ *for all* $i \in I$, *and every* $s \in \lim_{i \in I} s_i$, *it follows that* $s \geq r$.

Proof. Recall that in any topological space X, a subset $A \subseteq X$ is closed if and only if $\lim B \subseteq A$ for every prefilter B of X such that $\bigcup_{C \in B} C \subseteq A$, which is also equivalent to $\lim_{i \in I} x_i \subseteq B$ for every net $(x_i)_{i \in I} \subseteq A$. As a consequence, conditions (2) and (3) are equivalent to $\downarrow r$ being closed, and conditions (4) and (5) are equivalent to $\uparrow r$ being closed, so it only suffices to rely on Remark 2.13. □

We will now concentrate on providing necessary and sufficient conditions for a ring R to be a topological ordered ring. In this first result, we present a necessary condition for this fact.

Theorem 2.15. *Let* X *be a poset. Then* $\{\uparrow_\times x, \downarrow^\times x : x \in X\}$ *is a subbase for a topology on* X *if and only if for every* $x \in X$ *either* $\uparrow_\times x \neq \varnothing$ *or* $\downarrow^\times x \neq \varnothing$.

Proof. First, recall that if X is a set and $\mathcal{S}$ is a nonemtpy subset of $\mathcal{P}(X)$, then the set of finite intersections of $\mathcal{S}$, $\mathcal{B}(\mathcal{S}) := \left\{\bigcap_{T \in \mathcal{T}} T : \mathcal{T} \subseteq \mathcal{S} \text{ finite}\right\}$, is trivially closed under finite intersections, thus $\mathcal{B}(\mathcal{S})$ is a base for a topology on X if and only if $\bigcup_{S \in \mathcal{S}} S = X$. According to this observation, we have that $\{\uparrow_\times x, \downarrow^\times x : x \in X\}$ is a subbase for a topology on X if and only if $\left(\bigcup_{x \in X} \uparrow_\times x\right) \cup \left(\bigcup_{x \in X} \downarrow^\times x\right) = X$, that is, if for every $x \in X$ there exists $y \in X$ such that either $x \in \uparrow_\times y$ or $x \in \downarrow^\times y$, or equivalently, if and only if for every $x \in X$ there exists $y \in X$ such that either $y \in \downarrow^\times x$ or $y \in \uparrow_\times x$. □

Theorem 2.15 has important implications towards topological ordered rings. By definition, the order topology on a poset X is the topology generated by the subbase $\{\uparrow_\times x, \downarrow^\times x : x \in X\}$. According to Theorem 2.15, the order topology is only possible in those posets X satisfying that for every $x \in X$ either $\uparrow_\times x \neq \varnothing$ or $\downarrow^\times x \neq \varnothing$.

Corollary 2.16. *Let* R *be an ordered ring. Then* $\{\uparrow_\times r, \downarrow^\times r : r \in R\}$ *is a subbase for a topology on* R *if and only if* $R^+ \neq \{0\}$.

Proof. Simply observe that, by bearing in mind Remark 2.13, for every $r \in R$, $\downarrow r = r + R^-$ and $\uparrow r = r + R^+$. Finally, it only suffices to apply Theorem 2.15. □

Our next goal is to find sufficient conditions for the order topology in an ordered ring to become a ring topology. We will first find a sufficient condition for the order topology to be an additive group topology on the ordered ring. Before proving this result, we recall the following characterization of a ring topology which can be found in both [17, Theorem 11.4] and [18, Theorem 3.5] that will needed in the proof.

Theorem 2.17. *Let R be a ring. If τ is a ring topology on R and $\mathcal{B}$ is a basis of neighborhoods of* 0*, then the following is verified:*

(1) *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U + U \subseteq V$.*
(2) *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $-U \subseteq V$.*
(3) *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $UU \subseteq V$.*
(4) *For every $V \in \mathcal{B}$ and every $r \in R$, there exists $U \in \mathcal{B}$ with $rU \cup Ur \subseteq V$.*

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(R)$ verifying all four properties above, then there exists a unique ring topology on R such that $\mathcal{B}$ is a basis of 0-*neighborhoods. This topology is given by*

$$\tau := \{A \subseteq R : \forall a \in A\,, \exists U \in \mathcal{B} \text{ such that } a + U \subseteq A\} \cup \{\varnothing\}.$$

We can now state the following theorem that provides sufficient conditions for the order topology in an ordered ring to become an additive group topology. However, we will first rely on the following technical lemma.

Lemma 2.18. *Let R be an ordered ring. If $R^+ \neq \{0\}$ and $R^+\backslash\{0\}$ is downward directed, then $\{(-r, r) : r > 0\}$ is a base of neighborhoods of* 0 *for the order topology.*

Proof. According to Corollary 2.16, $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$ is a subbase for a topology on R, which is the order topology. Let $W \subseteq R$ be a 0-neighborhood for the order topology. There can be found $r_1, \dots, r_n, s_1, \dots, s_m \in R$ satisfying that $0 \in \uparrow_\times r_1 \cap \cdots \cap \uparrow_\times r_n \cap \downarrow^\times s_1 \cap \cdots \cap \downarrow^\times s_m \subseteq W$. Observe that $r_i < 0$ and $s_j > 0$ for all $i \in \{1, \dots, n\}$ and all $j \in \{1, \dots, m\}$. Since $R^+\backslash\{0\}$ is downward directed, there exists $r_0 \in R^+\backslash\{0\}$ such that $r_0 \leq -r_i$ for all $i \in \{1, \dots, n\}$ and $r_0 \leq s_j$ for all $j \in \{1, \dots, m\}$. Notice that $(-r_0, r_0) \subseteq \uparrow_\times r_1 \cap \cdots \cap \uparrow_\times r_n \cap \downarrow^\times s_1 \cap \cdots \cap \downarrow^\times s_m \subseteq W$. □

Theorem 2.19. *Let R be an ordered ring. Let $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$. If $R^+ \neq \{0\}$, $R^+\backslash\{0\}$ is downward directed, and $0 \in \mathrm{cl}\,(R^+\backslash\{0\})$, then the topology generated by the subbase $\mathcal{S}$ is an additive group topology.*

Proof. First off, by Corollary 2.16, $\mathcal{S}$ is a subbase for a topology on R. We will show that this topology satisfies the first two conditions of Theorem 2.17. Indeed, let $\mathcal{B}_0(\mathcal{S})$ denote the family of all finite intersections of $\mathcal{S}$ containing 0. Then:

- Let $W \in \mathcal{B}_0(\mathcal{S})$. In accordance with Lemma 2.18, there exists $r_0 \in R^+\backslash\{0\}$ such that $(-r_0, r_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(R^+\backslash\{0\})$, there exists $t_0 \in R$ such that $0 < t_0 < r_0$. By using again that $0 \in \mathrm{cl}\,(R^+\backslash\{0\})$, we can find $s_0 \in R$ such that $0 < s_0 < r_0 - t_0$. Finally, we will prove that $(-s_0, s_0) + (-t_0, t_0) \subseteq W$. Indeed, let us observe first that $(-s_0, s_0) +$

$(-t_0, t_0) \subseteq (-s_0 - t_0, s_0 + t_0)$. Next, take $a \in R$ with $-s_0 - t_0 < a < s_0 + t_0$. Note that $a < s_0 + t_0 < r_0$. Also, $-s_0 - t_0 < a$ so $-a < s_0 + t_0 < r_0$. We have just shown that $(-s_0, s_0) + (-t_0, t_0) \subseteq (-s_0 - t_0, s_0 + t_0) \subseteq (-r_0, r_0) \subseteq W$. We have proven condition (1) in Theorem 2.17.

- Let $W \in \mathcal{B}_0(\mathcal{S})$. By using again Lemma 2.18, we can find $r_0 \in R^+ \backslash \{0\}$ such that $(-r_0, r_0) \subseteq W$. Now, it only suffices to notice that $(-r_0, r_0)$ is an additively symmetric element of $\mathcal{B}_0(\mathcal{S})$ and then condition (2) in Theorem 2.17 holds.

□

Now, let us a find a stronger sufficient condition to assure that the order topology is a ring topology on an ordered ring. For that purpose, a new concept in Associative Ring Theory is introduced.

Definition 2.20. Let R be a ring. A ring order on R is called left-strong provided that $R^+ \neq \{0\}$ and for every $r \in R^+ \backslash \{0\}$ and every $s \in R$ there exists $t \in R^+ \backslash \{0\}$ such that $s(-t, t) \subseteq \downarrow^\times r$. In a similar way, right-strong ring order can be defined. A ring order is said to be strong if it is left- and right-strong.

Theorem 2.21. *Let R be an ordered ring. Let $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$. If $0, 1$ are comparable, the ring order is strong, $R^+ \backslash \{0\}$ is downward directed and $0 \in \mathrm{cl}\,(R^+ \backslash \{0\})$, then the topology generated by the subbase $\mathcal{S}$ is a ring topology.*

Proof. First of all, by definition of strong ring order, $R^+ \neq \{0\}$. Also, $1 \in R^+ \backslash \{0\}$. Therefore, Corollary 2.16 assures that $\mathcal{S}$ is a subbase for a topology on R, which is the order topology. In view of Theorem 2.19, the generated base $\mathcal{B}_0(\mathcal{S})$ satisfies the first two conditions of Theorem 2.17. Let us finally check that $\mathcal{B}_0(\mathcal{S})$ also satisfies the third and fourth condition of Theorem 2.17.

- Let $W \in \mathcal{B}_0(\mathcal{S})$. By bearing in mind Lemma 2.18, there exits $r_0 \in R^+ \backslash \{0\}$ such that $(-r_0, r_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(R^+ \backslash \{0\})$, there exists $t_0 \in R$ such that $0 < t_0 < r_0$. By using again that $0 \in \mathrm{cl}\,(R^+ \backslash \{0\})$, we can find $p_0 \in R$ such that $0 < p_0 < r_0 - t_0$. Again, by relying on the fact that $0 \in \mathrm{cl}\,(R^+ \backslash \{0\})$, there exists $q_0 \in R$ satisfying that $0 < q_0 < r_0 - t_0 - p_0$. Next, since $R^+ \backslash \{0\}$ is downward directed, $h_0 \in R^+ \backslash \{0\}$ can be found in such a way that $h_0 \leq q_0$, $h_0 \leq p_0$, and $h_0 \leq t_0$. Finally, we will prove that $(-h_0, h_0)(-1, 1) \subseteq (-3h_0, 3h_0) \subseteq (-r_0, r_0) \subseteq W$. Indeed, take arbitrary elements $a, b \in R$ with $-h_0 < a < h_0$ and $-1 < b < 1$. Note that $h_0 - a > 0$ and $1 - b > 0$, meaning that $h_0 - h_0 b - a + ab = (h_0 - a)(1 - b) \geq 0$. Then $ab \geq -h_0 + h_0 b + a$. Since $b > -1$ and $h_0 > 0$, we have that $h_0 b \geq -h_0$. Also, $-h_0 < a$. As a consequence, $ab \geq -h_0 + h_0 b + a > -h_0 - h_0 - h_0$. By repeating the same reasoning for a and $-b$, we conclude that $-ab = a(-b) > -h_0 - h_0 - h_0$, that is, $ab < 3h_0$. We have just shown that $(-h_0, h_0)(-1, 1) \subseteq (-3h_0, 3h_0)$. Let us finally prove that $(-3h_0, 3h_0) \subseteq (-r_0, r_0) \subseteq W$. Indeed, if $c \in R$ and $-3h_0 < c < 3h_0$, then $c < h_0 + h_0 + h_0 \leq q_0 + p_0 + t_0 < r_0$, and

$-c < h_0 + h_0 + h_0 \leq q_0 + p_0 + t_0 < r_0$, so $-r_0 < c$. Consequently, condition (3) of Theorem 2.17 holds.

- Let $W \in \mathcal{B}_0(\mathcal{S})$ and fix an arbitrary element $r \in R$. If we keep in mind Lemma 2.18 again, we can find $r_0 \in R^+ \backslash \{0\}$ such that $(-r_0, r_0) \subseteq W$. Since the ring order is strong, we can find $t, s \in R^+ \backslash \{0\}$ such that $r(-t,t) \cup (-s,s)r \subseteq \downarrow^\times r_0$. Since $r(-t,t)$ and $(-s,s)r$ are both additively symmetric, we then have that $r(-t,t) \cup (-s,s)r \subseteq (-r_0, r_0) \subseteq W$, which proves condition (4) in Theorem 2.17 and then the conclusion holds.

□

The following result shows that the converse of Theorem 2.21 does not hold.

Theorem 2.22. *Let R be a ring of characteristic 0. Consider the ring ordered set $S := \{0\} \cup \{\overbrace{1 + \cdots + 1}^{n} : n \in \mathbb{N}\}$. The ring order induced by S is strong and the order topology generated by the subbase $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$ is the discrete topology, hence a ring topology.*

Proof. Notice that $\uparrow_\times (-1) = -1 + (S \backslash \{0\})$ and $\downarrow^\times 1 = 1 - (S \backslash \{0\})$. If $r \in \uparrow_\times (-1) \cap \downarrow^\times 1$, then there are $s_1, s_2 \in S \backslash \{0\}$ such that $r = -1 + s_1$ and $r = 1 - s_2$. Then $s_1 + s_2 = 1 + 1$. There are $n_1, n_2 \in \mathbb{N}$ such that $s_1 = \overbrace{1 + \cdots + 1}^{n_1}$ and $s_2 = \overbrace{1 + \cdots + 1}^{n_2}$. Thus, $\overbrace{1 + \cdots + 1}^{n_1 + n_2} = 1 + 1$. Since $\mathrm{char}(R) = \{0\}$, we necessarily have that $n_1 = n_2 = 1$, hence $s_1 = s_2 = 1$, meaning that $r = 0$. As a consequence, $\{0\} = \uparrow_\times (-1) \cap \downarrow^\times 1$. Therefore, the ring ordering induced by S is trivially strong since for every $r \in R^+ \backslash \{0\}$ and every $s \in R$, $s(-1,1) = (-1,1)s = \{0\} \subseteq \downarrow^\times r$ because $(-1,1) = \uparrow_\times (-1) \cap \downarrow^\times 1 = \{0\}$. Next, by Corollary 2.16, $\mathcal{S}$ is a subbase for a topology on R. We will prove that this topology is the discrete topology. Let fix any $r \in R$. We will prove that $\{r\} = \uparrow_\times (r-1) \cap \downarrow^\times (r+1)$. Indeed, $0 < 1$ and $-1 < 0$, so $r < r+1$ and $r - 1 < r$, meaning that $r \in \uparrow_\times (r-1) \cap \downarrow^\times (r+1)$. Conversely, take any $s \in \uparrow_\times (r-1) \cap \downarrow^\times (r+1)$. Notice that $r - 1 < s < r + 1$, hence $-1 < s - r < 1$, so $s - r \in (-1,1) = \{0\}$, meaning that $s = r$. □

We conclude this section by pointing out that Remark 2.13, Proposition 2.14, Theorems 2.19 and 2.21 can be easily adapted to the scope of modules.

3. Radon Measures

In this section, by using the concepts of topological ordered rings and modules, we will extend the notions of Radon and regular measures to the scope of module-valued measures. Module measures are those measures with values on a topological module. Let $\mathcal{B}(X)$ be the algebra of Borel subsets of a topological space X, M a topological module over a topological ring R and $\mu : \mathcal{B}(X) \to M$ a map. We recall that μ is a measure if it is an additive map; that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \mathcal{B}(X)$ such that $A \cap B = \emptyset$. It is trivial to check that $\mu(\emptyset) = 0$, $\mu(B \backslash A) = \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{B}(X)$, and $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{B}(X)$.

The following remark is well known in the literature of Classical Measure Theory for positive measures [12]. However, we make use of it for measures valued on ordered modules. The proof follows similarly as for classical positive measures.

Remark 3.1. Let M be an ordered module over an ordered ring R. Let $\mu : \mathcal{B}(X) \to M^+$ be a measure. If $A, B \in \mathcal{B}(X)$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$. If $A, B, C \in \mathcal{B}(X)$ with $B \subseteq C$, then $\mu(A \cap C) - \mu(A \cap B) \leq \mu(C) - \mu(B)$.

Definition 3.2. Let X be a topological space. Let M be a topological module over a topological ring R. Consider a measure $\mu : \mathcal{B}(X) \to M$ and a Borel subset $A \in \mathcal{B}(X)$. Then:

(1) A is called a μ-Radon subset if for every 0-neighborhood $W \subseteq M$ there exists a compact subset $F \subseteq A$, such that for every Borel subset $B \subseteq A \backslash F$, $\mu(B) \in W$.
(2) μ is called a Radon measure if every Borel subset is μ-Radon.

The notion of regular measure can also be transported to this scope by considering ordered modules.

Definition 3.3. Let X be a topological space. Let M be an ordered module over an ordered ring R. Consider a measure $\mu : \mathcal{B}(X) \to M^+$ and a Borel subset $A \in \mathcal{B}(X)$. Then:

(1) A is called an inner μ-regular subset if $\mu(A) = \sup\{\mu(F) : F \subseteq A \text{ compact}\}$.
(2) μ is called inner regular if every Borel subset is inner μ-regular.
(3) A is called an outer μ-regular subset if $\mu(A) = \inf\{\mu(U) : U \supseteq A \text{ open}\}$.
(4) μ is called outer regular if every Borel subset is outer μ-regular.
(5) A is called a μ-regular subset if it is inner and outer μ-regular.
(6) μ is called regular if it is inner and outer regular.

Remark 3.4. Let X be a topological space. Let M be an ordered module over an ordered ring R. Let $\mu : \mathcal{B}(X) \to M^+$ be a measure. Let $B \in \mathcal{B}(X)$. According to Lemma 3.1, if $\mu(B) = 0$, then B is trivially inner μ-regular. If $\mu(B) = \mu(X)$, then B is trivially outer μ-regular.

Our first result of this section establishes a relationship between Radon and regular measures. Notice that by faithful topological ordered module/ring we mean an ordered module/ring such that the order is faithful and the order topology is a module/ring topology.

Theorem 3.5. *Let X be a topological space. Let M be a faithful topological ordered module over a faithful topological ordered ring R. Consider a measure*

$\mu : \mathcal{B}(X) \to M^+$. A Borel subset $A \in \mathcal{B}(X)$ is μ-Radon if and only if A is inner μ-regular.

Proof. Suppose first that A is μ-Radon. Notice that $\mu(A) \geq \mu(F)$ for all $F \subseteq A$ compact in view of Remark 3.1. Take an upper bound $m \in M$ for $\{\mu(F) : F \subseteq A \text{ compact}\}$. Notice that $m \geq \mu(\varnothing) = 0$. Since the module order is faithful, either $m - \mu(A) \geq 0$ or $\mu(A) - m \geq 0$. Suppose on the contrary that $m < \mu(A)$. Let $W := (-\infty, \mu(A) - m)$, which is a 0-neighborhood in M. By hypothesis, there exists a compact subset $F \subseteq A$ such that, for every Borel subset $B \subseteq A \backslash F$, $\mu(B) \in W$. In particular, by Remark 3.1, $\mu(A) - \mu(F) = \mu(A \backslash F) \in W$, meaning the contradiction that $m < \mu(F)$. Conversely, suppose that A is inner μ-regular. If $\mu(A) = 0$, then A is clearly μ-Radon, so let us assume that $\mu(A) > 0$. Take $W \subseteq M$ a 0-neighborhood. Since $M^+ \backslash \{0\}$ is downward directed, there exists $m \in M$, $m > 0$, such that $(-m, m) \subseteq W$. By using again that $M^+ \backslash \{0\}$ is downward directed, we can find $p > 0$ such that $p \leq \mu(A)$ and $p \leq m$. Note that $0 \leq \mu(A) - p < \mu(A)$, so $\mu(A) - p$ is not an upper bound for $\{\mu(F) : F \subseteq A \text{ compact}\}$. Since the module order is faithful, $\mu(A) - p$ is comparable with $\mu(F)$ for each $F \subseteq A$ compact. As a consequence, there exists a compact subset $F \subseteq A$ such that $\mu(F) > \mu(A) - p$. Finally, for every Borel subset $B \subseteq A \backslash F$, $0 \leq \mu(B) \leq \mu(A \backslash F) = \mu(A) - \mu(F) < p \leq m$, thus $\mu(B) \in W$. □

As a consequence of Theorem 3.5, a positive measure is Radon if and only if it is inner regular.

Theorem 3.6. *Let X be a topological space. Let M be a faithfully ordered module over a faithfully ordered ring R. Consider a measure $\mu : \mathcal{B}(X) \to M^+$. Let $A \in \mathcal{B}(X)$ be a Borel subset. Then:*

1. *If X is Hausdorff and A is inner μ-regular, then $X \backslash A$ is outer μ-regular.*
2. *If X is compact and A is outer μ-regular, then $X \backslash A$ is inner μ-regular.*

Proof. (1) Observe that $\mu(X \backslash A) \leq \mu(U)$ for all $U \supseteq X \backslash A$ open in view of Remark 3.1. Let $m \in M$ be a lower bound for $\{\mu(U) : U \supseteq X \backslash A \text{ open}\}$. We will prove that $\mu(X \backslash A) \geq m$. Notice that $\mu(X) - m \geq 0$ and $\mu(X) - \mu(X \backslash A) \geq 0$, therefore, since the module order is faithful, we have that either $\mu(X \backslash A) - m = (\mu(X) - m) - (\mu(X) - \mu(X \backslash A)) \geq 0$ or $\mu(X \backslash A) - m = (\mu(X) - m) - (\mu(X) - \mu(X \backslash A)) \leq 0$. If $\mu(X \backslash A) - m \geq 0$, then we are done. So, let us assume that $\mu(X \backslash A) - m < 0$, that is, $m > \mu(X \backslash A) \geq 0$. Let $e := m - \mu(X \backslash A) > 0$. Note that $\mu(A) - e = \mu(A) - m + \mu(X \backslash A) = \mu(A) - m + \mu(X) - \mu(A) = \mu(X) - m \geq 0$. Then $0 \leq \mu(A) - e < \mu(A)$, so $\mu(A) - e$ is not an upper bound for $\{\mu(F) : F \subseteq A \text{ compact}\}$. Since the module order is faithful, $\mu(A) - e$ is comparable with $\mu(F)$ for each $F \subseteq A$ compact. As a consequence, there exists a compact subset $F \subseteq A$ such that $\mu(A) - e < \mu(F) \leq \mu(A)$. Note that F is closed in X because X is Hausdorff. Take $U := X \backslash F$, which is open and contains $X \backslash A$. Then we reach the contradiction that $\mu(U) = \mu(X\backslash F) = \mu(X) - \mu(F) < \mu(X) - \mu(A) + e = \mu(X\backslash A) + m - \mu(X\backslash A) = m$.

(2) According to Remark 3.1, $\mu(X\backslash A) \geq \mu(F)$ for all $F \subseteq X\backslash A$ compact. Let $m \in M$ be an upper bound for $\{\mu(F) : F \subseteq X\backslash A \text{ compact}\}$. We will prove that $\mu(X\backslash A) \leq m$. Notice that $m \geq \mu(\varnothing) = 0$, thus, since the module order is faithful, we have that either $\mu(X\backslash A) - m \geq 0$ or $\mu(X\backslash A) - m \leq 0$. If $\mu(X\backslash A) - m \leq 0$, then we are done. So, let us assume that $\mu(X\backslash A) - m > 0$. Let $e := \mu(X\backslash A) - m > 0$. Note that $\mu(A) + e > \mu(A)$, so $\mu(A) + e$ is not a lower bound for $\{\mu(U) : U \supseteq A \text{ open}\}$. Since the module order is faithful, $\mu(A) + e$ is comparable with $\mu(U)$ for each $U \supseteq A$ open. As a consequence, there exists an open subset $U \supseteq A$ such that $\mu(A) + e > \mu(U) \geq \mu(A)$. Take $F := X\backslash U \subseteq X\backslash A$, which is closed, hence compact because so is X. Then we reach the contradiction that $\mu(F) = \mu(X\backslash U) = \mu(X) - \mu(U) > \mu(X) - \mu(A) - e = \mu(X\backslash A) - \mu(X\backslash A) + m = m$.

□

If $A \subset \mathcal{B}(X)$, R is a ring and $\mu : \mathcal{B}(X) \to R$ is a measure, then every element $A \subset \mathcal{B}(X)$ such that $\mu(A) \in \mathcal{U}(R)$, where $\mathcal{U}(R)$ denotes the set of invertibles of R, induces a conditional measure:

$$\begin{aligned} \mu_A : \mathcal{B}(X) &\to R \\ B &\mapsto \mu_A(B) := \mu(A)^{-1}\mu(A \cap B). \end{aligned} \tag{3.1}$$

Theorem 3.7. *Let X be a topological space. Let R be an ordered ring. Consider a measure $\mu : \mathcal{B}(X) \to R^+$. Fix $A \in \mathcal{B}(X)$ with $\mu(A) \in \mathcal{U}(R)$ and $B \in \mathcal{B}(X)$. Then:*

(1) *If $A \cap B$ is inner μ-regular, then B is inner μ_A-regular.*
(2) *If $A \cap B$ is outer μ-regular and A is closed, then B is outer μ_A-regular.*
(3) *If the ring order of R is faithful and B is outer μ-regular, then B is outer μ_A-regular.*

Proof. (1) Notice that $\mu_A(F) \leq \mu_A(B)$ for all $F \subseteq B$ compact in accordance with Lemma 3.1. Let $r \in R$ be an upper bound for $\{\mu_A(F) : F \subseteq B \text{ compact}\}$. We will prove that $\mu_A(B) \leq r$. For every $F \subseteq A \cap B$ compact, we have that $\mu(F) = \mu(F \cap A) = \mu(A)\mu_A(F) \leq \mu(A)r$, that is, $\mu(A)r$ is an upper bound for $\{\mu(F) : F \subseteq A \cap B \text{ compact}\}$. Therefore, $\mu(A \cap B) \leq \mu(A)r$, hence $\mu_A(B) = \mu(A)^{-1}\mu(A \cap B) \leq \mu(A)^{-1}\mu(A)r = r$.

(2) By relying again of Lemma 3.1, we have that $\mu_A(U) \geq \mu_A(B)$ for all $U \supseteq B$ open. Let $r \in R$ be a lower bound for $\{\mu_A(U) : U \supseteq B \text{ open}\}$. We will prove that $\mu_A(B) \geq r$. For every $U \supseteq A \cap B$ open, we have that $\mu(U) \geq \mu(U \cap A) = \mu((U \cup (X\backslash A)) \cap A) = \mu(A)\mu_A(U \cup (X\backslash A)) \geq \mu(A)r$, that is, $\mu(A)r$ is a lower bound for $\{\mu(U) : U \supseteq A \cap B \text{ open}\}$. Therefore, $\mu(A \cap B) \geq \mu(A)r$, hence $\mu_A(B) = \mu(A)^{-1}\mu(A \cap B) \geq \mu(A)^{-1}\mu(A)r = r$.

(3) As a consequence of Lemma 3.1, we have that $\mu_A(U) \geq \mu_A(B)$ for all $U \supseteq B$ open. Let $r \in R$ be a lower bound for $\{\mu_A(U) : U \supseteq B \text{ open}\}$. We will prove that $\mu_A(B) \geq r$. Suppose on the contrary that $e := r - \mu_A(B) > 0$.

Observe that $\mu(B)+\mu(A)e > \mu(B)$, so $\mu(B)+\mu(A)e$ is not a lower bound for $\{\mu(U) : U \supseteq B \text{ open}\}$. Since the ring order is faithful, $\mu(B)+\mu(A)e$ is comparable with $\mu(U)$ for each $U \supseteq B$ open. As a consequence, there exists an open subset $U \supseteq B$ such that $\mu(B)+\mu(A)e > \mu(U) \geq \mu(B)$. Next, by Remark 3.1, $\mu_A(U) - \mu_A(B) = \mu(A)^{-1}(\mu(A \cap U) - \mu(A \cap B)) \leq \mu(A)^{-1}(\mu(U) - \mu(B)) < \mu(A)^{-1}\mu(A)e = e = r - \mu_A(B)$, reaching the contradiction that $\mu_A(U) < r$.

□

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