Transposed Poisson Structures on Generalized Witt Algebras and Block Lie Algebras

Ivan Kaygorodov and Mykola Khrypchenko

Abstract. We describe transposed Poisson structures on generalized Witt algebras $W(A, V, \langle \cdot, \cdot \rangle)$ and Block Lie algebras $L(A, g, f)$ over a field F of characteristic zero, where $\langle \cdot, \cdot \rangle$ and f are non-degenerate. More specifically, if $\dim(V) > 1$, then all the transposed Poisson algebra structures on $W(A, V, \langle \cdot, \cdot \rangle)$ are trivial; and if $\dim(V) = 1$, then such structures are, up to isomorphism, mutations of the group algebra structure on FA. The transposed Poisson algebra structures on $L(A, g, f)$ are in a one-to-one correspondence with commutative and associative multiplications defined on a complement of the square of $L(A, g, f)$ with values in the center of $L(A, g, f)$. In particular, all of them are usual Poisson structures on $L(A, g, f)$. This generalizes earlier results about transposed Poisson structures on Block Lie algebras $\mathcal{B}(q)$.

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Introduction

Poisson algebras originated from the Poisson geometry in the 1970s and have shown their importance in several areas of mathematics and physics, such as

Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics. One of the popular topics in the theory of Poisson algebras is the study of all possible Poisson algebra structures with fixed Lie or associative part [1,9,10,17]. Recently, Bai, Bai, Guo, and Wu [2] have introduced a dual notion of the Poisson algebra, called *transposed Poisson algebra*, by exchanging the roles of the two binary operations in the Leibniz rule defining the Poisson algebra. They have shown that a transposed Poisson algebra defined this way not only shares common properties of a Poisson algebra, including the closedness under tensor products and the Koszul self-duality as an operad, but also admits a rich class of identities. More significantly, a transposed Poisson algebra naturally arises from a Novikov-Poisson algebra by taking the commutator Lie algebra of the Novikov algebra. Thanks to [3], any unital transposed Poisson algebra is a particular case of a "contact bracket" algebra and a quasi-Poisson algebra. Later, in a recent paper by Ferreira, Kaygorodov, and Lopatkin a relation between $1/2$-derivations of Lie algebras and transposed Poisson algebras has been established [7]. These ideas were used to describe all transposed Poisson structures on Witt and Virasoro algebras in [7]; on twisted Heisenberg-Virasoro, Schrödinger-Virasoro and extended Schrödinger-Virasoro algebras in [19]; on oscillator algebras in [3]; Witt type Lie algebras in [12]. It was proved that each complex finite-dimensional solvable Lie algebra has a non-trivial transposed Poisson structure [13]. The Hom- and BiHom-versions of transposed Poisson algebras and transposed Poisson bialgebras have been considered in [14,15]. The algebraic and geometric classification of 3-dimensional transposed Poisson algebras is given in [4]. For the list of actual open questions on transposed Poisson algebras see [3].

The first non-trivial example of a transposed Poisson algebra was constructed on the Witt algebra with the multiplication law $[e_i, e_j] = (i - j)e_{i+j}$ for $i, j \in \mathbb{Z}$ (see, [7]). This attracted certain interest to the description of transposed Poisson structures on Lie algebras related to the Witt algebra. Thus, all transposed Poisson structures on the Virasoro algebra [7], Block type Lie algebras and Block type Lie superalgebras [11], Witt type Lie algebras [12] have been described. In the last years, the concept of Witt type and Block type Lie algebra has been enlarged and generalized by various authors, such as Kawamoto, Osborn, Đoković, Zhao, Xu, Passman, Jordan, etc. (see, for example, [11,12] and references therein). In the present paper, we study transposed Poisson structures on the class of generalized Witt algebras defined by Đoković and Zhao in [6] and general Block algebras defined by Block in [5]. We use the standard method of characterization of transposed Poisson algebra structures on a fixed Lie algebra $\mathfrak{L}$ based on the description of the space of $\frac{1}{2}$-derivations of $\mathfrak{L}$.

Our work consists of two main parts. Section 2 is devoted to a description of $\frac{1}{2}$-derivations and transposed Poisson structures on generalized Witt algebras $W(A, V, \langle \cdot, \cdot \rangle)$, which result in the following theorem.

Theorem A (Theorem 13). *Let $W(A, V, \langle \cdot, \cdot \rangle)$ be a generalized Witt algebra with non-degenerate $\langle \cdot, \cdot \rangle$ and $\mathrm{char}(F) = 0$.*

(i) *If $\dim(V) > 1$, then all the transposed Poisson algebra structures on $W(A, V, \langle \cdot, \cdot \rangle)$ are trivial.*

(ii) *If $\dim(V) = 1$, say, $V = \mathrm{span}_F\{v\}$, then the transposed Poisson algebra structures on $W(A, V, \langle \cdot, \cdot \rangle)$ are exactly mutations of the product $(a \otimes v) \cdot (b \otimes v) = (a + b) \otimes v$.*

In Sect. 3 we study the same questions on general Block Lie algebras $L(A, g, f)$ and obtain the following result.

Theorem B (Theorems 22 and 29). *Let $L(A, g, f)$ be a general Block Lie algebra with non-degenerate f and $\mathrm{char}(F) = 0$.*

(i) *If $g = 0$, then there is only one non-trivial transposed Poisson algebra structure $\cdot$ on $L(A, 0, f)$. It is given by $u_0 \cdot u_0 = u_0$.*

(ii) *If $g \neq 0$ and $(g(a), h(a)) \neq (0, -1)$ for all $a \in A$, then all the transposed Poisson algebra structures on $L(A, g, f)$ are trivial.*

(iii) *If $g \neq 0$ and there is $a \in A$, such that $(g(a), h(a)) \neq (0, -1)$, then the transposed Poisson algebra structures on $L(A, g, f)$ are usual Poisson algebra structures that are extensions by zero of commutative associative products $*$ on the complement $V = \mathrm{span}_F\{u_a \mid g(a) = h(a) + 2 = 0\}$ of $[L, L]$ with values in $Z(L) = \mathrm{span}_F\{u_a \mid g(a) = h(a) + 1 = 0\}$.*

1. Definitions and Preliminaries

All the algebras below will be over a field F of characteristic zero and all the linear maps will be F-linear, unless otherwise stated.

Definition 1. Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot, \cdot]$. The triple $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ is called a *transposed Poisson algebra* if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [\cdot, \cdot])$ is a Lie algebra that satisfies the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]. \tag{1}$$

Transposed Poisson algebras were first introduced in a paper by Bai, Bai, Guo, and Wu [2].

Definition 2. Let $(\mathfrak{L}, [\cdot, \cdot])$ be a Lie algebra. A *transposed Poisson algebra structure* on $(\mathfrak{L}, [\cdot, \cdot])$ is a commutative associative multiplication $\cdot$ on $\mathfrak{L}$ which makes $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ a transposed Poisson algebra.

Definition 3. Let $(\mathfrak{L}, [\cdot, \cdot])$ be an algebra and $\varphi : \mathfrak{L} \to \mathfrak{L}$ a linear map. Then φ is a $\frac{1}{2}$-*derivation* if it satisfies

$$\varphi([x, y]) = \frac{1}{2}\big([\varphi(x), y] + [x, \varphi(y)]\big). \tag{2}$$

Observe that $\frac{1}{2}$-derivations are a particular case of δ-derivations introduced by Filippov in [8] (see also [20] and references therein). The space of all $\frac{1}{2}$-derivations of an algebra $\mathfrak{L}$ will be denoted by $\Delta(\mathfrak{L})$.

Definitions 1 and 3 immediately imply the following key Lemma.

Lemma 4. *Let $(\mathfrak{L}, [\cdot, \cdot])$ be a Lie algebra and $\cdot$ a new binary (bilinear) operation on $\mathfrak{L}$. Then $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ is a transposed Poisson algebra if and only if $\cdot$ is commutative and associative and for every $z \in \mathfrak{L}$ the multiplication by z in $(\mathfrak{L}, \cdot)$ is a $\frac{1}{2}$-derivation of $(\mathfrak{L}, [\cdot, \cdot])$.*

The basic example of a $\frac{1}{2}$-derivation is the multiplication by a field element. Such $\frac{1}{2}$-derivations will be called *trivial*.

Theorem 5. *Let $\mathfrak{L}$ be a Lie algebra without non-trivial $\frac{1}{2}$-derivations. Then all transposed Poisson algebra structures on $\mathfrak{L}$ are trivial.*

Given a Lie algebra $(\mathfrak{L}, [\cdot, \cdot])$ denote by $Z(\mathfrak{L})$ its *center*, i.e. $Z(\mathfrak{L}) = \{a \in \mathfrak{L} \mid [a, b] = 0, \ \forall b \in \mathfrak{L}\}$, and by $[\mathfrak{L}, \mathfrak{L}]$ its square, i.e. $[\mathfrak{L}, \mathfrak{L}] = \mathrm{span}_F\{[a, b] \mid a, b \in \mathfrak{L}\}$. Fix a complement V of $[\mathfrak{L}, \mathfrak{L}]$ in $\mathfrak{L}$. Then any commutative associative product $* : V \times V \to Z(\mathfrak{L})$ defines a transposed Poisson algebra structure $\cdot$ on $\mathfrak{L}$ by means of

$$(a_1 + a_2) \cdot (b_1 + b_2) = a_1 * b_1, \tag{3}$$

where $a_1, b_1 \in V$ and $a_2, b_2 \in [\mathfrak{L}, \mathfrak{L}]$. Indeed, the right-hand side of (1) is zero, because $z \cdot x, z \cdot y \in Z(\mathfrak{L})$, and the left-hand side of (1) is zero by (3), because $[x, y] \in [\mathfrak{L}, \mathfrak{L}]$. We say that $\cdot$ is *the extension by zero* of $*$. Observe that $\cdot$ is at the same time a usual Poisson structure on $(\mathfrak{L}, [\cdot, \cdot])$.

2. Transposed Poisson Structures on Generalized Witt Algebras

2.1. Generalized Witt Algebras

Đoković and Zhao [6] introduced the following generalization of the classical Witt algebra.

Definition 6. Let F be a field, $(A, +)$ a non-trivial abelian group, $V \neq \{0\}$ a vector space and $\langle \cdot, \cdot \rangle : V \times A \to F$ a map linear in the first variable and additive in the second one. Denote $W := FA \otimes_F V$ and define the product $[\cdot, \cdot]$ on W by means of

$$[a \otimes v, b \otimes w] = (a + b) \otimes (\langle v, b \rangle w - \langle w, a \rangle v). \tag{4}$$

Then $(W, [\cdot, \cdot])$ is a Lie algebra called a *generalized Witt algebra*.

When it is necessary to specify A, V and $\langle \cdot, \cdot \rangle$, one writes $W = W(A, V, \langle \cdot, \cdot \rangle)$. We assume that $\langle \cdot, \cdot \rangle$ is *non-degenerate*, i.e.

$$\langle V, a \rangle = \{0\} \Leftrightarrow a = 0. \tag{5}$$

We also assume that $\mathrm{char}(F) = 0$. Then it follows from 5 that A is torsion-free.

The algebra $W(A, V, \langle \cdot, \cdot \rangle)$ is a generalization of the so-called *Witt type Lie algebra* $V(f)$ (corresponding to an additive map f) introduced[1] by Yu in [18]. We recall its definition using the notation from the present paper. Given an abelian group A, a field F and a function $f : A \to F$, define $V(f)$ to be a vector space with basis $\{e_a\}_{a \in A}$ and multiplication

$$[e_a, e_b] = (f(b) - f(a))e_{a+b}. \tag{6}$$

Without loss of generality, one assumes that $f(0) = 0$. Then $V(f)$ is a Lie algebra if and only if

$$\big(f(a+b) - f(a) - f(b)\big)\big(f(a) - f(b)\big) = 0$$

for all $a, b \in A$. Observe that in general one does not require that f be additive. However, it turns out to be so if $|f(A)| \geq 4$ by [18, Lemma 4.6].

Lemma 7. *Let* $\dim(V) = 1$ *and* $\langle \cdot, \cdot \rangle$ *be non-degenerate. Then* $W(A, V, \langle \cdot, \cdot \rangle)$ *is isomorphic to the Witt type Lie algebra* $V(f)$ *for some additive injective* $f : A \to F$ *with* $|f(A)| = \infty$.

Proof. Choose $v \in V \setminus \{0\}$. Then $\{a \otimes v \mid a \in A\}$ is a basis of W. Define an additive map $f : A \to F$ by $f(a) = \langle v, a \rangle$ and a bijective linear map $\varphi : W \to V(f)$ by $\varphi(a \otimes v) = e_a$. Then by (6) and (4)

$$[\varphi(a \otimes v), \varphi(b \otimes v)] = [e_a, e_b] = (f(b) - f(a))e_{a+b} = (\langle v, b \rangle - \langle v, a \rangle)e_{a+b}$$
$$= (\langle v, b \rangle - \langle v, a \rangle)\varphi((a+b) \otimes v) = \varphi([a \otimes v, b \otimes v]).$$

Observe that f is injective by (5), because $\langle v, a \rangle = 0 \Leftrightarrow \langle V, a \rangle = \{0\}$. Since A is torsion-free, then $|A| = \infty$, whence $|f(A)| = \infty$ as well. □

Since transposed Poisson structures on $V(f)$ were described in [12], we only need to deal with the case $\dim(V) > 1$.

Lemma 8. *Let* $\dim(V) > 1$. *If* $a \neq 0$, *then there exist two linearly independent* $v', v'' \in V$ *such that* $\langle v', a \rangle \neq 0 \neq \langle v'', a \rangle$.

Proof. Denote $V_0 = \{v \in V \mid \langle v, a \rangle = 0\}$. If $V_0 = \{0\}$, there is nothing to prove. Otherwise, choose $0 \neq v_0 \in V_0$. Since $a \neq 0$, by (5) there is $v' \in V$ such that $\langle v', a \rangle \neq 0$. Observe that v' and v_0 are linearly independent, since otherwise $v' = kv_0$ and $\langle v', a \rangle = k\langle v_0, a \rangle = 0$. Then v' and $v' + v_0$ are also linearly independent and $\langle v' + v_0, a \rangle = \langle v', a \rangle \neq 0$. So, we may choose $v'' = v' + v_0$. □

2.2. $\frac{1}{2}$-Derivations of Generalized Witt Algebras

Observe that W is an A-graded algebra, namely

$$W = \bigoplus_{a \in A} W_a, \text{ where } W_a = a \otimes V = \{a \otimes v \mid v \in V\}.$$

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[1]Notice that 'V' in $V(f)$ is not the same space V from Definition 6.

For all $a \in A$ and $v \in V$ denote, for simplicity,

$$v_a := a \otimes v. \tag{7}$$

Any linear map $\varphi : W \to W$ decomposes as

$$\varphi = \sum_{a \in A} \varphi_a,$$

where $\varphi_a : W \to W$ is a linear map such that $\varphi_a(W_b) \subseteq W_{a+b}$ for all $b \in A$. In particular, $\varphi \in \Delta(W)$ if and only if $\varphi_a \in \Delta(W)$ for all $a \in A$. We write

$$\varphi_a(v_b) = d_a(v_b)_{a+b}, \tag{8}$$

where $d_a : W \to V$.

Lemma 9. *Let $\varphi_a : W \to W$ be a linear map satisfying 8. Then $\varphi_a \in \Delta(W)$ if and only if for all $x, y \in A$ and $v, w \in V$*

$$2d_a(\langle v, y \rangle w_{x+y} - \langle w, x \rangle v_{x+y}) = \langle d_a(v_x), y \rangle w - \langle w, a + x \rangle d_a(v_x)$$
$$+ \langle v, a + y \rangle d_a(w_y) - \langle d_a(w_y), x \rangle v. \tag{9}$$

Proof. By (4), (8) and (7) we have

$$2\varphi_a([v_x, w_y]) = 2\varphi_a(\langle v, y \rangle w_{x+y} - \langle w, x \rangle v_{x+y})$$
$$= 2d_a(\langle v, y \rangle w_{x+y} - \langle w, x \rangle v_{x+y})_{a+x+y}$$

and

$$[\varphi_a(v_x), w_y] + [v_x, \varphi_a(w_y)] = [d_a(v_x)_{a+x}, w_y] + [v_x, d_a(w_y)_{a+y}]$$
$$= \langle d_a(v_x), y \rangle w_{a+x+y} - \langle w, a + x \rangle d_a(v_x)_{a+x+y}$$
$$+ \langle v, a + y \rangle d_a(w_y)_{a+x+y}$$
$$- \langle d_a(w_y), x \rangle v_{a+x+y}.$$

$\square$

Lemma 10. *Let $\dim(V) > 1$, $a \neq 0$ and $\varphi_a \in \Delta(W)$ satisfying (8). Then $\varphi_a = 0$.*

Proof. Substitute $y = 0$ into (9):

$$-2\langle w, x \rangle d_a(v_x) = -\langle w, a + x \rangle d_a(v_x) + \langle v, a \rangle d_a(w_0) - \langle d_a(w_0), x \rangle v,$$

that is

$$\langle w, a - x \rangle d_a(v_x) = \langle v, a \rangle d_a(w_0) - \langle d_a(w_0), x \rangle v. \tag{10}$$

Then setting $x = a$ in (10) we obtain

$$\langle v, a \rangle d_a(w_0) = \langle d_a(w_0), a \rangle v. \tag{11}$$

By 8 there are two linearly independent $v', v'' \in V$ such that $\langle v', a \rangle, \langle v'', a \rangle \neq 0$. Choosing consecutively $v = v'$ and $v = v''$ in (11) we have

$$d_a(w_0) = \frac{\langle d_a(w_0), a \rangle}{\langle v', a \rangle} v' = \frac{\langle d_a(w_0), a \rangle}{\langle v'', a \rangle} v''.$$

By the linear independence of v' and v'',

$$d_a(w_0) = 0 \qquad (12)$$

for all $w \in V$. It follows from (10) that

$$\langle w, a - x \rangle d_a(v_x) = 0.$$

If $x \neq a$, then $\langle w, a - x \rangle \neq 0$ for some $w \in V$ by (5). Thus,

$$d_a(v_x) = 0, \ \text{if } x \neq a. \qquad (13)$$

Now substitute $x = a$ and $y = -a$ into (9) and use (12):

$$0 = \langle d_a(v_a), a \rangle w + 2\langle w, a \rangle d_a(v_a) + \langle d_a(w_{-a}), a \rangle v.$$

Since A is torsion-free, then $a \neq -a$, so $d_a(w_{-a}) = 0$ by (13). Taking consecutively $w = v'$ and $w = v''$ we have

$$d_a(v_a) = -\frac{\langle d_a(v_a), a \rangle}{2\langle v', a \rangle} v' = -\frac{\langle d_a(v_a), a \rangle}{2\langle v'', a \rangle} v'',$$

whence

$$d_a(v_a) = 0 \qquad (14)$$

by the linear independence of v' and v''. Combining (13) and (14), we conclude that $\varphi_a = 0$. $\qquad \square$

Lemma 11. *Let $\varphi_0 \in \Delta(W)$ satisfying (8) with $a = 0$. Then $\varphi_0 \in \operatorname{span}_F\{\mathrm{id}\}$.*

Proof. For $a = 0$, equality (9) takes the form

$$2d_0(\langle v, y \rangle w_{x+y} - \langle w, x \rangle v_{x+y}) = \langle d_0(v_x), y \rangle w - \langle w, x \rangle d_0(v_x)$$
$$+ \langle v, y \rangle d_0(w_y) - \langle d_0(w_y), x \rangle v. \qquad (15)$$

Then $y = 0$ in (15) gives

$$\langle w, x \rangle d_0(v_x) = \langle d_0(w_0), x \rangle v.$$

If $x \neq 0$, then choosing $w \in V$ with $\langle w, x \rangle \neq 0$, we obtain

$$d_0(v_x) = \frac{\langle d_0(w_0), x \rangle}{\langle w, x \rangle} v =: k_x v, \ \text{if } x \neq 0. \qquad (16)$$

In particular, $d_0(v_x) = d_0(v_{-x})$ for all $x \neq 0$. On the other hand, taking $y = -x \neq 0$ in (15), we have

$$2d_0(\langle v, x \rangle w_0 + \langle w, x \rangle v_0) = \langle d_0(v_x), x \rangle w + \langle w, x \rangle d_0(v_x)$$
$$+ \langle v, x \rangle d_0(w_x) + \langle d_0(w_x), x \rangle v, \qquad (17)$$

which for $w = v$ gives

$$2\langle v, x \rangle d_0(v_0) = \langle d_0(v_x), x \rangle v + \langle v, x \rangle d_0(v_x).$$

Choosing $v \in V$ with $\langle v, x \rangle \neq 0$ and applying (16), we conclude that

$$d_0(v_0) = k_x v = d_0(v_x), \ \text{if } \langle v, x \rangle \neq 0. \qquad (18)$$

If $x \neq 0$ and $\langle v, x \rangle = 0$, then, thanks to (16), equality (17) becomes

$$2\langle w, x \rangle d_0(v_0) = \langle w, x \rangle d_0(v_x) + \langle d_0(w_x), x \rangle v = \langle w, x \rangle k_x v + \langle k_x w, x \rangle v$$
$$= 2k_x \langle w, x \rangle v.$$

Choosing an arbitrary $w \in V$ with $\langle w, x \rangle \neq 0$, we arrive at

$$d_0(v_0) = k_x v = d_0(v_x), \text{ if } x \neq 0 \text{ and } \langle v, x \rangle = 0. \tag{19}$$

Combining (18), (19) and (8), we finally prove the desired fact.   □

Proposition 12. *If* $\dim(V) > 1$, *then* $\Delta(W) = \operatorname{span}_F\{\operatorname{id}\}$.

Proof. The inclusion $\Delta(W) \subseteq \operatorname{span}_F\{\operatorname{id}\}$ is Lemmas 10 and 11. The converse inclusion is trivial.   □

Theorem 13. *Let* $\operatorname{char}(F) = 0$ *and* $\langle \cdot, \cdot \rangle$ *be non-degenerate.*
  (i) *If* $\dim(V) > 1$, *then all the transposed Poisson algebra structures on* $W(A, V, \langle \cdot, \cdot \rangle)$ *are trivial.*
  (ii) *If* $\dim(V) = 1$, *say,* $V = \operatorname{span}_F\{v\}$, *then the transposed Poisson algebra structures on* $W(A, V, \langle \cdot, \cdot \rangle)$ *are exactly mutations of the product* $(a \otimes v) \cdot (b \otimes v) = (a + b) \otimes v$.

Proof. ((i)) is an immediate consequence of Proposition 12 and [7, Theorem 8], while ((ii)) follows from Lemma 7 and [12, Proposition 26].

    □

3. Transposed Poisson Structures on General Block Lie Algebras

3.1. General Block Lie Algebras

Another generalization of the Witt algebra is the class of Lie algebras studied by Block in [5].

Definition 14. Let F be a field, $(A, +)$ a non-trivial abelian group, $g : A \to F$ an additive map and $f : A \times A \to F$ an anti-symmetric biadditive map. The *Block algebra* $L(A, g, f)$ is the F-vector space with basis $\{u_a\}_{a \in A}$ and product

$$[u_a, u_b] = (f(a, b) + g(a - b))u_{a+b}. \tag{20}$$

It is known [5] (and, in fact, easy to see) that $L(A, g, f)$ is a Lie algebra if and only if either $g = 0$ or there exists an additive map $h : A \to F$ such that for all $a, b \in A$:

$$f(a, b) = g(a)h(b) - g(b)h(a). \tag{21}$$

We will write $L = L(A, g, f)$ for the simplicity of notation. We will also assume that $\operatorname{char}(F) = 0$ and f is *non-degenerate* in the sense that

$$f(a, A) = \{0\} \Leftrightarrow a = 0. \tag{22}$$

Then, as in Sect. 2.1, this implies that A is torsion-free.

Observe that $L(A, g, f)$ is a generalization of the *Block Lie algebra* $\mathcal{B}(q)$ studied in [11] (it had been introduced in [16] under slightly different assumptions on q and on the basis). Recall that $\mathcal{B}(q)$, where $q \in \mathbb{C}$, is the complex Lie algebra with a basis $\{L_{m,i} \mid m, i \in \mathbb{Z}\}$, where

$$[L_{m,i}, L_{n,j}] = (n(i+q) - m(j+q))L_{m+n,i+j}$$

for all $i, j, m, n \in \mathbb{Z}$. It is immediately seen that $\mathcal{B}(q) = L(A, g, f)$, where $F = \mathbb{C}$, $A = \mathbb{Z} \times \mathbb{Z}$,

$$g(m, i) = -qm \text{ and } f((m, i), (n, j)) = ni - mj \tag{23}$$

for $(m, i), (n, j) \in \mathbb{Z} \times \mathbb{Z}$. If $q \neq 0$, then the corresponding map h from (21) can be chosen to be

$$h(m, i) = i/q. \tag{24}$$

Observe that f is non-degenerate, because $f((m, i), \ (0, -1)) = m$ and $f((m, i), (1, 0)) = i$.

We will need descriptions of $Z(L)$ and $[L, L]$ in the general case.

Lemma 15. *Let f be non-degenerate.*

  (i) *If $g = 0$, then $Z(L) = \mathrm{span}_F\{u_0\}$. Otherwise, $Z(L) = \mathrm{span}_F\{u_a \mid g(a) = h(a) + 1 = 0\}$.*

  (ii) *If $g = 0$, then $[L, L] = \mathrm{span}_F\{u_a \mid a \neq 0\}$. Otherwise, $[L, L] = \mathrm{span}_F\{u_a \mid g(a) \neq 0 \text{ or } h(a) + 2 \neq 0\}$.*

Proof. (i). Let $g = 0$. The inclusion $\mathrm{span}_F\{u_0\} \subseteq Z(L)$ is trivial. Conversely, if $x = \sum x_a u_a \in Z(L)$ and $x_a \neq 0$ for some $a \neq 0$, then choose $b \in A$ such that $f(a, b) \neq 0$ (it exists due to non-degeneracy of f) and calculate

$$[x, u_b] = \sum_{c \neq a} x_c f(c, b)u_{c+b} + x_a f(a, b)u_{a+b} \neq 0.$$

Let $g \neq 0$. If $g(a) = h(a) + 1 = 0$, then for all $b \in A$

$$f(a, b) + g(a - b) = g(a)h(b) - g(b)h(a) + g(a) - g(b) = g(b) - g(b) = 0,$$

so $[u_a, u_b] = 0$. This proves the inclusion $\mathrm{span}_F\{u_a \mid g(a) = h(a) + 1 = 0\} \subseteq Z(L)$. Conversely, assume that $x = \sum x_a u_a \in Z(L)$. Then $[x, u_0] = 0$ implies $g(a) = 0$ for all a with $x_a \neq 0$. Consequently, $[x, u_b] = -g(b) \sum x_a (h(a) + 1)u_{a+b}$. Choosing $b \in A$ with $g(b) \neq 0$, we conclude that $h(a) + 1 = 0$ whenever $x_a \neq 0$. Thus, $Z(L) \subseteq \mathrm{span}_F\{u_a \mid g(a) = h(a) + 1 = 0\}$.

(ii). Let $g = 0$. If $[u_a, u_b] \neq 0$, then $b \neq -a$, since otherwise $f(a, b) = f(a, -a) = 0$. Hence, $[u_a, u_b] = f(a, b)u_{a+b} \in \mathrm{span}_F\{u_a \mid a \neq 0\}$. This proves $[L, L] \subseteq \mathrm{span}_F\{u_a \mid a \neq 0\}$. Conversely, for any $a \neq 0$ there exists $b \in A$ such that $f(a, b) \neq 0$. Then $[u_{a-b}, u_b] = f(a - b, b)u_a = f(a, b)u_a \neq 0$, whence $u_a \in [L, L]$.

Let $g \neq 0$. If $[u_a, u_b] \neq 0$, then either $g(a+b) \neq 0$ or $h(a+b) + 2 \neq 0$, since otherwise

$$f(a,b) + g(a-b) = g(a)h(b) - g(b)h(a) + g(a) - g(b)$$
$$= g(a)(-h(a) - 2) + g(a)h(a) + g(a) + g(a) = 0.$$

Hence, $[L, L] \subseteq \operatorname{span}_F\{u_a \mid g(a) \neq 0 \text{ or } h(a) + 2 \neq 0\}$. Conversely, take $a \in A$ with $g(a) \neq 0$ or $h(a) + 2 \neq 0$. If $g(a) \neq 0$, then $[u_a, u_0] = g(a)u_a \neq 0$, so $u_a \in [L, L]$. Otherwise, $h(a) + 2 \neq 0$ and

$$f(a-b, b) + g(a-b-b) = f(a,b) + g(a-2b)$$
$$= g(a)h(b) - g(b)h(a) + g(a) - 2g(b)$$
$$= -g(b)h(a) - 2g(b) = -g(b)(h(a) + 2),$$

so choosing $b \in A$ with $g(b) \neq 0$ we have $[u_{a-b}, u_b] = -g(b)(h(a) + 2)u_a \neq 0$, whence $u_a \in [L, L]$. $\square$

We will also need the following technical lemma.

Lemma 16. *Let $\alpha, \beta : A \to F$ two non-zero additive functions. Then there exists $a \in A$ such that $\alpha(a) \neq 0 \neq \beta(a)$.*

Proof. Assume that for any $a \in A$ either $\alpha(a) = 0$ or $\beta(a) = 0$. Then $A = \ker \alpha \cup \ker \beta$. Since $\ker \alpha$ and $\ker \beta$ are subgroups of A, then either $\ker \alpha \subseteq \ker \beta$, in which case $A = \ker \beta$, or $\ker \beta \subseteq \ker \alpha$, in which case $A = \ker \alpha$. Hence, either $\alpha = 0$ or $\beta = 0$, a contradiction. $\square$

3.2. $\frac{1}{2}$-Derivations of General Block Lie Algebras

It follows from (20) that $L = \bigoplus_{a \in A} Fu_a$ is an A-grading, so any linear map $\varphi : L \to L$ decomposes into the direct sum of linear maps

$$\varphi = \sum_{a \in A} \varphi_a,$$

where $\varphi_a(u_b) \in Fu_{a+b}$ for all $b \in A$. Moreover, $\varphi \in \Delta(L)$ if and only if $\varphi_a \in \Delta(L)$ for all $a \in A$. As usual, we write

$$\varphi_a(u_b) = d_a(u_b)u_{a+b}, \tag{25}$$

where $d_a : L \to F$.

Lemma 17. *Let $\varphi_a : L \to L$ be a linear map satisfying (25). Then $\varphi_a \in \Delta(L)$ if and only if for all $x, y \in A$*

$$2(f(x,y) + g(x-y))d_a(x+y) = (f(a+x, y) + g(a+x-y))d_a(x)$$
$$+ (f(x, a+y) + g(x-a-y))d_a(y). \tag{26}$$

Proof. By (20) and (25) we have

$$2\varphi_a([u_x, u_y]) = 2\varphi_a((f(x,y) + g(x-y))u_{x+y})$$
$$= 2(f(x,y) + g(x-y))d_a(x+y)u_{a+x+y}$$

and

$$[\varphi_a(u_x), u_y] + [u_x, \varphi_a(u_y)] = [d_a(x)u_{a+x}, u_y] + [u_x, d_a(y)u_{a+y}]$$
$$= (f(a+x,y) + g(a+x-y))d_a(x)u_{a+x+y}$$
$$+ (f(x,a+y) + g(x-a-y))d_a(y)u_{a+x+y}.$$

$\square$

3.2.1. The case $g = 0$. Assume first that $g = 0$.

Lemma 18. *Let $a \neq 0$ and $\varphi_a \in \Delta(L)$ satisfying (25). Then $\varphi_a = 0$.*

Proof. Taking $y = -x$ in (26) and using anti-symmetry of f, we have

$$0 = f(a+x, -x)d_a(x) + f(x, a-x)d_a(-x) = -f(a,x)(d_a(x) + d_a(-x)).$$

Hence,

$$d_a(-x) = -d_a(x), \text{ if } f(a,x) \neq 0. \tag{27}$$

Now, substitute $y = -a$ into (26):

$$2f(x, -a)d_a(x-a) = f(a+x, -a)d_a(x) = f(x, -a)d_a(x),$$

whence

$$d_a(x) = 2d_a(x-a), \text{ if } f(a,x) \neq 0.$$

Since $f(a, a+x) = f(a, x)$, the latter is equivalent to

$$d_a(a+x) = 2d_a(x), \text{ if } f(a,x) \neq 0. \tag{28}$$

On the other hand, $y = a$ in (26) gives

$$2f(x,a)d_a(x+a) = f(a+x, a)d_a(x) + f(x, 2a)d_a(a)$$
$$= f(x,a)d_a(x) + 2f(x,a)d_a(a).$$

If $f(a,x) \neq 0$, then using (28), we come to $4d_a(x) = d_a(x) + 2d_a(a)$. Consequently,

$$3d_a(x) = 2d_a(a), \text{ if } f(a,x) \neq 0. \tag{29}$$

However, $f(a,x) \neq 0 \Leftrightarrow f(a, -x) \neq 0$, so replacing x by $-x$ in (29) and taking into account $\text{char}(F) = 0$, we have

$$d_a(-x) = d_a(x), \text{ if } f(a,x) \neq 0. \tag{30}$$

Combining (27) and (30), we conclude that

$$d_a(x) = 0, \text{ if } f(a,x) \neq 0. \tag{31}$$

Now assume that $f(a,x) = 0$. Since $a \neq 0$, by (22) there exists $y \in A$ such that $f(a,y) \neq 0$. Observe that $f(a, x+y) = f(a,y) \neq 0$. Then $d_a(y) = d_a(x+y) = 0$ thanks to (31), so (26) takes the form

$$0 = f(a+x, y)d_a(x).$$

By Lemma 16 applied to $f(a + x, -)$ and $f(a, -)$, whenever $x \neq -a$, the element y can be chosen in a way that $f(a + x, y) \neq 0 \neq f(a, y)$. Thus, we have proved

$$d_a(x) = 0, \text{ if } f(a, x) = 0 \text{ and } x \neq -a. \tag{32}$$

Finally, taking $y = -a - x$ in (26) we see that the right-hand side is zero, while the left-hand side equals $2f(a, x)d_a(-a)$. Choosing $x \in A$ such that $f(a, x) \neq 0$, we show that $d_a(-a) = 0$. Combining this with (31) and (32), we get the desired fact. $\qquad\square$

Lemma 19. *Let $\varphi_0 \in \Delta(L)$ satisfying (25) with $a = 0$. Then $\varphi_0(x) = \varphi_0(y)$ for all $x, y \neq 0$.*

Proof. Write (26) with $a = 0$:

$$2f(x, y)d_0(x + y) = f(x, y)d_0(x) + f(x, y)d_0(y).$$

Consequently,

$$2d_0(x + y) = d_0(x) + d_0(y), \text{ if } f(x, y) \neq 0. \tag{33}$$

Observe that $f(x, y) = f(x + y, -y)$, so applying (33) with (x, y) replaced by $(x + y, -y)$, we have

$$2d_0(x) = d_0(x + y) + d_0(-y), \text{ if } f(x, y) \neq 0. \tag{34}$$

Combining (33) and (34), we come to

$$3d_0(x) = d_0(y) + 2d_0(-y), \text{ if } f(x, y) \neq 0. \tag{35}$$

However, $f(x, -y) = -f(x, y)$, so replacing y by $-y$ in (35), we obtain

$$3d_0(x) = d_0(-y) + 2d_0(y), \text{ if } f(x, y) \neq 0. \tag{36}$$

It follows from (35) and (36) that $d_0(y) = d_0(-y)$, so

$$d_0(x) = d_0(y), \text{ if } f(x, y) \neq 0, \tag{37}$$

because $\text{char}(F) = 0$.

Now let $x, y \neq 0$. By Lemma 16 applied to $f(x, -)$ and $f(y, -)$ there exists $z \in A$ such that $f(x, z) \neq 0 \neq f(y, z)$. Then (37) gives

$$d_0(x) = d_0(z) = d_0(y), \text{ if } x, y \neq 0,$$

as needed. $\qquad\square$

Lemma 20. *The linear map $\alpha : L \to L$ given by*

$$\alpha(u_a) = \begin{cases} u_0, & a = 0, \\ 0, & a \neq 0, \end{cases}$$

is a $\frac{1}{2}$-derivation of L.

Proof. Observe by Lemma 15(i) that $\alpha(L) \subseteq Z(L)$, so the right-hand side of (2) is always zero for $\varphi = \alpha$. Now, $\alpha([L, L]) = \{0\}$ by Lemma 15(ii). Thus, the left-hand side of (2) is always zero as well. $\qquad\square$

Proposition 21. *We have* $\Delta(L) = \operatorname{span}_F\{\mathrm{id}, \alpha\}$.

Proof. The inclusion $\Delta(L) \subseteq \operatorname{span}_F\{\mathrm{id}, \alpha\}$ is Lemmas 18 and 19. The converse inclusion is Lemma 20.

Theorem 22. *Let* $\operatorname{char}(F) = 0$ *and* f *be non-degenerate. Then there is only one non-trivial transposed Poisson algebra structure* $\cdot$ *on* $L(A, 0, f)$. *It is given by*

$$u_0 \cdot u_0 = u_0. \tag{38}$$

Proof. Let $\cdot$ be a non-trivial transposed Poisson algebra structure on $L(A, 0, f)$. By Proposition 21 and Lemma 4 for any $a \in A$ there are $k_a, l_a \in F$ such that

$$u_a \cdot u_b = k_a u_b + l_a \alpha(u_b) = \begin{cases} (k_a + l_a)u_0, & b = 0, \\ k_a u_b, & b \neq 0. \end{cases} \tag{39}$$

Since $|A| > 2$ (A is torsion-free), for any $a \neq 0$ there exists $b \notin \{0, a\}$. Then by (39) and commutativity of $\cdot$ we have $k_a u_b = u_a \cdot u_b = u_b \cdot u_a = k_b u_a$. Consequently, $k_a = 0$ for $a \neq 0$. Similarly, $(k_a + l_a)u_0 = u_a \cdot u_0 = u_0 \cdot u_a = k_0 u_a$ gives $k_0 = l_a = 0$ for $a \neq 0$. Thus, the only non-zero product $u_a \cdot u_b$ is $u_0 \cdot u_0 = l_0 u_0$. So, up to isomorphism, $\cdot$ is of the form (38).

Conversely, in view of Lemma 15 the product (38) is of the form (3), so $(L, \cdot, [\cdot, \cdot])$ is a transposed Poisson (and usual Poisson) algebra. $\square$

Remark 23. Consider $\mathcal{B}(0)$ as the complex Block algebra $L(\mathbb{Z} \times \mathbb{Z}, 0, f)$, where f is given by (23). Then we obtain the description of transposed Poisson algebra structures on $\mathcal{B}(0)$ given in [11, Theorem 2.14] as a particular case of Lemma 22.

3.2.2. The case $g \neq 0$. In this case, as it was commented above, there exists an additive map $h : A \to F$ such that (21) holds.

Lemma 24. *Let* $a \neq 0$ *and* $\varphi_a \in \Delta(L)$ *satisfying* (25). *If* $g(x)f(a, x) \neq 0$, *then* $\varphi_a(x) = 0$.

Proof. Consider first $y = 0$ in 26

$$2g(x)d_a(x) = g(a + x)d_a(x) + (f(x, a) + g(x - a))d_a(0).$$

Then

$$g(a - x)d_a(x) = (f(a, x) + g(a - x))d_a(0). \tag{40}$$

Replacing x by $-x$, we obtain

$$g(a + x)d_a(-x) = (-f(a, x) + g(a + x))d_a(0). \tag{41}$$

On the other hand, $y = -x$ in (26) gives

$$4g(x)d_a(0) = (-f(a, x) + g(a + 2x))d_a(x) + (-f(a, x) + g(2x - a))d_a(-x). \tag{42}$$

Multiplying both sides of (42) by $g(a-x)g(a+x) = g(a)^2 - g(x)^2$ and using (41) and (40), we get

$$\begin{aligned} 4g(x)(g(a)^2 - g(x)^2)d_a(0) = (-f(a,x) + g(a+2x))g(a+x)(f(a,x) \\ + g(a-x))d_a(0) \\ + (-f(a,x) + g(2x-a))g(a-x)(-f(a,x) \\ + g(a+x))d_a(0). \end{aligned} \tag{43}$$

We have

$$\begin{aligned} (-f(a,x) + g(a+2x))(f(a,x) + g(a-x)) \\ = (g(a) + g(x)/2)^2 - (f(a,x) - 3g(x)/2)^2, \\ (-f(a,x) + g(2x-a))(-f(a,x) + g(a+x)) \\ = (f(a,x) - 3g(x)/2)^2 - (g(a) - g(x)/2)^2. \end{aligned}$$

Since

$$\begin{aligned} (g(a) + g(x)/2)^2 - (g(a) - g(x)/2)^2 = 2g(a)g(x), \\ (g(a) + g(x)/2)^2 + (g(a) - g(x)/2)^2 = 2g(a)^2 + g(x)^2/2, \end{aligned}$$

the coefficient of $d_a(0)$ on the right-hand side of (43) equals

$$\begin{aligned} g(a) \cdot 2g(a)g(x) + g(x)(2g(a)^2 + g(x)^2/2) - 2g(x)(f(a,x) - 3g(x)/2)^2 \\ = g(x)(4g(a)^2 + g(x)^2/2 - 2(f(a,x) - 3g(x)/2)^2). \end{aligned}$$

Subtracting the coefficient of $d_a(0)$ on the left-hand side of (43), we obtain

$$g(x)(9g(x)^2/2 - 2(f(a,x) - 3g(x)/2)^2) = 2g(x)f(a,x)(3g(x) - f(a,x)).$$

Thus, under the assumption $g(x)f(a,x) \neq 0$, (43) is equivalent to

$$(3g(x) - f(a,x))d_a(0) = 0. \tag{44}$$

Case 1. $f(a,x) \neq 3g(x)$. Then (44) gives

$$d_a(0) = 0. \tag{45}$$

Case 1.1. $g(a) \neq g(x)$. It follows from (45) and (40) that $d_a(x) = 0$.

Case 1.2. $g(a) = g(x)$. Then $g(a+x) = 2g(x) \neq 0$, so $d_g(-x) = 0$ by (41). Moreover, $-f(a,x) + g(a+2x) = -f(a,x) + 3g(x) \neq 0$, so (45) and (42) yield $d_a(x) = 0$.

Case 2. $f(a,x) = 3g(x) \neq 0$. Then (40) becomes

$$g(a-x)d_a(x) = (3g(x) + g(a-x))d_a(0) = (2g(x) + g(a))d_a(0). \tag{46}$$

Since $f(a,x) = 3g(x)$ is invariant under the replacement of x by kx, then (46) implies

$$g(a-kx)d_a(kx) = (2kg(x) + g(a))d_a(0). \tag{47}$$

On the other hand, $y = 2x$ in (26) gives

$$-2g(x)d_a(3x) = (5g(x) + g(a))d_a(x) - (4g(x) + g(a))d_a(2x). \qquad (48)$$

Multiplying both sides of this equality by $g(a-x)g(a-2x)g(a-3x)$ and using (47) we get

$$
\begin{aligned}
- 2g(x)g(a-x)&g(a-2x)(6g(x) + g(a))d_a(0) \\
= (5g(x) + g(a))&g(a-2x)g(a-3x)(2g(x) + g(a))d_a(0) \\
- (4g(x) + g(a))&g(a-x)g(a-3x)(4g(x) + g(a))d_a(0). \qquad (49)
\end{aligned}
$$

Comparing the coefficients of $g(a)^i g(x)^j d_a(0)$, $0 \leq i + j \leq 4$, in (49), we see that (49) is equivalent to $36g(x)^4 d_a(0) = 0$. Hence, we again have (45).

Case 2.1. $g(a) \neq g(x)$. Then (45) and (40) yield $d_a(x) = 0$.

Case 2.2. $g(a) = g(x)$. Then $g(a - kx) = (1 - k)g(x) \neq 0$ for $k \neq 1$, so $d_a(2x) = d_a(3x) = 0$ by (47). Moreover, $5g(x) + g(a) = 6g(x) \neq 0$, so (45) and (48) imply $d_a(x) = 0$. □

Lemma 25. *Let $a \neq 0$ and $\varphi_a \in \Delta(L)$ satisfying 25. If $g(x)f(a,x) = 0$, then $d_a(x) = 0$, unless $g(a) = g(x) = 0$, $h(a) = 1$ and $h(x) = -2$.*

Proof. Case 1. $g(a+x) \neq 0$. Since $a \neq 0$, by Lemma 16 applied to $f(a, -)$ and g there exists $y \in A$ such that $f(a, y) \neq 0 \neq g(y)$. Observe that for any $k \neq 0$ we have

$$f(a, ky) \neq 0 \neq g(ky), \qquad (50)$$

because $\mathrm{char}(F) = 0$. We affirm that k can be chosen in a way that

$$f(a, x + ky) \neq 0 \neq f(a + x, ky) + g(a + x - ky). \qquad (51)$$

Indeed, $f(a, y) \neq 0$, so there exists at most one k such that $f(a, x) + kf(a, y) = 0$. If $f(a+x, y) - g(y) = 0$, then $f(a+x, ky) + g(a+x-ky) = g(a+x) \neq 0$ for all k. Otherwise, there exists at most one k such that $k(f(a+x, y) - g(y)) + g(a+x) = 0$. Thus, (51) and (50) hold for infinitely many integer $k \neq 0$ (recall that $\mathrm{char}(F) = 0$). Then $d_a(ky) = d_a(x + ky) = 0$ for any such k by Lemma 24. Consequently, applying (26) with y replaced by ky and using (51) we prove $d_a(x) = 0$.

Case 2. $g(a + x) = 0$. Then

$$f(a+x, y) + g(a+x-y) = -g(y)(h(a+x) + 1).$$

Case 2.1. $h(a + x) + 1 \neq 0$. Then the same argument as in Case 1 gives $d_a(x) = 0$.

Case 2.2. $h(a + x) + 1 = 0$. Then $0 = g(x)f(a, x) = g(a)^2 = g(x)^2$, so (26) simplifies to

$$2g(y)h(a)d_a(x + y) = g(y)h(a)d_a(y).$$

Observe that $h(a) \neq 0$, since otherwise $f(a, y) = 0$ for all $y \in A$ contradicting (22). Hence,

$$2d_a(x + y) = d_a(y), \text{ if } g(y) \neq 0. \tag{52}$$

On the other hand, since $g(a) = 0$, then substituting $x = 0$ in (26) we get

$$d_a(y) = (h(a) + 1)d_a(0), \text{ if } g(y) \neq 0. \tag{53}$$

Observe that $g(x + y) = g(y)$ whenever $g(x) = 0$, so (53) and (52) yield

$$2(h(a) + 1)d_a(0) = (h(a) + 1)d_a(0).$$

It follows that $d_a(0) = 0$, unless $h(a) + 1 = 0$. In any case, (53) implies

$$d_a(y) = 0, \text{ if } g(y) \neq 0. \tag{54}$$

Write (26) replacing x by $x - y$, where $g(y) \neq 0$. Since $d_a(y) = d_a(x - y) = 0$ by (54), we come to

$$(h(x) + 2)d_a(x) = 0.$$

Thus, $d_a(x) = 0$, unless $h(x) = -2$ (in which case $h(a) = 1$). $\qquad\square$

Lemma 26. *Let* $\varphi_0 \in \Delta(L)$ *satisfying* (25) *with* $a = 0$. *Then* $\varphi_0 \in \mathrm{span}_F\{\mathrm{id}\}$.

Proof. Writing (26) with $a = 0$, we get

$$2(f(x, y) + g(x - y))d_0(x + y) = (f(x, y) + g(x - y))(d_0(x) + d_0(y)). \tag{55}$$

Case 1. $g(x) \neq 0$. Then put $y = 0$ in (55) to get:

$$d_0(x) = d_0(0), \text{ if } g(x) \neq 0. \tag{56}$$

Case 2. $g(x) = 0$ and $h(x) \neq -1$. Then $f(x, y) = -g(y)h(x)$, and (55) is equivalent to

$$2g(y)d_0(x + y) = g(y)(d_0(x) + d_0(y)). \tag{57}$$

Choose $y \in A$ such that $g(y) \neq 0$. Then $d_0(y) = d_0(x + y) = d_0(0)$ by (56). Hence, (57) yields $d_0(x) = d_0(0)$.

Case 3. $g(x) = 0$ and $h(x) = -1$. Let us use (55) with (x, y) replaced by $(x + y, -y)$. Observe that $f(x + y, -y) = -f(x, y) = -g(y)$, so we get

$$6g(y)d_0(x) = 3g(y)(d_0(x + y) + d_0(-y)). \tag{58}$$

Choosing $y \in A$ with $g(y) \neq 0$ we have $d_0(x + y) = d_0(-y) = d_0(0)$ by (56). Thus, $d_0(x) = d_0(0)$ due to (58). $\qquad\square$

Given $\lambda, \mu \in F$, we introduce the following notation:

$$A_{(\lambda, \mu)} := \{a \in A \mid g(a) = \lambda \text{ and } h(a) = \mu\}.$$

Lemma 27. *Let $a \in A_{(0,-2)}$ and $b \in A_{(0,-1)}$. Then the linear map $\alpha_{(a,b)} : L \to L$ given by*

$$\alpha_{(a,b)}(u_c) = \begin{cases} u_b, & c = a, \\ 0, & otherwise, \end{cases}$$

is a $\frac{1}{2}$-derivation of L.

Proof. Observe by Lemma 15(i) that $\alpha_{(a,b)}(L) \subseteq Z(L)$, showing that the right-hand side of (2) is always zero for $\varphi = \alpha_{(a,b)}$. Furthermore, $\alpha_{(a,b)}([L,L]) = \{0\}$ by Lemma 15(ii), and the left-hand side of (2) is always zero as well. $\square$

Proposition 28. *We have $\Delta(L) = \operatorname{span}_F(\{\mathrm{id}\} \cup \{\alpha_{(a,b)} \mid a \in A_{(0,-2)} \text{ and } b \in A_{(0,-1)}\})$.*

Proof. The fact that any $\varphi \in \Delta(L)$ is a linear combination of id and $\alpha_{(a,b)}$ follows from Lemma 24–26. Conversely, the inclusion $\mathrm{id} \in \Delta(L)$ is trivial, and $\alpha_{(a,b)} \in \Delta(L)$ is Lemma 27. $\square$

Theorem 29. *Let $\operatorname{char}(F) = 0$, $g \neq 0$ and f be non-degenerate. If $(g(a), h(a)) \neq (0, -1)$ for all $a \in A$, then all the transposed Poisson algebra structures on $L(A, g, f)$ are trivial. Otherwise, the transposed Poisson algebra structures on $L(A, g, f)$ are exactly extensions by zero of commutative associative products $*$ on the complement $V = \operatorname{span}_F\{u_a \mid g(a) = h(a) + 2 = 0\}$ of $[L, L]$ with values in $Z(L) = \operatorname{span}_F\{u_a \mid g(a) = h(a) + 1 = 0\}$.*

Proof. Let $\cdot$ be a transposed Poisson algebra structure on $L(A, g, f)$. If $(g(a), h(a)) \neq (0, -1)$ for all $a \in A$, then $A_{(0,-1)} = \emptyset$, so by Proposition 28 we have $\Delta(L) = \{\mathrm{id}\}$. It follows from [7, Theorem 8] that $\cdot$ is trivial.

Assume that $(g(a), h(a)) = (0, -1)$ for some $a \in A$. Then $(g(2a), h(2a)) = (0, -2)$, so both $A_{(0,-1)}$ and $A_{(0,-2)}$ are non-empty. By Proposition 28 and Lemma 4 for any $a \in A$ there are $k_a \in F$ and $\{l_a^{(x,y)}\}_{x \in A_{(0,-2)}, y \in A_{(0,-1)}} \subseteq F$, such that

$$u_a \cdot u_b = k_a u_b + \sum_{x \in A_{(0,-2)}, y \in A_{(0,-1)}} l_a^{(x,y)} \alpha_{(x,y)}(u_b)$$

$$= \begin{cases} k_a u_b + \sum_{y \in A_{(0,-1)}} l_a^{(b,y)} u_y, & b \in A_{(0,-2)}, \\ k_a u_b, & b \notin A_{(0,-2)}. \end{cases} \tag{59}$$

Since $\operatorname{char}(F) = 0$ and A is torsion-free, then $A \backslash A_{(0,-1)}$ and $A \backslash A_{(0,-2)}$ are infinite (indeed, if $a \in A_{(\lambda,\mu)}$ with $(\lambda, \mu) \neq (0,0)$, then $ka \notin A_{(\lambda,\mu)}$ for all $k \neq 1$). So, for any $a \notin A_{(0,-2)}$ there exists $b \notin A_{(0,-2)}$, $b \neq a$. Then by (59) and commutativity of $\cdot$ we have $k_a u_b = u_a \cdot u_b = u_b \cdot u_a = k_b u_a$. Consequently,

$$k_a = 0 \text{ for } a \notin A_{(0,-2)}. \tag{60}$$

Now let $a \in A_{(0,-2)}$ and $b \notin A_{(0,-1)} \cup A_{(0,-2)}$. Then $k_b u_a + \sum_{y \in A_{(0,-1)}} l_b^{(a,y)} u_y = u_b \cdot u_a = u_a \cdot u_b = k_a u_b$ gives

$$k_a = 0 \text{ for } a \in A_{(0,-2)}. \tag{61}$$

It follows from (60) and (61) and the commutativity of $\cdot$ that

$$u_a \cdot u_b = \begin{cases} \sum_{y \in A_{(0,-1)}} l_a^{(b,y)} u_y, & a, b \in A_{(0,-2)}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\cdot$ is of the form (3) for the commutative associative product $u_a * u_b = \sum_{y \in A_{(0,-1)}} l_a^{(b,y)} u_y \in Z(L)$ on $V = \operatorname{span}_F\{u_a \mid a \in A_{(0,-2)}\}$.

     Conversely, in view of Lemma 15 the product (38) is of the form (3), so $(L, \cdot, [\cdot, \cdot])$ is a transposed Poisson (and usual Poisson) algebra.

Remark 30. Consider $\mathcal{B}(q)$ with $q \neq 0$ as the complex Block algebra $L(\mathbb{Z} \times \mathbb{Z}, g, f)$, where g, f and h are given by (23) and (24). Then $(g(m,i), h(m,i)) = (0,-2) \Leftrightarrow (m,i) = (0,-2q)$ and $(g(m,i), h(m,i)) = (0,-1) \Leftrightarrow (m,i) = (0,-q)$, so we again obtain the description of transposed Poisson algebra structures on $\mathcal{B}(q)$ given in [11, Theorem 2.14] as a particular case of Theorem 29.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest There is no potential conflict of ethical approval, conflict of interest, and ethical standards.

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Ivan Kaygorodov
CMA-UBI, Universidade da Beira Interior
Covilhã
Portugal
e-mail: `kaygorodov.ivan@gmail.com`

Mykola Khrypchenko
Departamento de Matemática, Universidade Federal de Santa Catarina
Florianopolis
Brazil
e-mail: `nskhripchenko@gmail.com`

and

CMUP, Departamento de Matemática, Faculdade de Ciências
Universidade do Porto
Rua do Campo Alegre s/n
4169–007 Porto
Portugal