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# Achievement Sets of Reciprocals of Complete Sequences

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**Abstract.** The paper is dedicated to answer Question 1 from the preprint of the paper (Jones in Am Math Mon 118: 508-521, 2011). We show that it is possible to construct a complete sequence $(x_n)$ such that the achievement set $A(\frac{1}{x_n})$ is not an interval. In particular it can be a Cantorval.

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**Keywords.** purely atomic measure, achievement set, set of subsums, absolutely convergent series, complete sequence, Cantorval.

## 1. Introduction

For a sequence $(x_n)$ of real numbers we study its achievement set defined as the set of subsums of the series $\sum_{n=1}^{\infty} x_n$, that is

$$A(x_n) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}} \right\} = \left\{ \sum_{n \in A} x_n : A \subset \mathbb{N} \right\}$$

The first paper where the achievement set was considered is that of Kakeya, see [12]. The Author proved the following

**Theorem 1.1.** *If $\sum_{n=1}^{\infty} x_n$ is absolutely convergent with infinite many nonzero terms and $(x_n)$ is non-increasing, then*

*(1) $A(x_n)$ is a finite union of compact intervals iff $x_k \leq r_k := \sum_{n=k+1}^{\infty} x_n$ for all but finitely many $k \in \mathbb{N}$*
*(2) $A(x_n)$ is homeomorphic to a Cantor set, if $x_k > r_k$ for all but finitely many $k \in \mathbb{N}$*

The series $\sum_{n=1}^{\infty} x_n$, whose terms satisfy the first condition for all natural numbers is called slowly convergent or interval-filling [5,7,8] or high achiever [11], while the second condition define quick or fast convergence. Kakeya claimed that for an absolutely convergent series with infinitely many nonzero terms the set $A(x_n)$ is either a finite union of compact intervals or a set homeomorphic to a Cantor set. Due to Guthrie and Nymann [10] we know that there is one more possible form.

**Theorem 1.2.** *For an absolutely convergent series $\sum_{n=1}^{\infty} x_n$ with infinite many nonzero terms, the achievement set $A(x_n)$ has one of the following fashions: a finite union of compact intervals, a set homeomorphic to a Cantor set or a Cantorval, that is a set homeomorphic to $A(y_n)$ for $y_{2n-1} = \frac{3}{4^n}$, $y_{2n} = \frac{2}{4^n}$ for all $n \in \mathbb{N}$.*

Let us consider the ternary Cantor set $C$, that is $C = A(\frac{2}{3^n})$. By its classical construction we know that $C = [0,1] \setminus \cup_{n=1}^{\infty} G_n$, where $G_n$ is a sum of $2^{n-1}$ removed open intervals of the length $\frac{1}{3^n}$ each. A Cantorval is a set homemorphic to the set $[0,1] \setminus \cup_{n=1}^{\infty} G_{2n}$. In particular it is regular-closed. Theorem 1.2 was first published in [10], but the correct proof was given in [14]. The set $GN = A(y_n)$ is called the Guthrie Nymann Cantorval. It is obtained for a series belonging to a multigeometric class, that is of the form

$$(x_n) = (a_1, a_2, \ldots, a_m; q) := (a_1 q, a_2 q, \ldots, a_m q, a_1 q^2, a_2 q^2, \ldots, a_m q^2, a_1 q^3, \ldots).$$

Using that notation the Guthrie Nymann Cantorval can be described as $A(3, 2; \frac{1}{4})$. If we denote $\Sigma := A(a_1, \ldots, a_m) = \{\sum_{n=1}^{m} \varepsilon_n a_n : (\varepsilon_n) \in \{0,1\}^m\}$ then $A(a_1, a_2, \ldots, a_m; q) = \{\sum_{n=1}^{\infty} x_n q^n : (x_n) \in \Sigma^{\infty}\}$. Multigeometric series were considered in [1,2,4]. It is probably the most well known and studied class, since any possible form of the achievement set can be obtained and the regularity makes calculations simpler or even possible. Moreover the set $A(a_1, a_2, \ldots, a_m; q)$ is attractor of the iterated function system $\{f_\sigma(x) = q \cdot x + q \cdot \sigma : \sigma \in \Sigma\}$.

It is worth to mention that the negative answer for Kakeya's hypothesis was obtained before Guthrie and Nymann's result. First counterexample was given without proof by Weinstein and Shapiro [17]. Ferens [9] constructed a purely atomic finite measure $\mu$ and proved that its range is a Cantorval. The theory of achievement sets and ranges of purely atomic finite measure coincide. Indeed we may assume that $\mu$ is defined on $\mathbb{N}$. Then $rng(\mu) = \{\mu(A) : A \subset \mathbb{N}\} = A(x_n)$, where the terms of our series are the values of measure on atoms, that is $x_n = \mu(\{n\})$ for all $n \in \mathbb{N}$. Hence we may say that Ferens observed that $A(7, 6, 5, 4, 3; \frac{2}{27})$ is a Cantorval.

Note that if $(x_n)$ has no subsequence $(x_{n_k})$ which tends to 0, then $A(x_n)$ is reduced to finite sums and hence countable.

**Definition 1.3.** A sequence $(x_n)$ of natural numbers is called complete iff $A(x_n) = \mathbb{N}_0$.

There are many well known and important Examples of complete sequences, for instance consequtive powers of two $1, 2, 4, 8, \ldots, 2^n, \ldots$ is one of them. Other and trivial ones are the constant sequence $x_n = 1$ for each $n \in \mathbb{N}$

or shifted constant $x_1 = 1$, $x_n = 2$ for $n \geq 2$. Probably the easiest way to check if the sequence is complete is to use Brown's characterization, given in [6].

**Theorem 1.4.** *The non-decreasing sequence* $(x_n)$ *is complete if and only if* $x_{k+1} \leq \sum_{n=1}^{k} x_n + 1$ *for each* $k \in \mathbb{N}$.

Jones [11] asked the following question:

**Question 1.** If $(x_n)$ is complete, what can we say about $A(\frac{1}{x_n})$ ?

The Author was inspired by the sequence of Fibonacci numbers $F_n$. It is the example of complete sequence and such that the achievement set of its reciprocals $A(\frac{1}{F_n})$ is interval. The Author compare the notions of completeness and slow convergence. Both of them are described by the inequalities. In the completeness we use the preeceding terms, while in the slow convergence the tails. Both notions mean that $A(x_n)$ is as large as possible for the considered kinds of sequences. For that reason Jones [11] admits that high achievers are analogous to the completeness.

By our knowledge there is no Example of a complete sequence $(x_n)$ such that $A(\frac{1}{x_n})$ is not an interval except some trivial cases which should be omitted. One of them is a complete sequence, which is constant for large enough indexes. Then $(\frac{1}{x_n})$ does not tend to 0 and $A(\frac{1}{x_n})$ is a countable set, which contains all finite subsums. We will also not consider the sequence for which $\sum_{n=1}^{\infty} \frac{1}{x_n} = \infty$ and $(\frac{1}{x_n}) \to 0$. Then we have $A(\frac{1}{x_n}) = [0, \infty)$. Hence we are interested in complete sequences $(x_n)$ such that $\sum_{n=1}^{\infty} \frac{1}{x_n}$ is absolutely convergent.

## 2. Main results

In the 'typical' case the set $A(\frac{1}{x_n})$ is interval. Indeed, let us consider the following Examples.

*Example 2.1.* Let $x_n = 2^{n-1}$. Then we have $A(\frac{1}{2^{n-1}}) = [0, 2]$.

*Example 2.2.* From Steinhaus Theorem we know that $C + C = [0, 2]$ for the ternary Cantor set $C$. But $\frac{3}{2}C = A(\frac{1}{3^{n-1}})$, so $\frac{3}{2}C + \frac{3}{2}C = A(\frac{1}{x_n}) = [0, 3]$ for $(x_n) = (1, 1, 3, 3, 9, 9, 27, \ldots)$, which is complete.

Example 2.2 is connected with the famous result proved by Steinhaus [18]. The Author showed that the algebraic sum of two meager or null sets can be large in both senses of category and measure. The result was rediscovered many times, for instance see [16]. Now the equality $C+C = [0, 2]$ is rather well-known fact, however it is still popular and being developed in many aspects. Nymann [13] counted the numbers of subsums representations for points in achievement set, while Pavone [15] found another and completely different method of proving the equality $C + C = [0, 2]$.

Moreover note that if $(x_n)$ is complete, then $x_1 = 1$ and $x_n \le 2^{n-1}$ for all $n \in \mathbb{N}$. Thus $\sum_{n=2}^{\infty} \frac{1}{x_n} \ge \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = 1 = \frac{1}{x_1}$, so the sequence $(\frac{1}{x_n})$ can not be quickly convergent. Note that if we assume a weaker condition than the quick convergence, that is $x_n > r_n$ for all $n \ge k$, where $k \in \mathbb{N}$, then its achievement set is still a Cantor set as a finite sum of Cantor sets. However as we show in Theorem 2.3 this kind of sequences are not possible for the reciprocals of a complete sequence.

**Theorem 2.3.** *Let $(x_n)$ be a complete sequence. Then the inequality $\frac{1}{x_n} \le \sum_{k=n+1}^{\infty} \frac{1}{x_k}$ holds for infinite many indexes $n$.*

*Proof.* Let us consider $\delta_n = \sum_{k=1}^{n-1} x_k - x_n$. By the completeness of the sequence $(x_n)$ we obtain that $\delta_n \ge -1$ for each $n \in \mathbb{N}$. Suppose that there exists $r \in \mathbb{N}$ such that $\frac{1}{x_n} > \sum_{k=n+1}^{\infty} \frac{1}{x_k}$ for all $n > r$. We will show it implies that there exists an decreasing subsequence $(\delta_{m_n})$ of $(\delta_n)$, that is for some $n_0$ we have $\delta_{n_0} < -1$, which is impossible for complete sequence $(x_n)$. Fix any $n$ larger than $r$ and put $m_1 = n$. We have

$$\begin{aligned} \delta_{n+1} - \delta_n &= \sum_{k=1}^{n} x_k - x_{n+1} - \left( \sum_{k=1}^{n-1} x_k - x_n \right) = 2x_n - x_{n+1} \\ \delta_{n+2} - \delta_n &= (\delta_{n+2} - \delta_{n+1}) + (\delta_{n+1} - \delta_n) \\ &= (2x_{n+1} - x_{n+2}) + (2x_n - x_{n+1}) \\ &= 2x_n + x_{n+1} - x_{n+2} \end{aligned}$$

Thus for each $p \in \mathbb{N}$ we have:

$$\delta_{n+p} - \delta_n = 2x_n + x_{n+1} + x_{n+2} + x_{n+3} + \ldots + x_{n+p-2} + x_{n+p-1} - x_{n+p}$$

We will show that $\delta_{n+p_0} - \delta_n < 0$ for some $p_0$. Suppose that $\delta_{n+p} - \delta_n \ge 0$ for all $p \in \mathbb{N}$. Then

$$\delta_{n+p} - \delta_n = 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-1} x_i - x_{n+p} \ge 0 \text{ for every natural } p$$

Hence

$$\begin{aligned} x_{n+p} &\le 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-1} x_i = 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-2} x_i + x_{n+p-1} \\ &\le 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-2} x_i + 2x_n + x_{n+1} \\ &\quad + \sum_{i=n+2}^{n+p-2} x_i = 2 \cdot \left( 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-2} x_i \right) \end{aligned}$$

$$= 2 \cdot \left( 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-3} x_i + x_{n+p-2} \right)$$

$$\leq 4 \left( 2x_n + x_{n+1} + \sum_{i=n+2}^{n+p-3} x_i \right) \leq \ldots \leq 2^{p-2}(2x_n + x_{n+1}) \leq 2^p x_n,$$

so $\frac{1}{2^p}\frac{1}{x_n} \leq \frac{1}{x_{n+p}}$ for all $p$. But then $\sum_{p=1}^{\infty} \frac{1}{x_{n+p}} \geq \frac{1}{x_n} \sum_{p=1}^{\infty} \frac{1}{2^p} = \frac{1}{x_n}$, which contradicts with the quick convergence of the series $\sum_{n>r} \frac{1}{x_n}$. Hence $\delta_{n+p_0} < \delta_n$ for some $p_0$. We put $m_2 = n + p_0$ and continue the construction in the same way for $m_2$ and so on. Thus we obtain a decreasing sequence $(\delta_{m_n})$ of natural numbers, which contradicts with the completeness of the sequence $(x_n)$. □

By Theorem 2.3 we see that if it is possible to obtain that $A(\frac{1}{x_n})$ is a Cantor set, then it will not be the trivial case observed by Kakeya. Now we show that one can find a complete sequence $(x_n)$ such that $A(\frac{1}{x_n})$ is a Cantorval.

**Definition 2.4.** By a gap in the achievement set $A(x_n)$ we mean any open interval $(a, b)$ such that $a, b \in A(x_n)$ and $(a, b) \cap A(x_n) = \emptyset$.

In particular $A(x_n)$ is an interval if and only if it has no gaps and $A(x_n)$ is a finite union of compact intervals iff it has finitely many gaps. The following lemma can be found in [3].

**Lemma 2.5** (First Gap Lemma) r If $x_k > r_k$ then $(r_k, x_k)$ is a gap in $A(x_n)$.

**Theorem 2.6.** *There exists a complete sequence $(x_n)$ such that $A(\frac{1}{x_n})$ is a Cantorval.*

*Proof.* Let us start with the Guthrie-Nymann's Cantorval $GN = A(3, 2; \frac{1}{4})$. Then $\frac{2}{3}GN = A(\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{12}, \frac{1}{32}, \ldots)$ is a Cantorval. The problem now is the sequence $(2, 3, 8, 12, 32, \ldots)$ lacks completeness. We will add more terms to fulfill the completeness, but not too much because we do not want to obtain an interval as achievement set. Let us consider

$$x_1 = 1, \quad x_{6n-4} = 2 \cdot 16^{n-1}, \quad x_{6n-3} = x_{6n-2} = x_{6n-1} = 3 \cdot 16^{n-1},$$
$$x_{6n} = 8 \cdot 16^{n-1}, x_{6n+1} = 12 \cdot 16^{n-1}$$

for all $n \in \mathbb{N}$. The sequence $(x_n)$ satisfies the following inequalities for every $n \in \mathbb{N}$:

$$x_{6n+1} = 12 \cdot 16^{n-1} < 14 \cdot 16^{n-1} = x_{6n} + x_{6n-1} + x_{6n-2}$$
$$x_{6n} = 8 \cdot 16^{n-1} < 9 \cdot 16^{n-1} = x_{6n-1} + x_{6n-2} + x_{6n-3}$$
$$x_{6n-3} = 3 \cdot 16^{n-1} = 2 \cdot 16^{n-1} + 16 \cdot 16^{n-2} < 2 \cdot 16^{n-1} + 20 \cdot 16^{n-2}$$
$$= x_{6n-4} + x_{6n-5} + x_{6n-6}$$

Moreover $x_2 = x_1 + 1$ and for $n \geq 2$ we have:

$$\sum_{k=1}^{6n-5} x_k = 1 + \sum_{k=1}^{n-1} x_{6k-4} + \sum_{k=1}^{n-1} x_{6k-3} + \sum_{k=1}^{n-1} x_{6k-2} + \sum_{k=1}^{n-1} x_{6k-1} + \sum_{k=1}^{n-1} x_{6k}$$
$$+ \sum_{k=1}^{n-1} x_{6k+1} = 1 + \sum_{k=1}^{n-1} (2+3+3+3+8+12) \cdot 16^{k-1}$$
$$= 1 + 31 \cdot \frac{1-16^{n-1}}{1-16}$$
$$= \frac{31}{15} \cdot 16^{n-1} - \frac{16}{15} \geq 2 \cdot 16^{n-1} = x_{6n-4}.$$

Hence the sequence $(x_n)$ is complete. Note that for all $n \geq 2$ we have $(\frac{1}{x_n}) = \frac{2}{3}(12, 8, 8, 8, 3, 2; \frac{1}{16})$, so the sequence of reciprocals of $(x_n)$ is summable. We have

$$\sum_{k=6n}^{\infty} \frac{1}{x_k} = \left(\frac{1}{8 \cdot 16^{n-1}} + \frac{1}{12 \cdot 16^{n-1}} + \frac{1}{2 \cdot 16^{n}} + \frac{3}{3 \cdot 16^{n}}\right) \cdot \frac{1}{1-\frac{1}{16}}$$
$$= \frac{1}{16^{n-1}} \left(\frac{1}{8} + \frac{1}{12} + \frac{1}{32} + \frac{1}{16}\right) \cdot \frac{16}{15}$$

Thus for each $n \in \mathbb{N}$ we get

$$\sum_{k=6n}^{\infty} \frac{1}{x_k} = \frac{1}{16^{n-1}} \cdot \frac{29}{96} \cdot \frac{16}{15} = \frac{1}{16^{n-1}} \cdot \frac{29}{90} < \frac{1}{16^{n-1}} \cdot \frac{1}{3} = \frac{1}{x_{6n-1}},$$

which by First Gap Lemma means that $(\sum_{k=6n}^{\infty} \frac{1}{x_k}, \frac{1}{x_{6n-1}})$ is a gap in $A(\frac{1}{x_n})$. Hence $A(\frac{1}{x_n})$ is not a finite union of compact intervals. Moreover $A(\frac{1}{x_n}) \supset \frac{2}{3}GN$, so the achievement set is Cantorval. □

*Remark 2.7.* Note that there is a correspondence between the sequences of reciprocals of natural numbers and specific multigeometric sequences. Indeed if we have a multigeometric sequence with ratio $\frac{1}{p}$, which is a reciprocal of a natural number, then $(a_1, a_2, \ldots, a_m; \frac{1}{p})$ can be scaled into the sequence of reciprocals of natural numbers. We have

$$\frac{p}{a_1 a_2 \ldots a_m} \left(a_1, a_2, \ldots, a_m; \frac{1}{p}\right) = \left(\frac{1}{a_2 a_3 \ldots a_m}, \frac{1}{a_1 a_3 \ldots a_m},\right.$$
$$\left.\frac{1}{a_1 a_2 a_4 \ldots a_m}, \ldots, \frac{1}{a_1 a_2 \ldots a_{m-1}}, \frac{1}{p a_2 a_3 \ldots a_m}, \ldots\right)$$

**Problem 2.8.** *Let $(x_n)$ be a complete sequence. Is it possible to construct $A(\frac{1}{x_n})$, which is a Cantor set ?*

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