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Results in Mathematics

On κ-Pseudocompactess and Uniform Homeomorphisms of Function Spaces

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Abstract. A Tychonoff space X is called *κ-pseudocompact* if for every continuous mapping f of X into $\mathbb{R}^\kappa$ the image $f(X)$ is compact. This notion generalizes pseudocompactness and gives a stratification of spaces lying between pseudocompact and compact spaces. It is well known that pseudocompactness of X is determined by the uniform structure of the function space $C_p(X)$ of continuous real-valued functions on X endowed with the pointwise topology. In respect of that A.V. Arhangel'skii asked if analogous assertion is true for κ-pseudocompactness. We provide an affirmative answer to this question.

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1. Introduction

In this note, by a space we mean a Tychonoff topological space. For a space X, by $C_p(X)$ we denote the space of all continuous real-valued functions on X endowed with the pointwise topology. The symbol $C_p^*(X)$ stands for the subspace of $C_p(X)$ consisting of all *bounded* continuous functions. Recall that X is *pseudocompact* if $C_p(X) = C_p^*(X)$, i.e. every real-valued continuous function on X is bounded. In 1962 J.F. Kennison [7] introduced the following generalization of psudocompactness. Let κ be an infinite cardinal. A space X is called *κ-pseudocompact* if for every continuous mapping f of X into $\mathbb{R}^\kappa$ the image $f(X)$ is compact. Clearly, κ-pseudocompactness implies λ-pseudocompactness

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for every infinite cardinal $\lambda \le \kappa$. Since metrizable pseudocompact spaces are compact, it is easy to see that ω-pseudocompactness is precisely pseudocompactness. In particular any κ-pseudocompact space is pseudocompact. It was established by Uspenskiĭ in [12] that pseudocompactness of X is determined by the uniform structure of the function space $C_p(X)$ (see [4] for a different proof of this result). In respect of that A.V. Arhangel'skii asked in 1998 if analogous result holds for κ-pseudocompactness (see [3, Question 13] or [11, Problem 4.4.2]).

The aim of the present note is to provide an affirmative answer to this question by proving the following extension of Uspenskiĭ's theorem:

Theorem 1.1. *For any infinite cardinal κ, if $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic, then X is κ-pseudocompact if and only if Y is κ-pseudocompact.*

Let us recall that that a map $\varphi : C_p(X) \to C_p(Y)$ is *uniformly continuous* if for each open neighborhood U of the zero function in $C_p(Y)$, there is and open neighborhood V of the zero function in $C_p(X)$ such that $(f - g) \in V$ implies $(\varphi(f) - \varphi(g)) \in U$. Spaces $C_p(X)$ and $C_p(Y)$ are *uniformly homeomorphic* if there is a homeomorphism φ between them such that both φ and φ^{-1} are uniformly continuous.

The proof of Theorem 1.1 is inspired by author's recent work [8] concerned with linear homeomorphisms of function spaces. The basic idea in [8] relies on the fact that certain topological properties of a space X can be conveniently characterized by the way X is positioned in its Čech-Stone compactification βX; κ-pseudocompactess is one of such properties. Indeed, Hewitt [6] gave the following description of pseudocompactness (cf. [1, Theorem 1.3.3]).

Theorem 1.2 (Hewitt). *A space X is pseudocompact if and only if every nonempty G_δ-subset of βX meets X.*

It was noted by Retta in [10] that the above result easily extends to κ-pseudocompactness. We need the following notation. Let κ be an infinite cardinal. A subset A of a space Z is a G_κ-*set* if it is an intersection of at most κ-many open subsets of Z. The G_ω-sets are called G_δ-sets and the complement of a G_δ-set is called F_σ-set. We have (see [10, Theorem 1]):

Theorem 1.3 (Retta). *Let κ be an infinite cardinal. A space X is κ-pseudo compact if and only if every nonempty G_κ-subset of βX meets X.*

The uniform structure of spaces of continuous functions was studied by many authors; the interested reader should consult the book [11]. For our purposes, the most important are some ideas developed by Gul'ko in [5].

2. Results

For a space Z and a function $f \in C_p^*(Z)$ the function $\widetilde{f} : \beta Z \to \mathbb{R}$ is the unique continuous extension of f over the Čech-Stone compactification βZ of Z. Let

$\varphi : C_p^*(X) \to C_p^*(Y)$ be a uniformly continuous surjection. For $y \in \beta Y$ and a subset K of βX we define

$$a(y, K) = \sup\{|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| : \ f, g \in C_p^*(X) \text{ such that } |\widetilde{f}(x) - \widetilde{g}(x)| < 1 \text{ for every } x \in K\}$$

Note that $a(y, \emptyset) = \infty$ since φ is onto.

For $y \in \beta Y$ define the family

$$\mathscr{A}(y) = \{K \subseteq \beta X : K \text{ is compact and } a(y, K) < \infty\}.$$

Similarly, for $y \in \beta Y$ and $n \in \mathbb{N}$ let

$$\mathscr{A}_n(y) = \{K \subseteq \beta X : K \text{ is compact and } a(y, K) \le n\}.$$

It may happen that for some n the family $\mathscr{A}_n(y)$ is empty. However, we have the following:

Proposition 2.1. *For every $y \in Y$, there exists n for which $\mathscr{A}_n(y)$ contains a nonempty finite subset of X. In particular, for this n the family $\mathscr{A}_n(y)$ is nonempty.*

Proof. By uniform continuity of φ, there is $n \in \mathbb{N}$ and a finite subset F of X such that

$$\begin{aligned} &\text{if } |f(x) - g(x)| < 1/n \text{ for every } x \in F, \\ &\text{then } |\varphi(f)(y) - \varphi(g)(y)| < 1. \end{aligned} \tag{1}$$

We claim that $F \in \mathscr{A}_n(y)$. To see this, take arbitrary functions $f, g \in C_p^*(X)$ such that $|f(x) - g(x)| < 1$, for every $x \in F$. Put $f_k = f + \frac{k}{n}(g - f)$, for $k = 0, 1, \ldots, n$. Then $f_0 = f$, $f_n = g$ and $|f_k(x) - f_{k+1}(x)| < 1/n$ for $x \in F$. Hence, by (1) we get

$$\begin{aligned} &|\varphi(f)(y) - \varphi(g)(y)| \\ &\quad \le |\varphi(f_0)(y) - \varphi(f_1)(y)| + \cdots + |\varphi(f_{n-1})(y) - \varphi(f_n)(y)| < n, \end{aligned}$$

as required. □

Clearly, for every $y \in \beta Y$ we have $\mathscr{A}(y) = \bigcup_{n \in \mathbb{N}} \mathscr{A}_n(y)$. In particular, for $y \in Y$ the family $\mathscr{A}(y)$ is always nonempty.

For $n \in \mathbb{N}$ we set

$$Y_n = \{y \in \beta Y : \mathscr{A}_n(y) \text{ is nonempty}\}.$$

Note that $y \in Y_n$ if and only if $\beta X \in \mathscr{A}_n(y)$. Using this observation it is easy to show the following:

Lemma 2.2. *For every $n \in \mathbb{N}$ the set Y_n is closed in βY; hence compact.*

Proof. Pick $y \in \beta Y \setminus Y_n$. Since $\beta X \notin \mathscr{A}_n(y)$, there are functions $f, g \in C_p^*(X)$ satisfying $|\widetilde{f}(x) - \widetilde{g}(x)| < 1$ for every $x \in \beta X$, and $|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| > n$. The set

$$U = \{z \in \beta Y : |\widetilde{\varphi(f)}(z) - \widetilde{\varphi(g)}(z)| > n\}$$

is an open neighborhood of y in βY. Moreover if $z \in U$, then f and g witness that $\beta X \notin \mathscr{A}_n(z)$; thus $U \cap Y_n = \emptyset$. □

For a space X and a positive integer m, we denote by $[X]^{\leq m}$ the space of all nonempty at most m-element subsets of X endowed with the Vietoris topology, i.e. basic open sets in $[X]^{\leq m}$ are of the form

$$\langle U_1, \ldots, U_k\rangle = \left\{ F \in [X]^{\leq m} : \forall i \leq k \quad F \cap U_i \neq \emptyset \text{ and } F \subseteq \bigcup_{i=1}^{k} U_i \right\},$$

where $\{U_1, \ldots, U_k\}$ is a finite collection of open subset of X.

For any positive integers n, m we define

$$Y_{n,m} = \{y \in \beta Y : \mathscr{A}_n(y) \cap [\beta X]^{\leq m} \neq \emptyset\}$$

Note that $Y_{n,m} \subseteq Y_n$ and by Proposition 2.1 we have

$$Y \subseteq \bigcup_{n,m} Y_{n,m} \tag{2}$$

We claim that $Y_{n,m}$ is closed in Y_n and hence it is compact:

Lemma 2.3. *The set $Y_{n,m}$ is closed in Y_n, hence it is compact.*

Proof. Consider the following subset Z of the product $Y_n \times [\beta X]^{\leq m}$

$$Z = \{(y, F) \in Y_n \times [\beta X]^{\leq m} : F \in \mathscr{A}_n(y)\}.$$

We show that Z is closed. Pick $(y, F) \in (Y_n \times [\beta X]^{\leq m}) \setminus Z$. Then $F \notin \mathscr{A}_n(y)$ and thus there are $f, g \in C_p^*(X)$ satisfying $|\widetilde{f}(x) - \widetilde{g}(x)| < 1$ for every $x \in F$, and $|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| > n$. Let $U = \{x \in \beta X : |\widetilde{f}(x) - \widetilde{g}(x)| < 1\}$ and $V = \{z \in Y_n : |\widetilde{\varphi(f)}(z) - \widetilde{\varphi(g)}(z)| > n\}$. The set $V \times \langle U\rangle$ is an open neighborhood of (y, F) in $Y_n \times [\beta X]^{\leq m}$ disjoint from Z.

The set Z, being is closed in the compact space $Y_n \times [\beta X]^{\leq m}$, is compact. Since the set $Y_{n,m}$ is the image of Z under the projection map it must be compact. □

Corollary 2.4. *Suppose that Y is pseudocompact and let $\varphi : C_p^*(X) \to C_p^*(Y)$ be a uniformly continuous surjection. For every $y \in \beta Y$, there exist n and m such that $y \in Y_{n,m}$.*

Proof. By Lemma 2.3, the set $\bigcup_{n,m} Y_{n,m}$ is F_σ in βY and contains Y, by (2). It follows from Theorem 1.2 that $\bigcup_{n,m} Y_{n,m} = \beta Y$. □

For $y \in \bigcup_{n,m} Y_{n,m}$ we define

$$K(y) = \bigcap \mathscr{A}(y).$$

Remark. For $y \in Y$ the set $K(y)$ is the support introduced by Gul'ko in [5] (see also [2,4,9]).

Lemma 2.5. *For every $y \in \bigcup_{n,m} Y_{n,m}$ the set $K(y)$ is a nonempty finite subset of βX. Moreover, $K(y) \in \mathscr{A}(y)$. If $y \in Y$, then $K(y)$ is a subset of X.*

Proof. We show that the family $\mathscr{A}(y)$ is closed under finite intersections. Pick $K_1, K_2 \in \mathscr{A}(y)$ and let $f, g \in C_p^*(X)$ be such that $|\widetilde{f}(x) - \widetilde{g}(x)| < 1$ for every $x \in K_1 \cap K_2$. Let

$$U = \{x \in \beta X : |\widetilde{f}(x) - \widetilde{g}(x)| < 1\}.$$

The set U is open in βX and $K_1 \cap K_2 \subseteq U$.

Since K_1 and $K_2 \backslash U$ are disjoint closed subsets of the compact space βX, by Urysohn's lemma there is a continuous function $u : \beta X \to [0, 1]$ such that

$$u(x) = \begin{cases} 1 & \text{for } x \in K_1 \\ 0 & \text{for } x \in K_2 \setminus U \end{cases}$$

Let

$$\widetilde{h} = u \cdot (\widetilde{f} - \widetilde{g}) + \widetilde{g}$$

and let $h \in C_p^*(X)$ be the restriction of $\widetilde{h}$ to X. We have:

- $\widetilde{h}(x) = \widetilde{f}(x)$ for $x \in K_1$,
- $\widetilde{h}(x) = \widetilde{g}(x)$ for $x \in K_2 \backslash U$ and
- if $x \in U$, then $|\widetilde{h}(x) - \widetilde{g}(x)| = |u(x)| \cdot |\widetilde{f}(x) - \widetilde{g}(x)| < 1$, by definition of U and the fact that u maps into $[0, 1]$.

In particular, since $K_1 \cap K_2 \subseteq U$, we get

- $|\widetilde{h}(x) - \widetilde{g}(x)| < 1$ for $x \in K_2$.

Since $K_1 \in \mathscr{A}(y)$ and $\widetilde{h}(x) = \widetilde{f}(x)$ for $x \in K_1$, we get $|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(h)}(y)| \leq a(y, K_1) < \infty$. Similarly, since $|\widetilde{h}(x) - \widetilde{g}(x)| < 1$ for $x \in K_2$ and $K_2 \in \mathscr{A}(y)$, we have $|\widetilde{\varphi(g)}(y) - \widetilde{\varphi(h)}(y)| \leq a(y, K_2) < \infty$. Hence,

$$\begin{aligned} |\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| &\leq |\widetilde{\varphi(f)}(y) - \widetilde{\varphi(h)}(y)| + |\widetilde{\varphi(h)}(y) - \widetilde{\varphi(g)}(y)| \\ &\leq a(y, K_1) + a(y, K_2). \end{aligned}$$

So $a(y, K_1 \cap K_2) \leq a(y, K_1) + a(y, K_2)$ and thus $K_1 \cap K_2 \in \mathscr{A}(y)$. By induction the result follows for any finite intersection.

The family $\mathscr{A}(y)$, consisting of closed subsets of βX, is closed under finite intersections and $\emptyset \notin \mathscr{A}(y)$ so by compactness the intersection $\bigcap \mathscr{A}(y)$ must be nonempty. It is finite because $y \in Y_{n,m}$ guarantees that the family $\mathscr{A}(y)$ contains a subset of βX which is at most m-element.

Since $\mathscr{A}(y)$ contains a finite subset F of βX, the set $K(y) \subseteq F$ is an intersection of finitely many elements of $\mathscr{A}(y)$ so the first part of the proof implies that $K(y) \in \mathscr{A}(y)$.

Finally, if $y \in Y$ then $y \in \bigcup_{n,m} Y_{n,m}$, by (2). So $K(y)$ is well defined. The inclusion $K(y) \subseteq X$ follows from Proposition 2.1. □

For $y \in \bigcup_{n,m} Y_{n,m}$ we define

$$a(y) = a(y, K(y))$$

By Lemma 2.5, $K(y) \in \mathscr{A}(y)$ so $a(y) < \infty$. For a subset A of βX we set

$$K^{-1}(A) = \left\{ y \in \bigcup_{n,m} Y_{n,m} : K(y) \cap A \neq \emptyset \right\}.$$

Combining Corollary 2.4 and Lemma 2.5 we get

Proposition 2.6. *Suppose that $\varphi : C_p^*(X) \to C_p^*(Y)$ is a uniformly continuous surjection. If Y is pseudocompact then, for every $y \in \beta Y$, the set $K(y)$ is a well-defined nonempty finite subset of βX that belongs to the family $\mathscr{A}(y)$. Also, $a(y)$ is a well-defined number, for every $y \in \beta Y$.*

The proof of the next lemma is analogous to the proof of [2, Lemma 1.3].

Lemma 2.7. *Suppose that $U \subseteq \beta X$ is open and let n be a positive integer. For every $y \in K^{-1}(U) \cap Y_n$ there exists an open neighborhood V of y in βY such that for every $z \in V \cap Y_n$ and every $A \in \mathscr{A}_n(z)$ we have $A \cap U \neq \emptyset$.*

Proof. Fix $x_0 \in K(y) \cap U$ witnessing $y \in K^{-1}(U)$. Since $K(y)$ is finite, shrinking U if necessary we can assume that $U \cap K(y) = \{x_0\}$. Note that $\beta X \setminus U \notin \mathscr{A}(y)$ for otherwise $K(y)$ would be a subset of $\beta X \setminus U$ and this is not the case because $x_0 \in K(y) \cap U$. It follows that there are $f, g \in C_p^*(X)$ such that

$$|\widetilde{f}(x) - \widetilde{g}(x)| < 1 \text{ for every } x \in \beta X \setminus U \quad \text{and} \tag{3}$$

$$|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| > a(y) + n \tag{4}$$

Let $\widetilde{h} \in C_p(\beta X)$ be a function satisfying

$$\widetilde{h}(x) = \widetilde{f}(x) \text{ for every } x \in \beta X \setminus U, \quad \text{and} \quad \widetilde{h}(x_0) = \widetilde{g}(x_0), \tag{5}$$

and let $h \in C_p^*(X)$ be the restriction of $\widetilde{h}$.

Note that by (3) and (5), $|\widetilde{h}(x) - \widetilde{g}(x)| < 1$ for every $x \in K(y)$. Therefore,

$$|\widetilde{\varphi(h)}(y) - \widetilde{\varphi(g)}(y)| \leq a(y). \tag{6}$$

According to (4) and (6) we have

$$|\widetilde{\varphi(f)}(y)-\widetilde{\varphi(h)}(y)|\geq|\widetilde{\varphi(f)}(y)-\widetilde{\varphi(g)}(y)|-|\widetilde{\varphi(h)}(y)-\widetilde{\varphi(g)}(y)|>n. \tag{7}$$

Let

$$V=\{z\in\beta Y:|\widetilde{\varphi(f)}(z)-\widetilde{\varphi(h)}(z)|>n\}.$$

The set V is open and $y\in V$, by (7). We show that V is as required.

Take $z\in V\cap Y_n$ and let $A\in\mathscr{A}_n(z)$. If $A\cap U=\emptyset$ then $\widetilde{h}(x)=\widetilde{f}(x)$ for every $x\in A$, by (5). So $|\widetilde{\varphi(f)}(z)-\widetilde{\varphi(h)}(z)|\leq n$, contradicting $z\in V$. □

Proposition 2.8. *Suppose that Y is pseudocompact. If $U\subseteq\beta X$ is open, then the set $K^{-1}(U)$ is a G_δ-subset of βY.*

Proof. By Proposition 2.6, for every $y\in\beta Y$, the set $K(y)$ is a nonempty finite subset of βX. For $n=1,2,\ldots$, let

$$L_n=K^{-1}(U)\cap Y_n.$$

For $y\in L_n$ let V_n^y be an open neighborhood of y in βY provided by Lemma 2.7, i.e.

$$\text{if } z\in V_n^y\cap Y_n \text{ and } A\in\mathscr{A}_n(z),\ \text{then } A\cap U\neq\emptyset. \tag{8}$$

Let

$$V_n=\bigcup\{V_n^y:y\in L_n\}.$$

We claim that

$$K^{-1}(U)=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}V_n.$$

Indeed, pick $y\in K^{-1}(U)$ and fix an arbitrary $m\geq 1$. Since $\beta Y=\bigcup_{n=1}^\infty Y_n$ (cf. Corollary 2.4), there is i such that $y\in Y_i$. Since $Y_n\subseteq Y_{n+1}$, we can assume that $i>m$. We have $y\in L_i$ whence $y\in V_i^y\subseteq V_i\subseteq\bigcup_{n=m}^\infty V_n$, because $i>m$.

To prove the opposite inclusion, take $z\in\bigcap_{m=1}^\infty\bigcup_{n=m}^\infty V_n$. Again, there is i such that $z\in Y_i$. Let j be a positive integer satisfying $j>\max\{a(z),i\}$. By our assumption, $z\in\bigcup_{n=j}^\infty V_n$, so there is $k\geq j$ such that $z\in V_k$. Clearly, $z\in Y_k$ and since $k>a(z)$ we have

$$K(z)\in\mathscr{A}_k(z). \tag{9}$$

By definition of V_k, there is $y\in L_k$ such that $z\in V_k^y$. Now, from (8) and (9) we get $z\in K^{-1}(U)$. □

Remark 2.9. If $\varphi:C_p^*(X)\to C_p^*(Y)$ is a uniform homeomorphism, we may consider the inverse map $\varphi^{-1}:C_p^*(Y)\to C_p^*(X)$ and apply all of the above results to φ^{-1}. In particular, if X is pseudocompact, then for every $x\in\beta X$ we can define the set $K(x)\subseteq\beta Y$ and the real number $a(x)$ simply by interchanging the roles of X and Y above.

Lemma 2.10. *Suppose that both X and Y are pseudocompact spaces. Let $\varphi : C_p^*(X) \to C_p^*(Y)$ be a uniform homeomorphism. For any $x \in \beta X$ there is $y \in K(x)$ such that $x \in K(y)$.*

Proof. Let $x \in \beta X$. Applying Proposition 2.6, first to φ^{-1} and then to φ, we infer that the set $F = \bigcup\{K(y) : y \in K(x)\} \subseteq \beta X$ is finite being a finite union of finite sets. Let M be a positive integer such that

$$M > \max\{a(y) : y \in K(x)\}.$$

Striving for a contradiction, suppose that $x \notin F$. Let $\widetilde{f}, \widetilde{g} \in C_p(\beta X)$ be functions satisfying

$$\widetilde{f}(z) = \widetilde{g}(z) \text{ for every } z \in F \text{ and } |\widetilde{f}(x) - \widetilde{g}(x)| > M \cdot a(x). \tag{10}$$

Let $f \in C_p^*(X)$ and $g \in C_p^*(X)$ be the restrictions of $\widetilde{f}$ and $\widetilde{g}$, respectively. Since for every $y \in K(x)$ the functions $\widetilde{f}$ and $\widetilde{g}$ agree on $K(y) \subseteq F$, we have

$$|\widetilde{\varphi(f)}(y) - \widetilde{\varphi(g)}(y)| \leq a(y) < M, \text{ for every } y \in K(x). \tag{11}$$

For $k \in \{0, 1, \ldots, M\}$ define a function $\widetilde{h_k} \in C_p(\beta Y)$ by the formula

$$\widetilde{h_k} = \widetilde{\varphi(f)} + \tfrac{k}{M}\left(\widetilde{\varphi(g)} - \widetilde{\varphi(f)}\right).$$

Obviously, $\widetilde{h_0} = \widetilde{\varphi(f)}$ and $\widetilde{h_M} = \widetilde{\varphi(g)}$. Moreover, by (11), we have

$$|\widetilde{h_{k+1}}(y) - \widetilde{h_k}(y)| = \tfrac{1}{M}|\widetilde{\varphi(g)}(y) - \widetilde{\varphi(f)}(y)| < 1, \text{ for every } y \in K(x). \tag{12}$$

For $k \in \{0, 1, \ldots, M\}$ let $h_k \in C_p^*(Y)$ be the restriction of $\widetilde{h_k}$. Using (12) we get:

$$\begin{aligned}|\widetilde{f}(x) - \widetilde{g}(x)| &= |\widetilde{\varphi^{-1}(\varphi(f))}(x) - \widetilde{\varphi^{-1}(\varphi(g))}(x)| = |\widetilde{\varphi^{-1}(h_0)}(x) - \widetilde{\varphi^{-1}(h_M)}(x)| \\ &\leq |\widetilde{\varphi^{-1}(h_0)}(x) - \widetilde{\varphi^{-1}(h_1)}(x)| + \cdots \\ &\quad + |\widetilde{\varphi^{-1}(h_{M-1})}(x) - \widetilde{\varphi^{-1}(h_M)}(x)| \leq M \cdot a(x)\end{aligned}$$

This however contradicts (10). □

Now we are ready to prove of our main result.

Proof of Theorem 1.1. Let κ be an infinite cardinal and let $\varphi : C_p(X) \to C_p(Y)$ be a uniform homeomorphism. By symmetry it is enough to show that if Y is κ-pseudocompact then so is X. So let us assume that Y is κ-pseudocompact. Then, in particular Y is pseudocompact and hence by Uspenskiĭ's theorem [12, Corollary] (cf. [11, V.136]), so is X. Hence, $C_p(Y) = C_p^*(Y)$ and $C_p(X) = C_p^*(X)$. In order to prove that X is κ-pseudocompact we will employ Theorem 1.3. For this purpose fix a nonempty G_κ-subset G of βX. It suffices to prove that $G \cap X \neq \emptyset$.

Claim 1. The set $K^{-1}(G) = \{y \in \beta Y : K(y) \cap G \neq \emptyset\}$ is nonempty.

Proof. The set G is nonempty so let us fix $x \in G$. According to Lemma 2.10 there is $y \in K(x)$ such that $x \in K(y)$. In particular, $y \in K^{-1}(G)$. □

Claim 2. The set $K^{-1}(G) = \{y \in \beta Y : K(y) \cap G \neq \emptyset\}$ is a G_κ-set in βY.

Proof. Write $G = \bigcap\{U_\alpha : \alpha < \kappa\}$, where each U_α is an open subset of βX. We can also assume that the family $\{U_\alpha : \alpha < \kappa\}$ is closed under finite intersections. It follows from Proposition 2.8 that for each $\alpha < \kappa$, the set $K^{-1}(U_\alpha)$ is G_δ in βY. Thus, it is enough to show that

$$K^{-1}(G) = \bigcap_{\alpha<\kappa} K^{-1}(U_\alpha).$$

To this end, take $y \in \bigcap_{\alpha<\kappa} K^{-1}(U_\alpha)$. According to Proposition 2.6, the set $K(y)$ is a nonempty finite subset of βX. Enumerate $K(y) = \{x_1, \ldots, x_k\}$, where k is a positive integer. If $y \notin K^{-1}(G)$, then for every $i \leq k$ there is $\alpha_i < \kappa$ such that

$$x_i \notin U_{\alpha_i}. \tag{13}$$

The family $\{U_\alpha : \alpha < \kappa\}$ is closed under finite intersections, so there is $\gamma < \kappa$ with $U_\gamma = U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$. But $y \in \bigcap_{\alpha<\kappa} K^{-1}(U_\alpha) \subseteq K^{-1}(U_\gamma)$. Hence, there is $j \leq k$ such that $x_j \in U_\gamma \subseteq U_{\alpha_j}$, which is a contradiction with (13). Therefore, we must have $y \in K^{-1}(G)$. This provides the inclusion $K^{-1}(G) \supseteq \bigcap_{\alpha<\kappa} K^{-1}(U_\alpha)$. The opposite inclusion is immediate. □

It follows from Claims 1 and 2 that the $K^{-1}(G)$ is a nonempty G_κ-subset of βY. Hence, by Theorem 1.3, there exists $p \in K^{-1}(G) \cap Y$. We have $K(p) \cap G \neq \emptyset$ and since $p \in Y$, we infer from Lemma 2.5 that $K(p)$ is a nonempty finite subset of X. Therefore, $\emptyset \neq K(p) \cap G \subseteq X \cap G$. □

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