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Results in Mathematics

The $\mathcal{X}$-series of a p-group and Complements of Abelian Subgroups

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Abstract. Let G be a finite p-group. We denote by $\mathcal{X}_i(G)$ the intersection of all subgroups of G having index p^i in G. In this paper, the newly introduced series $\{\mathcal{X}_i(G)\}_i$ is investigated and a number of results concerning its behaviour are proved. As an application of these results, we show that if an abelian subgroup A of G intersects each one of the subgroups $\mathcal{X}_i(G)$ at $\mathcal{X}_i(A)$, then A has a complement in G. Conversely if an arbitrary subgroup H of G has a normal complement, then $\mathcal{X}_i(H) = \mathcal{X}_i(G) \cap H$.

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1. Introduction

In this paper, we shall only be concerned with finite groups. We begin with some general remarks on complementation. For a fuller discussion in textbook format, the reader is invited to consult the recent monograph by Kirtland [6].

The (rather broad) question we ask is the following:

> Let G be a group and let N be a normal subgroup of G. Under which circumstances is N guaranteed to have a complement in G?

Perhaps the most well-known and most general answer is the celebrated Schur–Zassenhaus theorem which asserts that if N is a Hall subgroup of G (so that $\gcd(|N|, |G : N|) = 1$), then N does indeed have a complement. In fact, the theorem asserts more: all complements of N in G are G-conjugate. The conjugacy part of the theorem relies on a solubility assumption: either N or G/N

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must be soluble for it to admit a proof. This, in turn, can only be made to hold unconditionally by an appeal to the deep Odd Order Theorem of Feit and Thompson [1].

Another answer to the question above (relaxing the requirement in the abelian case of the Schur–Zassenhaus theorem) is furnished by a theorem of Gaschütz [3]. For recent work discussing Gaschütz's theorem, see [8].

Theorem 1.1 (Gaschütz). *Let N be an abelian normal subgroup of a group G. Let $N \leq H \leq G$ such that N has a complement in H and* $\gcd(|N|, |G : H|) = 1$. *Then N has a complement in G.*

In particular, if N is normal in G and abelian then N has a complement in G if and only if for each prime p dividing $|N|$ the unique Sylow p-subgroup of N (which is normal in G) has a complement in some (and thus in every) Sylow p-subgroup of G, and this is a Reduktionsatz in Gaschütz's paper.

This global-to-local reduction that Gaschütz's theorem achieves reaches an obvious and natural limit if G is itself a p-group. Thus we are led to ask:

> Let G be a p-group and N be a (possibly normal) abelian subgroup of G. Is there a necessary and sufficient condition for N to have a complement in G?

In the case that G itself is an abelian group, a condition which is both necessary and sufficient for a subgroup A of G to have a complement in G exists and is based on Prüfer's notion of a **pure subgroup**; see for example [2, Chap. 5]. A subgroup A of an abelian group G is called pure provided that if $g^n \in A$ for some $g \in G$ and $n \in \mathbb{N}$ then there exists $a \in A$ such that $a^n = g^n$. Equivalently, A is pure in G if and only if $A^n = G^n \cap A$, for all $n \in \mathbb{N}$, and the above mentioned necessary and sufficient condition can be stated as:

Theorem 1.2 . *Let G be a abelian group and $A \leq G$. Then A has a complement in G if and only if $A^n = A \cap G^n$, for all $n \in \mathbb{N}$.*

This theorem was the exact point of departure for the work reported here.

Given a p-group G, we define $\mathcal{X}_i(G)$ to be the intersection of all subgroups of G having index p^i in G for $i \leq \log_p(|G|)$ and (for technical reasons) extend its definition to all non-negative integers. We call the associated series $\{\mathcal{X}_i(G)\}_i$ the $\mathcal{X}$-series of G. This series has several nice properties:

- It is subgroup-monotone; cf. Proposition 2.4.
- It respects direct products; cf. Theorem 2.8.
- For every $i, j \geq 0$ we have $\mathcal{X}_i(\mathcal{X}_j(G)) \leq \mathcal{X}_{i+j}(G)$. Moreover, successive quotients of the $\mathcal{X}$-series of G are elementary abelian p-groups; cf. Lemma 2.5.
- For every $i \geq 0$ we have $\gamma_{p^{i-1}+1}(G) \leq \mathcal{X}_i(G)$; cf. Proposition 2.11.

Furthermore, the use of the $\mathcal{X}$-series enables us to generalize Theorem 1.2 to any p-group G and give a condition which is sufficient (and in some cases necessary) for an abelian subgroup A of G to have a complement in G. This is our main result, given in Sect. 3:

Theorem 1.3 . *Let A be an abelian subgroup of the p-group G and suppose that*

$$A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$$

for all indices $i \geq 0$. Then A has a complement in G.

This is indeed a generalization of Theorem 1.2 for the following reason: In the case of an abelian p-group G, the only relevant exponents n in the equation

$$A^n = A \cap G^n \qquad (\dagger)$$

are the powers of p, thus (†) is essentially equivalent to

$$\mho_i(A) = A \cap \mho_i(G)$$

for all non-negative integers i. On the other hand, the $\mho_i$ subgroup of an abelian p-group is its $\mathcal{X}_i$ subgroup, as we will see in Proposition 2.9. Thus equation (†) is equivalent to $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all indices $i \geq 0$.

A sort of partial converse of Theorem 1.3 holds and its proof is given in Sect. 2:

Theorem 1.4 . *Let H be an arbitrary subgroup of the p-group G and assume that H has a normal complement in G. Then*

$$H \cap \mathcal{X}_i(G) = \mathcal{X}_i(H)$$

for all indices $i \geq 0$.

1.1. Notation

We outline below some notational conventions that we will use throughout the paper.

(i) $[n]$ denotes the set of the first n positive integers $\{1, 2, \ldots, n\}$.

(ii) Let G be a p-group and let i be a non-negative integer. We will write

$$\mho_i(G) = \left\langle g^{p^i} : g \in G \right\rangle$$

for the subgroup generated by the p^i-powers of elements of G. Clearly, $\mho_0(G) = G$. The index $i = 1$ is sometimes dropped, so that $\mho_1(G) = \mho(G)$. These subgroups, for various indices i, are referred to as the agemo subgroups of G. We remark that G^{p^i} is sometimes used instead of $\mho_i(G)$ in the literature.

(iii) We will have occasion to refer to the Frattini series of a group. For an arbitrary group, the Frattini series is defined by $\Phi_0(G) = G$ and $\Phi_i(G) = \Phi(\Phi_{i-1}(G))$, where $\Phi(G)$ is the Frattini subgroup of G, i.e. the intersection of all maximal subgroups of G.

(iv) We will also write $\gamma_i(G)$ for the lower central series of an arbitrary group G, defined inductively by means of $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for all $i \geq 1$.

2. Properties of the $\mathcal{X}$-series

We begin in earnest by defining the $\mathcal{X}$-series of a p-group.

Definition 2.1. Given a p-group G and a positive integer $i \leq \log_p |G|$ we write

$$\mathcal{P}_i = \mathcal{P}_i(G) := \left\{T \leq G \,:\, |G : T| = p^i\right\}$$

for the collection of subgroups of G of index p^i in G. We also write

$$\mathcal{X}_i(G) := \bigcap_{T \in \mathcal{P}_i(G)} T;$$

that is, $\mathcal{X}_i(G)$ is the intersection of all subgroups of G that have index p^i in G for all such i. We call the associated series the $\mathcal{X}$-series of G and by convention we write $\mathcal{X}_0(G) = G$. Finally, if $j > \log_p |G|$ then $\mathcal{X}_j(G) = 1$ by assumption.

Some first properties following the definition of the $\mathcal{X}$-series are collected in the following lemma.

Lemma 2.2. *If G is a p-group then the subgroups $\mathcal{X}_i(G)$ are characteristic subgroups of G and*

(i) $\mathcal{X}_1(G) = \Phi(G)$ *;*
(ii) For every $j \in [k]$ *we have* $\mathcal{X}_k(G) = \bigcap_{T \in \mathcal{P}_{k-j}(G)} \mathcal{X}_j(T)$*;*
(iii) $\mathcal{X}_{i+1}(G) \leq \mathcal{X}_i(G)$ *;*
(iv) $\Phi_i(G) \leq T$*, for every* $T \in \mathcal{P}_i(G)$ *;*
(v) $\Phi_i(G) \leq \mathcal{X}_i(G)$ *for every* i *;*
(vi) If $N \trianglelefteq G$ *and* $N \leq \mathcal{X}_i(G)$ *for some index* i*, then* $\mathcal{X}_i(G/N) = \mathcal{X}_i(G)N/N$*.*

Proof. (i) Clearly the $\mathcal{X}_i(G)$ are characteristic subgroups of G, for all i, and

$$\mathcal{X}_1(G) = \bigcap_{T \in \mathcal{P}_1(G)} T = \Phi(G),$$

as $\mathcal{P}_1(G)$ is exactly the set of maximal subgroups of G.
(ii) To see this observe that for every $j \in [k]$ we have

$$\mathcal{P}_k(G) = \bigcup_{T \in \mathcal{P}_{k-j}(G)} \mathcal{P}_j(T),$$

due to the fact that for every subgroup S of index p^k in a p-group G there exists a subgroup $T \leq G$ of index p^{k-j} that contains S.
(iii) Now this part follows from (ii) as

$$\mathcal{X}_{i+1}(G) = \bigcap_{T \in \mathcal{P}_i(G)} \mathcal{X}_1(T) \leq \bigcap_{T \in \mathcal{P}_i(G)} T = \mathcal{X}_i(G).$$

(iv) We induce on i. For $i = 1$ we clearly have $\Phi(G) \leq M$ for every maximal subgroup M of G. Assuming it holds for i, we will prove it for $i+1$. Let $T \in \mathcal{P}_{i+1}(G)$ and pick $K \in \mathcal{P}_i(G)$ with $T \leq K$. So T is a maximal subgroup of K and $\Phi(K) \leq T$. Furthermore, the inductive hypothesis implies that $\Phi_i(G) \leq K$ and therefore

$$\Phi_{i+1}(G) = \Phi(\Phi_i(G)) \leq \Phi(K) \leq T.$$

(v) The proof of this part follows directly from (iv) and the definition of $\mathcal{X}_i(G)$.
(vi) Since $N \leq \mathcal{X}_i(G)$, it follows from the correspondence theorem that

$$\mathcal{P}_i(G/N) = \{H/N : H \in \mathcal{P}_i(G)\};$$

this clearly proves the claim, and the proof of the lemma is complete. □

Remark 2.3. Firstly, observe that equality really is possible in (iii), since in Q_8, for example, we have $\mathcal{X}_1(Q_8) = \mathcal{X}_2(Q_8)$. Secondly, we comment on the behaviour of the $\mathcal{X}$-series under quotients. Assuming that G is a p-group, as usual, and that N is an arbitrary normal subgroup of G, we have $\mathcal{X}_1(G/N) = \mathcal{X}_1(G)N/N$ since $\mathcal{X}_1(G) = \Phi(G) = G'\mho(G)$ and each of G', $\mho(G)$ behaves well under quotients. However, things change for larger-index terms of the $\mathcal{X}$-series. It is not true, for instance, that $\mathcal{X}_2(G/N) = \mathcal{X}_2(G)N/N$ for arbitrary G, N. An example where this fails to be true is the group of order 16 with presentation

$$G = C_4 \rtimes C_4 = \left\langle a, b \,\middle|\, a^4 = b^4 = 1, a^b = a^{-1} \right\rangle.$$

The subgroup $N = \langle a^2b^2 \rangle$ is central of order 2 and affords the quotient $G/N \cong Q_8$. To see this, observe that G/N is non-abelian (since, for example, $abN \neq baN$) and $a^2N = b^2N$ is the unique element of order 2 in G/N. Now $\mathcal{X}_2(G)$ is trivial, as $\langle a \rangle \cap \langle b \rangle = 1$, and thus $\mathcal{X}_2(G)N/N$ is trivial, but $\mathcal{X}_2(G/N)$ is not.

Next, we prove a useful property of the $\mathcal{X}$-series.

Proposition 2.4. *Let G be a p-group and let $M \leq G$. Then $\mathcal{X}_i(M) \leq \mathcal{X}_i(G)$.*

Proof. Fix a subgroup $M \leq G$ and an index $i \geq 0$. To prove the desired containment, it suffices to show that $\mathcal{X}_i(M) \leq K$ for every $K \in \mathcal{P}_i(G)$. The equality $|MK|/|K| = |M|/|M \cap K|$ and the fact that K has index p^i in G together imply that the index of $K \cap M$ in M is at most p^i. Thus there exists a subgroup T of $K \cap M$ whose index in M is exactly p^i. Therefore

$$\mathcal{X}_i(M) \leq T \leq K \cap M \leq K.$$

As this holds for every $K \in \mathcal{P}_i(G)$, the proposition follows.

We will refer to this property of the $\mathcal{X}$-series as the subgroup-monotonicity of $\mathcal{X}$. Moreover, the same will be said for a subgroup series that satisfies the conclusion of Proposition 2.4 (i.e. that it is subgroup-monotone).

Lemma 2.5. *Let G be a p-group. Then*

$$\mathcal{X}_i(\mathcal{X}_j(G)) \leq \mathcal{X}_{i+j}(G).$$

In addition, $\mathcal{X}_i(G)/\mathcal{X}_{i+1}(G)$ is an elementary abelian p-group for every non-negative integer i.

Proof. In view of Lemma 2.2 we have $\mathcal{X}_{i+j}(G) = \bigcap_{T \in \mathcal{P}_j(G)} \mathcal{X}_i(T)$. Let

$$K := \bigcap_{T \in \mathcal{P}_j(G)} T = \mathcal{X}_j(G).$$

Then $\mathcal{X}_i(K) \leq \mathcal{X}_i(T)$ for every $T \in \mathcal{P}_j(G)$ by Proposition 2.4. Hence

$$\mathcal{X}_{i+j}(G) = \bigcap_{T \in \mathcal{P}_j(G)} \mathcal{X}_i(T) \geq \mathcal{X}_i(K) = \mathcal{X}_i(\mathcal{X}_j(G)).$$

Thus, the first part of the lemma holds, from which we deduce that $\mathcal{X}_i(G)/\mathcal{X}_{i+1}(G)$ is an elementary abelian group seeing as

$$\Phi(\mathcal{X}_i(G)) = \mathcal{X}_1(\mathcal{X}_i(G)) \leq \mathcal{X}_{i+1}(G).$$

This completes the induction and the proof of the lemma. □

We now look at the situation where some normal subgroup N of G is complemented in G. Theorem 1.4 is essentially the second part of the following:

Theorem 2.6. *Let G be a p-group and let N be a normal subgroup of G. Suppose that N is complemented, by H say, in G. Then for all $i \geq 0$ we have:*

(i) $\mathcal{X}_i(N)\,\mathcal{X}_i(H) \leq \mathcal{X}_i(G) \leq N\,\mathcal{X}_i(H)$;

(ii) $\mathcal{X}_i(H) = \mathcal{X}_i(G) \cap H$.

Proof. (i) The leftmost inclusion is a consequence of the fact that the product is well-defined (since $\mathcal{X}_i(N)$ is characteristic in N and the latter is normal in G) and the subgroup-monotonicity of the $\mathcal{X}$-series.

As regards the right-hand-side inclusion, assume first that $i \leq \log_p(|H|)$. Observe that for all $S, T \leq H$, Dedekind's lemma yields

$$NS \cap NT = N(NS \cap T) \tag{2.1}$$

and

$$NS \cap T = NS \cap H \cap T = S(N \cap H) \cap T = S \cap T. \tag{2.2}$$

By (2.1) and (2.2) we get $NS \cap NT = N(S \cap T)$ and repeatedly applying this fact gives us

$$\bigcap_{T \in \mathcal{P}_i(H)} NT = N \bigcap_{T \in \mathcal{P}_i(H)} T = N\,\mathcal{X}_i(H).$$

But if $T \in \mathcal{P}_i(H)$ then $NT \in \mathcal{P}_i(G)$, thus

$$\mathcal{X}_i(G) = \bigcap_{S \in \mathcal{P}_i(G)} S \leq \bigcap_{T \in \mathcal{P}_i(H)} NT = N\,\mathcal{X}_i(H).$$

In particular, for $i_0 = \log_p(|H|)$ we get $\mathcal{X}_{i_0}(G) \leq N\,\mathcal{X}_{i_0}(H) = N$. Hence for $i \geq i_0$ we see that $\mathcal{X}_i(G) \leq \mathcal{X}_{i_0}(G) \leq N$. Therefore the inclusion $\mathcal{X}_i(G) \leq N\,\mathcal{X}_i(H)$ is valid for all non-negative integers i, completing the proof.
(ii) We have that

$$\mathcal{X}_i(H) \leq H \cap \mathcal{X}_i(G) \leq H \cap N\,\mathcal{X}_i(H) = \mathcal{X}_i(H)(N \cap H) = \mathcal{X}_i(H),$$

where the first inclusion follows from the subgroup-monotonicity of the $\mathcal{X}$-series, the second inclusion from part (i) and the penultimate equality by Dedekind's lemma. This double inclusion forces $\mathcal{X}_i(H) = \mathcal{X}_i(G) \cap H$, as desired. □

Corollary 2.7. *Let G be a p-group and let N be a normal subgroup of G that is complemented in G by H, say. Then $N \cap \mathcal{X}_i(G)$ has a complement in $\mathcal{X}_i(G)$ for all $i \geq 0$ and, in fact, $\mathcal{X}_i(H)$ is one such complement.*

Proof. Fix an index $i \geq 0$. We show that $\mathcal{X}_i(H)$ is a complement for $N \cap \mathcal{X}_i(G)$ in $\mathcal{X}_i(G)$ by observing the following:

$$(N \cap \mathcal{X}_i(G))\,\mathcal{X}_i(H) = \mathcal{X}_i(G) \cap N\,\mathcal{X}_i(H) = \mathcal{X}_i(G).$$

The first equality is yet another application of Dedekind's lemma and the second equality follows directly from Theorem 2.6 (i). The proof is complete.

□

Theorem 2.6 will also aid us in the proof of the next result.

Theorem 2.8. *Given p-groups G and H, we have $\mathcal{X}_i(G \times H) = \mathcal{X}_i(G) \times \mathcal{X}_i(H)$ for all non-negative integers i.*

Proof. Fix an index i. Proposition 2.4 clearly implies that $\mathcal{X}_i(G) \times 1 = \mathcal{X}_i(G \times 1) \leq \mathcal{X}_i(G \times H)$ and similarly for H. Hence

$$\mathcal{X}_i(G) \times \mathcal{X}_i(H) \leq \mathcal{X}_i(G \times H).$$

For the other direction, note that by Theorem 2.6 we have

$$\mathcal{X}_i(G \times H) \leq G\,\mathcal{X}_i(H) \tag{2.3}$$

and

$$\mathcal{X}_i(G \times H) \leq \mathcal{X}_i(G)H. \tag{2.4}$$

Combining (2.3) and (2.4) gives us

$$\begin{aligned}\mathcal{X}_i(G \times H) \leq G\,\mathcal{X}_i(H) \cap \mathcal{X}_i(G)H &= \mathcal{X}_i(G)\,(G\,\mathcal{X}_i(H) \cap H)\\ &= \mathcal{X}_i(G)\,[\mathcal{X}_i(H)(G \cap H)] = \mathcal{X}_i(G) \times \mathcal{X}_i(H),\end{aligned}$$

where all equalities are applications of Dedekind's lemma. Since both inclusions are valid, the proof of the theorem is complete. □

Proposition 2.9. *Assume that G is an abelian p-group. Then $\mathcal{X}_i(G) = \Phi_i(G) = \mho_i(G)$ and $\mathcal{X}_j(\mathcal{X}_i(G)) = \mathcal{X}_{i+j}(G)$, for all integers $i, j \geq 1$.*

Proof. Assume first that G is a cyclic group. Then $\mathcal{P}_i(G) = \mho_i(G)$ as $\mho_i(G)$ is the unique subgroup of G of index p^i in G and thus $\mathcal{X}_i(G) = \mho_i(G) = \Phi_i(G)$.

In the general case, let $G = C_1 \times C_2 \times \cdots \times C_t$, where C_j are cyclic subgroups of G. Then in view of Theorem 2.8 we get $\mathcal{X}_i(G) = \prod_{j=1}^{t} \mathcal{X}_i(C_j)$. But for any cyclic subgroup C we have $\mathcal{X}_i(C) = \mho_i(C) = \Phi_i(C)$. Hence

$$\mathcal{X}_i(G) = \prod_{j=1}^{t} \mathcal{X}_i(C_j) = \prod_{j=1}^{t} \mho_i(C_j) = \prod_{j=1}^{t} \Phi_i(C_j) = \Phi_i(G) = \mho_i(G),$$

where the penultimate equality follows from the fact that the Frattini subgroup respects direct products and an easy induction argument (see [4, Satz 6]). The last part of the proposition follows directly from the corresponding relations that the agemo subgroups of an abelian group[1] satisfy:

$$\mho_i(\mho_j(G)) = \mho_{i+j}(G).$$

The proof of the proposition is now complete. □

Proposition 2.10. *Let C be a cyclic subgroup of G of order p^n. Then the following are equivalent:*

(i) $C \cap \mathcal{X}_n(G) = 1$;
(ii) C *has a complement in* G;
(iii) $C \cap \mathcal{X}_i(G) = \mathcal{X}_i(C)$ *for every non-negative integer* i.

Proof. (i) → (ii) We assume that $C \cap \mathcal{X}_n(G) = 1$ and wish to establish the existence of a complement for C in G. Indeed, if $C \cap T > 1$ for all subgroups T of index p^n in G, then $C \cap T \geq \Omega_1(C) > 1$, as $\Omega_1(C)$ is the unique subgroup of order p in C. Therefore $C \cap \mathcal{X}_n(G) \geq \Omega_1(C) > 1$, and this is clearly a contradiction. It follows that there exists at least one subgroup T of index p^n in G whose intersection with C is trivial. That subgroup T serves as the desired complement.
(ii) → (iii) Let T be, as in the previous paragraph, a complement for C in G. As C is a cyclic group, we see that $\mathcal{X}_i(C) = \mho_i(C)$ is the unique subgroup of order p^{n-i} in C. Now we pick a subgroup series of G

$$T = K_n < K_{n-1} < \ldots < K_1 < G = K_0$$

with $|K_i/K_{i+1}| = p$ and we claim that $K_i = T\,\mathcal{X}_i(C)$ and $C \cap K_i = \mathcal{X}_i(C)$. We clearly have $K_i = T(C \cap K_i)$, while

$$|C \cap K_i| = \frac{|K_i|}{|T|} = \frac{|G|}{p^i|T|} = p^{n-i}.$$

Since $\mathcal{X}_i(C)$ is the unique subgroup of C of order p^{n-i}, we deduce that $C \cap K_i = \mathcal{X}_i(C)$, and the claim follows.

[1] In fact, this relation among the agemo subgroups is satisfied more generally by regular p-groups; see Lemma 1.2.12 (ii) in [7], for example.

Now, K_i has index p^i in G and therefore $\mathcal{X}_i(G) \leq K_i$. We conclude that

$$C \cap \mathcal{X}_i(G) \leq C \cap K_i = \mathcal{X}_i(C) \leq C \cap \mathcal{X}_i(G),$$

and thus $\mathcal{X}_i(G) \cap C = \mathcal{X}_i(C)$, as desired.
(iii) $\rightarrow$ (i) This follows easily from the fact that $C \cap \mathcal{X}_i(G) = \mathcal{X}_i(C)$ for all $i \geq 0$ upon taking $i = n$ and observing that $\mathcal{X}_n(C) = 1$. $\square$

Our next result outlines which members of the lower central series of G are guaranteed to be contained in some term of the $\mathcal{X}$-series of G. As we will see, our result is essentially best possible.

Proposition 2.11. *If G is a p-group and i is a positive integer then*

$$\gamma_{p^{i-1}+1}(G) \leq \mathcal{X}_i(G)$$

and the index $p^{i-1}+1$ is optimal.

Proof. Let H be an arbitrary subgroup of G of index p^i in G. The regular action of G on the left cosets of H in G gives rise to a homomorphism $\phi : G \rightarrow S_{p^i}$ whose kernel N is the core of H in G; that is

$$N = \bigcap_{g \in G} H^g \leq H.$$

Let L_i be a Sylow p-subgroup of S_{p^i}. Then G/N is isomorphic to a subgroup of L_i. Now L_i is an iterated wreath product

$$\underbrace{C_p \wr \ldots \wr C_p}_{i \text{ terms}}$$

having order $p^{\frac{p^i-1}{p-1}}$ and class $c = p^{i-1}$ (cf. [5, Satz 15.3]). Hence $\gamma_{p^{i-1}+1}(L_i) = 1$, so $\gamma_{p^{i-1}+1}(G/N) = 1$. But the lower central series behaves well with respect to quotients and therefore

$$1 = \gamma_{p^{i-1}+1}(G/N) = \gamma_{p^{i-1}+1}(G)N/N.$$

We conclude that $\gamma_{p^{i-1}+1}(G) \leq N \leq H$ and as H was arbitrary we see that

$$\gamma_{p^{i-1}+1}(G) \leq \bigcap_{H \in \mathcal{P}_i(G)} H = \mathcal{X}_i(G).$$

To see that the index $p^{i-1}+1$ is optimal, observe that $\gamma_{p^{i-1}+1}(L_i) = 1$ and the index $j = p^{i-1}+1$ is the least positive integer such that $\gamma_j(L_i) = 1$ (since L_i has class p^{i-1}). In proof of the second claim of the proposition therefore, it will be sufficient to establish that $\mathcal{X}_i(L_i) = 1$ for all positive integers i. We argue by induction on i, noting that the base case $i = 1$ is clearly valid since L_1 has prime order p and thus its Frattini subgroup is trivial. Assume validity of the claim for $i-1$. Note that $L_i = L_{i-1} \wr C_p$ and that the base of L_i

$$\underbrace{B = L_{i-1} \times \ldots \times L_{i-1}}_{p \text{ factors}}$$

is a maximal subgroup of L_i.

On the other hand, we have by Lemma 2.2 (ii) (with $k = i$ and $j = i-1$)

$$\mathcal{X}_i(L_i) = \bigcap_{M \in \mathcal{P}_1(L_i)} \mathcal{X}_{i-1}(M) \leq \mathcal{X}_{i-1}(B).$$

But

$$\mathcal{X}_{i-1}(B) = \mathcal{X}_{i-1}(L_{i-1}) \times \ldots \times \mathcal{X}_{i-1}(L_{i-1}) = 1,$$

where the first equality follows from Theorem 2.8 while the second equality is valid by the inductive hypothesis. Thus, $\mathcal{X}_i(L_i) = 1$, completing the induction and the proof. □

Corollary 2.12. *Let G be a 2-group. Then $[G, \Phi(G)] \leq \mathcal{X}_2(G)$.*

Proof. For 2-groups we have $\Phi(G) = \mho(G)$ and thus it suffices to prove that $[G, \mho(G)] \leq \mathcal{X}_2(G)$. Observe that $[G, \mho(G)] \leq \mho([G,G])[G,G,G]$, as for every $x, y \in G$ a direct calculation yields

$$[x, y^2] = [x,y][x,y]^y = [x,y]^2[x,y,y].$$

Furthermore, $\gamma_3(G) \leq \mathcal{X}_2(G)$ according to Proposition 2.11 with $p = i = 2$. Therefore,

$$[G, \mho(G)] \leq \mho([G,G])[G,G,G] = \mho(G')\gamma_3(G) \leq \mho(G')\,\mathcal{X}_2(G).$$

In addition,

$$\mho(G') \leq \mho(\Phi(G)) \leq \Phi(\Phi(G)) = \Phi_2(G) \leq \mathcal{X}_2(G),$$

where the first containment is justified by the subgroup-monotonicity of the agemo subgroup (cf. Lemma 1.2.7 in [7]) and the last by Lemma 2.2 (v). Hence the claim has been proved. □

The content of Corollary 2.12 is no longer valid if $p > 2$. Indeed, for any odd prime p the Sylow p-subgroup $C_p \wr C_p$ of S_{p^2} satisfies $\mathcal{X}_2(C_p \wr C_p) = 1$, since it has a maximal subgroup which is elementary abelian (in particular, its base). But $C_p \wr C_p$ has class $p \geq 3$ which implies that $G/Z(G)$ is non-abelian and thus $\Phi(G) \not\leq Z(G)$.

3. Complements and Abelian Subgroups

For the convenience of the reader we present here an independent proof of Theorem 1.2 stated in the introduction. Note that no originality is claimed here. The result itself should be well-known to abelian group theorists (and, in fact, holds more generally than we have stated), but perhaps not so much to group theorists in general.

Proof of Theorem 1.2. Suppose first that A is complemented in G and let C be a complement. Then G is the (internal) direct product $G = A \times C$. Thus, for each $g \in G$ there exist unique $a \in A$ and $c \in C$ such that $g = ac$. Therefore, if $g^n \in A$ for some $n \in \mathbb{N}$ then $a^n c^n \in A$ thus $c^n \in A \cap C = 1$. So $g^n = a^n$ and thus A is pure in G.

For the converse, suppose that A is pure in G and that $A \le C \le G$, where C/A is cyclic. Let $n = |C : A|$ and write $\langle cA \rangle = C/A$. As $c^n \in A$, there exists $a \in A$ such that $c^n = a^n$. Setting $g := ca^{-1}$, we have $g^n = 1$. Since $gA = cA$, we see that $n = o(gA)$ in C/A. It follows that $C = A \times \langle g \rangle$ and thus A is a direct factor of C.

Now for the general case, write G/A as a direct product of cyclic groups C_i/A. Let X_i be a complement for A in C_i and note that $|X_i| = |C_i : A|$. Write $D = \prod_i X_i$.

We have $AD \ge AX_i = C_i$ for all i, and thus $AD = \prod_i C_i = G$. Also,

$$|D| \le \prod_i |X_i| = \prod_i |C_i : A| = |G/A|,$$

and thus $G = A \times D$. Our proof is complete. □

We are now ready to prove our main theorem.

Proof of Theorem 1.3. First notice that if A is cyclic, then the equivalence between (ii) and (iii) in Proposition 2.10 ensures that A is complemented in G.

Now we argue by induction on $|A|$, noting that we have addressed the base case of the induction in the previous paragraph. We may also assume that A is not cyclic. So let C be a cyclic subgroup of maximal order in A and note that $|C| = \exp(A) = p^r$, say. By assumption, $A \cap \mathcal{X}_r(G) = \mathcal{X}_r(A) = 1$. Then $C \cap \mathcal{X}_r(G) \le A \cap \mathcal{X}_r(G) = 1$, thus $C \cap \mathcal{X}_r(G) = 1$ and the equivalence between (i) and (ii) in Proposition 2.10 suffices to guarantee that C has a complement, say H, in G.

By Dedekind's lemma, $A = C \times B$, where $B := A \cap H$. Observe that since B is complemented by C in A and taking into account Theorem 1.4, we have $B \cap \mathcal{X}_i(A) = \mathcal{X}_i(B)$ for all $i \le t$, where $p^t = \exp(B)$.

Since the $\mathcal{X}$-series is subgroup-monotone, we have for each $i \le t$ the following:

$$B \cap \mathcal{X}_i(H) \le B \cap \mathcal{X}_i(G) = B \cap (A \cap \mathcal{X}_i(G)) = B \cap \mathcal{X}_i(A) = \mathcal{X}_i(B).$$

Also, again by subgroup-monotonicity we have $\mathcal{X}_i(B) \le B$ and $\mathcal{X}_i(B) \le \mathcal{X}_i(H)$ for all appropriate indices i. Thus, $\mathcal{X}_i(B) \le B \cap \mathcal{X}_i(H)$ and since the reverse containment also holds we have $\mathcal{X}_i(B) = B \cap \mathcal{X}_i(H)$. The induction hypothesis applied to B relative to the subgroup H now ensures that there exists a subgroup D in H which complements B. Then

$$A \cap D = (A \cap H) \cap D = B \cap D = 1.$$

Moreover,

$$|A| = |C||B| = |G : H||H : D| = |G : D|.$$

We conclude that D complements A in G thus our induction is complete. □

We follow up with a few remarks on the content and proof of Theorem 1.3.

Remark 3.1. It should be noted that Theorem 1.3 works for a class of p-groups which is strictly larger than the class of abelian p-groups. For A to have a complement in G it is sufficient, as the proof of the theorem indicates, that A have the following property:

for every $H \leq A$ there exists a normal (in H) cyclic subgroup $1 < C \trianglelefteq H$ that admits a complement in H,

For example, the property above is enjoyed by the dihedral 2-groups D_{2^n} which are non-abelian for $n \geq 3$.

On the other hand, Theorem 1.3 is not true unconditionally; some assumption about A *must* be made. Take $H = C_4 \times \mathcal{G}$, where $\mathcal{G}$ is as in Remark 2.3. Then there exists a maximal subgroup M of H which satisfies the condition of Theorem 1.3, i.e. $\Phi(M) = \Phi(H)$, yet M has no complement in H since $\Omega_1(H) = \Phi(H)$. We have used [9] to find this group.

Remark 3.2. On the surface, it looks as though a literal interpretation of purity might be enough to guarantee the existence of a complement in Theorem 1.3. That is to say, under the assumptions of the theorem, perhaps it is true that an abelian subgroup A of G satisfying $A \cap \mho_i(G) = \mho_i(A)$ for all $i \geq 0$ is necessarily complemented in G. However, this is not so; purity *really must* be stated in an equivalent form. To see why, consider an odd prime p and let G be a non-abelian p-group of exponent p. Then it is easy to see that the equality $A \cap \mho_i(G) = \mho_i(A)$ is valid for every (abelian) subgroup of G, but clearly not every abelian subgroup can be complemented; simply consider a subgroup of $\mathcal{X}_1(G) = \Phi(G)$ of order p.

Corollary 3.3. *Let G be a p-group and let A be a central subgroup of G. Then A has a complement in G if and only if $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$.*

Proof. Since A is central in G, it is abelian. Thus Theorem 1.3 implies that A is complemented in G. Conversely, suppose that A has a complement in G, say H. Then H is normalised by both A and H, thus $G = A \times H$ and A is, in fact, normally complemented in G. Now Theorem 1.4 implies that $\mathcal{X}_i(A) = A \cap \mathcal{X}_i(G)$, as desired.

In fact, looking carefully at the proof of Theorem 1.3 and extracting the abstract properties of the $\mathcal{X}$-series used in the proof allows us to state a generalisation of Theorem 1.3. We need a definition first.

Definition 3.4. Given a p-group G we call the series

$$\mathcal{W}_i(G) = \mathcal{X}_i(G) \cap \mathcal{B}_i(G),$$

a **good series** for G if the following four conditions are satisfied:

(i) $\{\mathcal{B}_i(G)\}$ is a descending normal series of G;
(ii) $\mathcal{B}_i(G)$ (and thus $\mathcal{W}_i(G)$ as well) is subgroup-monotone;
(iii) $\mathcal{B}_i(G)$ (and thus $\mathcal{W}_i(G)$ as well) respects direct products;
(iv) If G is a cyclic group, then $\Omega_1(G) \leq \mathcal{B}_1(G)$ (and thus $\Omega_1(G) \leq \mathcal{W}_1(G)$ as well).

Clearly, $\mathcal{X}_i(G)$ itself is a good series for G. Other examples of good series are the following:

(1) $\mathcal{W}_i(G) = \mathcal{X}_i(G) \cap \Omega_1(G)$;
(2) $\mathcal{W}_i(G) = \mathcal{X}_i(G) \cap \mho_{i-1}(G)$;
(3) $\mathcal{W}_i(G) = \mathcal{X}_i(G) \cap \mho_{i-1}(G) \cap \Omega_1(G)$.

The generalisation asserted earlier now reads as follows.

Theorem 3.5. *Let A be an abelian subgroup of the p-group G and suppose that $\mathcal{W}_i$ is a good series for G so that*

$$A \cap \mathcal{W}_i(G) = \mathcal{W}_i(A)$$

for all integers $i \geq 1$. Then A has a complement in G.

Since the proof is nearly identical to that of Theorem 1.3, we have elected to omit it.

We remark that the following result, although resembling Corollary 2.7, is ultimately independent, since no normality assumption is made.

Proposition 3.6. *Let G be a p-group and A an abelian subgroup of G such that*

$$A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$$

for all non-negative integers i. Then $A \cap \mathcal{X}_i(G)$ has a complement in $\mathcal{X}_i(G)$ for all $i \geq 0$.

Proof. By Theorem 1.3 we have already established the case $i = 0$, as $\mathcal{X}_0(G) = G$ and A has a complement in G. According to the same theorem, the abelian group $A \cap \mathcal{X}_i(G)$ has a complement in $\mathcal{X}_i(G)$ provided we can show that

$$(A \cap \mathcal{X}_i(G)) \cap \mathcal{X}_t(\mathcal{X}_i(G)) = \mathcal{X}_t(A \cap \mathcal{X}_i(G)) \tag{3.1}$$

for every integer $t \geq 1$.

One inclusion is straightforward, as the $\mathcal{X}$-series is subgroup-monotone and therefore

$$\mathcal{X}_t(A \cap \mathcal{X}_i(G)) \leq (A \cap \mathcal{X}_i(G)) \cap \mathcal{X}_t(\mathcal{X}_i(G)).$$

For the other inclusion, we first observe that

$$\mathcal{X}_t(A \cap \mathcal{X}_i(G)) = \mathcal{X}_t(\mathcal{X}_i(A)) = \mathcal{X}_{t+i}(A),$$

where the last equality follows from Proposition 2.9. Hence

$$(A \cap \mathcal{X}_i(G)) \cap \mathcal{X}_t(\mathcal{X}_i(G)) = A \cap \mathcal{X}_t(\mathcal{X}_i(G)) \leq A \cap \mathcal{X}_{t+i}(G)$$
$$= \mathcal{X}_{t+i}(A) = \mathcal{X}_t(A \cap \mathcal{X}_i(G)).$$

So we have shown that equation (3.1) is valid for every $t \geq 1$, and the proposition follows. □

Corollary 3.7. *Let G be a p-group and let A be a normal abelian subgroup of G with the property*

$$A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$$

for all non-negative integers i. Then A has a complement in G and for every complement H the analogous property

$$H \cap \mathcal{X}_i(G) = \mathcal{X}_i(H)$$

holds for all $i \geq 0$. As a consequence $\mathcal{X}_i(G) = \mathcal{X}_i(A) \rtimes \mathcal{X}_i(H)$ for all $i \geq 0$.

Proof. Since $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$, Theorem 1.3 implies that A is complemented in G. If H is any such complement for A, then the fact that $H \cap \mathcal{X}_i(G) = \mathcal{X}_i(H)$ holds for all $i \geq 0$ is a consequence of Theorem 1.4. Moreover, Corollary 2.7 implies that $A \cap \mathcal{X}_i(G)$, which by assumption is $\mathcal{X}_i(A)$, is complemented in $\mathcal{X}_i(G)$ by $\mathcal{X}_i(H)$ and this holds, of course, for all $i \geq 0$. □

We conclude our work by pointing out the limitations of Theorem 1.3.

First of all, by virtue of Proposition 2.10 it is the case that if A is cyclic then it is complemented in G if and only if $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$. Moreover, the condition works as "if and only if" in case A is central in G owing to Corollary 3.3.

On the other hand, observe that an abelian subgroup A of a p-group G can be complemented in G without satisfying the condition $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$. Concrete examples abound; one possibility is to take A to be an elementary abelian group containing the Frattini subgroup. If A is complemented in G (and this happens for many groups, e.g. D_8 with A either of the two non-cyclic maximal subgroups), where G is non-abelian, then it cannot possibly be the case that $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$. Things go awry as quickly as possible, i.e. for $i = 1$.

In fact, an elementary abelian subgroup A satisfies $A \cap \mathcal{X}_i(G) = \mathcal{X}_i(A)$ for all $i \geq 0$ if and only if $A \cap \Phi(G) = 1$, thus Theorem 1.3 is only useful when A avoids the Frattini subgroup of G. However, in that case Theorem 1.3 tells us nothing new since Satz 7 in Gaschütz's paper [4] guarantees that A has a complement in G regardless of whether G is a p-group or not, as long as G is finite.

What is perhaps not so obvious is the following.

Proposition 3.8. *Let G be a p-group and let A be a normal subgroup of G such that $A \cap \Phi(G) = 1$. Then A has a normal complement in G and is thus a direct factor of G.*

Proof. First of all, observe that we do not need to assume that A is abelian since it follows from

$$\Phi(A) \leq A \cap \Phi(G) = 1$$

that A must, in fact, be elementary abelian.

Now observe that $G/\Phi(G)$ is elementary abelian thus every subgroup of this quotient is complemented. Then there exists a (necessarily normal) subgroup B such that

$$G/\Phi(G) = A\Phi(G)/\Phi(G) \times B/\Phi(G)$$

and thus $G = A\Phi(G)B = AB$. Moreover

$$A \cap B \leq A \cap A\Phi(G) \cap B = A \cap \Phi(G) = 1,$$

competing the proof. □

It is clear that the same proof would work just as well if we had assumed more generally that G is a nilpotent group instead of a p-group. We preferred to state the proposition in a more restricted setting to remain in the geist of the rest of our paper.

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