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Results in Mathematics

Integral Transforms Characterized by Convolution

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Abstract. Inspired by Jaming's characterization of the Fourier transform on specific groups via the convolution property, we provide a novel approach which characterizes the Fourier transform on any locally compact abelian group. In particular, our characterization encompasses Jaming's results. Furthermore, we demonstrate that the cosine transform as well as the Laplace transform can also be characterized via a suitable convolution property.

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1. Introduction

It is well-known that the Fourier transform satisfies the convolution property. More precisely, let G be a locally compact abelian group with a Haar measure μ and a dual group $\widehat{G}$. The Fourier transform is a map $\mathcal{F} : L^1(G) \longrightarrow C_0(\widehat{G})$ given by the formula

$$\mathcal{F}(f)(\chi) := \int_G f(x)\overline{\chi(x)}\ d\mu(x), \tag{1}$$

whereas the Fourier convolution $\star_{\mathcal{F}} : L^1(G) \times L^1(G) \longrightarrow L^1(G)$ is given by

$$f \star_{\mathcal{F}} g(x) := \int_G f(u)g(x-u)\ d\mu(u). \tag{2}$$

For the sake of convenience, from this point onwards we will write dx and du instead of "$d\mu(x)$" and "$d\mu(u)$", respectively. The following Fourier convolution property (see Lemma 1.7.2 in [3], p. 30) holds for every $f, g \in L^1(G)$:

$$\mathcal{F}(f \star_{\mathcal{F}} g) = \mathcal{F}(f)\mathcal{F}(g).$$

Jaming (partly inspired by the works of Alesker, Artstein-Avidan, Faifman and Milman on product preserving maps, see [1]) proved that the convolution property characterizes (to a certain degree) the Fourier transform if $G = \mathbb{R}, S^1, \mathbb{Z}$ or $\mathbb{Z}_n$ (see [5]). His paper inspired Lavanya and Thangavelu to show that any continuous *-homomorphism of $L^1(\mathbb{C}^d)$ (with twisted convolution as multiplication) into $B(L^2(\mathbb{R}^d))$ is essentially a Weyl transform and deduce a similar characterization for the group Fourier transform on the Heisenberg group (see [8,9]). Furthermore, Kumar and Sivananthan went on to demonstrate that the convolution property characterizes the Fourier transform on compact groups (see [7]).

Studying the topic we realized two things that prompted us to write this paper. Firstly, Jaming's characterization of the Fourier transform need not be restricted to particular cases $G = \mathbb{R}, S^1, \mathbb{Z}$ or $\mathbb{Z}_n$. There is a unified approach for all locally compact abelian groups, which we demonstrate in the first part of Sect. 2.

Secondly, we discovered that the Fourier transform is not the only integral transform that can be characterized via the convolution property. In the second part of Sect. 2 and in Sect. 3 we explain that cosine and Laplace convolutions characterize cosine and Laplace transforms, respectively.

2. Fourier and Cosine Transform

As we have agreed in the Introduction, let G stand for a locally compact abelian group with Haar measure μ and dual group $\widehat{G}$. The formula for the Fourier transform and Fourier convolution are given by (1) and (2), respectively. We start off with a well-known fact:

Lemma 1. *For every function $g \in L^1(G)$ and $x, y \in G$ the following equality holds*

$$L_x g \star_{\mathcal{F}} L_y g = g \star_{\mathcal{F}} L_{x+y} g,$$

where for every $z \in G$ the operator $L_z : L^1(G) \longrightarrow L^1(G)$ is given by

$$L_z f(u) := f(u - z).$$

The following theorem generalizes Theorems 2.1 and 3.1 in Jaming's paper (see [5]).

Theorem 1. *Let $T : L^1(G) \longrightarrow C^b(\widehat{G})$ be a linear and bounded operator. If it satisfies the Fourier convolution property*

$$T(f \star_{\mathcal{F}} g) = T(f)T(g)$$

for every $f, g \in L^1(G)$*, then there exists an open set* $E \subset \widehat{G}$ *and a continuous function* $\theta_{\mathcal{F}} : E \longrightarrow \widehat{G}$ *such that*

$$T(f)(\phi) = \begin{cases} \mathcal{F}(f) \circ \theta_{\mathcal{F}}(\phi) & \text{if } \phi \in E, \\ 0 & \text{if } \phi \in \widehat{G} \backslash E. \end{cases} \tag{3}$$

Furthermore, if we assume that T *satisfies the additional property*

$$T(L_x f)(\phi) = \overline{\phi(x)} \cdot T(f)(\phi) \tag{4}$$

for every $f \in L^1(G)$, $\phi \in \widehat{G}$ *and* $x \in G$*, then* $\theta_{\mathcal{F}}(\phi) = \overline{\phi}$*.*

Proof. We begin by defining

$$D := \bigcap_{f \in L^1(G)} T(f)^{-1}(\{0\}) = \bigcap_{f \in L^1(G)} \left\{ \phi \in \widehat{G} \;\; T(f)(\phi) = 0 \right\},$$

which is obviously a closed set. Next, we define

$$E := \widehat{G} \backslash D = \left\{ \phi \in \widehat{G} \;\; \exists_{f \in L^1(G)} \; T(f)(\phi) \neq 0 \right\},$$

which is open. If $\phi \in D = \widehat{G} \backslash E$ then the equality (3) is obvious. Hence, we choose $\phi \in E$ and consider a nonzero, linear functional $T_\phi : L^1(G) \longrightarrow \mathbb{C}$ given by $T_\phi(f) := T(f)(\phi)$. We pick $g_* \in L^1(G)$ such that $T_\phi(g_*) = 1$ and define a function $\chi_\phi : G \longrightarrow \mathbb{C}$ by the formula

$$\chi_\phi(x) := \overline{T_\phi(L_x g_*)}.$$

Let us remark, that the choice of g_* need not be unique, so there might be many functions χ_ϕ corresponding to ϕ.

We will now focus on proving various properties of the function χ_ϕ. To begin with, we observe that for every $x, y \in G$ we have

$$\begin{aligned} T_\phi(L_x g_*) T_\phi(L_y g_*) &= T_\phi(L_x g_* \star_{\mathcal{F}} L_y g_*) \overset{\text{Lemma 1}}{=} T_\phi(g_* \star_{\mathcal{F}} L_{x+y} g_*) \\ &= T_\phi(g_*) T_\phi(L_{x+y} g_*) = T_\phi(L_{x+y} g_*). \end{aligned}$$

Taking the complex conjugate reveals the equation

$$\chi_\phi(x)\chi_\phi(y) = \chi_\phi(x+y),$$

which holds for every $x, y \in G$. Furthermore, since T_ϕ is a continuous linear functional, then Lemma 1.4.2 in [3], p. 18 implies that χ_ϕ is continuous. It is also nonzero (as $\chi_\phi(0) = 1$) and bounded, since for every $x \in G$ we have

$$|\chi_\phi(x)| \leqslant |T_\phi(L_x g_*)| \leqslant \|T_\phi\| \cdot \|L_x g_*\| = \|T_\phi\| \cdot \|g_*\|. \tag{5}$$

Next, we argue that $|\chi_\phi(x)| = 1$ for every $x \in G$. Indeed, suppose there exists $\bar{x} \in G$ such that $|\chi_\phi(\bar{x})| \neq 1$. Since

$$1 = |\chi_\phi(0)| = |\chi_\phi(\bar{x} - \bar{x})| = |\chi_\phi(\bar{x})||\chi_\phi(-\bar{x})|$$

then either $|\chi_\phi(\bar{x})| > 1$ or $|\chi_\phi(-\bar{x})| > 1$. Without loss of generality, we may assume that the former is true. Consequently, we have

$$\lim_{n\to\infty} |\chi_\phi(n\bar{x})| = |\chi_\phi(\bar{x})|^n = \infty,$$

which contradicts boundedness of χ_ϕ. Hence, we conclude that $|\chi_\phi(x)| = 1$ for every $x \in G$. This means that $\chi_\phi \in \widehat{G}$. We put $\theta_{\mathcal{F}}(\phi) := \chi_\phi$ and observe that by Lemma 11.45 in [2], p. 427 (or Proposition 7 in [4], p. 123) we have

$$\begin{aligned}\mathcal{F}(f) \circ \theta_{\mathcal{F}}(\phi) &= \int_G f(x)\overline{\chi_\phi(x)}\, dx = \int_G f(x)T_\phi(L_x g_*)\, dx \\ &= T_\phi\left(\int_G f(x)L_x g_*\, dx\right) = T_\phi(f \star_{\mathcal{F}} g_*) \\ &= T_\phi(f)T_\phi(g_*) = T_\phi(f)\end{aligned}$$

for every $f \in L^1(G)$. This is the equality (3).

In order to see that $\theta_{\mathcal{F}}$ is continuous we invoke (again) Lemma 1.4.2 in [3], p. 18 and conclude that the map $x \mapsto T(L_x g_*)$ is continuous. Consequently, the family $(T(L_x g_*))_{x\in K}$ is compact in $C^b(\widehat{G})$ for every compact $K \subset G$. In particular, by the Arzelà–Ascoli theorem, the family $(T(L_x g_*))_{x\in K}$ is equicontinuous at every $\phi_* \in E$, i.e., for every $\varepsilon > 0$ there exists an open neighbourhood U_* of ϕ_* such that

$$|T(L_x g_*)(\phi) - T(L_x g_*)(\phi_*)| < \varepsilon$$

for every $\phi \in U_*$ and $x \in K$. This implies that

$$\sup_{x\in K} |\chi_\phi(x) - \chi_{\phi_*}(x)| \leqslant \varepsilon$$

for every $\phi \in U_*$. In other words, as ϕ converges to ϕ_*, χ_ϕ converges to χ_{ϕ_*}, which proves continuity of $\theta_{\mathcal{F}}$ and concludes the first part of the proof.

For the second part, we assume that the operator T satisfies (4). Hence, we compute that

$$\chi_\phi(x) = \overline{T(L_x g_*)(\phi)} \overset{(4)}{=} \overline{\phi(x)} \cdot \overline{T(g_*)(\phi)} = \overline{\phi(x)} \cdot \chi_\phi(0) = \overline{\phi(x)}$$

for every $\phi \in E$ and $x \in G$, which concludes the proof. □

We move on to show that the cosine transform admits a similar convolution characterization. The cosine transform is a map $\mathcal{C} : L^1(G) \longrightarrow C_0(\mathcal{COS}(G))$ given by the formula

$$\mathcal{C}(f)(\chi) := \int_G f(x)\chi(x)\, dx, \tag{6}$$

where

$$\mathcal{COS}(G) := \left\{\chi \in C^b(G)\ \ \chi \neq 0,\ \forall_{x,y\in G}\ \chi(x)\chi(y) = \frac{\chi(x+y) + \chi(x-y)}{2}\right\}. \tag{7}$$

Functions in $\mathcal{COS}(G)$ are called cosine functions on group G, since for $G = \mathbb{R}$ we have

$$\mathcal{COS}(G) = \left\{ x \mapsto \cos(yx) \;\; y \in \mathbb{R} \right\}.$$

The cosine convolution $\star_{\mathcal{C}} : L^1(G) \times L^1(G) \longrightarrow L^1(G)$ is given by

$$f \star_{\mathcal{C}} g(x) := \int_G f(u) \cdot \frac{g(x+u) + g(x-u)}{2} \, du. \tag{8}$$

Let us recall a property of the cosine convolution which will be crucial in the sequel (see Theorem 2 in [6]):

Lemma 2. *Let $x, y \in G$. If $g \in L^1(G)$ is an even function, then*

$$L_y g \star_{\mathcal{C}} L_x g = g \star_{\mathcal{C}} \frac{L_{x+y} g + L_{x-y} g}{2}. \tag{9}$$

Before we present a characterization of the cosine transform we need one more technical result:

Lemma 3. *Let $T : L^1(G) \longrightarrow C^b(\mathcal{COS}(G))$ be a linear and bounded operator, which satisfies the cosine convolution property*

$$T(f \star_{\mathcal{C}} g) = T(f)T(g) \tag{10}$$

for every $f, g \in L^1(G)$. If $\phi \in \mathcal{COS}(G)$ is such that $T(f_*)(\phi) \neq 0$ for some $f_* \in L^1(G)$, then there exists an even function $g_* \in L^1(G)$ such that $T(g_*)(\phi) \neq 0$.

Proof. Let $\iota : G \longrightarrow G$ be the inverse function $\iota(x) := -x$ and let $T_\phi : L^1(G) \longrightarrow \mathbb{C}$ be a nonzero, linear functional given by $T_\phi(f) := T(f)(\phi)$. Then

$$\begin{aligned}(f \circ \iota) \star_{\mathcal{C}} f(x) &= \int_G f \circ \iota(u) \cdot \frac{f(x+u) + f(x-u)}{2} \, du \\ &\overset{u \mapsto -u}{=} \int_G f(u) \cdot \frac{f(x-u) + f(x+u)}{2} \, du = f \star_{\mathcal{C}} f\end{aligned}$$

for every $f \in L^1(G)$. This leads to

$$T_\phi(f \circ \iota) T_\phi(f) = T_\phi((f \circ \iota) \star_{\mathcal{C}} f) = T_\phi(f \star_{\mathcal{C}} f) = T_\phi(f)^2$$

for every $f \in L^1(G)$. Consequently, we have $T_\phi(f_* \circ \iota) = T_\phi(f_*)$. Finally, we put $g_* := f_* + f_* \circ \iota$ and observe that

$$T_\phi(g_*) = T_\phi(f_* + f_* \circ \iota) = T_\phi(f_*) + T_\phi(f_* \circ \iota) = 2T_\phi(f_*) \neq 0,$$

which concludes the proof. $\square$

We are now ready to demonstrate a counterpart of Theorem 1 for the cosine transform:

Theorem 2. *Let $T : L^1(G) \longrightarrow C^b(\mathcal{COS}(G))$ be a linear and bounded operator. If it satisfies the cosine convolution property (10), then there exists an open set $E \subset \mathcal{COS}(G)$ and a continuous function $\theta_{\mathcal{C}} : E \longrightarrow \mathcal{COS}(G)$ such that*

$$T(f)(\phi) = \begin{cases} \mathcal{C}(f) \circ \theta_{\mathcal{C}}(\phi) & \text{if } \phi \in E, \\ 0 & \text{if } \phi \in \widehat{G} \backslash E. \end{cases} \tag{11}$$

Furthermore, if we assume that T satisfies the additional property

$$T\left(\frac{L_x f + L_{-x} f}{2}\right)(\phi) = \phi(x) \cdot T(f)(\phi) \tag{12}$$

for every $f \in L^1(G)$, $\phi \in E$ and $x \in G$, then $\theta_{\mathcal{C}}(\phi) = \phi$.

Proof. The proof follows the same lines as the proof of Theorem 1. We define

$$E := \left\{\phi \in \mathcal{COS}(G) \;\; \exists_{f \in L^1(G)} \; T(f)(\phi) \neq 0\right\}$$

and (as before) argue that it is an open set. For $\phi \in \widehat{G} \backslash E$ we have $T(f)(\phi) = 0$ for every $f \in L^1(G)$, i.e., (11) is satisfied. Thus, for the rest of the proof we choose $\phi \in E$.

Next, we consider a nonzero, linear functional $T_\phi : L^1(G) \longrightarrow \mathbb{C}$ given by $T_\phi(f) := T(f)(\phi)$. By Lemma 3 there exists an even function $g_* \in L^1(G)$ such that $T_\phi(g_*) = 1$, which we use to define a function $\chi_\phi : G \longrightarrow \mathbb{C}$ by the formula

$$\chi_\phi(x) := T_\phi(L_x g_*).$$

As in Theorem 1 we study the properties of χ_ϕ and see that

- it is nonzero due to $\chi_\phi(0) = 1$,
- it is bounded due to (5),
- it is continuous due to Lemma 1.4.2 in [3], p. 18.

Furthermore, we have

$$\begin{aligned} T_\phi(L_x g_*) T_\phi(L_y g_*) &= T_\phi(L_x g_* \star_c L_y g_*) \overset{\text{Lemma 2}}{=} T_\phi\left(g_* \star_{\mathcal{C}} \frac{L_{x+y} g_* + L_{x-y} g_*}{2}\right) \\ &= T_\phi(g_*) \cdot \frac{T_\phi(L_{x+y} g_*) + T_\phi(L_{x-y} g_*)}{2} \\ &= \frac{T_\phi(L_{x+y} g_*) + T_\phi(L_{x-y} g_*)}{2} \end{aligned}$$

for every $x, y \in G$. This can be written as

$$\chi_\phi(x)\chi_\phi(y) = \frac{\chi_\phi(x+y) + \chi_\phi(x-y)}{2} \tag{13}$$

for every $x, y \in G$. We conclude that $\chi_\phi \in \mathcal{COS}(G)$, which in particular means that

$$\chi_\phi(y) = \frac{\chi_\phi(y) + \chi_\phi(-y)}{2} \tag{14}$$

for every $y \in G$. We put $\theta_{\mathcal{C}}(\phi) := \chi_\phi$ and observe that by Lemma 11.45 in [2], p. 427 (or Proposition 7 in [4], p. 123) we have

$$\begin{aligned}
\mathcal{C}(f) \circ \theta_{\mathcal{C}}(\phi) &= \int_G f(x)\chi_\phi(x)\,dx \overset{(14)}{=} \int_G f(x) \cdot \frac{\chi_\phi(x) + \chi_\phi(-x)}{2}\,dx \\
&= \int_G f(x) \cdot \frac{T_\phi(L_x g_*) + T_\phi(L_{-x} g_*)}{2}\,dx \\
&= T_\phi\left(\int_G f(x) \cdot \frac{L_x g_* + L_{-x} g_*}{2}\,dx\right) \\
&= T_\phi(f \star_{\mathcal{C}} g_*) = T_\phi(f)T_\phi(g_*) = T_\phi(f)
\end{aligned}$$

for every $f \in L^1(G)$. This demonstrates the equality (11). To conclude the first part of the theorem we observe that $\theta_{\mathcal{C}}$ is continuous due to the same reasoning as in Theorem 1.

For the second part of the theorem we compute that

$$\begin{aligned}
\phi(x)T_\phi(g_*) &\overset{(12)}{=} T_\phi\left(\frac{L_x g_* + L_{-x} g_*}{2}\right) = \int_G \frac{L_x g_*(y) + L_{-x} g_*(y)}{2} \cdot \chi_\phi(y)\,dy \\
&= \int_G g_*(y) \cdot \frac{\chi_\phi(x+y) + \chi_\phi(y-x)}{2}\,dy \\
&\overset{(13)}{=} \chi_\phi(x) \int_G g_*(y)\chi_\phi(y)\,dy = \chi_\phi(x)T_\phi(g_*)
\end{aligned}$$

for every $\phi \in E$ and $x \in G$. Dividing by $T_\phi(g_*)$ we conclude the proof. □

3. Laplace Transform

In the previous section we focused on Fourier and cosine transforms, characterizing them via suitable convolution properties. The purpose of the current section is to demonstrate that the Laplace transform enjoys a similar characterization. Let us recall that the Laplace transform is a map $\mathcal{L} : L^1(\mathbb{R}_+) \longrightarrow C_0(\mathbb{R}_+)$ given by

$$\mathcal{L}(f)(y) := \int_0^\infty e^{-yx} f(x)\,dx.$$

Unlike in the previous section, when discussing the Laplace transform we assume that both spaces $L^1(\mathbb{R}_+)$ and $C^b(\mathbb{R}_+)$ comprise of real-valued (not complex-valued) functions.

If the Laplace convolution $\star_{\mathcal{L}} : L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+) \longrightarrow L^1(\mathbb{R}_+)$ is given by

$$f \star_{\mathcal{L}} g(x) := \int_0^x f(u)g(x-u)\,du \tag{15}$$

then it is well-known that following equality (see Theorem 2.39 in [10], p. 92) holds for every $f, g \in L^1(\mathbb{R}_+)$:

$$\mathcal{L}(f \star_{\mathcal{L}} g) = \mathcal{L}(f)\mathcal{L}(g). \tag{16}$$

Similarly to the previous section, our goal is to "reverse the implication" and ask whether there are any other operators satisfying (16). A (almost) negative answer will constitute a characterization of the Laplace transform in terms of the Laplace convolution (15). To this end, we need the following auxilary lemma:

Lemma 4. *Let $g \in L^\infty(\mathbb{R}_+)$ and let D be a dense subset in $L^1(\mathbb{R}_+)$. If*

$$\int_{\mathbb{R}_+} f(x)g(x)\ dx = 0$$

for every $f \in D$, then $g = 0$.

At last, we are ready to characterize the Laplace transform in terms of the Laplace convolution property:

Theorem 3. *Let $T : L^1(\mathbb{R}_+) \longrightarrow C^b(\mathbb{R}_+)$ be a linear and bounded operator. If it satisfies the Laplace convolution property*

$$T(f \star_{\mathcal{L}} g) = T(f)T(g) \tag{17}$$

for every $f, g \in L^1(\mathbb{R}_+)$, then there exists an open set $E \subset \mathbb{R}_+$ and a continuous function $\theta_{\mathcal{L}} : E \longrightarrow \mathbb{R}_+ \cup \{0\}$ such that

$$T(f)(y) = \begin{cases} \mathcal{L}(f) \circ \theta_{\mathcal{L}}(y) & \text{if } y \in E, \\ 0 & \text{if } y \in \mathbb{R}_+ \backslash E. \end{cases} \tag{18}$$

Furthermore, if we assume that T satisfies the additional property that there exists $t_* \in \mathbb{R}_+$ such that

$$T(L_{t_*} f)(y) = e^{-yt_*} T(f)(y) \tag{19}$$

for every $f \in L^1(\mathbb{R}_+)$ and $y \in E$, where

$$L_{t_*} f(x) = \begin{cases} f(x - t_*) & \text{if } x > t_*, \\ 0 & \text{if } 0 < x \leqslant t_*, \end{cases}$$

then $\theta_{\mathcal{L}}(y) = y$.

Proof. Just as in Theorems 1 and 2 we commence the proof by defining

$$E := \Big\{ y \in \mathbb{R}_+ \ \ \exists_{f \in L^1(\mathbb{R}_+)}\ T(f)(y) \neq 0 \Big\}.$$

If $y \in \mathbb{R}_+ \backslash E$ then $T(f)(y) = 0$ for every $f \in L^1(\mathbb{R}_+)$ and thus the equality (18) holds. Hence, we focus on $y \in E$ and define a nonzero, linear functional $T_y : L^1(\mathbb{R}_+) \longrightarrow \mathbb{R}$ given by $T_y(f) := T(f)(y)$. As the dual of $L^1(\mathbb{R}_+)$ is $L^\infty(\mathbb{R}_+)$ then there exists a nonzero function $\chi_y \in L^\infty(\mathbb{R}_+)$ such that

$$T_y(f) = \int_0^\infty f(x)\chi_y(x)\ dx$$

for every $f \in L^1(\mathbb{R}_+)$.

By Fubini's theorem we have

$$\begin{aligned} T_y(f \star_{\mathcal{L}} g) &= \int_0^\infty f \star_{\mathcal{L}} g(u)\chi_y(u)\ du = \int_0^\infty \int_0^u f(v)g(u-v)\chi_y(u)\ dvdu \\ &= \int_0^\infty \int_v^\infty f(v)g(u-v)\chi_y(u)\ dudv \\ &\overset{u\mapsto u+v}{=} \int_0^\infty \int_0^\infty f(v)g(u)\chi_y(u+v)\ dudv \end{aligned}$$

for every $f, g \in L^1(\mathbb{R}_+)$. Since T_y is nonzero we pick $g_* \in C_c(\mathbb{R}_+)$ such that $T_y(g_*) = 1$. By the Laplace convolution property (17) we have

$$T_y(f \star_{\mathcal{L}} g_*) = T_y(f)T_y(g_*) = T_y(f)$$

for every $f \in L^1(\mathbb{R}_+)$. This implies that we have

$$\int_0^\infty \int_0^\infty f(v)g_*(u)\chi_y(u+v)\ dudv = \int_0^\infty f(v)\chi_y(v)\ dv,$$

or equivalently

$$\int_0^\infty f(v)\left(\int_0^\infty g_*(u)\chi_y(u+v)\ du - \chi_y(v)\right) dv = 0$$

for every $f \in L^1(\mathbb{R}_+)$. Since

$$v \mapsto \int_0^\infty g_*(u)\chi_y(u+v)\ du - \chi_y(v)$$

is a $L^\infty-$function then by Lemma 4 we conclude that

$$\chi_y(v) = \int_0^\infty g_*(u)\chi_y(u+v)\ du$$

for every $v \in \mathbb{R}_+$. We use this integral functional equation to establish continuity of χ_y. For a fixed $v_* \in \mathbb{R}_+$ and $h \in (-v_*, v_*)$ we have

$$\begin{aligned} |\chi_y(v_*+h)-\chi_y(v_*)| &= \left|\int_0^\infty g_*(u)\chi_y(u+v_*+h)\ du - \int_0^\infty g_*(u)\chi_y(u+v_*)\ du\right| \\ &= \left|\int_{v_*+h}^\infty g_*(u-v_*-h)\chi_y(u)\ du - \int_{v_*}^\infty g_*(u-v_*)\chi_y(u)\ du\right| \\ &\leqslant \int_{v_*+h}^\infty |g_*(u-v_*-h) - g_*(u-v_*)||\chi_y(u)|\ du \\ &\quad + \int_{v_*}^{v_*+h} |g_*(u-v_*)||\chi_y(u)|\ du \\ &\leqslant \|\chi_y\|_\infty \left(\int_{v_*+h}^\infty |g_*(u-v_*-h)-g_*(u-v_*)|\ du + \|g_*\|_\infty h\right). \end{aligned}$$

Since g_* is continuous and compactly supported and $h \in (-v_*, v_*)$ then there exists some constant C such that the function

$$h \mapsto \int_{v_*+h}^{\infty} |g_*(u - v_* - h) - g_*(u - v_*)| \, du$$
$$= \int_{v_*+h}^{C} |g_*(u - v_* - h) - g_*(u - v_*)| \, du$$

is continuous and takes the value 0 for $h = 0$. Consequently, we have

$$\lim_{h \to 0} |\chi_y(v_* + h) - \chi_y(v_*)| = 0,$$

establishing the continuity of χ_y.

For the second time we use the Laplace convolution property (17) to obtain

$$\int_0^{\infty} \int_0^{\infty} f(v) g(u) \chi_y(u + v) \, du \, dv = \int_0^{\infty} \int_0^{\infty} f(v) g(u) \chi_y(v) \chi_y(u) \, du \, dv,$$

which we rewrite as

$$\int_0^{\infty} \int_0^{\infty} f(v) g(u) \Big(\chi_y(u + v) - \chi_y(v) \chi_y(u) \Big) \, du \, dv = 0$$

for every $f, g \in L^1(\mathbb{R}_+)$. Since $(u, v) \mapsto \chi_y(u+v) - \chi_y(v)\chi_y(u)$ is a $L^\infty-$function, then by Lemma 4 we have

$$\chi_y(u + v) = \chi_y(v) \chi_y(u)$$

for every $u, v \in \mathbb{R}_+$.

Suppose, for the sake of contradiction, that there exists some $\bar{u} \in \mathbb{R}_+$ such that $\chi_y(\bar{u}) = 0$. It immediately follows that $\chi_y(v) = 0$ for every $v > \bar{u}$. Next, since

$$0 = \chi_y(\bar{u}) = \chi_y \left(\frac{\bar{u}}{2} \right)^2$$

then $\chi_y\left(\frac{\bar{u}}{2}\right) = 0$. An inductive reasoning establishes that $\chi_y\left(\frac{\bar{u}}{2^n}\right) = 0$ for every $n \in \mathbb{N}$. This means that $\chi_y = 0$ and consequently, $T_y = 0$. This contradicts our choice of $y \in E$ (made at the beginning of the proof) so we assume that $\chi_y(u) \neq 0$ for every $u \in \mathbb{R}_+$. By a standard (an easy) argument we establish that $\chi_y(u) = e^{-zu}$ for some $z \in \mathbb{R}_+ \cup \{0\}$. We set $\theta_{\mathcal{L}}(y) := z$.

To conclude the first part of the proof we need to demonstrate that $\theta_{\mathcal{L}}$ is continuous. We fix $y_* \in E$ and observe that since T_{y_*} is a nonzero functional, then there exist $a, b \in \mathbb{R}_+$ such that $T(\mathbb{1}_{(a,b)})(y_*) \neq 0$. By continuity of $T(\mathbb{1}_{(a,b)})$ there exists an open neighbourhood $U_* \subset E$ of y_* such that $T(\mathbb{1}_{(a,b)})(y) \neq 0$ for $y \in U_*$. Next, we define $h : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ with the formula

$$h(z) := \int_a^b e^{-zt} \, dt. \tag{20}$$

This function is clearly continuous on $\mathbb{R}_+$ and (as strictly monotone) has a continuous inverse. Consequently, for every $y \in U_*$ we have

$$T(\mathbb{1}_{(a,b)})(y) = \int_a^b e^{-\theta_{\mathcal{L}}(y)t}\, dt = h(\theta_{\mathcal{L}}(y)).$$

This implies that $\theta_{\mathcal{L}} = h^{-1} \circ T(\mathbb{1}_{(a,b)})|_{U_*}$ is continuous at y_* as a composition of two continuous functions.

For the second part of the proof, we have

$$\begin{aligned} e^{-yt_*}T(f)(y) \overset{(19)}{=} T(L_{t_*}f)(y) &= \int_0^\infty L_{t_*}f(x)e^{-\theta_{\mathcal{L}}(y)x}\, dx \\ &= \int_t^\infty f(x-t_*)e^{-\theta_{\mathcal{L}}(y)x}\, dx \\ &= \int_0^\infty f(x)e^{-\theta_{\mathcal{L}}(y)(x+t_*)}\, dx \\ &= e^{-\theta_{\mathcal{L}}(y)t_*}T(f)(y) \end{aligned}$$

for every $f \in L^1(\mathbb{R}_+)$ and $y \in E$. For every $y \in E$ we choose $f \in L^1(\mathbb{R}_+)$ such that $T(f)(y) \neq 0$ to conclude that $\theta_{\mathcal{L}}(y) = y$. □

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