Results Math (2023) 78:114

1422-6383/23/030001-9
published online March 30, 2023
https://doi.org/10.1007/s00025-023-01891-w

Results in Mathematics

Calibrations for the Volume of Unit Vector Fields in Dimension 2

Rui Albuquerque

Abstract. We use the theory of calibrations to write an equation of a minimal volume vector field on a given Riemann surface.

Mathematics Subject Classification. Primary 53C38, 57M50, 57R25; Secondary 58A15.

Keywords. Vector field, minimal volume, calibration.

1. Introduction

Gluck and Ziller have used the theory of calibrations to prove that the minimal volume unit vector fields defined on the 3-dimensional sphere are the Hopf vector fields, [10]. Inspired by this celebrated result we try to parallel its main ideas on the setting of an oriented Riemannian 2-manifold.

In [10] an appropriate calibration 3-form φ is found on the total space of the unit tangent sphere bundle $\pi : T^1\mathbb{S}^3 \longrightarrow \mathbb{S}^3$. Sections of this bundle are the unit vector fields. Applying the theory of calibrations of Harvey and Lawson ([11]), the corresponding embedded 3-dimensional submanifolds calibrated by φ are precisely the Hopf vector fields. They minimize volume globally in a unique homology class, namely the canonical class of the base $\mathbb{S}^3$ which lies in $H_3(T^1\mathbb{S}^3)$.

The question of minimality in dimension 2 has been studied before and there are several important results eg. in [4–7, 9, 14]. A simple differential equation characterizing the 2-dimensional variational problem, ie. whose solutions are precisely the germs of minimal volume vector fields on a Riemannian surface, is partly missing. Such equation must of course have a space of solutions compatible with the base manifold isometries. Not to mention the topological obstructions to the existence of a unit vector field, such as Euler characteristic zero.

Let M denote a Riemann surface endowed with a unit norm class C^2 vector field X. By the original definition in [10], or [9], we have

$$\mathrm{vol}(X)=\mathrm{vol}(M,X^*g^S)=\int_M\sqrt{1+\|\nabla_{e_0}X\|^2+\|\nabla_{e_1}X\|^2}\,\mathrm{vol}_M \tag{1}$$

where g^S is the Sasaki metric on T^1M and e_0,e_1 is any local orthonormal frame on M. Indeed, $\|\nabla_{e_0}X\|^2+\|\nabla_{e_1}X\|^2$ is a frame invariant quantity.

We denote by $\pi:T^1M\longrightarrow M$ the unit tangent sphere bundle of M, perhaps with boundary. T^1M is a Riemannian submanifold of TM of metric contact type with contact 1-form e^0, this is, a contact manifold with the metric g^S and contact structure induced from the geodesic spray. N.B.: the present e^0 is a 1-form on the manifold T^1M.

For M oriented, there exists a natural differential system of 1-forms e^0,e^1,e^2 globally defined on T^1M—the well-known Cartan structural equations, which we like to see as the simplest case of a fundamental differential system introduced in [2].

It is clear how to find the global frame e_0,e_1,e_2 at each point $u\in T^1M$ such that $\pi(u)=x\in M$. The global vector field e_0 is the tautologial horizontal vector field, ie. the horizontal lift of $u\in T_xM$ to $T_u(T^1M)$, also known as geodesic spray vector field. Then e_1 is such that e_0,e_1 is a well-defined direct orthonormal basis of horizontal vector fields. Finally e_2 is the vertical dual of e_1, tangent to the S^1 fibres. Let us remark the dual of e^0 is the tautologial vertical vector field ξ, which gives $T(T^1M)=\xi^\perp\subset TTM$.

In this article we study the 2-forms $\varphi=b_2\,e^0\wedge e^1+b_1\,e^2\wedge e^0+b_0\,e^1\wedge e^2$ on T^1M which define a calibration, ie. a comass 1 and closed 2-form, seemingly appropriate for the study of unit vector fields on M.

Next, we endeavour to identify the existence of φ with that of a minimal vector field X. Since M is not required to satisfy any further condition, our main theorem becomes a local result; we deduce an equation of a minimal volume vector field in any bounded domain: letting A denote the $\mathbb{C}$-valued function given essentially by the components of $\nabla.X$, we must have, on a conformal chart z of M,

$$\frac{\partial}{\partial\overline{z}}\frac{A}{\sqrt{1+|A|^2}}=0. \tag{2}$$

The function A is indeed globally defined.

In another article, [3], we have shown that the imaginary part of this Cauchy-Riemann equation is indeed the necessary condition for minimal volume, deduced by Gil-Medrano and Llinares-Fuster in [8] to coincide with the critical points of the functional (1). Our equation is a sufficient condition for minimality arising from a certain type of calibrations.

A second main result is the solution of (2) on a manifold of constant negative sectional curvature $K<0$.

Let us further remark that the existence of a parallel vector field, clearly an absolute minima for the volume functional, starts as a local metric issue. More precisely, the Riemann curvature tensor applied to X would imply flatness. On the other end, the theory of calibrations applies to manifolds with boundary and thus there is a path through geometry and topology to pursue.

2. Minimal Volume Over a Surface

We start by recalling some general ideas in any dimension.

Let $(M,\langle\ \rangle)$ be an oriented Riemannian manifold of dimension $n+1$. Recall the well-known metric and contact structure e^0 on the total space of $\pi : T^1M \longrightarrow M$. As usual, we let e_0 denote the geodesic spray, i.e. the unit norm horizontal vector field such that $\mathrm{d}\pi_u(e_0) = u \in T_{\pi(u)}M,\ \forall u \in T^1\,M$. Using duality of the Sasaki metric yields that, for any $v \in T_u(T^1\,M)$, we have $e^0(v) = \langle e_0, v\rangle = \langle u, \mathrm{d}\pi(v)\rangle$. It is thus easy to prove that e^0 is the restriction of the Liouville form pulled back from T^*M to TM (we use musical isomorphism notation throughout; eg. $e^0 = {e_0}^\flat$).

Let φ be a degree $n+1$ calibration defined on the manifold T^1M.

Let $X \in \mathfrak{X}_M$ be a class C^2 unit norm vector field on M and let us fix the $H_{n+1}(T^1M,\mathbb{R})$ homology class of $X(M)$. Since $\varphi \le \mathrm{vol}_X$, the minimal volume unit vector fields, within the same homology class, are those for which $\varphi = \mathrm{vol}_X$, ie. restricted to the Riemannian submanifold $X(M)$ the calibration coincides with the submanifold Riemannian volume. In other words, recalling vol_X from [9,10], such unit vector fields are those for which $X^*\varphi = \mathrm{vol}_X$; corresponding to the so-called φ-submanifolds which are sections of $\pi : T^1M \longrightarrow M$. Then the fundamental relation from [11] follows: for any unit $X' \in \mathfrak{X}_M$,

$$\int_M \mathrm{vol}_X = \int_{X(M)} \varphi = \int_{X'(M)} \varphi \le \int_M \mathrm{vol}_{X'}. \tag{3}$$

The theory of calibrations holds for submanifolds-with-boundary of the calibrated manifold. So we may well focus on a fixed open subset, a domain $\Omega \subset M$ perhaps with non-empty boundary, and seek an immersion $X : \Omega \to T^1M$ giving a φ-submanifold. We remark that prescribing boundary values for X on a compact $\partial\Omega$ implies that certain *moment* conditions are satisfied, cf. [11, Eq. 6.9].

Recalling a useful notation $\pi^*, \pi^\star$ for the horizontal, respectively vertical, canonical lift, cf. [1], we have the 'horizontal plus vertical' decomposition $\mathrm{d}X(Y) = \pi^*Y + \pi^\star(\nabla_Y X)$ in TTM. Also, we may find local adapted frames $e_0, e_1, \ldots, e_n, e_{1+n}, \ldots, e_{2n}$, indeed local oriented orthonormal moving frames on T^1M with the e_{i+n} the vertical *mirror* of the horizontal e_i, $i = 1, \ldots, n$.

π^*X is the horizontal lift of X and thus $\pi^*X = e_0$ restricted to the submanifold $X(M) \subset T^1M$. The horizontal e_i project through $\mathrm{d}\pi$ to a frame

$e_i \in TM$ (we use the same notation). Hence, we may write

$$\mathrm{d}X(e_i) = e_i + \sum_{j=1}^{n} A_{ij} e_{j+n} \tag{4}$$

for $i = 0, 1 \ldots, n$, where $A_{ij} = \langle \nabla_{e_i} X, e_j \rangle$. Since $\|X\| = 1$, $A_{i0} = 0$. This implies that $X^* e^0 = X^\flat$.

We now suppose M is a Riemann surface and $\pi : T^1M \longrightarrow M$ is the unit circle tangent bundle. Let us search for the calibration φ.

As it is well-known, T^1M is parallelizable: we have the global direct orthonormal frame e_0, e_1, e_2, with e_2 the vertical mirror of e_1. In particular $\pi^* \mathrm{vol}_M = e^0 \wedge e^1$.

The following formulas are well-known, cf. [2] and the references therein:

$$\mathrm{d}e^0 = e^2 \wedge e^1, \qquad \mathrm{d}e^1 = e^0 \wedge e^2, \qquad \mathrm{d}e^2 = K\, e^1 \wedge e^0 \tag{5}$$

where $K = \langle R(e_0, e_1)e_1, e_0 \rangle$ is the Gauss curvature. Notice K is the pullback of a function on M and it is not necessarily a constant.

Let us assume the abbreviation $b = \pi^* b$ for any given real function b on M; this gives a function on T^1M of course constant along the fibres.

Given $b_0, b_1, b_2 \in \mathrm{C}^1_M(\mathbb{R})$, we have a 2-form on T^1M:

$$\varphi = b_2\, e^0 \wedge e^1 + b_1\, e^2 \wedge e^0 + b_0\, e^1 \wedge e^2. \tag{6}$$

This is a 2-calibration if it has comass 1 and $\mathrm{d}\varphi = 0$. Recall from [11] that comass 1 is defined by

$$\sup\{\|\varphi\|^*_u : \ u \in T^1M\} = 1 \tag{7}$$

where

$$\|\varphi\|^*_u = \sup\big\{\langle \varphi_u, \zeta \rangle : \ \zeta \ \text{ is a unit simple 2-vector at } \ u\big\}. \tag{8}$$

Proposition 1. *The 2-form φ on T^1M has comass 1 if and only if*

$$\sup\big\{{b_0}^2 + {b_1}^2 + {b_2}^2 : \ x \in M\big\} = 1. \tag{9}$$

The form φ is closed if and only if the function $b_1 + \sqrt{-1}b_0$ is holomorphic.

Proof. For the first part, first, it is easy to deduce $\varphi(\mathfrak{u}, \mathfrak{v}) = \langle b_0 e_0 + b_1 e_1 + b_2 e_2, \mathfrak{u} \times \mathfrak{v} \rangle$, for any $\mathfrak{u}, \mathfrak{v}$ tangent to T^1M. We then recall that $\|\mathfrak{u} \times \mathfrak{v}\| = \|\mathfrak{u} \wedge \mathfrak{v}\|$. The definition of comass 1 together with Cauchy inequality yields $|(b_0, b_1, b_2)| \leq 1$ and the requirement that the above supremum is 1. For the second part of the theorem, we note that $\mathrm{d}b_i(e_2) = 0, \forall i = 0, 1, 2$, by construction. And therefore $\mathrm{d}\varphi = 0$ is equivalent to the condition $\mathrm{d}b_1(e_1) + \mathrm{d}b_0(e_0) = 0$. As the frame varies along a single fibre we find Cauchy-Riemann equations. Hence the result. □

Remark. There seems to be no advantage, later on, in considering general functions on T^1M; even if the equation $\mathrm{d}b_2(e_2) + \mathrm{d}b_1(e_1) + \mathrm{d}b_0(e_0) = 0$ looks quite charmful. It is interesting to observe, by the way, that any two functions

f, g on M, such that $\sup\{f^2 + |\nabla g|^2\} = 1$, define a calibration 2-form by $f\, e^0 \wedge e^1 + \mathrm{d}g \wedge e^2$.

Let us now seek for a calibration φ on T^1M, intended for the new study on M.

Again let $X \in \mathfrak{X}_M$ have unit norm and be defined over (a domain contained in) M. We then have a unique vector field Y on the same domain such that X, Y is a direct orthonormal frame.

The differential of the map X is given by the identities $\mathrm{d}X(e_0) = e_0 + A_{01}e_2$, $\mathrm{d}X(e_1) = e_1 + A_{11}e_2$, with usual notation $A_{ij} = \langle \nabla_{e_i} X, e_j \rangle$. In other words, abbreviating $A_{i1} = A_i$,

$$X^*e^0 = e^0, \qquad X^*e^1 = e^1, \qquad X^*e^2 = A_0e^0 + A_1e^1. \tag{10}$$

Recalling definition (1), we find

$$\begin{aligned} \mathrm{vol}_X &= \|\mathrm{d}X(e_0) \wedge \mathrm{d}X(e_1)\|\, e^0 \wedge e^1 \\ &= \|e_0 \wedge e_1 + A_1 e_0 \wedge e_2 + A_0 e_2 \wedge e_1\|\, e^0 \wedge e^1 \\ &= \sqrt{1 + {A_1}^2 + {A_0}^2}\, e^0 \wedge e^1. \end{aligned} \tag{11}$$

On the other hand,

$$X^*\varphi = (-b_0A_0 - b_1A_1 + b_2)\, e^0 \wedge e^1. \tag{12}$$

Theorem 1. *Suppose there exists a unit vector field X on M such that the $\mathbb{C}$-valued function $A = A_1 + \sqrt{-1}A_0$ satisfies the following equation, on a conformal chart z of M:*

$$2(1 + |A|^2)\frac{\partial A}{\partial \overline{z}} - A\frac{\partial |A|^2}{\partial \overline{z}} = 0, \tag{13}$$

corresponding to $A/\sqrt{1 + |A|^2}$ being holomorphic. Then there exists a calibration φ on the total space of T^1M for which X is a φ-submanifold. In particular, X is a unit vector field on M of minimal volume.

Proof. By Proposition 1, we search for a map $\vec{b} = (b_0, b_1, b_2)$ from M into the Euclidean ball of radius 1 and having a limit value in the $\mathbb{S}^2$ boundary. Let us also denote $\vec{A} = (-A_0, -A_1, 1)$.

Now, by (11) and (12), condition $X^*\varphi \leq \mathrm{vol}_X$ is equivalent to

$$\langle \vec{b}, \vec{A} \rangle \leq |\vec{A}|.$$

Since we wish equality and since $|\vec{b}| \leq 1 \leq |\vec{A}|$, there is a unique solution:

$$\vec{b} = \frac{\vec{A}}{|\vec{A}|}.$$

The corresponding φ is globally defined, with the same domain as X. Finally, one must have φ closed. Hence the function $A/\sqrt{1 + |A|^2}$ must be holomorphic; and a straightforward computation leads to (13). □

Remark. Seeing A as ∇X, one certainly finds inspiration for (13) from the minimal surface $u = u(x, y)$ graph equation in $\mathbb{R}^3$, due to Lagrange, cf. [13, Eq. 1]:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Corollary 1. *Suppose X is a solution of (13) such that the function $|A|$ is constant. Then A is constant and the Riemann surface has constant sectional curvature $K = -|A|^2 \leq 0$. In particular,*

$$\mathrm{vol}(X) = \sqrt{1-K}\,\mathrm{vol}(M). \tag{14}$$

Proof. Let us use the notation $e_0 = X$, $e_1 = Y$ on M, as before. In general context, we have $\nabla_0 e_0 = A_0 e_1$, $\nabla_1 e_0 = A_1 e_1$ and so $\nabla_0 e_1 = -A_0 e_0$, $\nabla_1 e_1 = -A_1 e_0$. Hence $[e_0, e_1] = -A_0 e_0 - A_1 e_1$ and then

$$\begin{aligned} R(e_0, e_1)e_1 &= \nabla_{e_0}\nabla_{e_1}e_1 - \nabla_{e_1}\nabla_{e_0}e_1 - \nabla_{[e_0,e_1]}e_1 \\ &= -\nabla_0(A_1 e_0) + \nabla_1(A_0 e_0) + A_0\nabla_0 e_1 + A_1\nabla_1 e_1 \\ &= -\mathrm{d}A_1(e_0)e_0 - A_1A_0e_1 + \mathrm{d}A_0(e_1)e_0 + A_0A_1e_1 - {A_0}^2 e_0 - {A_1}^2 e_0 \\ &= \mathrm{d}A_0(e_1)e_0 - \mathrm{d}A_1(e_0)e_0 - {A_0}^2 e_0 - {A_1}^2 e_0. \end{aligned}$$

Now, if $|A|$ is constant, then from (13) it follows that A is holomorphic. Henceforth A is constant. And thus $K = \langle R(e_0, e_1)e_1, e_0\rangle = -|A|^2$. □

Here follows a non-trivial complete example to which Corollary 1 applies. It is the Lie group of affine transformations $M = \mathrm{Aff}(\mathbb{R})$ with left invariant metric, together with any unit left invariant vector field X. It is easy to prove that A is a constant.

$\mathrm{Aff}(\mathbb{R})$ is indeed a constant curvature hyperbolic surface, it is the 2-dimensional case of Special Example 1.7 from [12], which is deduced there to be hyperbolic. Moreover, we know there are no other Lie groups of dimension 2 with the same constant curvature $K < 0$ up to isometry.

Equation (13) proves quite hard to solve, be it for constant $K < 0$ or > 0. In the hyperbolic case, we cannot be sure about uniqueness of the solutions given by invariant theory.

3. On a Conformal Chart

We seek further general understanding of (13). Let us recall that a complex chart $z = x + iy$ corresponds with isothermal coordinates, ie. a chart such that the metric is given by $\lambda|\mathrm{d}z|^2$ for some function $\lambda > 0$.

A real vector field X is given by $X = a\partial_x + b\partial_y = f\partial_z + \overline{f}\partial_{\overline{z}}$ where $f = a + ib$. If $Z = h\partial_z + \overline{h}\partial_{\overline{z}}$ is another vector field, then

$$\langle X, Z\rangle = (f\overline{h} + \overline{f}h)\frac{\lambda}{2} \tag{15}$$

so that $\|X\|^2 = f\overline{f}\lambda$. We have $Y = if\partial_z - i\overline{f}\partial_{\overline{z}} = \overline{Y}$.

Recall the Levi-Civita connection, a real operator, is given by $\nabla_z\partial_z = \Gamma\partial_z$ where $\Gamma = \frac{1}{\lambda}\frac{\partial\lambda}{\partial z}$, $\nabla_z\partial_{\overline{z}} = \nabla_{\overline{z}}\partial_z = 0$, $\nabla_{\overline{z}}\partial_{\overline{z}} = \frac{1}{\lambda}\frac{\partial\lambda}{\partial\overline{z}}\partial_{\overline{z}}$. In particular we have $R(\partial_z, \partial_{\overline{z}})\partial_z = -\frac{\partial\Gamma}{\partial\overline{z}}\partial_z$ and hence

$$K = \frac{\langle R(\partial_z, \partial_{\overline{z}})\partial_z, \partial_{\overline{z}}\rangle}{\langle\partial_z, \partial_{\overline{z}}\rangle^2} = -\frac{2}{\lambda}\frac{\partial\Gamma}{\partial\overline{z}} = -\frac{2}{\lambda}\frac{\partial^2\log\lambda}{\partial z\partial\overline{z}}. \tag{16}$$

Therefore $\nabla_X X = \varepsilon_0\partial_z + \overline{\varepsilon}_0\partial_{\overline{z}}$ and $\nabla_Y X = i\varepsilon_1\partial_z - i\overline{\varepsilon}_1\partial_{\overline{z}}$ where

$$\varepsilon_0 = ff'_z + \frac{f^2}{\lambda}\lambda'_z + \overline{f}f'_{\overline{z}} \qquad \text{and} \qquad \varepsilon_1 = ff'_z + \frac{f^2}{\lambda}\lambda'_z - \overline{f}f'_{\overline{z}}. \tag{17}$$

We have $A_1 = \langle\nabla_Y X, Y\rangle = (\varepsilon_1\overline{f} + \overline{\varepsilon}_1 f)\frac{\lambda}{2}$ and $A_0 = \langle\nabla_X X, Y\rangle = (-i\varepsilon_0\overline{f} + i\overline{\varepsilon}_0 f)\frac{\lambda}{2}$. Now for a unit vector we have the identity $f'_z\overline{f}\lambda + f\overline{f}'_z\lambda + f\overline{f}\lambda'_z = 0$ and its conjugate. This yields $A_0 = i\lambda(f^2\overline{f}'_z - \overline{f}^2 f'_{\overline{z}})$ and $A_1 = -\lambda(f^2\overline{f}'_z + \overline{f}^2 f'_{\overline{z}})$, finally giving a simple and noteworthy result.

Proposition 2. *$A = -2\lambda f^2\overline{f}'_z = 2(\Gamma f + f'_z)$.*

We note that $|A| = 2|\overline{f}'_z|$ and that a holomorphic unit vector field is just a parallel vector field.

It is an interesting exercise to see from the last identities that A is defined globally, independently of the choice of conformal chart.

Finding f from equation (13) in Theorem 1 together with Proposition 2 proves quite difficult, even for the trivial non-flat metrics.

On the round $\mathbb{S}^2$ punctured at two antipodal points, it is stated and proved in [5] that a minimum of $\mathrm{vol}(X)$ is attained: a solution X_0 is given, for instance, by the directed meridians unit tangent vector field, invariant by parallel transport between poles. However, this solution does not solve *our* equation—which is not surprising, for we have found vector fields with even less volume than X_0 in a smaller open region of $\mathbb{S}^2$. Indeed, the integrand function is smaller in the region. Such result is shown in a proper article, [3].

Funding Open access funding provided by FCT—FCCN (b-on). The research leading to these results has received funding from Fundação para a Ciência e a Tecnologia. Project Ref. UIDB/04674/2020

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author has no conflict of interest to declare.

References

[1] Albuquerque, R.: Notes on the Sasaki metric. Expo. Math. **37**(2), 207–224 (2019)

[2] Albuquerque, R.: A fundamental differential system of Riemannian geometry. Rev. Mat. Iberoam. **35**(7), 2221–2250 (2019)

[3] Albuquerque, R.: Vector fields with big and small volume on $\mathbb{S}^2$, to appear in Hiroshima Mathematical Journal **53**(2), (2023)

[4] Borrelli, V., Gil-Medrano, O.: Area-minimizing vector fields on round 2-spheres. J. Reine Angew. Math. **640**, 85–99 (2010)

[5] Brito, F., Chacón, P., Johnson, D.: Unit vector fields on antipodally punctured spheres: big index, big volume. Bull. Soc. Math. France **136**(1), 147–157 (2008)

[6] Brito, F., Chacón, P., Naveira, A.: On the volume of unit vector fields on spaces of constant sectional curvature. Comment. Math. Helv. **79**, 300–316 (2004)

[7] Brito, F., Gomes, A., Gonçalves, I.: Poincaré index and the volume functional of unit vector fields on punctured spheres. Manuscr. Math. **161**, 487–499 (2020)

[8] Gil-Medrano, O., Llinares-Fuster, E.: Second variation of volume and energy of vector fields. Stability of Hopf vector fields. Math. Ann. **320**, 531–545 (2001)

[9] Gil-Medrano, O., Llinares-Fuster, E.: Minimal unit vector fields. Tohoku Math. J. **54**, 71–84 (2002)

[10] Gluck, H., Ziller, W.: On the volume of a unit vector field on the three-sphere. Comment. Math. Helv. **61**, 177–192 (1986)

[11] Harvey, R., Lawson, H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)

[12] Milnor, J.W.: Curvatures of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)

[13] Meeks, W.H., Pérez, J.: The classical theory of minimal surfaces. Bull. Am. Math. Soc. **48**(3), 325–407 (2011)

[14] Wiegmink, G.: Total bending of vector fields on Riemannian manifolds. Math. Ann. **303**(2), 325–344 (1995)

Rui Albuquerque
Centro de Investigação em Matemática e Aplicações
Rua Romão Ramalho, 59
7000-671 Évora
Portugal
e-mail: rpa@uevora.pt

Received: September 10, 2022.
Accepted: March 4, 2023.