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Results in Mathematics

More on Kakeya Conditions for Achievement Sets

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Abstract. We identify an incorrect estimate in the proof of one of principal theorems from Marchwicki and Miska (Results Math, 2021. https://doi.org/10.1007/s00025-021-01479-2) and demonstrate that the original construction of a special series with unique subsums remains valid when using a weaker estimate that we prove to be true. Additionally, we present a weaker version—without the uniqueness of subsums—of the Thm. 2.1 from Marchwicki and Miska (2021), but with a very simple proof based on the concept of semi-fast convergent series.

Mathematics Subject Classification. 40A05.

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The main goal of this short note is to expose an incorrect estimate in the intricate proof of one of the principal theorems of the recently published paper [2]. The proof is based on a very delicate and complicated construction of a special convergent series and we have decided to write this note as an extension of the forementioned paper, assuming in particular that the reader will be familiar with all preliminaries and notation used in the paper [2] which allows us to use identical notation and definitions without adding any preliminaries to this note. The theorem whose proof requires an amendment is the following [2, Thm. 2.1].

Theorem 1. *Let* $C = \{n_1 < n_2 < n_3 < \cdots\}$ *be an infinite subset of* $\mathbb{N}$. *Then there exists a non-increasing sequence* (x_n) *of positive real numbers such that the series* $\sum_{n=1}^{\infty} x_n$ *is convergent,* $\{n \in \mathbb{N} : \ x_n > r_n\} = C$, $\{n \in \mathbb{N} : \ x_n < r_n\} = C^c$ *and* $A(x_n) = U(x_n)$. *In particular,* $A(x_n)$ *is a Cantor set.*

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The first part of the original proof [2, pp. 4–6] is devoted to the definition an auxiliary sequence (y_n), the main sequence (x_n) and to some crucial properties of these sequences. The second part of the proof [2, pp. 6–7] deals with uniqueness of subsums which is then used to conclude that the achievement set $A(x_n)$ necessarily is a Cantor set. However, it came to our attention that one estimate in the second part is incorrect. Namely, in the second line from the top on the seventh page of the article the Authors claim that

$$\left|\sum_{n\in F\cap I_k} x_n - \sum_{n\in G\cap I_k} x_n\right| \in \left[2^{n_k-n_{k+1}}y_{k+1},\, 2^{-1}y_{k+1}\right]$$

when

$$|F\cap I_k| = |G\cap I_k| \qquad \text{and} \qquad F\cap I_k \neq G\cap I_k. \tag{1}$$

However, under the assumptions (1), we have for any m between 1 and $\left\lfloor \frac{n_{k+1}-n_k}{2} \right\rfloor$

$$\begin{aligned} &\max\left\{\left|\sum_{n\in A} x_n - \sum_{n\in B} x_n\right| : \quad A,B\subset I_k,\ A\cap B=\emptyset,\ |A|=|B|=m\right\} \\ &= \sum_{i=1}^{m} 2^{-i}y_{k+1} - \sum_{i=n_{k+1}-n_k-m+1}^{n_{k+1}-n_k} 2^{-i}y_{k+1} \end{aligned} \tag{2}$$

and hence, in the case (1), $\left|\sum_{n\in F\cap I_k} x_n - \sum_{n\in G\cap I_k} x_n\right|$ can take the value

$$\begin{aligned} &\max\left\{\left|\sum_{n\in A} x_n - \sum_{n\in B} x_n\right| : \quad \emptyset\neq A,B\subset I_k,\ A\cap B=\emptyset,\ |A|=|B|\right\} \\ &= \max_{1\le m\le \left\lfloor \frac{n_{k+1}-n_k}{2}\right\rfloor}\left(\sum_{i=1}^{m} 2^{-i}y_{k+1} - \sum_{i=n_{k+1}-n_k-m+1}^{n_{k+1}-n_k} 2^{-i}y_{k+1}\right) \\ &> y_{k+1}\max_{1\le m\le \left\lfloor \frac{n_{k+1}-n_k}{2}\right\rfloor}\left(\sum_{i=1}^{m-1} 2^{-i} + \frac{1}{2^m} - \frac{1}{2^{n_{k+1}-n_k-m}}\right) \\ &> y_{k+1}\left(1-\frac{1}{2^{m-1}}\right), \end{aligned}$$

that is, the estimate

$$\left|\sum_{n\in F\cap I_k} x_n - \sum_{n\in G\cap I_k} x_n\right| \le 2^{-1}y_{k+1}$$

could fail in the case (1) when $n_{k+1}-n_k > 5$.

Fortunately, the original proof can be rescued by observing that the equality (2) yields—in the discussed subcase—the following estimate

$$\left|\sum_{n\in F\cap I_k} x_n - \sum_{n\in G\cap I_k} x_n\right| \le (1-2^{n_k-n_{k+1}})y_{k+1}$$

and hence (see [2], lines 6–9 from the top of the page 7)

$$\begin{aligned}
&\left|\sum_{n\in F\cap I_{k+1}} y_{k+1} - \sum_{n\in G\cap I_{k+1}} y_{k+1}\right| - \left|\sum_{n\in F\cap I_k} x_n - \sum_{n\in G\cap I_k} x_n\right| \\
&- \left|\sum_{n\in F\cap I_{k+1}} 2^{n_{k+1}-n} y_{k+2} - \sum_{n\in G\cap I_{k+1}} 2^{n_{k+1}-n} y_{k+2}\right| \\
&- \left|\sum_{\substack{n\in F\\ n>n_{k+2}}} x_n - \sum_{\substack{n\in G\\ n>n_{k+2}}} x_n\right| \\
&> y_{k+1} - (1-2^{n_k-n_{k+1}})y_{k+1} - y_{k+2} - r_{n_{k+2}} = 2^{n_k-n_{k+1}} y_{k+1} \\
&- y_{k+2} - r_{n_{k+2}} > 0,
\end{aligned}$$

and the last inequality is exactly(!) the inequality $2^{n_{k-1}-n_k} y_k > r_{n_{k+1}} + y_{k+1}$ (with k replaced by $k+1$) that was proved on page 5 of [2].

We conclude this note with a weak version of the Thm. 2.1 from [2] that, firstly, preserves the crucial part of the thesis of the theorem and, secondly, admits a rather simple proof based on the omission of the argument of $A(x_n) = U(x_n)$ and on the use of the concept of semi-fast convergent series instead.

A series $\sum x_n$ with monotonic and positive terms convergent to 0 is called semi-fast convergent [1] if it satisfies the condition

$$x_n > \sum_{k\colon x_k < x_n} x_k.$$

If $\sum x_n$ is a semi-fast convergent series, then there exist two uniquely determined sequences, (α_k) of positive numbers decreasing to 0 and (N_k) of positive integers such that

$$x_i = \alpha_k \qquad \text{for} \quad \sum_{j=0}^{k-1} N_j < i \le \sum_{j=0}^{k} N_j,$$

where we put $N_0 := 0$. The numbers α_k are the values of the terms of the series $\sum x_n$ and N_k is the multiplicity of the value α_k in the series. Thus, we can identify

$$\sum x_n = \sum(\alpha_k, N_k) \qquad \text{or} \qquad (x_n) = (\alpha_k, N_k)$$

and the sum of the series is $\sum x_n = \sum(\alpha_k, N_k) = \sum_{k=1}^{\infty} \alpha_k N_k$.

Below is the weaker version of the Thm. 2.1 from [2] that can be proven by means of semi-fast convergent series.

Theorem 2. *Let* $C = \{n_1 < n_2 < n_3 < \ldots\}$ *be an infinite subset of* $\mathbb{N}$. *Then there is a non-increasing sequence* (x_n) *of positive real numbers such that the series* $\sum_{n=1}^{\infty} x_n$ *converges,* $\{n \in \mathbb{N} : \ x_n > r_n\} = C$, $\{n \in \mathbb{N} : \ x_n < r_n\} = C^c$ *and* $A(x_n)$ *is a Cantor set.*

Proof. Define $n_0 := 0$ and choose any $x_0 > 0$. Further, let $I_k := \{n_{k-1} + 1, n_{k-1} + 2, \ldots, n_k\}$ for $k \in \mathbb{N}$. Then we define by induction a sequence $(x_i)_{i \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N} \ \forall i \in I_k \qquad x_i \ = \ \frac{1}{3} \frac{1}{n_k - n_{k-1}} x_{n_{k-1}}.$$

Then for all k

$$\sum_{i \in I_k} x_i \ = \ \frac{1}{3} x_{n_{k-1}} \qquad \text{and} \qquad x_{n_k+1} \ \leq \ \frac{1}{3} x_{n_k},$$

and hence by induction on m

$$\forall m \in \mathbb{N} \qquad \sum_{i \in I_{k+m}} x_i \ \leq \ \frac{1}{3^m} x_{n_k}.$$

The following estimate holds for the n_k-th remainder of the series $\sum_{i=1}^{\infty} x_i$:

$$r_{n_k} \ = \ \sum_{m=1}^{\infty} \sum_{i \in I_{k+m}} x_i \ \leq \ \sum_{m=1}^{\infty} \frac{1}{3^m} x_{n_k} \ = \ \frac{x_{n_k}}{2} \ < \ x_{n_k}.$$

Thus $(x_i)_{i \in \mathbb{N}} = (\alpha_k, N_k)_{k \in \mathbb{N}}$ with $\alpha_k = x_{n_k}$ and $N_k = n_k - n_{k-1}$, that is, $\sum_{i=1}^{\infty} x_i$ is a semi-fast convergent series and hence $A(x_i)$ is a Cantor set by the Thm. 16 from [1].

Clearly, the equalities $\{n \in \mathbb{N} : \ x_n > r_n\} = C$, $\{n \in \mathbb{N} : \ x_n < r_n\} = C^c$ hold for the series. $\square$

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Declarations

Conflict of interest There are neither conflicts of interest nor competing interests.

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