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Results in Mathematics

New q-Analogues of Van Hamme's (E.2) Supercongruence and of a Supercongruence by Swisher

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Abstract. In this paper, a couple of q-supercongruences for truncated basic hypergeometric series are proved, most of them modulo the cube of a cyclotomic polynomial. One of these results is a new q-analogue of the (E.2) supercongruence by Van Hamme, another one is a new q-analogue of a supercongruence by Swisher, while the other results are closely related q-supercongruences. The proofs make use of special cases of a very-well-poised ${}_6\phi_5$ summation. In addition, the proofs utilize the method of creative microscoping (which is a method recently introduced by the first author in collaboration with Wadim Zudilin), and the Chinese remainder theorem for coprime polynomials.

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1. Introduction

In 1997, Van Hamme [24] presented 13 remarkable supercongruences corresponding to Ramanujan's or to Ramanujan-like formulas for $1/\pi$. For instance, the two infinite series expansions

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi},$$
$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}=\frac{3\sqrt{3}}{2\pi},$$

correspond to the following two supercongruences for truncated hypergeometric series:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}\equiv(-1)^{(p-1)/2}p \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/3}(-1)^k(6k+1)\frac{(\frac13)_k^3}{k!^3}\equiv p \pmod{p^3}, \quad \text{for} \quad p\equiv 1 \pmod 3, \tag{1.2}$$

where p is an odd prime, and $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. The supercongruence (1.1) was first proved by Mortenson [18] using a technical evaluation of gamma functions, and later reproved by Zudilin [28] and Long [16]. Swisher [23] employed Long's method to prove four supercongruences of Van Hamme, including (1.2) (i.e., the (E.2) supercongruence in [24]). He [11] also gave a generalization of (1.2). In 2016, Osburn and Zudilin [21] confirmed the last supercongruence conjecture of Van Hamme.

During the past few years, q-analogues of supercongruences have been investigated by many authors (see, for example, [3–10, 12–15, 19, 20, 22, 25–27, 29]). In particular, the first author [3,4] gave q-analogues of (1.1) and (1.2) as follows: for any odd integer n,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}q^{k^2}\equiv(-1)^{(n-1)/2}q^{(n-1)^2/4}[n] \pmod{[n]\Phi_n(q)^2},$$

and for any positive integer n with $n\equiv 1 \pmod 3$,

$$\sum_{k=0}^{(n-1)/3}(-1)^k[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3}q^{(3k^2+k)/2}\equiv(-1)^{(n-1)/3}q^{(n-1)(n-2)/6}[n] \pmod{[n]\Phi_n(q)^2}.$$

Here and in what follows, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, $[n]=[n]_q=(1-q^n)/(1-q)$ is the *q-integer*, and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. The first author and Zudilin [10, Theorem 3.5 with $r=1$] also gave another q-analogue of (1.2): for any positive integer n with $n\equiv 1 \pmod 6$,

$$\sum_{k=0}^{(n-1)/3}(-1)^k[6k+1]_{q^2}\frac{(q^2;q^6)_k^3(-q^3;q^6)_k}{(q^6;q^6)_k^3(-q^5;q^6)_k}q^k\equiv q^{1-n}[n]_{q^2} \pmod{[n]\Phi_n(q)^2}. \tag{1.3}$$

One of the aims of this paper is to establish the following new q-analogue of (1.2).

Theorem 1.1. *Let $n \equiv 1 \pmod 6$ be a positive integer. Then*

$$\sum_{k=0}^{M}(-1)^k[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv q^{2(1-n)/3}[n]\frac{(-q^3;q^3)_{(n-1)/3}}{(-q^2;q^3)_{(n-1)/3}} \pmod{[n]\Phi_n(q)^2}, \tag{1.4}$$

where $M=(n-1)/3$ or $M=n-1$.

We shall also give the following similar result.

Theorem 1.2. *Let $n \equiv 1 \pmod 3$ be a positive integer. Then*

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv q^{2(1-n)/3}[n]\frac{(q^3;q^3)_{(n-1)/3}}{(q^2;q^3)_{(n-1)/3}} \pmod{\Phi_n(q)^3}, \tag{1.5}$$

where $M=(n-1)/3$ or $M=n-1$.

Note that the supercongruence (1.5) does not hold modulo $[n]\Phi_n(q)^2$ in general, even for $n \equiv 1 \pmod 6$. We take this opportunity to point out that Theorems 1 and 2 in [8] only hold modulo $\Phi_n(q)^3$ and $\Phi_n(q)^2$, respectively, but do not hold modulo $[n]$, since Lemma 3 in [8] is not true (it only holds for even integers d).

Swisher [23] also proved that, for any prime $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(2p-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv -2p \pmod{p^3}. \tag{1.6}$$

A q-analogue of (1.6) was given by the first author [4, Theorem 1.5 with $(d,r)=(3,1)$]: for any positive integer $n \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(2n-1)/3}(-1)^k q^{(3k^2+k)/2}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv -[2n]q^{(n-1)(2n-1)/3} \pmod{[n]\Phi_n(q)^2}.$$

In this paper, we shall give a new q-analogue of (1.6).

Theorem 1.3. *Let $n \equiv 5 \pmod 6$ be a positive integer. Then*

$$\sum_{k=0}^{M}(-1)^k[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv -q^{2(1-2n)/3}[2n]\frac{(-q^3;q^3)_{(2n-1)/3}}{(-q^2;q^3)_{(2n-1)/3}} \pmod{[n]\Phi_n(q)^2}, \tag{1.7}$$

where $M=(2n-1)/3$ or $M=n-1$.

Similarly, we have the following result.

Theorem 1.4. *Let $n \equiv 2 \pmod 3$ be a positive integer. Then*

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv q^{2(1-2n)/3}[2n]\frac{(q^3;q^3)_{(2n-1)/3}}{(q^2;q^3)_{(2n-1)/3}} \pmod{\Phi_n(q)^2}, \quad (1.8)$$

where $M=(2n-1)/3$ or $M=n-1$.

Note that the q-supercongruence (1.8) does not hold modulo $\Phi_n(q)^3$ for $n>2$. We shall prove Theorems 1.1, 1.2, and 1.3 modulo $\Phi_n(q)^3$ and Theorem 1.4 by using a summation for a very-well-poised ${}_6\phi_5$ series and the 'creative microscoping' method introduced by the first author in collaboration with Zudilin [9]. The proof of Theorems 1.1 and 1.3 also requires the use of a lemma previously given by the present authors.

From Theorems 1.2 and 1.4, we can deduce the following supercongruences.

Corollary 1.5. *Let $p \equiv 1 \pmod 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/3}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv p\Gamma_p(\tfrac{2}{3})^3 \pmod{p^3}, \quad (1.9)$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function.

Corollary 1.6. *Let $p \equiv 2 \pmod 3$ be an odd prime. Then*

$$\sum_{k=0}^{(2p-1)/3}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv -6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^2}.$$

2. Proof of Theorem 1.1

We first give the following result, which is due to the present authors [6, Lemma 2.1].

Lemma 2.1. *Let d, m and n be positive integers with $m \leqslant n-1$. Let r be an integer satisfying $dm \equiv -r \pmod n$. Then, for $0 \leqslant k \leqslant m$ and any indeterminate a, we have*

$$\frac{(aq^r;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k}\frac{(aq^r;q^d)_k}{(q^d/a;q^d)_k}q^{m(dm-d+2r)/2+(d-r)k} \pmod{\Phi_n(q)}.$$

If $\gcd(d,n)=1$, then the above q-congruence also holds for $a=1$.

We also need the following result to prove the truth of (1.4) modulo $[n]$.

Lemma 2.2. *Let n be a positive integer coprime with 6, and let a be an indeterminate. Then*

$$\sum_{k=0}^{m}(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k} \equiv 0 \pmod{[n]}, \quad (2.1)$$

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k}\equiv 0 \pmod{[n]}, \tag{2.2}$$

where $m=(n-1)/3$ *if* $n\equiv 1 \pmod 6$, *and* $m=(2n-1)/3$ *if* $n\equiv 5 \pmod 6$.

Proof. Clearly, Lemma 2.2 is true for $n=1$. We now assume that $n>1$. By Lemma 2.1, we can easily deduce that the k-th and $(m-k)$-th terms on the left-hand side of (2.1) cancel each other modulo $\Phi_n(q)$, i.e.,

$$\begin{aligned}(-1)^{m-k}&\frac{[6(m-k)+1](aq;q^3)_{m-k}(q/a;q^3)_{m-k}(q;q^3)_{m-k}}{(aq^3;q^3)_{m-k}(q^3/a;q^3)_{m-k}(q^3;q^3)_{m-k}}\\ &\equiv -(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k} \pmod{\Phi_n(q)}.\end{aligned}$$

Thus, we have proved that the q-congruence (2.1) holds modulo $\Phi_n(q)$. Since the numerator contains the factor $(q;q^3)_k$, it is easy to see that the k-th summand in (2.2) is congruent to 0 modulo $\Phi_n(q)$ for $m<k\leqslant n-1$. This proves the q-congruence (2.2) modulo $\Phi_n(q)$.

Now we can prove (2.1) and (2.2) modulo $[n]$. Let $\zeta\neq 1$ be an n-th root of unity, not necessarily primitive. In other words, ζ is a primitive root of unity of degree s satisfying $s\mid n$ and $s>1$. Let $c_q(k)$ stand for the k-th term on the left-hand side of (2.2), i.e.,

$$c_q(k)=(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k}.$$

Taking $n=s$ in the q-congruences (2.1) and (2.2) modulo $\Phi_n(q)$, we get

$$\sum_{k=0}^{s_1}c_\zeta(k)=\sum_{k=0}^{s-1}c_\zeta(k)=0,$$

where $s_1=(s-1)/3$ if $s\equiv 1 \pmod 6$, and $s_1=(2s-1)/3$ if $s\equiv 5 \pmod 6$. It is not difficult to see that

$$\lim_{q\to\zeta}\frac{c_q(\ell s+k)}{c_q(\ell s)}=\frac{c_\zeta(\ell s+k)}{c_\zeta(\ell s)}=c_\zeta(k).$$

Therefore,

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/s-1}\sum_{k=0}^{s-1}c_\zeta(\ell s+k)=\sum_{\ell=0}^{n/s-1}c_\zeta(\ell s)\sum_{k=0}^{s-1}c_\zeta(k)=0, \tag{2.3}$$

and

$$\sum_{k=0}^{m}c_\zeta(k)=\sum_{\ell=0}^{(m-s_1)/s-1}c_\zeta(\ell s)\sum_{k=0}^{s-1}c_\zeta(k)+c_\zeta(m-s_1)\sum_{k=0}^{s_1}c_\zeta(k)=0.$$

This proves that both of the sums $\sum_{k=0}^{n-1} c_q(k)$ and $\sum_{k=0}^{m} c_q(k)$ are divisible by $\Phi_s(q)$ for any divisor $s>1$ of n. Since

$$\prod_{s|n,\, s>1} \Phi_s(q)=[n],$$

we complete the proof of (2.1) and (2.2). $\square$

Like most of the q-supercongruences in [9], we have the following parametric generalization of Theorem 1.1.

Theorem 2.3. *Let $n\equiv 1 \pmod 6$ be a positive integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{M}(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k}\equiv q^{2(n-1)/3}[n]\frac{(-q^3;q^3)_{(n-1)/3}}{(-q^2;q^3)_{(n-1)/3}}, \tag{2.4}$$

where $M=(n-1)/3$ or $M=n-1$.

Proof. We start with the following summation for a very-well-poised ${}_6\phi_5$ series (see [2, Appendix (II.20)]):

$$\sum_{k=0}^{\infty}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(d;q)_k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq/d;q)_k}\left(\frac{aq}{bcd}\right)^k$$
$$=\frac{(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty(aq/cd;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/bcd;q)_\infty}. \tag{2.5}$$

(The infinite series in (2.5) converges for $|q|<1$ and $|aq/bcd|<1$.) Specializing (2.5) by letting $q\mapsto q^3$, $a=q$, $b=q^{1-n}$, $c=q^{1+n}$, and $d=-q^2$, we have

$$\sum_{k=0}^{(n-1)/3}(-1)^k[6k+1]\frac{(q^{1-n};q^3)_k(q^{1+n};q^3)_k(q;q^3)_k}{(q^{3-n};q^3)_k(q^{3+n};q^3)_k(q^3;q^3)_k}$$
$$=\frac{(q^4;q^3)_{(n-1)/3}(-q^{1-n};q^3)_{(n-1)/3}}{(q^{3-n};q^3)_{(n-1)/3}(-q^2;q^3)_{(n-1)/3}}$$
$$=q^{2(1-n)/3}[n]\frac{(-q^3;q^3)_{(n-1)/3}}{(-q^2;q^3)_{(n-1)/3}}.$$

This shows that both sides of (2.4) are equal for $a=q^{-n}$ and $a=q^n$. This means that the congruence (2.4) holds modulo $1-aq^n$ and $a-q^n$.

Moreover, by Lemma 2.2, the left-hand side of (2.4) is congruent to 0 modulo $[n]$. Since $1-q^n$ (n is odd) is relatively prime to $1+q^k$, we see that the right-hand side of (2.4) is also congruent to 0 modulo $[n]$. Noticing that $1-aq^n$, $a-q^n$, and $[n]$ are pairwise coprime polynomials in q, we finish the proof of the theorem. $\square$

Proof of Theorem 1.1. Since $(1-q^n)^2$ contains the factor $\Phi_n(q)^2$ and $(q^3;q^3)_M$ is coprime with $\Phi_n(q)$, letting $a=1$ in (2.4), we conclude that (1.4) is true modulo $\Phi_n(q)^3$. Note that Lemma 2.2 also holds for $a=1$. Namely, the q-congruence (1.4) is true modulo $[n]$ and is therefore also true modulo $[n]\Phi_n(q)^2$. This completes the proof. $\square$

3. Proof of Theorem 1.2

We first give the following parametric generalization of Theorem 1.2: for $n\equiv 1 \pmod 3$, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{M}[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k}\equiv q^{2(n-1)/3}[n]\frac{(q^3;q^3)_{(n-1)/3}}{(q^2;q^3)_{(n-1)/3}}, \tag{3.1}$$

where $M=(n-1)/3$ or $M=n-1$. The proof of (3.1) is analogous to that of (2.4). This time, we make the substitutions $q\mapsto q^3$, $a=q$, $b=q^{1-n}$, $c=q^{1+n}$, and $d=q^2$ in (2.5) to obtain

$$\begin{aligned}\sum_{k=0}^{(n-1)/3}&[6k+1]\frac{(q^{1-n};q^3)_k(q^{1+n};q^3)_k(q;q^3)_k}{(q^{3-n};q^3)_k(q^{3+n};q^3)_k(q^3;q^3)_k}\\ &=\frac{(q^4;q^3)_{(n-1)/3}(q^{1-n};q^3)_{(n-1)/3}}{(q^{3-n};q^3)_{(n-1)/3}(q^2;q^3)_{(n-1)/3}}\\ &=q^{2(1-n)/3}[n]\frac{(q^3;q^3)_{(n-1)/3}}{(q^2;q^3)_{(n-1)/3}}.\end{aligned}$$

Thus, the two sides of (3.1) are equal for $a=q^{-n}$ and $a=q^n$. This means that the congruence (3.1) is true modulo $1-aq^n$ and $a-q^n$.

Moreover, by Lemma 2.1, for $m=(n-1)/3$ we can deduce that the k-th and $(m-k)$-th terms on the left-hand side of (3.1) cancel each other modulo $\Phi_n(q)$, i.e.,

$$\begin{aligned}&\frac{[6(m-k)+1](aq;q^3)_{m-k}(q/a;q^3)_{m-k}(q;q^3)_{m-k}}{(aq^3;q^3)_{m-k}(q^3/a;q^3)_{m-k}(q^3;q^3)_{m-k}}\\ &\quad\equiv -[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k}\pmod{\Phi_n(q)}.\end{aligned}$$

(Note that we have utilized the fact that $q^{n/2}\equiv -1 \pmod{\Phi_n(q)}$ for even n.) This proves (3.1) modulo $\Phi_n(q)$.

Finally, letting $a=1$ in (3.1), we arrive at the q-supercongruence (1.5).

4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. We first give a parametric generalization of Theorem 1.4: for $n \equiv 5 \pmod 6$, modulo $[n](1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{M}(-1)^k[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k} \equiv -q^{2(1-2n)/3}[2n] \frac{(-q^3;q^3)_{(2n-1)/3}}{(-q^2;q^3)_{(2n-1)/3}}, \tag{4.1}$$

where $M=(2n-1)/3$ or $n-1$. The proof of (4.1) is very similar to that of (2.4). Specializing (2.5) by $q\mapsto q^3$, $a=q$, $b=q^{1-2n}$, $c=q^{1+2n}$, and $d=-q^2$, we have

$$\begin{aligned}\sum_{k=0}^{(2n-1)/3}(-1)^k[6k+1]\frac{(q^{1-2n};q^3)_k(q^{1+2n};q^3)_k(q;q^3)_k}{(q^{3-2n};q^3)_k(q^{3+2n};q^3)_k(q^3;q^3)_k} &= \frac{(q^4;q^3)_{(2n-1)/3}(-q^{1-2n};q^3)_{(2n-1)/3}}{(q^{3-2n};q^3)_{(2n-1)/3}(-q^2;q^3)_{(2n-1)/3}} \\ &= -q^{2(1-2n)/3}[2n]\frac{(-q^3;q^3)_{(2n-1)/3}}{(-q^2;q^3)_{(2n-1)/3}}.\end{aligned}$$

This proves the congruence (4.1) modulo $1-aq^{2n}$ and $a-q^{2n}$. Moreover, the proof of (4.1) modulo $[n]$ follows from Lemma 2.2. $\square$

Finally, taking $a=1$ in (4.1), we arrive at the desired q-supercongruence (1.7).

Proof of Theorem 1.4. We have the following congruence with a parameter a: for $n \equiv 5 \pmod 6$, modulo $(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{M}[6k+1]\frac{(aq;q^3)_k(q/a;q^3)_k(q;q^3)_k}{(aq^3;q^3)_k(q^3/a;q^3)_k(q^3;q^3)_k} \equiv q^{2(1-2n)/3}[2n]\frac{(q^3;q^3)_{(2n-1)/3}}{(q^2;q^3)_{(2n-1)/3}}, \tag{4.2}$$

where $M=(2n-1)/3$ or $M=n-1$. The congruence (4.2) is equivalent to say that both sides are equal for $a=q^{2n}$ and $a=q^{-2n}$. But this again follows from (2.5) by performing the parameter substitutions $q\mapsto q^3$, $a=q$, $b=q^{1-2n}$, $c=q^{1+2n}$, and $d=q^2$. At last, letting $a=1$ in (4.2), we get (1.8). $\square$

5. Proof of Corollaries 1.5 and 1.6

Proof of Corollary 1.5. Letting $n=p$, where p is a prime congruent to $1 \pmod 3$, and $q\to 1$ in (1.5), we obtain

$$\sum_{k=0}^{(p-1)/3}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv p\frac{(\frac{p-1}{3})!}{(\frac{2}{3})_{(p-1)/3}} \pmod{p^3}.$$

Recall that the p-adic Gamma function has the properties: for any p-adic integer x,

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & p \nmid x, \\ -1, & p \mid x, \end{cases}$$
$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)},$$

where $a_0(x) \in \{1, 2, \dots, p\}$ satisfies $a_0(x) \equiv x \pmod p$. Let $\Gamma(x)$ be the classical Gamma function. Then

$$\begin{aligned}\frac{(\frac{p-1}{3})!}{(\frac{2}{3})_{(p-1)/3}} &= \frac{\Gamma(\frac{p+2}{3})\Gamma(\frac{2}{3})}{\Gamma(1)\Gamma(\frac{p+1}{3})} = \frac{\Gamma_p(\frac{p+2}{3})\Gamma_p(\frac{2}{3})}{\Gamma_p(1)\Gamma_p(\frac{p+1}{3})} \\ &= (-1)^{(2p+1)/3}\frac{\Gamma_p(\frac{p+2}{3})\Gamma_p(\frac{2-p}{3})\Gamma_p(\frac{2}{3})}{\Gamma_p(1)}.\end{aligned}$$

By [17, Theorem 14]), for $p \geqslant 5$, we have

$$\Gamma_p(a+mp) \equiv \Gamma_p(a) + \Gamma_p'(a)mp \pmod{p^2}, \tag{5.1}$$

and so $\Gamma_p(\frac{p+2}{3})\Gamma_p(\frac{2-p}{3}) \equiv \Gamma_p(\frac{2}{3})^2 \pmod{p^2}$. The proof then follows from the fact $\Gamma_p(1) = (-1)^{(2p+1)/3} = -1$. □

Proof of Corollary 1.6. Letting $n = p$, where p is an odd prime congruent to $2 \pmod 3$, and $q \to 1$ in (1.8), we obtain

$$\sum_{k=0}^{(2p-1)/3}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv 2p\frac{(\frac{2p-1}{3})!}{(\frac{2}{3})_{(2p-1)/3}} \pmod{p^2}.$$

Further,

$$\frac{p(\frac{2p-1}{3})!}{(\frac{2}{3})_{(2p-1)/3}} = p\frac{\Gamma(\frac{2p+2}{3})\Gamma(\frac{2}{3})}{\Gamma(1)\Gamma(\frac{2p+1}{3})} = 3\frac{\Gamma_p(\frac{2p+2}{3})\Gamma_p(\frac{2}{3})}{\Gamma_p(1)\Gamma_p(\frac{2p+1}{3})} = 3\frac{\Gamma_p(\frac{2p+2}{3})\Gamma_p(\frac{2-2p}{3})\Gamma_p(\frac{2}{3})}{\Gamma_p(1)},$$

and by (5.1), $\Gamma_p(\frac{2p+2}{3})\Gamma_p(\frac{2-2p}{3}) \equiv \Gamma_p(\frac{2}{3})^2 \pmod{p^2}$. □

6. Some Open Problems

Although the q-supercongruence (1.5) is not true modulo $[n]$ in general, using the same arguments as in the proof of Theorem 1.1, we can show that, for $n \equiv 1 \pmod 3$ and $n > 1$,

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv 0 \pmod{\prod_{\substack{j\mid n,\ j>1 \\ j\equiv 1 \bmod 3}} \Phi_j(q)}, \tag{6.1}$$

where $M=(n-1)/3$ or $M=n-1$. Letting $n=p^r$ and $q\to 1$ in the above q-congruence, we obtain the following result: for any prime $p\equiv 1 \pmod 3$ and integer $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/d}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv 0 \pmod{p^r}, \tag{6.2}$$

where $d=1,3$. Inspired by Dwork's work [1] and Swisher's conjectures [23, (A.3)–(L.3)], we propose the following conjecture on Dwork-type supercongruences, which is a uniform generalization of (1.9) and (6.2).

Conjecture 6.1. *Let $p\equiv 1 \pmod 3$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/d}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p\Gamma_p(\tfrac{2}{3})^3\sum_{k=0}^{(p^{r-1}-1)/d}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \pmod{p^{3r}},$$

where $d=1,3$.

Note that the following Dwork-type supercongruence (see [23, (E.3)] and [4, Conjecture 5.3]) has been proved by the first author and Zudilin [9, Theorem 3.5] by establishing its q-analogue:

For any prime $p\equiv 1 \pmod 3$ and integer $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p\sum_{k=0}^{(p^{r-1}-1)/d}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \pmod{p^{3r}}, \tag{6.3}$$

where $d=1,3$.

We believe that the following new q-analogue of (6.3), which is also a generalization of Theorem 1.1, should be true.

Conjecture 6.2. *Let $n>1$ be an integer with $n\equiv 1 \pmod 6$ and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}(-1)^k[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3}&\equiv q^{2(1-n)/3}[n]\frac{(-q^3;q^3)_{(n^r-1)/3}(-q^{2n};q^{3n})_{(n^{r-1}-1)/3}}{(-q^2;q^3)_{(n^r-1)/3}(-q^{3n};q^{3n})_{(n^{r-1}-1)/3}}\\&\quad\times\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[6k+1]_{q^n}\frac{(q^n;q^{3n})_k^3}{(q^{3n};q^{3n})_k^3},\end{aligned}$$

where $d=1,3$.

Likewise, we conjecture a Dwork-type generalization of Theorem 1.2 as follows.

Conjecture 6.3. *Let $n>1$ be an integer with $n\equiv 1 \pmod 3$ and let $r\geqslant 1$. Then, modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$,*

$$\sum_{k=0}^{(n^r-1)/d}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3}\equiv q^{2(1-n)/3}[n]\frac{(q^3;q^3)_{(n^r-1)/3}(q^{2n};q^{3n})_{(n^{r-1}-1)/3}}{(q^2;q^3)_{(n^r-1)/3}(q^{3n};q^{3n})_{(n^{r-1}-1)/3}}$$

$$\times \sum_{k=0}^{(n^{r-1}-1)/d} [6k+1]_{q^n} \frac{(q^n;q^{3n})_k^3}{(q^{3n};q^{3n})_k^3},$$

where $d = 1, 3$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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