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Results in Mathematics

On Invertible m-isometrical Extension of an m-isometry

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Abstract. We give necessary and sufficient conditions on an m-isometry to have an invertible m-isometrical extension. As particular cases, we give a useful characterization for a general m-isometrical unilateral weighted shift and for ℓ-Jordan isometries. In particular, every ℓ-Jordan isometry operator has an invertible $(2\ell - 1)$-isometrical extension.

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1. Introduction

In the last twenty years there has been an intense research activity on m-isometries. In this paper, we focus our attention on characterizing m-isometries that have an invertible extension that is also m-isometry.

The notion of m-isometric operator on a Hilbert space was introduced by J. Agler [2] and studied in detail shortly after by J. Agler and M. Stankus in three papers [4–6]. These publications can be considered the first ones to initiate this topic of study.

An operator $T \in L(H)$, the algebra of all bounded linear operators acting on a Hilbert space H, is called an *m-isometry*, for some positive integer m, if

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^k T^{*k} T^k = 0,$$

where T^* denotes the adjoint operator of T. When $m = 1$, we obtain an isometry. It is said that T is a *strict m-isometry* if either $m = 1$ or T is an m-isometry with $m > 1$ but it is not $(m - 1)$-isometry.

As one should expect, m-isometries share many important properties with isometries. For example, the following dichotomy property: the spectrum of an

m-isometry is the closed unit disc if it is not invertible or a closed subset of the unit circle if it is invertible [4]. Also, if T is an m-isometry, then T is bounded below; that is, there exists $M > 0$ such that $\|Tx\| \geq M\|x\|$ for every $x \in H$.

Given an m-isometry $T \in L(H)$, we are interested in research conditions which guarantee the existence of a Hilbert space K and an operator $S \in L(K)$, which is an extension of T, such that S is an invertible m-isometry. To say that $S \in L(K)$ is an *extension* of $T \in L(H)$ means that K contains an isometric subspace to H, which we denote also by H, and the restriction $S_{|H}$ from H to H coincides with T.

Problem 1.1. Characterize those m-isometric operators which have an invertible m-isometrical extension.

In 1969 Douglas [13] obtained that any isometry in a Banach space has an invertible isometric extension, also valid in a Hilbert space context. So, the case $m = 1$ holds. For $m \geq 2$, first immediate consideration is that m must be odd, since every invertible m-isometry with even m is an $(m-1)$-isometry by [4, Proposition 1.23].

Our problem is similar to others that arise naturally in Operator Theory and can be formulated in very general terms as follows. Given a class $\mathcal{C}$ of operators, for example defined on Hilbert spaces, and given a property P relative to those operators, we wish to characterize the operators that have an extension in the class $\mathcal{C}$ with property P.

Let $T \in L(H)$ and $S \in L(K)$ with H a closed subspace of K. Denote by P_H the orthogonal projection of K onto H and by J the inclusion of H into K. It is said that

- S is a *lifting* of T if $P_H S = T P_H$.
- S is a *dilation* of T if $T^n = P_H S^n J$, for every $n \in \mathbb{N}$.

Many authors have studied, for a given bounded linear operator $T \in L(H)$, some additional properties of extension, lifting, or dilation of the operator T. The following results are known and respond to these problems:

- Every contraction has an extension which in turn has a unitary lifting. Thus, every contraction has a unitary dilation. Also, a contraction has a lifting which is an isometry. See [16].
- Every isometry has a unitary extension. See [13].
- Every operator T such that the norms of its powers grow polynomially has an m-isometric lifting for some integer $m \geq 1$. This lifting can be also extended to an invertible m-isometry. See [9].

Notice that the norms of the powers of an m-isometry have a polynomial behaviour (see part (1) of Proposition 2.1). However, there are operators such that those norms have a polynomial behaviour that are not m-isometries. In [9], the authors study lifting and dilations which are m-isometries. In particular, they obtain that if T is an m-isometry, then T has an $(m+3)$-isometric lifting with other additional properties.

A special class of m-isometric operators is the ℓ-Jordan isometries; that is, operators which are the sum of an isometry and an ℓ-nilpotent operator which commute. It is known that every ℓ-Jordan isometry is a strict $(2\ell-1)$-isometry, but the converse is not valid. However, every strict m-isometry on a finite dimensional Hilbert space is an $\frac{(m+1)}{2}$-Jordan isometry operator. See [3,12,17] for more details.

Another natural and important examples of m-isometries are certain weighted shift operators. In [1,11], the authors obtained a characterization of weighted shift which are m-isometric.

We summarize the contents of the paper. In Sect. 2, we define a bilateral sequence of operators associated to an m-isometry that allow us to transfer important information of the m-isometry to the bilateral sequence, that it will be an important tool in the paper. In Sect. 3, we present some necessary conditions to obtain an invertible m-isometrical extension. The main results are given in Sect. 4 where we obtain characterizations for an m-isometry to have an invertible m-isometrical extension. Finally, in Sect. 5, we present particular classes of m-isometries for which one can obtain nice results. In particular, we give a useful characterization for a general m-isometrical unilateral weighted shift and for ℓ-Jordan isometries. In particular, every ℓ-Jordan isometry operator has an invertible $(2\ell-1)$-isometrical extension.

2. Some Previous Results

In this section, we define a bilateral sequence of operators associated to an m-isometry, that allow us to transfer important information of the m-isometry to the bilateral sequence that it will be relevant for obtaining necessary conditions for having an invertible m-isometrical extension.

Any polynomial of degree less or equal to $m-1$ is uniquely determined by its values at m distinct points. If $a_0, a_1, \ldots, a_{m-1}$ are given real (or complex) numbers, then the unique polynomial p of degree less or equal to $m-1$ satisfying $p(k) = a_k$ for all $k \in \{0, 1, \ldots, m-1\}$ is giving by Lagrange interpolating polynomial

$$p(z) = \sum_{k=0}^{m-1} a_k \prod_{\substack{0 \le j \le m-1 \\ j \ne k}} \frac{z-j}{k-j}.$$

Note that

$$p(n) = \sum_{k=0}^{m-1} a_k b_k(n)$$

with

$$b_k(n) := \prod_{\substack{0\le j\le m-1\\ j\ne k}} \frac{n-j}{k-j} = (-1)^{m-k-1}\frac{n(n-1)\dots\widehat{(n-k)}\cdots(n-m+1)}{k!(m-k-1)!} \tag{2.1}$$

where $\widehat{(n-k)}$ means that the factor $(n-k)$ is omitted.

Given $T \in L(H)$, define the bilateral sequence by

$$D_n := \sum_{k=0}^{m-1} b_k(n) T^{*k}T^k, \tag{2.2}$$

for every $n \in \mathbb{Z}$. Clearly $D_n \in L(H)$ and it is self adjoint operator for every $n \in \mathbb{Z}$.

Denote $p_x(k) := \langle D_k x, x\rangle$ for every $x \in H$ and $k \in \mathbb{Z}$.

Given $T \in L(H)$, denote $T > 0$ if $\langle Tx, x\rangle > 0$ for every $x \in H \setminus \{0\}$ and we call it *strictly positive operator*.

We concentrate now on the family $(D_n)_{n\in\mathbb{Z}}$ of operators which arise from a fixed m-isometry. Indeed, the bilateral sequence $(D_n)_{n\in\mathbb{Z}}$ has some interesting properties that will be important tools to solve Problem 1.1.

Proposition 2.1. *Let $T \in L(H)$ be an m-isometry and $(D_n)_{n\in\mathbb{Z}}$ be operators defined by* (2.2). *Then*

1. [11, Theorem 2.1] *& [4] $D_n = T^{*n}T^n$ and $p_x(n) = \langle D_n x, x\rangle = \|T^n x\|^2 > 0$ for every $x \in H\setminus\{0\}$ and $n \in \mathbb{N}\cup\{0\}$. Henceforth, there exists the square root $D_n^{1/2}$ of D_n, for every $n \in \mathbb{N}\cup\{0\}$.*
2. *D_n is invertible for every $n \in \mathbb{N}\cup\{0\}$.*
3. *$T^{*k}D_nT^k = D_{n+k}$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.*
4. *Let $y \in R(T^k)$ for some $k \in \mathbb{N}$. Then $p_y(-k) = \|x\|^2$, where $y = T^k x$.*
5. *If $D_{-n} > 0$ and invertible, then $D_{-k} > 0$ and invertible for every $k \in \{1, 2, \cdots, n-1\}$.*

Proof. (2) Let $n \in \mathbb{N}$. By [[14] Theorem 2.3] & [[10], Theorem 3.1] any power of T, T^n is an m-isometry, so, T^n is bounded below. Hence

$$\|D_n x\|\|x\| \ge |\langle D_n x, x\rangle| = \langle D_n x, x\rangle = \|T^n x\|^2 \ge M(n)^2\|x\|^2,$$

where $M(n) > 0$. That is, D_n is bounded below. Then trivially D_n is invertible since D_n is self adjoint operator. *(3)* It is enough to prove the required equality for $k = 1$. Observe that

$$p_{Tx}(n) = \|T^n T x\|^2 = \|T^{n+1}x\|^2 = p_x(n+1),$$

for every $n \in \mathbb{N}$ and

$$\langle D_{n+1}x, x\rangle = p_x(n+1) = p_{Tx}(n) = \langle D_n Tx, Tx\rangle = \langle T^* D_n Tx, x\rangle$$

for every $n \in \mathbb{Z}$. *(4)* Let $y = T^k x$ for some $k \in \mathbb{N}$ and $x \in H$. Then

$$p_y(n) = p_{T^k x}(n) = p_x(k+n),$$

for every $n \in \mathbb{N}$. Therefore $p_y(n) = p_x(k+n)$ for every $n \in \mathbb{Z}$. *(5)* Let $k \in \{1, 2, \cdots, n-1\}$ and $x \in H\backslash\{0\}$. If $D_{-n} > 0$, then by part (3),

$$\langle D_{-k}x, x\rangle = \langle T^{*n-k}D_{-n}T^{n-k}x, x\rangle = \langle D_{-n}T^{n-k}x, T^{n-k}x\rangle > 0. \quad (2.3)$$

Since T^{n-k} is bounded below and by (2.3), we have that

$$\|D_{-k}^{1/2}x\|^2 = \|D_{-n}^{1/2}T^{n-k}x\|^2 \geq M\|x\|^2.$$

So, the result is obtained since D_{-k} is a self adjoint operator. □

We close this section by studying the bilateral sequence $(D_n)_{n\in\mathbb{Z}}$ associated to unilateral weighted shift which are m-isometries.

Let H be a Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Recall that the unilateral weighted shift given by $S_w e_n = w_n e_{n+1}$ on H, where $w_n = \sqrt{\dfrac{p(n+1)}{p(n)}}$ with p a polynomial of degree $m-1$, is a non invertible strict m-isometry, [1]. Also

$$p_{e_j}(n) = \|S_w^n e_j\|^2 = |w_j w_{j+1}\cdots w_{n+j-1}|^2 = \frac{p(j+n)}{p(j)}. \quad (2.4)$$

The following proposition gives an explicit expression of the operator D_n, when T is an m-isometrical unilateral weighted shift operator.

Proposition 2.2. *Let H be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $S_w \in L(H)$ be an m-isometrical unilateral weighted shift with weight sequence $w = (w_n)_{n\in\mathbb{N}}$. Then*

(1) D_n is a diagonal operator for every $n \in \mathbb{Z}$, with diagonal

$$\lambda_n(j) := \sum_{k=0}^{m-1} b_k(n) \prod_{\ell=j}^{j+k-1} |w_\ell|^2,$$

where $b_k(n)$ is giving by (2.1).

(2) Let $n \in \mathbb{Z}$. The following conditions are equivalent
- *(a) D_n is invertible.*
- *(b) $D_n > 0$.*
- *(c) $\lambda_n(j) > 0$ for every $j \in \mathbb{N}$.*

Proof. (1) By [1], there exists a polynomial p of degree $m-1$, such that the weights are given by $w_n = \sqrt{\frac{p(n+1)}{p(n)}}$. So,

$$\begin{aligned} D_n e_j &= \sum_{k=0}^{m-1} b_k(n) S_w^{*k} S_w^k e_j = \sum_{k=0}^{m-1} b_k(n) \prod_{\ell=j}^{j+k-1} |w_\ell|^2 e_j \\ &= \sum_{k=0}^{m-1} b_k(n) \frac{p(j+k)}{p(j)} e_j = \lambda_n(j) e_j\ , \end{aligned} \quad (2.5)$$

where

$$\lambda_n(j) = \sum_{k=0}^{m-1} b_k(n) \frac{p(j+k)}{p(j)}. \tag{2.6}$$

(2) It is immediate by (1). □

In general, the converse of part (5) of Proposition 2.1 is not valid. A suitable choose of the weight sequence gives an example such that $D_{-q} > 0$ and $D_{-(q+1)}$ is not positive for some $q \in \mathbb{N}$.

Example 2.3. Let $q \in \mathbb{N}$ and define $p_q(n) := (n+q)(n+q+1)$. Then S_w with weight $w_n = \sqrt{\frac{p_q(n+1)}{p_q(n)}}$ is a 3-isometry and it satisfies that $D_{-n} > 0$ and invertible for $n \in \{1, \cdots, q\}$ and $D_{-(q+1)}$ is not. In fact,

$$\lambda_{-n}(j) := \frac{p_q(j-n)}{p_q(j)} = \frac{(j+q-n)(j+q-n+1)}{(j+q)(j+q+1)},$$

for $n \in \mathbb{N}$. If $n \in \{1, \cdots, q\}$, then we have that $-q-1+n < -q+n < 0$. Hence, $\lambda_{-n}(j) > 0$, for every $j \in \mathbb{N}$. If $n = q+1$,

$$\lambda_{-(q+1)}(j) = \frac{j(j-1)}{(j+q)(j+q+1)}.$$

Hence $\lambda_{-(q+1)}(1) = 0$ and consequently $\langle D_{-(q+1)} e_1, e_1 \rangle = 0$.

3. Necessary Conditions of Having an Invertible m-isometrical Extension

In an attempt towards solution of finding necessary conditions to obtain an invertible m-isometrical extension, we draw upon an interesting connection between $D_{-1} > 0$ and the invertibility of D_{-1} with the existence of a particular m-isometrical extension. Notice that in the following theorem we do not obtain an invertible m-isometrical extension.

Theorem 3.1. *Let $T \in L(H)$ be an m-isometry. The following statements are equivalent:*

(i) There exist a Hilbert space $K \supset H$ and an m-isometry $S \in L(K)$ such that $S_{|H} = T$ and $R(S) = H$.
(ii) $D_{-1} > 0$ and D_{-1} is invertible.

Proof. (i) ⇒ (ii): Let $x \in H$ and $y = S^{-1}x \in K$. For $n \in \mathbb{Z}$, denote

$$\widetilde{D}_n := \sum_{k=0}^{m-1} b_k(n) S^{*k} S^k, \quad D_n := \sum_{k=0}^{m-1} b_k(n) T^{*k} T^k$$

and for $n \in \mathbb{N}$

$$\widetilde{p}_x(n) := \|S^n x\|^2, \quad p_x(n) := \|T^n x\|^2,$$

where $b_k(n)$ is given by (2.1). Then

$$\begin{aligned}\langle \widetilde{D}_{-1}x, x\rangle &= \langle \widetilde{D}_{-1}Sy, Sy\rangle = \langle S^*\widetilde{D}_{-1}Sy, y\rangle = \langle \widetilde{D}_0 y, y\rangle = \|y\|^2\\ &= \sum_{k=0}^{m-1} b_k(-1)\langle S^{*k}S^k x, x\rangle = \sum_{k=0}^{m-1} b_k(-1)\langle T^k x, T^k x\rangle\\ &= \langle D_{-1}x, x\rangle.\end{aligned}$$

Then $\langle \widetilde{D}_{-1}x, x\rangle = \|y\|^2 = \langle D_{-1}x, x\rangle \geq 0$ for all $x \in H$. Also

$$\|D_{-1}x\|\|x\| \geq \langle D_{-1}x, x\rangle = \|y\|^2 \geq \frac{\|Sy\|^2}{\|S\|^2} = \frac{\|x\|^2}{\|S\|^2}.$$

So, $D_{-1} > 0$ and bounded below. Hence D_{-1} is invertible since D_{-1} is self adjoint operator.

$(ii) \Rightarrow (i)$: Consider the vector space $H \times H$ with a new seminorm

$$|||(h, h')||| := \|D_{-1}^{1/2}(Th + h')\|$$

and the subspace

$$N := \{(h, h') \in H \times H \ \ : \ \ |||(h, h')||| = 0\}.$$

Let $K := (H \times H)/N$ with the quotient norm

$$|||(h, h') + N||| := \|D_{-1}^{1/2}(Th + h')\|.$$

Then K is a normed space. Let us prove that $|||\cdot|||$ satisfies the parallelogram law. For $u = (h, h') + N$ and $v = (g, g') + N$ in K we have

$$\begin{aligned}|||u+v|||^2 + |||u-v|||^2 &= \langle D_{-1}(Th + h' + Tg + g'), Th + h' + Tg + g'\rangle\\ &\quad + \langle D_{-1}(Th + h' - Tg - g'), Th + h' - Tg - g'\rangle\\ &= 2\langle D_{-1}(Th + h'), Th + h'\rangle + 2\langle D_{-1}(Tg + g'), Tg + g'\rangle\\ &= 2|||u|||^2 + 2|||v|||^2.\end{aligned}$$

Henceforth, K is a pre-Hilbert space. The linear mapping $\phi : K \longrightarrow H$ defined by $\phi((h, h') + N) = Th + h'$ is an isomorphism. Indeed, ϕ is bounded since D_{-1} is an invertible operator. It is clear that ϕ is onto and bounded below since the square root of D_{-1} is a bounded operator. Hence K is complete and so it is a Hilbert space. Moreover,

$$\begin{aligned}|||(h, 0) + N|||^2 &= \|D_{-1}^{1/2}(Th)\|^2 = \langle D_{-1}Th, Th\rangle = \langle T^*D_{-1}Th, h\rangle\\ &= \|D_0 h\|^2 = \|h\|^2.\end{aligned}$$

So K contains H as a subspace and we identify $h \in H$ with $(h, 0) + N \in K$.

Define S on K by $\big((h,h')+N\big) := (Th+h',0)+N$. The operator S is well defined and bounded:

$$\begin{aligned}|||S\big((h,h')+N\big)|||^2 &= |||(Th+h',0)+N|||^2 = \|D_{-1}^{1/2}(T(Th+h'))\|^2\\ &= \langle D_{-1}(T(Th+h')), T(Th+h')\rangle = \langle D_0(Th+h'), Th+h'\rangle\\ &= \|Th+h'\|^2 \le \|D_{-1}^{-1/2}\|^2\|D_{-1}^{1/2}(Th+h')\|^2\\ &= \|D_{-1}^{-1/2}\|^2|||(h,h')+N|||^2.\end{aligned}$$

Clearly S is an extension of T. Let $h \in H$. We have identified h with $(h,0)+N \in K$ and $S((h,0)+N) = (Th,0)+N$. Also $SK = H$.

Let us prove that S is an m-isometry. Let $u = (h,h')+N \in K$ and write $y := Th+h' \in H$. We have that $Su = (y,0)+N$, $S^k u = (T^{k-1}y,0)+N$ and $|||S^k u|||^2 = \|D_{-1}^{1/2}(T^k y)\|^2 = \|T^{k-1}y\|^2$ for $k \in \mathbb{N}$. So

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k\binom{m}{k}|||S^k u|||^2 &= |||u|||^2 + \sum_{k=1}^{m}(-1)^k\binom{m}{k}|||S^k u|||^2\\ &= \langle D_{-1}y,y\rangle + \sum_{k=1}^{m}(-1)^k\binom{m}{k}\|T^{k-1}y\|^2\\ &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}p_y(k-1) = 0,\end{aligned}$$

since p_y has degree less or equal to $m-1$. Hence S is an m-isometry. □

The following result gives necessary conditions of having an invertible m-isometrical extension.

Proposition 3.2. *Let $T \in L(H)$ be a strict m-isometry.*

(1) If T is invertible, then $p_x(n) = \|T^n x\|^2 > 0$ for every $x \in H \setminus \{0\}$ and $n \in \mathbb{Z}$.
(2) If T has an invertible m-isometrical extension S, then $p_x(-k) := \|S^{-k}x\|^2 > 0$ for every $x \in H\setminus\{0\}$ and $k \in \mathbb{N}$, where $p_x(n) := \|T^n x\|^2$ for $n \in \mathbb{N}$. In particular, the degree of p_x is even for every $x \in H \setminus \{0\}$.
(3) If there exists an invertible m-isometrical extension of T, then $D_n > 0$ and invertible operator for every $n \in \mathbb{Z}$.

Proof. (1) Part (3) of Proposition 2.1 yields that $T^{*n}D_{-n}T^n = D_0 = I$ for $n \in \mathbb{N}$. So, for every $x \in H\setminus\{0\}$ and $n \in \mathbb{N}$,

$$p_x(-n) = \langle D_{-n}x,x\rangle = \langle T^{*-n}T^{-n}x,x\rangle = \|T^{-n}x\|^2 > 0,$$

since T^{-1} is an m-isometry.

(2) Let $x \in H$ and $n \in \mathbb{N}$. Denote by

$$p_x(n) := \langle D_n x, x\rangle := \sum_{k=0}^{m-1} b_k(n)\|T^k x\|^2$$
$$\widetilde{p}_x(n) := \langle \widetilde{D}_n x, x\rangle := \sum_{k=0}^{m-1} b_k(n)\|S^k x\|^2,$$

where S is an invertible m-isometrical extension of T. Clearly, $p_x(n) = \widetilde{p}_x(n)$ is a polynomial of degree less or equal to $m-1$. Observe that $p_x(-n) = \widetilde{p}_x(-n) = \|S^{-n}x\|^2$ for every $n \in \mathbb{N}$. □

Remark 3.3. (1) Observe that part (2) of the above Proposition implies that the degree of p_x is even if $p_x(n) > 0$ for every $n \in \mathbb{Z}$. Indeed, this is a different way to prove that there are no invertible strict m-isometries for even m. See also [4, Proposition 1.23].

(2) The conditions $D_n > 0$ and invertible operator for every $n \in \mathbb{Z}$ are not sufficient to define an invertible m-isometrical extension of T. Indeed, invertibility of D_n would suffice to construct an unbounded m-isometrical extension of T with dense range.

Proposition 3.2 allow us to obtain that some m-isometries have not an invertible m-isometrical extension.

Remark 3.4. Let $T \in L(H)$ be a strict m-isometry. Denote $p_x(n) := \|T^n x\|^2$, for $n \in \mathbb{N}$ and $x \in H \setminus \{0\}$. Then

(1) If $m = 1$, then $p_x(n) > 0$ for every $x \in H\setminus\{0\}$ and $n \in \mathbb{Z}$.
(2) If m is even, then there exist $x_0 \in H$ and $n_0 \in \mathbb{Z}$ with $n_0 < 0$ such that $p_{x_0}(n_0) \leq 0$.
(3) If m is odd, then it is possible that $p_x(n) > 0$ for every $x \in H \setminus \{0\}$ and $n \in \mathbb{Z}$ or there exist $x_0 \in H$ and $n_0 \in \mathbb{Z}$ with $n_0 < 0$ such that $p_{x_0}(n_0) \leq 0$.

In the following examples we present different behaviours of $p_x(n)$ with negative integer n for unilateral weighted shift.

Example 3.5. Let $p(n) = n^{m-1}$ with odd m. It is clear that $p_{e_j}(n) := \|S_w^n e_j\|^2 = \left(\frac{j+n}{j}\right)^{m-1}$ and $p_{e_j}(-j) = 0$. So, S_w can not have an invertible m-isometrical extension.

Example 3.6. Let $p(n) := \prod_{i=1}^{m-1}(mn+i)$ with odd m. It is clear that

$$p_{e_j}(n) := \|S_w^n e_j\|^2 = \frac{\prod_{i=1}^{m-1}(m(j+n)+i)}{\prod_{i=1}^{m-1}(mj+i)}.$$

If $j \geq n$, then $p_{e_j}(-n) > 0$. In other case, $p_{e_j}(-n) > 0$ since $m-1$ is even. As we will see later, S_w has an invertible m-isometrical extension by Theorem 5.1.

4. Characterization of Having an Invertible m-isometrical Extension

The main result of this paper is to obtain, for a fixed m-isometry, characterizations of having an invertible m-isometrical extension. In Proposition 3.2, we proved that a necessary condition is that the bilateral sequence of operators $(D_n)_{n\in\mathbb{Z}}$ must be strictly positive and invertible.

Now, we are in position to prove the main result.

Theorem 4.1. *Let $T \in L(H)$ be an m-isometry and let $(D_n)_{n\in\mathbb{Z}}$ be the bilateral sequence defined by (2.2). Denote $p_x(n) := \langle D_n x, x\rangle$ for every $x \in H\backslash\{0\}$ and $n \in \mathbb{Z}$. The following statements are equivalent:*

(i) There exist a Hilbert space $K \supset H$ and an invertible m-isometrical operator $S \in L(K)$ such that $S_{|H} = T$.

(ii) $p_x(j) > 0$ for every $x \in H\backslash\{0\}$, and $j \in \mathbb{Z}$ and

$$\sup\left\{\frac{p_x(j+1)}{p_x(j)} : x \in H \setminus \{0\},\ \ j \in \mathbb{Z}\right\} < \infty. \tag{4.7}$$

(iii) $D_n > 0$ and invertible for every $n \in \mathbb{Z}$, and

$$\sup\left\{\frac{\langle D_{-n+1}x, x\rangle}{\langle D_{-n}x, x\rangle} : x \in H,\ \ \|x\| = 1,\ \ n \in \mathbb{N}\right\} < \infty. \tag{4.8}$$

Proof. $(i) \Rightarrow (ii)$: Let $x \in H \setminus \{0\}$. Then

$$\|S^{j+1}x\|^2 = \|T^{j+1}x\|^2 = p_x(j+1) > 0$$

for $j \in \mathbb{Z}$ and

$$\frac{p_x(j+1)}{p_x(j)} = \frac{\|S^{j+1}x\|^2}{\|S^j x\|^2} \leq \|S\|^2.$$

So, we get (4.7).

$(ii) \Rightarrow (iii)$: By parts (1) and (2) of Proposition 2.1 we have that $D_n > 0$ and invertible for $n \in \mathbb{N}$. By hypothesis, $D_j > 0$ for $j \in \mathbb{Z}$ since $p_x(j) = \langle D_j x, x\rangle$. Let us prove that D_{-n} are bounded below for every $n \in \mathbb{N}$. The condition (4.7) yields that there exists $M > 0$ such that

$$p_x(-n) \geq \frac{p_x(-n+1)}{M} \geq \frac{p_x(0)}{M^n} = \frac{\|x\|^2}{M^n}$$

hence

$$\|D_{-n}^{1/2}x\|^2 \geq \frac{\|x\|^2}{M^n},$$

for every $x \in H \setminus \{0\}$ and $n \in \mathbb{N}$. Therefore D_{-n} is bounded below for $n \in \mathbb{N}$ and hence invertible.

It is remained to prove (4.8). Indeed, (4.8) is an immediate consequence of (4.7) using the identification $p_x(j) = \langle D_j x, x\rangle$ for every $x \in H\setminus\{0\}$ and $j \in \mathbb{Z}$.

$(iii) \Rightarrow (i)$: Let V be the vector space of all sequences $(h_1, h_2, \dots)$ of elements of H with finite support, that is, there exists $n \in \mathbb{N}$ such that $h_j = 0$ for $j > n$. Define a new seminorm on V by

$$|||(h_1, h_2, \dots)|||^2 := \langle D_{-n} y, y\rangle,$$

where $n \in \mathbb{N}$ is any integer satisfying $h_j = 0$ for $j > n$ and $y := \sum_{j=1}^{n} T^{n-j} h_j$.

The seminorm $|||\cdot|||$ does not depend on the choice of n. Indeed, if $h_j = 0$ for $j > n$, $r = n + n_0$ with $n_0 \in \mathbb{N}$, and $y = \sum_{j=0}^{n} T^{n-j} h_j$, then

$$\begin{aligned}
&\left\langle D_{-r} \sum_{j=1}^{r} T^{r-j} h_j, \sum_{i=1}^{r} T^{r-i} h_i \right\rangle \\
&\quad = \left\langle D_{-(n+n_0)} T^{n_0} \left(\sum_{j=1}^{n+n_0} T^{n-j} h_j \right), T^{n_0} \left(\sum_{i=1}^{n+n_0} T^{n-i} h_i \right) \right\rangle \\
&\quad = \left\langle T^{*n_0} D_{-(n+n_0)} T^{n_0} \left(\sum_{j=1}^{n} T^{n-j} h_j \right), \sum_{i=1}^{n} T^{n-i} h_i \right\rangle = \langle D_{-n} y, y\rangle
\end{aligned}$$

where the last equality is by part (3) of Proposition 2.1.

Let $N := \{(h_1, h_2, \dots) \in V \ : \ |||(h_1, h_2, \dots)||| = 0\}$ and let K be the completion of V/N.

Let us prove that K is a pre-Hilbert space. For that, it is enough to prove that $|||\cdot|||$ satisfies the parallelogram law. Let $u := (h_1, h_2, \cdots) + N$, $v := (g_1, g_2, \cdots) + N \in V/N$, $n \in \mathbb{N}$ such that $h_j = 0 = g_j$ for $j > n$ and $x := \sum_{j=1}^{n} T^{n-j} h_j$, $y := \sum_{j=1}^{n} T^{n-j} g_j$. Then

$$\begin{aligned}
|||u+v|||^2 + |||u-v|||^2 &= \langle D_{-n}(x+y), x+y\rangle + \langle D_{-n}(x-y), x-y\rangle \\
&= 2(|||u|||^2 + |||v|||^2).
\end{aligned}$$

For each $h \in H$ we have $|||(h, 0, 0, \dots) + N|||^2 = \langle D_{-1} Th, Th\rangle = \langle D_0 h, h\rangle = \|h\|^2$.

Let L be the closed subspace generated by $(h, 0, \cdots) + N$ with $h \in H$ and define ϕ on H taking values on L by $\phi(h) := (h, 0, \cdots) + N$. Then $\|h\|^2 = |||\phi(h)|||^2$ and $R(\phi) = L$. For each $h \in H$ we can identify h with $(h, 0, \dots) + N \in K$. So, K contains H as a subspace.

Define S on V/N by $S((h_1, h_2, \cdots) + N) := (Th_1 + h_2, h_3, \cdots) + N \in V/N$. Then the definition of S is correct and S is bounded. Indeed, let $u := (h_1, h_2, \cdots) + N \in V/N$, $n \in \mathbb{N}$ such that $h_j = 0$ for $j > n$ and $y :=$

$\sum_{j=1}^{n} T^{n-j} h_j$. Denote $(\widetilde{h}_1, \widetilde{h}_2, \cdots) := (Th_1 + h_2, h_3, \cdots)$. Then

$$|||Su|||^2 = |||(Th_1 + h_2, h_3, \cdots) + N|||^2 = \langle D_{-(n-1)}\widetilde{y}, \widetilde{y}\rangle$$

where

$$\widetilde{y} := \sum_{j=1}^{n-1} T^{n-1-j}\widetilde{h}_j = T^{n-1}(Th_1 + h_2) + \sum_{j=2}^{n-1} T^{n-1-j}\widetilde{h}_j = y\ .$$

Then $|||Su|||^2 = \langle D_{-(n-1)}y, y\rangle = p_y(-n+1)$. Repeating the process we have that

$$|||S^k u|||^2 = p_y(-n+k),$$

for $k = 0, \dots m$. Therefore

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} |||S^k u|||^2 = \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_y(-n+k) = 0,$$

since p_y has degree less or equal to $m-1$. By continuity, S is an m-isometry.

It is easy to see that $R(S) \supset V + N$. So the range of S is dense, and consequently S is an invertible m-isometry. □

Moreover, the invertible extension $S \in L(K)$ is defined uniquely (up to the unitary equivalence) if we assume that S is minimal, i.e., $K = \bigvee_{k\geq 0} S^{-k}H$.

We will prove that the converse of part (3) of Proposition 3.2 is not true in general, that is, if $D_n > 0$ and invertible for $n \in \mathbb{Z}$ are not sufficient to have an invertible m-isometrical extension of an m-isometry. Firstly, we need a previous result on m-isometries.

Proposition 4.2. *Let $(T_n)_{n\in\mathbb{N}} \subset L(H)$ be a uniformly bounded sequence of m-isometries. Then $T = T_1 \oplus T_2 \oplus \cdots$ is an m-isometry on $\ell^2(H)$.*

Proof. Since $(T_n)_{n\in\mathbb{N}}$ is a uniformly bounded, then $T = T_1 \oplus T_2 \oplus \cdots$ is well-defined on $\ell^2(H)$.

Let $x = (x_1, x_2, \cdots) \in \ell^2(H)$. Denote $p_{x_n}(k) := \|T_n^k x_n\|^2$. Since $(T_n)_{n\in\mathbb{N}}$ is a sequence of m-isometries, then $(p_{x_n}(k))_{n\in\mathbb{N}}$ is a sequence of polynomials of degree less or equal to $m-1$. Fixed $k \in \mathbb{N}$,

$$p_x(k) := \|T^k x\|^2 = \sum_{n=1}^{\infty} \|T_n^k x_n\|^2 = \sum_{n=1}^{\infty} p_{x_n}(k)$$

is a polynomial of degree less or equal to $m-1$. Hence T is an m-isometry. □

It is possible to exhibit an example of m-isometry with odd m such that $D_n > 0$ and invertible for every $n \in \mathbb{Z}$ but not fulfilling the hypothesis of Theorem 4.1. In order to simplify the presentation we include an example with a 3-isometry.

Example 4.3. Let $q_n(j) := j^2 + j(2 - \frac{1}{n}) + 1$. Let H be a Hilbert space with an orthonormal basis $(e_{n,j})_{n,j\in\mathbb{N}}$ and $K := \ell^2(H)$. Define $T \in L(K)$ by

$$Te_{n,j} := \sqrt{\frac{q_n(j+1)}{q_n(j)}}e_{n,j+1}$$

for any $n, j \in \mathbb{N}$. Then

(1) T is a 3-isometry on K.
(2) $p_x(k) > 0$ for every $x \in K\backslash\{0\}$ and $k \in \mathbb{Z}$, where $p_x(n) := \|T^n x\|^2$ for $n \in \mathbb{N}$.
(3) $D_n > 0$ and invertible for $n \in \mathbb{Z}$.
(4) There is no invertible 3-isometrical extension of T.

Proof: It is clear that $q_n(j) > 0$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$.
Let $x = (x_1, x_2, \cdots) = (\sum_{n=1}^{\infty} \alpha_{n,1}e_{n,1}, \sum_{n=1}^{\infty} \alpha_{n,2}e_{n,2}, \cdots) \in K$. Then

$$T(x_1, x_2, \cdots) := (0, T_1x_1, T_2, x_2, \cdots),$$

where

$$T_ix_i := T_i\left(\sum_{n=1}^{\infty} \alpha_{n,i}e_{n,i}\right) = \sum_{n=1}^{\infty} \alpha_{n,i}w_{n,i}e_{n,i+1}$$

and

$$w_{n,i} := \sqrt{\frac{q_n(i+1)}{q_n(i)}}.$$

By Proposition 4.2, the operator T is a 3-isometry, since T_n is a 3-isometry for every $n \in \mathbb{N}$ and also $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, that is

$$\sup_{n\in\mathbb{N}} \|T_n\| \leq \sup_{n,i\in\mathbb{N}} \sqrt{\frac{q_n(i+1)}{q_n(i)}} < M$$

for some positive constant M.

Let us prove that $p_x(k) > 0$ for every $x \in K\backslash\{0\}$ and $k \in \mathbb{Z}$. Let $x = (x_1, x_2, \cdots) = (\sum_{n=1}^{\infty} \alpha_{n,1}e_{n,1}, \sum_{n=1}^{\infty} \alpha_{n,2}e_{n,2}, \cdots) \in K\backslash\{0\}$ and $k \in \mathbb{N}$. Then

$$\begin{aligned} p_x(k) := \|T^kx\|^2 &= \|(0, \cdots, 0, T_kT_{k-1}\cdots T_1x_1, T_{k+1}T_k\cdots T_2x_2, \cdots\|^2 \\ &= \|[\| 3](0, \cdots, 0, \sum_{n=1}^{\infty} \alpha_{n,1}\sqrt{\frac{q_n(k+1)}{q_n(1)}}e_{n,k+1}, \cdots)^2 \\ &= \sum_{j=1}^{\infty} \|[\| 3]\sum_{n=1}^{\infty} \alpha_{n,j}\sqrt{\frac{q_n(k+j)}{q_n(j)}}e_{n,k+j}^2 \\ &= \sum_{n,j=1}^{\infty} |\alpha_{n,j}|^2\frac{q_n(k+j)}{q_n(j)} > 0 \end{aligned}$$

for $k \in \mathbb{N}$. Notice that

$$D_{-n} := \frac{(n+1)(n+2)}{2} I - n(n+2)T^*T + \frac{n(n+1)}{2} T^{*2}T^2,$$

is a diagonal operator given by $D_{-n}e_{m,j} = \lambda_{-n}(k,j)e_{k,j}$ where

$$\begin{aligned}\lambda_{-n}(k,j) &:= \frac{1}{2q_k(j)}\left((n+1)(n+2)q_k(j) - n(n+2)q_k(j+1) + n(n+1)q_k(j+2)\right)\\ &= \frac{1}{2q_k(j)}\left(j^2(n^2+2n+2) + j\left(-\frac{n^2}{k} + 4n^2 - 2\frac{n}{k} + 4n - \frac{2}{k} + 4\right)\right.\\ &\quad \left. -\frac{n^2}{k} + 6n^2 + 4n + 2\right) > 0,\end{aligned}$$

for $n, k, j \in \mathbb{N}$. So, it is immediate that D_{-n} is invertible for $n \in \mathbb{N}$.

In order to finish the proof, let us prove that there is no invertible 3-isometrical extension of T. Taking into account that

$$\frac{p_{e_{n,1}}(-1)}{p_{e_{n,1}}(-2)} = \frac{q_n(0)}{q_n(-1)} = n,$$

we have that

$$\sup\left\{\frac{p_x(j+1)}{p_x(j)} \;:\; x \in K \setminus \{0\},\;\; j \in \mathbb{Z}\right\} = \infty.$$

□

5. Some Particular Cases

In this section, the goal is to study two different examples of m-isometries, the ℓ-Jordan isometry and unilateral weighted shift that are m-isometries for some m.

In the case of unilateral weighted shift we can obtain a nice characterization of invertible m-isometrical extensions of an m-isometry, as a consequence of Theorem 4.1.

Theorem 5.1. *Let H be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $S_w \in L(H)$ be an m-isometrical unilateral weighted shift associated to the weight $w := (w_n)_{n\in\mathbb{N}}$. Then S_w has an invertible m-isometrical extension if and only if $p_{e_1}(n) > 0$ for every $n \in \mathbb{Z}$, where $p_{e_1}(n) := \|S_w^n e_1\|^2$ for $n \in \mathbb{N}$.*

Proof. If S_w has an invertible m-isometrical extension S, then $p_x(n) := \|S^n x\|^2 > 0$ for every $x \in H\setminus\{0\}$ and $n \in \mathbb{Z}$, by Proposition 3.2. Hence $p_{e_1}(n) > 0$ for $n \in \mathbb{Z}$.

Let us prove the sufficient condition. Suppose that $p_{e_1}(n) > 0$ for $n \in \mathbb{Z}$. A first consequence is that m is odd. By equality (2.4), $p_{e_1}(n)$ is a polynomial of degree $m-1$. Hence

$$\lim_{n\to\infty} \frac{p_{e_1}(-n+1)}{p_{e_1}(-n)} = 1,$$

and

$$\inf\left\{\frac{p_{e_1}(-n+1)}{p_{e_1}(-n)} \ : \ n\in\mathbb{N}\right\}>0.$$

Let K be a Hilbert space with $(e_n)_{n\in\mathbb{Z}}$ an orthonormal basis. Define $T_\beta \in L(K)$ by $T_\beta e_n = \beta_n e_{n+1}$ where $\beta_n = \sqrt{\frac{p_{e_1}(n)}{p_{e_1}(n-1)}}$ for $n \in \mathbb{Z}$. By [1, Theorem 19] we have that T_β is an m-isometry, since $p_{e_1}(n)$ is a polynomial of degree $m-1$ by (2.4). Moreover, T_β is an invertible extension of S_w and the desired result is proved. □

Remark 5.2. In the above theorem, it is possible to obtain the same information with different elements of the orthogonal basis, as a consequence of equality (2.4). Indeed, in the conditions of Theorem 5.1 the following statements are equivalent:

(1) S_w has an invertible m-isometrical extension.
(2) $p_{e_1}(n) > 0$ for $n \in \mathbb{Z}$.
(3) $p_{e_j}(n) > 0$ for $n \in \mathbb{Z}$ and some $j \in \mathbb{N}$.
(4) $p_{e_j}(n) > 0$ for $n \in \mathbb{Z}$ and $j \in \mathbb{N}$.

Let us obtain a first approach to ℓ-Jordan isommetries. In the next result we obtain that any 2-Jordan isometry operator admits an invertible 3-isometric extension, as a particular case of Theorem 4.1.

Corollary 5.3. *Let $T \in L(H)$ be a 2-Jordan isometry operator. Then T has an invertible 2-Jordan isometry extension.*

Proof. Let T be a 2-Jordan isometry operator, that is $T = A + Q$, where A is an isometry and Q is a 2-nilpotent operator such that $AQ = QA$. By (2.2) we obtain that

$$\begin{aligned} D_{-n} &= \frac{(n+1)(n+2)}{2}I - n(n+2)T^*T + \frac{n(n+1)}{2}T^{*2}T^2 \\ &= I - n(A^*Q + Q^*A) + n^2Q^*Q. \end{aligned}$$

Then

$$\langle D_{-n}x, x\rangle = \|x\|^2 - n(\langle Qx, Ax\rangle + \langle Ax, Qx\rangle) + n^2\|Qx\|^2.$$

Let us prove that $\langle D_{-n}x, x\rangle > 0$ for every $x \in H$ such that $\|x\| = 1$ and $n \in \mathbb{N}$. It is enough to prove that

$$n^2\|Qx\|^2 + 1 > 2nRe(\langle Ax, Qx\rangle), \tag{5.9}$$

where $Re(z)$ denotes the real part of z. If $Re(\langle Ax, Qx\rangle) \le 0$, then (5.9) is clear. Assume that $Re(\langle Ax, Qx\rangle) > 0$. Then

$$Re(\langle Ax, Qx\rangle) = |Re(\langle Ax, Qx\rangle)| \le |\langle Ax, Qx\rangle| \le \|Ax\|\|Qx\| \le \|Q\|.$$

If $|\langle Ax, Qx\rangle| = \|Ax\|\|Qx\|$, then the vectors Ax and Qx are linearly dependent, so there exists λ such that $Qx = \lambda Ax$. Then $\lambda = 0$, since $0 =$

$\|Q^2x\| = |\lambda|^2\|A^2x\| = |\lambda|^2$ and therefore $\|Qx\| = 0$, which is an absurd with $Re(\langle Ax, Qx\rangle > 0$. If $|\langle Ax, Qx\rangle| < \|Ax\|\|Qx\|$, then

$$2nRe(\langle Ax, Qx\rangle) < 2n\|Qx\| \leq n^2\|Qx\|^2 + 1.$$

So, $\langle D_{-n}x, x\rangle > 0$ for every $x \in H$ such that $\|x\| = 1$ and all $n \in \mathbb{N}$.

In order to get the result, it is enough to prove that (4.8) is bounded. Let $x \in H$ such that $\|x\| = 1$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\frac{\langle D_{-n+1}x, x\rangle}{\langle D_{-n}x, x\rangle} &= 1 + \frac{2Re(\langle Ax, Qx\rangle) + (-2n+1)\|Qx\|^2}{1 - 2nRe(\langle Ax, Qx\rangle) + n^2\|Qx\|^2} \\ &\leq 1 + \left|\frac{2Re(\langle Ax, Qx\rangle) + (-2n+1)\|Qx\|^2}{1 - 2nRe(\langle Ax, Qx\rangle) + n^2\|Qx\|^2}\right| \\ &\leq 1 + \frac{2\|Q\| + (2n-1)\|Q\|^2}{1 - 2n\|Q\| - n^2\|Q\|^2}\end{aligned}$$

converges to zero as n tends to infinity. Hence

$$\sup\left\{\frac{\langle D_{-n+1}x, x\rangle}{\langle D_{-n}x, x\rangle} \; : \; x \in H, \;\; \|x\| = 1, \;\; n \in \mathbb{N}\right\} < \infty.$$

□

Corollary 5.4. *Let T, $C \in L(H)$ such that $TC = CT$.*

(1) If T is an isometry, then $\widetilde{T} := \begin{pmatrix} T & C \\ 0 & T \end{pmatrix}$ has an invertible 3-isometric extension on $K \supset H \oplus H$.

(2) If λT is an isometry for some $\lambda \in \mathbb{C}$, then $\lambda\widetilde{T} = \lambda\begin{pmatrix} T & C \\ 0 & T \end{pmatrix}$ has an invertible 3-isometric extension on $K \supset H \oplus H$.

Proof. (1) It is clear that $\widetilde{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ is a 2-Jordan isometry operator. Therefore the result is consequence of Corollary 5.3.

Applying (1) to the operator λT we obtain (2). □

A similar result of part (1) of Corollary 5.4 was obtained in [8, Corollary 4.4]. That is, if $T \in L(H)$ is a contraction and $C \in L(H)$ such that $TC = CT$, then $\widetilde{T}$ has a 3-isometric lifting on $K \supset H \oplus H$.

In the next theorem we can improve Corollary 5.3. Indeed, we prove that every ℓ-Jordan isometry has an invertible ℓ-Jordan isometry extension. The first part of our proof is based in the construction by Douglas [13], as it is presented by Laursen and Neumann in the monograph [15, Proposition 1.6,6].

Theorem 5.5. *Let $T \in L(H)$ be an ℓ-Jordan isometry. Then there exist a Hilbert space K and $S \in L(K)$, such that H is isometrically embedded in K and S is an invertible ℓ-Jordan isometry extension of T.*

Proof. As T is an ℓ-Jordan isometry, there are an isometry $A \in L(H)$ and an ℓ-nilpotent operator $Q \in L(H)$ such that $AQ = QA$ and $T = A + Q$.

Let K_0 be the linear space of all the sequences $u = (u_n)_{n\in\mathbb{N}}$ in H such that there is $m \in \mathbb{N}$ satisfying $u_{m+k} = A^k u_m$, for $k \in \mathbb{N}$. Define, for $u, v \in K_0$,

$$\langle u, v\rangle_0 := \lim_{n\to\infty} \langle u_n, v_n\rangle \ ,$$

being $\langle\cdot,\cdot\rangle$ the inner product on H. Note that there exists $m \in \mathbb{N}$ such that $\langle u_m, v_m\rangle = \langle A^k u_m, A^k v_m\rangle = \langle u_{m+k}, v_{m+k}\rangle$, so the sequence $(\langle u_n, v_n\rangle)_{n\in\mathbb{N}}$ is eventually constant, that is, there exists $k_0 \in \mathbb{N}$ such that $\langle u_n, v_n\rangle$ is constant for $n > k_0$. It is routine to verify what $\langle\cdot,\cdot\rangle_0$ is a semi-inner product on K_0. Therefore K_0 is a semi pre-Hilbert space. Moreover,

$$\|u\|_0^2 := \langle u, u\rangle_0 = \lim_{n\to\infty} \langle u_n, u_n\rangle = \lim_{n\to\infty} \|u_n\|^2$$

defines a seminorm $\|\cdot\|_0$ on K_0.

Let $M := \{u \in K_0 : \langle u, u\rangle_0 = \|u\|_0^2 = 0\}$. Then M is a closed subspace of K_0 and we consider the quotient space K_0/M. In this space are defined, for $u, v \in K_0$,

$$\begin{aligned}&\langle u + M, v + M\rangle := \langle u, v\rangle_0 \quad \text{and}\\ &\|u + M\|^2 := \langle u + M, u + M\rangle = \langle u, u\rangle_0 = \|u\|_0^2 \ ,\end{aligned}$$

and we obtain that K_0/M is a pre-Hilbert space.

Denote by K the Hilbert space what it is the completion of K_0/M. The operator $J \in L(H, K)$, defined by $Jx := (A^n x)_{n\in\mathbb{N}} + M$ for $x \in H$, satisfies that

$$\|Jx\| = \|(A^n x)_{n\in\mathbb{N}} + M\| = \|(A^n x)_{n\in\mathbb{N}}\|_0 = \lim_{n\to\infty} \|A^n x\| = \|Ax\| = \|x\| \ ,$$

hence J is an isometry. So K contains an isometric copy of H. It is clear that $J(H)$ is a closed subspace of K.

In order to define $B \in L(K)$, we define an isometry on K_0/M by

$$B((u_n)_{n\in\mathbb{N}} + M) := (Au_n)_{n\in\mathbb{N}} + M \ ,$$

for every $(u_n)_{n\in\mathbb{N}} + M \in K_0/M$. Note that B is a linear isometry whose range contains K_0/M; in fact, given $(v_n)_{n\in\mathbb{N}} + M = (v_1, ..., v_m, Av_m, A^2 v_m, ...) + M$, we have that

$$\begin{aligned}B((\underbrace{0, ..., 0}_{m}, v_m, Av_m, A^2 v_m, ...) + M) &= (\underbrace{0, ..., 0}_{m}, Av_m, A^2 v_m, A^3 v_m, ...) + M\\ &= (v_1, \cdots, v_m, Av_m, A^2 v_m, \cdots) + M.\end{aligned}$$

As K_0/M is dense in K, we have that B can be extended to an invertible isometry defined on K. Moreover, B can be considered as an extension of A since, for $x \in H$,

$$BJx = B((A^n x)_{n\in\mathbb{N}} + M) = (A^{n+1} x)_{n\in\mathbb{N}} + M = JAx \ .$$

That is, $BJ = JA$.

Define $P \in L(K)$ in the following way

$$P((u_n)_{n\in\mathbb{N}} + M) = (Qu_n)_{n\in\mathbb{N}} + M \ ,$$

for every $(u_n)_{n\in\mathbb{N}} + M \in K_0/M$. It is clear that P is an ℓ-nilpotent. Let us prove that B and P commute. Taking into account that $AQ = QA$, we have that

$$\begin{aligned} BP((u_n)_{n\in\mathbb{N}} + M) &= B((Qu_n)_{n\in\mathbb{N}} + M) = (AQu_n)_{n\in\mathbb{N}} + M \\ &= (QAu_n)_{n\in\mathbb{N}} + M = P((Au_n)_{n\in\mathbb{N}} + M) \\ &= PB((u_n)_{n\in\mathbb{N}} + M) \ . \end{aligned}$$

for every $(u_n)_{n\in\mathbb{N}} + M \in K_0/M$. Therefore, $S := B + P \in L(K)$ is an ℓ-Jordan isometry that extends T. Moreover, S is an invertible since $\sigma(S) = \sigma(B)$ and B is an invertible isometry. So the proof is finished. □

An operator $T \in L(H)$ is a *doubly ℓ-Jordan isometry* if $T = A + Q$ is an ℓ-Jordan isometry operator such that the ℓ-nilpotent $Q \in L(H)$ which commutes with A also commutes with A^*. For all scalar λ with $|\lambda| = 1$ and an ℓ-nilpotent operator Q, we have that $\lambda I + Q$ is a doubly ℓ-Jordan isometry.

Corollary 5.6. *Let $T \in L(H)$ be a doubly ℓ-Jordan isometry. Then there exist a Hilbert space K, such that H is isometrically embedded in K and an invertible doubly ℓ-Jordan isometry extension $S \in L(K)$ of T.*

Remark 5.7. We use the notation of the proof of Theorem 5.5.

(1) It is easy to prove that the orthogonal subspace of $J(H)$, $J(H)^{\perp}$ is the closure of the subspace of all classes

$$(u_n)_{n\in\mathbb{N}} + M = (u_1, ..., u_m, Au_m, A^2u_m, ...) + M \in K_0/M$$

such that $u_m \in R(A^m)^{\perp}$.

(2) The decomposition $K = J(H) \oplus J(H)^{\perp}$ gives rise to the representation of B as a operator matrix:

$$B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \tag{5.10}$$

being $B_1 \in L(J(H))$, $B_2 \in L(J(H)^{\perp}, J(H))$ and $B_3 \in L(J(H)^{\perp})$. Notice that $J(H)$ is a closed invariant subspace of B.

(3) The operator P is defined by the following operator matrix, associated to the decomposition $K = J(H) \oplus J(H)^{\perp}$,

$$P = \begin{pmatrix} P_1 & P_2 \\ 0 & P_3 \end{pmatrix} \tag{5.11}$$

being $P_1 \in L(J(H))$, $P_2 \in L(J(H)^{\perp}, J(H))$ and $P_3 \in L(J(H)^{\perp})$. Notice that $J(H)$ is a closed invariant subspace of P.

(4) If T is a doubly ℓ-Jordan isometry, then $P_2 = 0$ in (5.11). For this purpose only it is necessary to prove that if $(u_n)_{n\in\mathbb{N}} + M \in J(H)^\perp$, then $P((u_n)_{n\in\mathbb{N}} + M) \in J(H)^\perp$, and that $BP^* = P^*B$. In fact, given $u = (u_1, ..., u_m, Au_m, A^2u_m, ...)$ such that $u_m \in R(A^m)^\perp$, we have that $Qu_m \in R(A^m)^\perp$ since, for all $x \in H$,

$$\langle Qu_m, A^m x\rangle = \langle u_m, Q^* A^m x\rangle = \langle u_m, A^m Q^* x\rangle = 0 \ ,$$

because $Q^*A = AQ^*$. Therefore $P((u_n)_{n\in\mathbb{N}} + M) = (Qu_1, ..., Qu_m, AQu_m, A^2Qu_m, ...) + M \in J(H)^\perp$. Hence $P(J(H)^\perp) \subset J(H)^\perp$.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest and the manuscript has no associated data.

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