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Results in Mathematics

On Topologies Generated by Lower Porosity

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Abstract. In the paper we study properties of a lower porosity of a set in a normed space $(X, \| \|)$. Two topologies $\underline{p}(X, \| \|)$ and $\underline{s}(X, \| \|)$ on X generated by the lower porosity are defined. Relationships between these topologies and, previously defined by V. Kelar and L. Zajíček, topologies $p(X, \| \|)$ and $s(X, \| \|)$ are studied. Applying topologies $\underline{p}(X, \| \|)$ and $\underline{s}(X, \| \|)$ we characterize maximal additive class of lower porouscontinuous functions. Some relevant properties of defined topologies are considered.

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Keywords. Porosity, lower porosity, topology generated by lower porosity, lower porouscontinuity, maximal additive class.

1. Preliminaries

Porosity of a set, defined in [4], is the notion of smallness more restrictive than nowhere density and meagerness. It can be defined in arbitrary metric space. The main idea is that we modify the "ball" definition of nowhere density by the request that the sizes of holes should be estimated. Usually, the notion of the (upper) porosity of sets is used in many aspects, see for example [4–6, 10, 12, 13]. We deal with the lower porosity, which also be considered in some papers, [11, 12]. It is known that there are big differences between the lower and the upper porosities. In [12, 13] some properties of the lower porosity in metric spaces are presented, whereas in [11] some properties of the lower porosity on $\mathbb{R}^2$ and of lower porouscontinuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ are studied.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. For $f : X \to Y$ and $Z \subset X$, by $f_{\restriction Z}$ we mean the restriction of f to Z. For the whole paper $(X, \| \|)$ denotes a normed space. The symbols $\mathrm{cl}_{\mathcal{T}}\, Z$, $\mathrm{int}_{\mathcal{T}}\, Z$ and $\mathrm{bd}_{\mathcal{T}}\, Z$ denote the closure, the interior and the boundary of $Z \subset X$ with respect to a topology $\mathcal{T}$ in X. The open ball in $(X, \| \ \|)$ with the center $x \in X$ and the radius $\varrho > 0$ is denoted by $B(x, \varrho)$. Similarly, by $S(x, \varrho)$ and $\overline{B}(x, \varrho)$ we denote the sphere and the closed ball with the center x and the radius ϱ, respectively. By 0_X we denote the zero element of X.

We will also consider spaces $(\mathbb{R}^n, \| \ \|_n)$, $(\mathbb{R}^n, \| \ \|_{\max})$ for $n \geq 1$ and $(l_\infty, \| \ \|_{\sup})$, where $\| \ \|_n$ is the natural norm in $\mathbb{R}^n$, $\| \ \|_{\max}$ is a norm in $\mathbb{R}^n$ defined by $\|(x_1, \ldots, x_n)\|_{\max} = \max\{|x_i| \colon i = 1, \ldots, n\}$, l_∞ is the space of all bounded real sequences and $\| \ \|_{\sup}$ is a norm in l_∞ defined by $\|(x_1, \ldots, x_n)\|_{\sup} = \sup\{|x_i| \colon i \geq 1\}$. Spaces $(\mathbb{R}^n, \| \ \|_{\max})$ and $(l_\infty, \| \ \|_{\sup})$ play important role in our paper and we will need some their properties.

For each $n \geq 1$ and $\zeta \in \{-1, 1\}^n$, $\zeta = (\zeta_1, \ldots, \zeta_n)$, we define

$$H^\zeta = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \forall i \leq n \ (x_i \leq 0 \text{ if } \ \zeta_i = -1 \ \text{ and } \ x_i \geq 0 \text{ if } \ \zeta_i = 1)\}.$$

For each $\zeta \in \{-1, 1\}^\omega$, $\zeta = (\zeta_1, \zeta_2, \ldots)$, we define

$$H^\zeta = \{x = (x_1, x_2, \ldots) \in l_\infty \colon \forall i \geq 1 \ (x_i \leq 0 \text{ if } \ \zeta_i = -1 \ \text{ and } \ x_i \geq 0 \text{ if } \ \zeta_i = 1)\}.$$

Lemma 1.1. *Let $(X, \| \ \|)$ be $(\mathbb{R}^n, \| \ \|_{\max})$ or $(l_\infty, \| \ \|_{\sup})$. For each $R > 0$ and for each ball $B(x, \eta) \subset B(0_X, R)$ such that $\eta \in \left(\frac{R}{4}, \frac{R}{2}\right)$ there exist $y \in X$ and $\zeta \in \{-1, 1\}^n$ or $\zeta \in \{-1, 1\}^\omega$ such that*

$$B(y, \eta - \tfrac{R}{4}) \subset B(x, \eta) \cap H^\zeta \cap B(0_X, \tfrac{R}{2}).$$

Proof. (See Fig. 1) it is enough to show that for each interval $[\alpha, \beta] \subset [-R, R]$, where $\beta - \alpha = 2\eta$, there exists (α_1, β_1) such that $\beta_1 - \alpha_1 = 2\eta - \frac{R}{2}$ and $(\alpha_1, \beta_1) \subset \left(\frac{R}{2}, 0\right) \cup \left(0, \frac{R}{2}\right)$.

If $0 \in [\alpha, \beta]$ then $\beta \geq \eta$ or $\alpha \leq -\eta$. Therefore as (α_1, β_1) we take $(0, \beta)$ or $(\alpha, 0)$, because $\eta \geq \eta + (\eta - \frac{R}{2}) = 2\eta - \frac{R}{2}$. If $0 \notin [\alpha, \beta]$ then $(R - 2\eta, \frac{R}{2}) \subset (\alpha, \beta)$ or $(-\frac{R}{2}, 2\eta - R) \subset (\alpha, \beta)$. Then $\frac{R}{2} - (R - 2\eta) = 2\eta - \frac{R}{2}$, which completes the proof. □

Let $(X, \| \ \|)$ be a normed space. By $\mathcal{T}_{\| \ \|}$ we denote a topology in X generated by $\| \ \|$. Sometimes we consider another topology $\mathcal{T}$ in X. We say that $f \colon X \to \mathbb{R}$ is $\mathcal{T}$-continuous at some $x \in X$ if f is continuous as a function $f \colon (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{\| \ \|_1})$.

Now, we recall definitions of the (upper) porosity and the lower porosity in a normed space. These notions can be defined in an arbitrary metric space but we present them only for a normed space $(X, \| \ \|)$, because only such the case will be considered in the paper. Let $U \subset X$, $x \in X$ and $R > 0$. Then, according to [4,12], by $\gamma(x, R, U)$ we denote the supremum of the set of all $\varrho > 0$ for which there exists $y \in X$ such that $B(y, \varrho) \subset B(x, R) \setminus U$. The number $p(U, x) = 2 \limsup_{R \to 0^+} \frac{\gamma(x, R, U)}{R}$ is called the (upper) porosity of U at x. Obviously, $p(U, x) = p(\mathrm{cl}_{\mathcal{T}_{\| \ \|}} U, x)$ for $U \subset X$ and $x \in X$.

Similarly, the number $\underline{p}(U, x) = 2 \liminf_{R \to 0^+} \frac{\gamma(x, R, U)}{R}$ is called the lower porosity of U at x. Clearly, $\underline{p}(U, x) = \underline{p}(\mathrm{cl}_{\mathcal{T}_{\| \ \|}} U, x)$ and $\underline{p}(U, x) \leq p(U, x)$ for

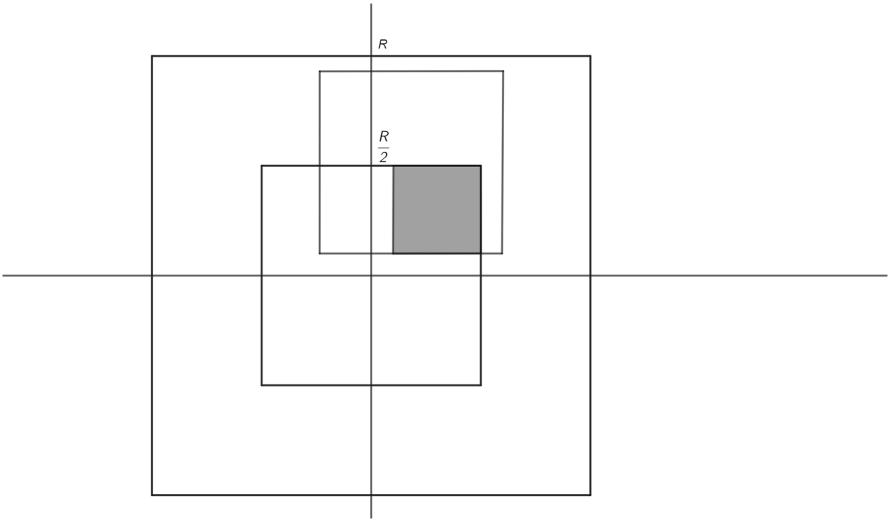

FIGURE 1. Construction in $(\mathbb{R}^2, \| \|_{\max})$

$U \subset X$ and $x \in X$. Moreover, for $U \subset V \subset X$ we have $\underline{p}(V,x) \le \underline{p}(U,x)$, $\underline{p}(U,x) \in [0,2]$ and $\underline{p}(U,x) \in [0,1]$ if $x \in \mathrm{cl}_{\mathcal{T}_{\| \|}} U$. We say that $U \subset X$ is (upper) porous (lower porous) at $x \in X$ if $p(U,x) > 0$ ($\underline{p}(U,x) > 0$). Similarly, we say that $A \subset X$ is (upper) strongly porous (lower strongly porous) at $x \in X$ if $p(U,x) = 1$ ($\underline{p}(U,x) = 1$).

Theorem 1.2. *Let $(X, \| \|)$ be a normed space, $A \subset X$, $x \in X$ and $\underline{p}(A,x) > 0$. Then there exists a sequence of closed balls $\left(\overline{B}(x_n, \varrho_n)\right)_{n \ge 1}$, not necessary pairwise disjoint, such that $\lim_{n\to\infty} x_n = x$, $\varrho_n \le \frac{1}{n}$ for $n \ge 1$, $\bigcup_{n=1}^{\infty} \overline{B}(x_n, \varrho_n) \cap A = \emptyset$ and*

$$\underline{p}(A,x) = \underline{p}\left(X \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, \varrho_n), x\right) = \liminf_{n\to\infty} 2n\varrho_n.$$

Proof. For every $n \ge 1$ put $\gamma_n = \sup\left\{\varrho \colon \exists_{y \in X} \ \left(\overline{B}(y,\varrho) \subset \overline{B}(x, \frac{1}{n}) \setminus A\right)\right\}$ and choose a closed ball $\overline{B}(x_n, \varrho_n) \subset \overline{B}(x, \frac{1}{n}) \setminus A$ such that $\varrho_n > \gamma_n \left(1 - \frac{1}{n^2}\right)$. Denote $B = X \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, \varrho_n)$. Since $A \subset B$, we get $\underline{p}(B,x) \le \underline{p}(A,x)$. Fix $n > 1$ and choose any $R \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Then

$$\frac{2\gamma(x,R,A)}{R} \le \frac{2\gamma_n}{\frac{1}{n+1}} < \frac{2\varrho_n}{\left(1 - \frac{1}{n^2}\right)\frac{1}{n+1}} = \frac{2\varrho_n n^2}{n-1}.$$

On the other hand,

$$\gamma(x,R,B) \ge \gamma(x, \tfrac{1}{n+1}, B) \ge \varrho_n - \tfrac{1}{2}(\tfrac{1}{n} - \tfrac{1}{n+1}) = \varrho_n - \frac{1}{2n(n+1)}$$

and

$$\frac{2\gamma(x,R,B)}{R} \geq \frac{2\varrho_n - \frac{1}{n(n+1)}}{\frac{1}{n}} = 2n\varrho_n - \frac{1}{n+1}.$$

We have showed that for each $n > 1$ and for each $R \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ the following inequalities

$$\frac{2\gamma(x,R,A)}{R} < 2\varrho_n n \frac{n}{n-1} \quad \text{and} \quad \frac{2\gamma(x,R,B)}{R} \geq 2n\varrho_n - \frac{1}{n+1}$$

are true. Hence

$$\underline{p}(A,x) \leq \liminf_{n\to\infty} 2n\varrho_n n \frac{n}{n-1} = \liminf_{n\to\infty} 2n\varrho_n \cdot \lim_{n\to\infty} \frac{n}{n-1} = \liminf_{n\to\infty} 2n\varrho_n$$

and

$$\underline{p}(B,x) = \liminf_{R\to 0} \frac{2\gamma(x,R,B)}{R} \geq \liminf_{n\to\infty} \left(2n\varrho_n - \frac{1}{n+1}\right) = \liminf_{n\to\infty} 2n\varrho_n.$$

Finally, $\underline{p}(A,x) = \underline{p}(B,x) = \liminf_{n\to\infty} 2n\varrho_n$. □

In [13] and [7] L. Zajíček and V. Kelar introduce two topologies using the notion of (upper) porosity and (upper) strong porosity.

Definition 1.3 [13]. Let $A \subset X$ and $x \in X$. We say that A is (upper) superporous at x if $A \cup B$ is (upper) porous at x whenever B is (upper) porous at x. A set A is said to be p-open (porosity open) if $X \setminus A$ is (upper) superporous at any point of A.

Definition 1.4 [7]. Let $A \subset X$ and $x \in X$. We say that A is (upper) strongly superporous at x if $A \cup B$ is (upper) porous at x whenever B is (upper) strongly porous at x. A set A is said to be s-open (strongly porosity open) if $X \setminus A$ is (upper) strongly superporous at any point of A.

The system of all p-open sets in $(X, \| \, \|)$ forms a topology $p(X, \| \, \|)$, which will also be called the p-topology or the porosity topology, [13]. The system of all s-open sets forms a topology $s(X, \| \, \|)$, which will be called s-topology or the strong porosity topology, [7]. Obviously $p(X, \| \, \|)$ and $s(X, \| \, \|)$ are finer than the initial topology. On a non-trivial normed space neither $s(X, \| \, \|)$ is finer than $p(X, \| \, \|)$ nor $p(X, \| \, \|)$ is finer than $s(X, \| \, \|)$, [7]. The both topologies are completely regular, [7].

The aim of our paper is to describe the properties of topologies $\underline{s}(X, \| \, \|)$ and $\underline{p}(X, \| \, \|)$ which are generated by the lower porosity in a similar way as $s(X, \| \, \|)$ and $p(X, \| \, \|)$ were generated by the standard (upper) porosity. Section 2 describes relationships between topologies $s(X, \| \, \|)$, $p(X, \| \, \|)$, $\underline{s}(X, \| \, \|)$, $\underline{p}(X, \| \, \|)$ and $\mathcal{T}_{\| \, \|}$, which are quite interesting. First of all, we show that $\mathcal{T}_{\| \, \|} \subset \underline{s}(X, \| \, \|)$. We give examples of spaces in which this inclusion is proper and examples of spaces in which we have equality. The more, we prove that there are two equivalent norms in $\mathbb{R}^n$ such that in the first we have a proper inclusion and in the second we have equality. Namely, $\mathcal{T}_{\| \|_n} = \underline{s}(\mathbb{R}^n, \| \|_n)$ and $\mathcal{T}_{\| \|_{\max}} \subsetneq \underline{s}(\mathbb{R}^n, \| \|_{\max})$. In particular, $\underline{s}(\mathbb{R}^n, \| \|_n) \neq \underline{s}(\mathbb{R}^n, \| \|_{\max})$ although $\mathcal{T}_{\| \|_{\max}} = \mathcal{T}_{\| \|_n}$. Then we show that the inclusion $p(X, \| \|) \subset \underline{p}(X, \| \|)$ holds in every normed space. Next, we define two geometrical conditions (A) and (B) such that the condition (B) implies the condition (A) and every considered by the authors normed space satisfies the condition (A). We prove that the inclusion $\underline{s}(X, \| \|) \subset p(X, \| \|)$ holds under the condition (A) and $\underline{s}(X, \| \|) \subset s(X, \| \|)$ holds under the condition (B). It turns out that the condition (B) is not necessary. There is no other general relationships between considered topologies. Some other examples and properties are presented. Crucial keys in presented examples play spaces $(\mathbb{R}^n, \| \|_{\max})$ and $(l_\infty, \| \|_{\sup})$.

The last section presents some applications of topologies $\underline{s}(X, \| \|)$ and $\underline{p}(X, \| \|)$. Namely, we define lower porouscontinuous functions, following ideas of J. Borsík and J. Holos from [1], and we describe maximal additive classes for some types of lower porouscontinuity in terms of topologies $\underline{s}(X, \| \|)$ and $\underline{p}(X, \| \|)$.

At the end of the section we prove a useful technical lemma.

Lemma 1.5. *Let $\beta > 0$ and $\delta \in \left(0, \frac{\beta}{4}\right)$. Then for every $\alpha > 0$ we have*

$$\sup\left\{\frac{b-a}{b} \colon [a,b] \subset [\delta, \beta] \setminus (\alpha, 2\alpha)\right\} \geq 1 - \sqrt{\tfrac{2\delta}{\beta}}.$$

Proof. Let us consider three cases. First, consider the case where $2\alpha \geq \beta$. Then $[\delta, \frac{\beta}{2}] \cap (\alpha, 2\alpha) = \emptyset$ and $\frac{\frac{\beta}{2} - \delta}{\frac{\beta}{2}} = 1 - \frac{2\delta}{\beta}$. Similarly, if $\alpha \leq \delta$ then $[2\alpha, \beta] \cap (\alpha, 2\alpha) = \emptyset$ and $\frac{\beta - 2\alpha}{\beta} \geq 1 - \frac{2\delta}{\beta}$. Finally, in the case where $\alpha \in (\delta, \frac{\beta}{2})$ we have $([\delta, \alpha] \cup [2\alpha, \beta]) \cap (\alpha, 2\alpha) = \emptyset$. Consider two functions $\varphi, \psi \colon (\delta, \frac{\beta}{2}) \to \mathbb{R}$ defined by $\varphi(\alpha) = \frac{\alpha - \delta}{\alpha} = 1 - \frac{\delta}{\alpha}$ and $\psi(\alpha) = \frac{\beta - 2\alpha}{\beta} = 1 - \frac{2\alpha}{\beta}$. Then φ is increasing, ψ is decreasing and $\varphi(\alpha) = \psi(\alpha)$ if $\frac{\delta}{\alpha} = \frac{2\alpha}{\beta}$, i.e. $\alpha = \sqrt{\frac{\delta\beta}{2}}$. Moreover, $\varphi\left(\sqrt{\frac{\delta\beta}{2}}\right) = \psi\left(\sqrt{\frac{\delta\beta}{2}}\right) = 1 - \frac{\delta}{\sqrt{\frac{\delta\beta}{2}}} = 1 - \sqrt{\frac{2\delta}{\beta}}$. Since $1 - \sqrt{\frac{2\delta}{\beta}} \leq 1 - \frac{2\delta}{\beta}$, the proof is completed. □

2. Relationships between topologies generated by porosities

Analogously, as in the case of the standard (upper) porosity we can define lower superporosity, lower strong superporosity and topologies $\underline{p}(X, \| \|)$ and $\underline{s}(X, \| \|)$.

Definition 2.1. Let $A \subset X$ and $x \in X$. We say that A is lower superporous at x if $A \cup B$ is lower porous at x whenever B is lower porous at x. A set A is said to be $\underline{p}$-lower open (lower porosity open) if $X \setminus A$ is lower superporous at any point of A.

Definition 2.2. Let $A \subset X$ and $x \in X$. We say that A is lower strongly superporous at x if $A \cup B$ is lower porous at x whenever B is lower strongly porous at x. A set A is said to be $\underline{s}$-lower open (lower strongly porosity open) if $X \setminus A$ is lower strongly superporous at any point of A.

A simple check shows that the system of all $\underline{p}$-lower open sets in $(X, \| \ \|)$ forms a topology $\underline{p}(X, \| \ \|)$, which will also be called the $\underline{p}$-topology or the lower porosity topology. The system of all $\underline{s}$-lower open sets forms a topology $\underline{s}(X, \| \ \|)$, which will be called $\underline{s}$-topology or the lower strong porosity topology.

Remark 2.3. Let $(X, \| \ \|)$ be a normed space. For every $E \subset [0, \infty)$ let $A_E = \{x \in X \colon \|x\| \in E\}$. Then $p(A_E, 0_X) = p(E \cup -E, 0)$ and $\underline{p}(A_E, 0_X) = \underline{p}(E \cup -E, 0)$.

Example 2.4. Let $(X, \| \ \|)$ be a normed space and $E = [0, \infty) \setminus \bigcup_{n=1}^{\infty} \{\frac{1}{2^n}\}$. We claim that $A_E \in (\underline{p}(X, \| \ \|) \cap p(X, \| \ \|)) \setminus (s(X, \| \ \|) \cup \underline{s}(X, \| \ \|) \cup \mathcal{T}_{\| \ \|})$. Clearly, $p(X \setminus A_E, 0_X) = \frac{1}{2}$ and $\underline{p}(X \setminus A_E, 0_X) < \frac{1}{2}$. The more, $X \setminus A_E$ is neither strongly superporous nor lower strongly superporous at 0_X. Hence, $A_E \notin s(X, \| \ \|) \cup \underline{s}(X, \| \ \|)$. Obviously, $A_E \notin \mathcal{T}_{\| \ \|}$.

On the other hand, it is clear that $X \setminus A_E$ is superporous and lower superporous at every $x \in A_E$, $x \neq 0_X$. Moreover, for every $B \subset X$ and $B(x, \eta) \subset X \setminus B$ we can find $y \in B(x, \eta)$ such that $B(y, \min\{\frac{\eta}{2}, \frac{1}{4}\|y\|\}) \subset (X \setminus B) \cap A_E$. Therefore, $X \setminus A_E$ is superporous and lower superporous at 0_X. Finally, $A_E \in \underline{p}(X, \| \ \|) \cap p(X, \| \ \|)$.

Example 2.5. Let $(X, \| \ \|)$ be a normed space and $E = [0, \infty) \setminus \bigcup_{n=2}^{\infty} [\frac{1}{n!}, \frac{2}{n!}]$. We claim that $A_E \in (s(X, \| \ \|) \cap \underline{p}(X, \| \ \|)) \setminus (p(X, \| \ \|) \cup \underline{s}(X, \| \ \|) \cup \mathcal{T}_{\| \ \|})$. Clearly,

$$\underline{p}(X \setminus A_E, 0_X) \leq \lim_{n \to \infty} \frac{2\gamma(0_X, \frac{2}{n!}, X \setminus A_E)}{\frac{2}{n!}} \leq \frac{1}{2} < 1.$$

Hence, $X \setminus A_E$ is not lower strongly superporous at 0_X and $A_E \notin \underline{s}(X, \| \ \|)$. Let $B = A_E$. Then $p(B, 0_X) = \lim_{n \to \infty} \frac{2\gamma(0_X, \frac{2}{n!}, B)}{\frac{2}{n!}} = \frac{1}{2}$ and $p(B \cup (X \setminus A_E), 0_X) = p(X, 0_X) = 0$. Thus $X \setminus A_E$ is not superporous at 0_X and $A_E \notin p(X, \| \ \|)$. Obviously, $A_E \notin \mathcal{T}_{\| \ \|}$.

It is clear that $X \setminus A_E$ is lower superporous and strongly superporous at every $x \in A_E$, $x \neq 0_X$. Take any $B \subset X$ satisfying $\underline{p}(B, 0_X) = 2c > 0$. Choose $n_0, k_0 \geq 2$ such that $\frac{4}{k_0} < c$, $k_0 < n_0$ and $\frac{2\gamma(0_X, R, B)}{R} > c$ for every $R < \frac{1}{(n_0 - 1)!}$. Take any $n > n_0$ and $R \in [\frac{1}{n!}, \frac{1}{(n-1)!}]$. If $R \in [\frac{1}{n!}, \frac{k_0}{n!}]$ then

$$\frac{2\gamma(0_X, R, B \cup (X \setminus A_E))}{R} \geq \frac{2\gamma(0_X, \frac{1}{n!}, B) - \frac{2}{(n+1)!}}{k_0 \frac{1}{n!}} > \frac{c}{k_0} - \frac{2}{k_0(n+1)}$$

and if $R \in [\frac{k_0}{n!}, \frac{1}{(n-1)!}]$ then

$$\frac{2\gamma(0_X, R, B \cup (X \setminus A_E))}{R} \geq \frac{2\gamma(0_X, R, B) - \frac{2}{n!}}{R} > c - \frac{2}{k_0} > \frac{c}{2}.$$

Therefore, $\underline{p}(B \cup (X \setminus A_E), 0_X) \geq \frac{c}{k_0}$, $X \setminus A_E$ is lower superporous at 0_X and $A_E \in \underline{p}(X, \| \, \|)$.

Take any $B \subset X$ satisfying $p(B, 0_X) = 1$ and $R \in (0,1)$. Let $R \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right)$. Choose $B(x_R, \eta_R) \subset X \setminus (B \cup \{0_X\})$ such that $\|x_R\| + \eta_R < R$, $\eta_R > \gamma(0_X, R, B) - \frac{R}{n}$ and $\|x_R\| - \eta_R > \frac{2}{(n+2)!}$. Since $p(B, 0_X) = 1$, we have $\limsup_{R\to 0^+} \frac{2\eta_R}{R} = 1$, $\limsup_{R\to 0^+} \frac{2\eta_R}{\|x_R\|+\eta_R} = 1$ and $\liminf_{R\to 0^+} \frac{\|x_R\|-\eta_R}{\|x_R\|+\eta_R} = 0$. Applying Lemma 1.5 with $\beta = \|x_R\| + \eta_R$ and $\delta = \|x_R\| - \eta_R$ we obtain $\frac{2\gamma(0_X, \|x_R\|+\eta_R, B\cup(X\setminus A_E))}{\|x_R\|+\eta_R} \geq 1 - \sqrt{\frac{2(\|x_R\|-\eta_R)}{\|x_R\|+\eta_R}}$. Therefore

$$p(B \cup (X \setminus A_E), 0_X) = \limsup_{R\to 0^+} \frac{2\eta_R}{\|x_R\| + \eta_R} \geq 1 - \liminf_{R\to 0^+} \sqrt{\frac{2(\|x_R\| - \eta_R)}{\|x_R\| + \eta_R}} = 1.$$

Hence $X \setminus A_E$ is strongly superporous at 0_X and $A_E \in s(X, \| \, \|)$.

Theorem 2.6. *Let $\| \, \|_{\max}$ be the maximum norm in $\mathbb{R}^n$ and $\mathcal{T}_N$ be the natural topology in $\mathbb{R}^n$. Then $\underline{s}(\mathbb{R}^n, \| \, \|_{\max}) \supsetneq \mathcal{T}_N$.*

Proof. The inclusion $\underline{s}(\mathbb{R}^n, \| \, \|_{\max}) \supset \mathcal{T}_{\| \, \|_{\max}} = \mathcal{T}_N$ is obvious. Let us take $U = (\mathbb{R}^n \setminus (\mathbb{R} \times \{0_{n-1}\}) \cup \{0_n\}$, where $0_n = (0, \ldots, 0) \in \mathbb{R}^n$. Certainly, $U \notin \mathcal{T}_N$. We claim that $U \in \underline{s}(\mathbb{R}^n, \| \, \|_{\max})$. It is easy to see that $\mathbb{R}^n \setminus U$ is lower strongly superporous at every $x \in U \setminus \{0_n\}$. Take any $V \subset \mathbb{R}^n$ such that V is lower strongly porous at 0_n. For every $R > 0$ choose $B(x_R, \eta_R) \subset B(0_n, 2R) \setminus V$ such that $\eta_R > \gamma(0_n, 2R, V) - (2R)^2$. By Lemma 1.1, we can find $y_R \in \mathbb{R}^n$ and $\xi \in \{-1, 1\}^n$ such that

$$B(y_R, \eta_R - \tfrac{R}{2}) \subset B(x_R, \eta_R) \cap H^\xi \cap B(0_n, R).$$

Therefore $\gamma(0_n, R, V \cup (\mathbb{R}^n \setminus U)) \geq \eta_R - \frac{R}{2}$. Hence

$$\begin{aligned} \underline{p}(V \cup (\mathbb{R}^n \setminus U), 0_n) &= \lim_{R\to 0^+} \frac{2\gamma(0_n, R, V \cup (\mathbb{R}^n \setminus U))}{R} \\ &\geq \lim_{R\to 0^+} \frac{4\gamma(0_n, 2R, V) - 4(2R)^2 - 2R}{2R} = 1 - \lim_{R\to 0^+} 8R = 1. \end{aligned}$$

Thus $\mathbb{R}^n \setminus U$ is lower strongly superporous at 0_n and $U \in \underline{s}(\mathbb{R}^n, \| \, \|_{\max})$, which completed the proof. □

Remark 2.7. Repeating arguments from the proof of Theorem 1.2 one can prove that $\mathcal{T}_{\| \, \|_{\sup}}$ is a proper subset of $\underline{s}(l_\infty, \| \, \|_{\sup})$.

We will show the equality $\underline{s}(\mathbb{R}^n, \mathcal{T}_N) = \mathcal{T}_N$, but first we need two technical lemmas.

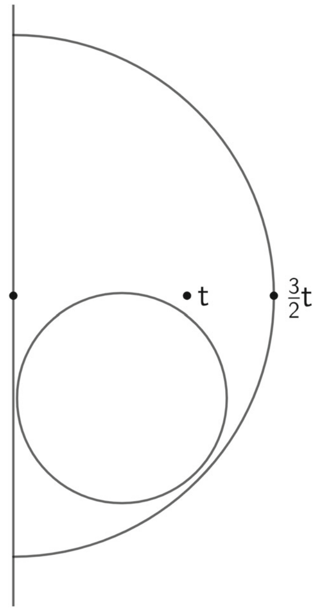

FIGURE 2. $B(x,\eta)$ in $(\mathbb{R}^2, \|\ \|_2)$

Lemma 2.8. *Let* $U = [0,\infty) \times \mathbb{R}^{n-1}$ *be a subset of* $(\mathbb{R}^n, \|\ \|_n)$*. Then* $\gamma(0_n, \frac{3}{2}t, (\mathbb{R}^n \setminus U) \cup \{(t,0,\ldots,0)\}) \leq \frac{2}{3}t$ *for every* $t \in (0,\infty)$*.*

Proof. Choose any $t \in (0,\infty)$ and let $y_t = (t,0,0,\ldots,0) \in \mathbb{R}^n$. Suppose to the contrary that there exists $B(x,\eta) \subset U \setminus \{y_t\}$ such that $\eta > \frac{2}{3}t$ (see Fig. 2). Then

(1) $x_1 > \frac{2}{3}t$,
(2) $\sqrt{\sum_{i=1}^n x_i^2} < \frac{3}{2}t - \frac{2}{3}t = \frac{5}{6}t$,
(3) $\|x - y_t\|_n > \frac{2}{3}t$,

where $x = (x_1, x_2, \ldots, x_n)$. Then $x_1^2 - (x_1 - t)^2 > \frac{4}{9}t^2 - \frac{1}{9}t^2 = \frac{1}{3}t^2$, by (1).

Therefore, by (1) and (3) we obtain

$$\sum_{i=1}^n x_i^2 = x_1^2 - (x_1 - t)^2 + (x_1 - t)^2 + \sum_{i=2}^n x_i^2 > \frac{1}{3}t^2 + \frac{4}{9}t^2 = \frac{28}{36}t^2 > \left(\frac{5}{6}t\right)^2,$$

which contradicts (2). □

In the sequel we will need the notions of cone and halfspace in $\mathbb{R}^n$. Let $a, b \in \mathbb{R}^n$, $a \neq b$ and $\varphi \in (0, \frac{\pi}{2})$. The cone $c(a,b,\varphi)$ with vertex a, angle φ and axis ab is defined as $c(a,b,\varphi) = \{x \in \mathbb{R}^n \colon |\sphericalangle(ab, ax)| < \varphi\}$. Moreover, by $h(a,b)$ we denote the halfspace $h(a,b) = \{x \in \mathbb{R}^n \colon |\sphericalangle(ab, ax)| > \frac{\pi}{2}\}$.

Lemma 2.9. *Let* $\varphi \in (0, \frac{\pi}{2})$ *satisfy* $\tan\varphi < \frac{1}{24}$. *If* $(a_k)_{k\in\mathbb{N}} \subset c(0_n, (1, 0, \dots, 0), \varphi)$ *is a sequence in* $(\mathbb{R}^n, \| \|_n)$ *converging to* 0_n *then*

$$\underline{p}\left((-\infty, 0) \times \mathbb{R}^{n-1} \cup \bigcup_{k=1}^{\infty}\{a_k\}, 0_n\right) < 1.$$

Proof. Let $a'_k = (a^k_1, 0, \dots, 0)$, where $a_k = (a^k_1, a^k_2, \dots, a^k_n)$ for $k \geq 1$. Then $\frac{\|a'_m - a_m\|_n}{\|a'_m\|_n} = \tan\sphericalangle(0_n a_m, 0_n a'_m) < \tan\varphi < \frac{1}{24}$ and

$$\begin{aligned}\gamma(0_n, \tfrac{3}{2}\|a'_m\|_n, &(-\infty, 0) \times \mathbb{R}^{n-1} \cup \textstyle\bigcup_{k=1}^{\infty}\{a_k\}) \\ &\leq \gamma(0_n, \tfrac{3}{2}\|a'_m\|_n, (-\infty, 0) \times \mathbb{R}^{n-1} \cup \{a_m\}) \\ &\leq \gamma(0_n, \tfrac{3}{2}\|a'_m\|_n, (-\infty, 0) \times \mathbb{R}^{n-1} \cup \{a'_m\}) + \|a'_m - a_m\|_n\end{aligned}$$

for every $m \geq 1$. By Lemma 2.8,

$$\frac{\gamma(0_n, \frac{3}{2}\|a'_m\|_n, (-\infty, 0) \times \mathbb{R}^{n-1} \cup \bigcup_{k=1}^{\infty}\{a_k\})}{\frac{3}{2}\|a'_m\|_n} \leq \frac{\frac{2}{3}\|a'_m\|_n + \frac{1}{24}\|a'_m\|_n}{\frac{3}{2}\|a'_m\|_n} = \frac{\frac{17}{24}}{\frac{3}{2}} = \frac{17}{36}$$

for every $m \geq 1$. Hence $\underline{p}\left((-\infty, 0) \times \mathbb{R}^{n-1} \cup \bigcup_{k=1}^{\infty}\{a_k\}, 0_n\right) \leq \frac{2\cdot 17}{36} < 1$, which completed the proof. □

Since porosity does not change under any isometry, we obtain the following corollary.

Corollary 2.10. *Let* $\varphi \in (0, \frac{\pi}{2})$ *satisfy* $\tan\varphi < \frac{1}{24}$ *and* $a, b \in \mathbb{R}^n$, $a \neq b$. *If* $(a_k)_{k\in\mathbb{N}} \subset c(a, b, \varphi)$ *is a sequence converging to* a *then*

$$\underline{p}\left(h(a, b) \cup \bigcup_{k=1}^{\infty}\{a_k\}, 0_n\right) < 1.$$

Theorem 2.11. *Let* $\| \|_n$ *and* $\mathcal{T}_N$ *be the natural norm and the natural topology in* $\mathbb{R}^n$, *respectively. Then* $\underline{s}(\mathbb{R}^n, \| \|_n) = \mathcal{T}_N$.

Proof. Obviously, $\mathcal{T}_N \subset \underline{s}(\mathbb{R}^n, \| \|_n)$. Let us take any $U \notin \mathcal{T}_N$. There exist $a_0 \in U$ and a sequence $(a_k)_{k\geq 1} \subset \mathbb{R}^n \setminus U$ converging to a_0. Then we can find $b \in \mathbb{R}^n$ and $\varphi \in (0, \frac{\pi}{2})$ such that $a \neq b$, $\tan\varphi < \frac{1}{24}$ and $c(a, b, \varphi)$ contains infinitely many elements of $(a_k)_{k\geq 1}$. Let $V = h(a, b)$. Obviously $\underline{p}(V, a) = 1$. But

$$\underline{p}\Big(V \cup (\mathbb{R}^n \setminus U), a\Big) \leq \underline{p}\left(h(a, b) \cup \bigcup_{k=1}^{\infty}\{a_k\}, a\right) < 1$$

by Corollary 2.10. Hence $U \notin \underline{s}(\mathbb{R}^n, \mathcal{T}_N)$, which completed the proof. □

Corollary 2.12. *For every* $n \geq 2$ *there exist equivalent norms* $\| \|^1, \| \|^2$ *in* $\mathbb{R}^n$ *such that* $\underline{s}(\mathbb{R}^n, \| \|^1) = \mathcal{T}_N$ *and* $\underline{s}(\mathbb{R}^n, \| \|^2) \supsetneq \mathcal{T}_N$. *In particular,* $\underline{s}(\mathbb{R}^n, \| \|^1) \subsetneq \underline{s}(\mathbb{R}^n, \| \|^2)$.

Question 2.13. For which norm $\| \|$ in $\mathbb{R}^n$ the equality $\underline{s}(\mathbb{R}^n, \| \|) = \mathcal{T}_N$ holds?

Question 2.14. For which normed space $(X, \| \|)$ the equality $\underline{s}(X, \| \|) = \mathcal{T}_{\| \|}$ holds?

Lemma 2.15. *Let $(X,\|\ \|)$ be a normed space. If $U\in p(X,\|\ \|)\cup s(X,\|\ \|)$ then $\underline{p}(X\setminus U,x)>0$ for every $x\in U$.*

Proof. Take any $U\subset X$ such that $\underline{p}(X\setminus U,x)=0$ for some $x\in U$. There exists a sequence $(R_n)_{n\geq 1}$ such that $\lim_{n\to\infty}\frac{2\gamma(x,R_n,X\setminus U)}{R_n}=0$. In particular, $R_n>4\gamma(x,R_n,X\setminus U)$ for almost all n. Without loss of generality we may assume that this is true for all $n\geq 1$. Take any $y\in X$ such that $\|x-y\|=1$. Let $x_n=x+\frac{R_n+4\gamma(x,R_n,X\setminus U)}{2}(y-x)$ and $\eta_n=\frac{R_n-4\gamma(x,R_n,X\setminus U)}{2}$ for $n\geq 1$. Define $A=X\setminus\bigcup_{n=1}^{\infty}B\left(x+\frac{3R_n}{4}(y-x),\frac{R_n}{4}\right)$ and $B=X\setminus\bigcup_{n=1}^{\infty}B\left(x_n,\eta_n\right)$. Taking a subsequence if necessary, we may assume that $X\setminus A$ and $X\setminus B$ consist of pairwise disjoint balls. Then $p(A,x)=\frac{1}{2}$ and $p(B,x)=1$. We claim that $p(A\cup(X\setminus U),x)=0$ and $p(B\cup(X\setminus U),x)<1$.

Fix $n\geq 1$ and $R\in[R_{n+1},R_n]$. If $R\in[\frac{R_n}{2},R_n]$ then

$$\frac{2\gamma(x,R,A\cup(X\setminus U))}{R}\leq\frac{2\gamma(x,R_n,A\cup(X\setminus U))}{\frac{R_n}{2}}\leq\frac{4\gamma(x,R_n,X\setminus U)}{R_n}$$

and if $R\in[R_{n+1},\frac{R_n}{2}]$ then

$$\frac{2\gamma(x,R,A\cup(X\setminus U))}{R}\leq\frac{2\gamma(x,R_{n+1},A\cup(X\setminus U))}{R_{n+1}}\leq\frac{2\gamma(x,R_{n+1},X\setminus U)}{R_{n+1}}.$$

Since $\lim_{n\to\infty}\frac{2\gamma(x,R_n,X\setminus U)}{R_n}=0$, we obtain $p(A\cup(X\setminus U),x)=0$.

Again, fix $n\geq 1$ and $R\in[R_{n+1},R_n]$. If $R\in[4\gamma(x,R_n,X\setminus U),R_n]$ then

$$\frac{2\gamma(x,R,B\cup(X\setminus U))}{R}\leq\frac{2\gamma(x,R_n,B\cup(X\setminus U))}{4\gamma(x,R_n,X\setminus U)}\leq\frac{2\gamma(x,R_n,X\setminus U)}{4\gamma(x,R_n,X\setminus U)}=\frac{1}{2}$$

and if $R\in[R_{n+1},4\gamma(x,R_n,X\setminus U)]$ then

$$\frac{2\gamma(x,R,B\cup(X\setminus U))}{R}\leq\frac{2\gamma(x,R_{n+1},B\cup(X\setminus U))}{R_{n+1}}\leq\frac{2\gamma(x,R_{n+1},X\setminus U)}{R_{n+1}}.$$

Since $\lim_{n\to\infty}\frac{2\gamma(x,R_n,X\setminus U)}{R_n}=0$, we obtain $p(B\cup(X\setminus U),x)\leq\frac{1}{2}<1$. □

Theorem 2.16. *Let $(X,\|\ \|)$ be a normed space. Then $p(X,\|\ \|)\subsetneq\underline{p}(X,\|\ \|)$.*

Proof. Let us take any $U\notin\underline{p}(X,\|\ \|)$. Then $X\setminus U$ is not lower superporous at some $x_0\in U$. Hence there is $V\subset X$ satisfying $\underline{p}(V,x_0)=2c>0$ and $\underline{p}(V\cup(X\setminus U),x_0)=0$. If $\underline{p}(X\setminus U,x_0)=0$ then, by Lemma 2.15, $U\notin p(X,\|\ \|)$ at once. Therefore we may assume $\underline{p}(X\setminus U,x_0)>0$. Hence $x_0\in\mathrm{cl}_{\mathcal{T}_{\|\ \|}}(V)$. Moreover, there exist a decreasing sequence of reals $(R_n)_{n\geq 1}$ tending to 0 such that

$$\underline{p}(V\cup(X\setminus U),x_0)=\lim_{n\to\infty}\frac{2\gamma(x_0,R_n,V\cup(X\setminus U))}{R_n}=0.$$

Since $\underline{p}(V,x_0)=c$, for every $n\geq 1$ we can find an open ball $B(x_n,\eta_n)$ for which $B(x_n,\eta_n)\subset B(x_0,R_n)\setminus V$ and $\eta_n>\gamma(x_0,R_n,V)-\frac{R_n}{n}$. Since $x_0\in\mathrm{cl}_{\mathcal{T}_{\|\ \|}}(V)$, we obtain $\eta_n\leq\|x_0-x_n\|$.

Define $A = X \setminus \bigcup_{n=1}^{\infty} B(x_n, \frac{\eta_n}{2})$. Then

$$p(A, x_0) \geq \limsup_{n\to\infty} \frac{\eta_n}{R_n} \geq \limsup_{n\to\infty} \frac{\gamma(x_0, R_n, V) - \frac{R_n}{n}}{R_n} \geq \frac{\underline{p}(V, x_0)}{2} = c.$$

Without loss of generality we may assume that $\frac{\eta_n}{R_n} > \frac{2c}{3}$ for every n. Thus $R_n < \frac{3\eta_n}{2c}$ and $\frac{\eta_n}{2} + \|x_n - x_0\| \geq \frac{3}{2}\eta_n > cR_n$. On the other hand, $V \subset A$ and

$$\lim_{n\to\infty} \frac{2\gamma(x_0, R_n, A \cup (X \setminus U))}{R_n} \leq \lim_{n\to\infty} \frac{2\gamma(x_0, R_n, V \cup (X \setminus U))}{R_n} = 0.$$

Therefore

$$\limsup_{n\to\infty} \frac{2\gamma(x_0, \frac{\eta_n}{2} + \|x_n - x_0\|, A\cup(X\setminus U))}{\frac{\eta_n}{2} + \|x_n - x_0\|} \leq \limsup_{n\to\infty} \frac{2\gamma(x_0, R_n, A\cup(X\setminus U))}{\frac{\eta_n}{2} + \|x_n - x_0\|} \leq$$
$$\leq \limsup_{n\to\infty} \frac{2\gamma(x_0, R_n, A\cup(X\setminus U))}{cR_n} = 0.$$

Fix $n \geq 1$ and take any $R \in [R_{n+1}, R_n)$. Then

$$\frac{2\gamma(x_0, R, A\cup(X\setminus U))}{R} = \frac{2\gamma(x_0, \frac{\eta_n}{2} + \|x_n - x_0\|, A\cup(X\setminus U))}{R} \leq \frac{2\gamma(x_0, \frac{\eta_n}{2} + \|x_n - x_0\|, A\cup(X\setminus U))}{\frac{\eta_n}{2} + \|x_n - x_0\|}$$

for $R \in [\frac{\eta_n}{2} + \|x_n - x_0\|, R_n]$,

$$\frac{2\gamma(x_0, R, A\cup(X\setminus U))}{R} \leq \frac{2\gamma(x_0, \frac{\eta_n}{2} + \|x_n - x_0\|, A\cup(X\setminus U))}{\|x_n - x_0\| - \frac{\eta_n}{2}} \leq \frac{2\gamma(x_0, \frac{\eta_n}{2} + \|x_n - x_0\|, A\cup(X\setminus U))}{\frac{1}{4}(\eta_n + \|x_n - x_0\|)}$$

for $R \in [\|x_n - x_0\| - \frac{\eta_n}{2}, \|x_n - x_0\| + \frac{\eta_n}{2}]$ and

$$\frac{2\gamma(x_0, R, A\cup(X\setminus U))}{R} \leq \frac{2\gamma(x_0, \frac{\eta_{n+1}}{2} + \|x_{n+1} - x_0\|, A\cup(X\setminus U))}{\|x_{n+1} - x_0\| + \frac{\eta_{n+1}}{2}}$$

for $R \in [R_{n+1}, \|x_n - x_0\| - \frac{\eta_n}{2}]$. Hence, $p(A \cup (X \setminus U), x_0) = 0$ and $X \setminus U$ is not superporous at x_0. Therefore $U \notin p(X, \|\ \|)$. Thus $p(X, \|\ \|) \subset \underline{p}(X, \|\ \|)$. By Example 2.5, $p(X, \|\ \|) \neq \underline{p}(X, \|\ \|)$. $\square$

Lemma 2.17. *Let $(X, \|\ \|)$ be a normed space and $x_0 \in X$ and let $(B(x_n, \eta_n))_{n\geq 1} \subset X \setminus \{x_0\}$ be a sequence of balls such that $\lim_{n\to\infty} x_n = x_0$, $\eta_n > \eta_{n+1}$ and $\|x_n - x_0\| > \|x_{n+1} - x_0\|$ for every $n \geq 1$. Then*

(1) if $\lim_{n\to\infty} \frac{\|x_n - x_0\| - \eta_n}{\|x_{n+1} - x_0\| + \eta_{n+1}} = 0$ then $\underline{p}(X \setminus \bigcup_{n=1}^{\infty} B(x_n, \eta_n), x_0) = 1$;

(2) if $\limsup_{n\to\infty} \frac{\|x_n - x_0\| - \eta_n}{\|x_{n+1} - x_0\| + \eta_{n+1}} < 1$ then $\underline{p}(X \setminus \bigcup_{n=1}^{\infty} B(x_n, \eta_n), x_0) > 0$.

Proof. Let $A = X \setminus \bigcup_{n=1}^{\infty} B(x_n, \eta_n)$, $\alpha_n = \|x_n - x_0\| - \eta_n$ and $\beta_n = \|x_n - x_0\| + \eta_n$ for $n \geq 1$. In both cases $\limsup_{n\to\infty} \frac{\alpha_n}{\beta_{n+1}} < 1$, by assumption. Therefore $\beta_{n+1} > \alpha_n$ for almost every n and we may assume that this is true for every n. Fix $n \geq 1$ and take any $R \in [\beta_{n+1}, \beta_n]$. Then $\gamma(x_0, \beta_{n+1}, A) \geq \frac{\beta_{n+1} - \alpha_n}{2}$ and $\gamma(x_0, R, A) \geq \frac{\beta_{n+1} - \alpha_n}{2} + \frac{R - \beta_{n+1}}{2}$. Therefore,

$$\frac{2\gamma(x_0, R, A)}{R} \geq \frac{\beta_{n+1} - \alpha_n + R - \beta_{n+1}}{\beta_{n+1} + (R - \beta_{n+1})} \geq \frac{\beta_{n+1} - \alpha_n}{\beta_{n+1}}.$$

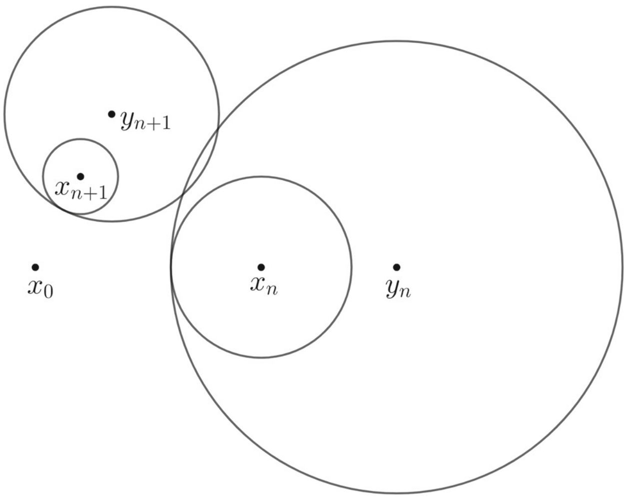

FIGURE 3. Construction of $B(y_n, \delta_n)$ in $(\mathbb{R}^2, \| \|_2)$

(Observe that inequality $\frac{a+c}{b+c} \geq \frac{a}{b}$ holds for every $0 < a \leq b$ and $c > 0$.) Hence

$$\underline{p}(A, x_0) = \liminf_{R \to 0^+} \frac{2\gamma(x_0, R, A)}{R} \geq \liminf_{n\to\infty} \frac{\beta_{n+1} - \alpha_n}{\beta_{n+1}} = 1 - \limsup_{n\to\infty} \frac{\alpha_n}{\beta_{n+1}},$$

which completed the proof. □

Lemma 2.18. *Let $(X, \| \|)$ be a normed space, $x_0 \in X$, $\beta \in (0, 1)$ and let $(B(x_n, \eta_n))_{n\geq 1} \subset X \setminus \{x_0\}$ be a sequence of balls such that $\lim_{n\to\infty} \frac{\eta_n}{\|x_n - x_0\|} = 1$ and $\beta\|x_n - x_0\| > \|x_{n+1} - x_0\| - \eta_{n+1}$ for every $n \geq 1$. Then we can find a sequence $(B(y_n, \delta_n))_{n\geq 1} \subset X \setminus \{x_0\}$ of balls satisfying the following three conditions:*

(1) $\underline{p}\left(X \setminus \bigcup_{n=1}^{\infty} B(y_n, \delta_n), x_0\right) = 1$,
(2) $\|y_{n+1} - x_0\| + \delta_{n+1} = \beta\|x_n - x_0\|$ for every $n > 1$,
(3) for every $n \geq 1$ points x_0, x_n and y_n are collinear and $\|y_n - x_0\| - \delta_n = \|x_n - x_0\| - \eta_n$.

Proof. Fix $\beta > 0$. Let $\delta_1 = \eta_1$ and $\delta_n = \frac{\beta\|x_{n-1}-x_0\|-\|x_n-x_0\|+\eta_n}{2}$ for $n > 1$. By assumptions, $\delta_n > 0$ for every n. Similarly, let $y_1 = x_1$ and $y_n = x_n + (\delta_n - \eta_n)\frac{x_n - x_0}{\|x_n - x_0\|}$ for $n > 1$, (see Fig. 3).

Clearly, x_0, x_n and y_n are collinear, $\|y_n - x_0\| - \delta_n = \|x_n - x_0\| - \eta_n$ and

$$\|y_{n+1} - x_0\| + \delta_{n+1} = \|x_{n+1} - x_0\| + 2\delta_{n+1} - \eta_n = \beta\|x_n - x_0\|$$

for every $n > 1$. Therefore

$$\lim_{n\to\infty} \frac{\|y_n - x_0\| - \delta_n}{\|y_{n+1} - x_0\| + \delta_{n+1}} = \lim_{n\to\infty} \frac{\|x_n - x_0\| - \eta_n}{\beta\|y_n - x_0\|} = 1.$$

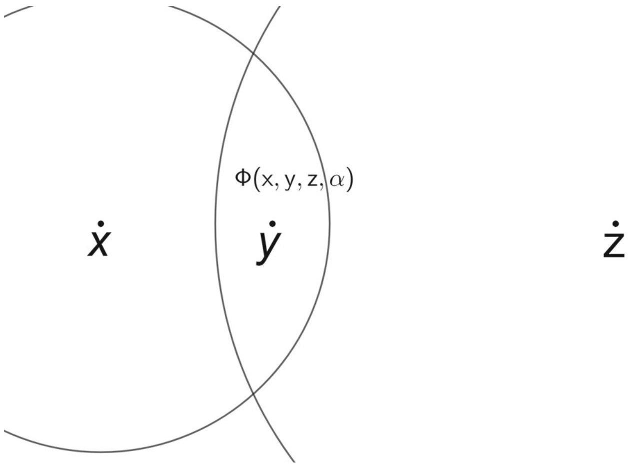

FIGURE 4. $\Phi(x, y, z, \alpha)$ in $(\mathbb{R}^2, \| \|_2)$

Hence $\underline{p}\left(X \setminus \bigcup_{n=1}^{\infty} B(y_n, \delta_n), x_0\right) = 1$, by Lemma 2.17, which completed the proof. □

Definition 2.19. Let $\alpha \in (0, \frac{1}{4})$ and x, y, z be collinear points in a normed space $(X, \| \|)$ such that y is between x and z. By $\Phi(x, y, z, \alpha)$ (see Fig. 4) we denote a set

$$\Phi(x, y, z, \alpha) = \\ = \{t \in X \colon \|t - x\| < \|x - y\| + \alpha\|x - y\| \text{ and } \|t - z\| < \|z - y\| + \alpha\|x - y\|\}.$$

Definition 2.20. We say that a normed space $(X, \| \|)$ satisfies condition (A) if for every $\varepsilon > 1$ there exists $\alpha \in (0, 1)$ such that for every collinear $x, y, z \in X$, where y is between x and z, we have

$$\Phi(x, y, z, \alpha) \subset B\left(y, \varepsilon\|x - y\|\right).$$

Remark 2.21. All considered by the authors normed spaces satisfy condition (A). Is there a normed space that does not satisfy condition (A)?

Theorem 2.22. *If a normed space* $(X, \| \|)$ *satisfies condition* (A) *then* $\underline{s}(X, \| \|) \subset p(X, \| \|)$.

Proof. Let us take any $U \notin p(X, \| \|)$. Then $X \setminus U$ is not superporous at some $x_0 \in U$. Hence there is $V \subset X$ satisfying $p(V, x_0) = c > 0$ and $p(V \cup (X \setminus U), x_0) = 0$. Since $p(V, x_0) = c$, there exist a sequence of pairwise disjoint open balls $(B(x_n, \eta_n))_{n \geq 1}$ such that $\lim_{n\to\infty} x_n = x_0$, $B(x_n, \eta_n) \cap V = \emptyset$ and $\lim_{n\to\infty} \frac{2\eta_n}{\|x_0 - x_n\| + \eta_n} = p(V, x_0) = c$ and since $p(V \cup (X \setminus U), x_0) = 0$, $\lim_{n\to\infty} \frac{2\gamma(x_0, \|x_0 - x_n\| + \eta_n, V \cup (X \setminus U))}{\|x_0 - x_n\| + \eta_n} = 0$. Without loss of generality we may assume that $\eta_n > \frac{3}{8}c\|x_n - x_0\|$ and $\frac{2\gamma(x_0,\|x_0-x_n\|+\eta_n,V\cup(X\setminus U))}{\|x_0-x_n\|+\eta_n} < \frac{c}{8}$ for every $n \geq 1$.

Since $(X, \| \, \|)$ satisfies condition (A), we can find $\alpha_0 \in (0, \frac{c}{8})$ such that for every collinear $x, y, z \in X$, where y lies between x and z, we have $\Phi(x, y, z, \alpha) \subset B\left(y, (1 + \frac{c}{8})\|x - y\|\right)$ for every $\alpha < \alpha_0$.

Let $\eta'_n = \|x_n - x_0\|(1 - \frac{1}{n+1})$. If need be taking a subsequence, we may assume that

$$\|x_{n+1} - x_0\| + \eta'_{n+1} < \tfrac{c}{4}\|x_n - x_0\| \qquad \text{for every } n. \tag{2.1}$$

Obviously, $p(X \setminus \bigcup_{n=1}^{\infty} B(x_n, \eta'_n), x_0) = 1$.

By Lemma 2.18, there exists a sequence $(B(y_n, \delta_n))_{n\geq 1} \subset X \setminus \{x_0\}$ of balls such that $\underline{p}\,(X \setminus \bigcup_{n=1}^{\infty} B(y_n, \delta_n), x_0) = 1$, x_0, x_n, y_n are collinear, x_n is between x_0 and y_n, $\|y_n - x_0\| - \delta_n = \|x_n - x_0\| - \eta'_n$ and

$$\|y_{n+1} - x_0\| + \delta_{n+1} = \tfrac{c}{4}\|x_n - x_0\| \quad \text{for every } n \geq 1. \tag{2.2}$$

Let $z_n = x_0 - (y_{n-1} - x_0)\frac{\|y_n-x_0\|}{\|y_{n-1}-x_0\|}$ for $n > 1$. Then x_0, z_n, y_{n-1} are collinear, x_0 lies between z_n and y_{n-1} and $\|x_0 - z_n\| = \|x_0 - y_n\|$. Let $A = X \setminus \bigcup_{n=1}^{\infty}(B(y_{2n-1}, \delta_{2n-1}) \cup B(z_{2n}, \delta_{2n}))$. Obviously, $\underline{p}(A, x_0) = 1$. Since x_0, z_{2n}, y_{2n-1} are collinear and x_0 is between y_{2n-1} and z_{2n}, $B(y_{2n-1}, \delta_{2n-1}) \cap B(z_{2n}, \delta_{2n}) = \emptyset$. Moreover, by (2.1) and (2.2) if any ball is contained in $(X \setminus A) \cap B(x_0, \|x_{2n-1} - x_0\| + \eta'_{2n-1})$ then it is contained either in $B(y_{2n-1}, \delta_{2n-1})$ or in $B(x_0, \frac{c}{4}\|x_{2n-1} - x_0\|)$.

We claim that $\underline{p}(A \cup (X \setminus U), x_0) < 1$. Let $R_n = \|x_n - x_0\| + \eta'_n$ for $n \geq 1$. Obviously, $\lim_{n\to\infty} \frac{R_n}{2\|x_n-x_0\|} = 1$. Fix $n \geq 1$. Assume that there exists a ball $B(z, \varrho)$ contained in $B(x_0, R_{2n-1})$, disjoint from $A \cup (X \setminus U)$ and such that $\varrho > \|x_{2n-1} - x_0\| - \frac{\alpha_0}{2}\|x_{2n-1} - x_0\|$. Then $B(z, \varrho) \not\subset B(x_0, \frac{c}{4}\|x_{2n-1} - x_0\|)$ and therefore, $B(z, \varrho) \subset B(y_{2n-1}, \delta_{2n-1}) \subset B(y_{2n-1}, \|y_{2n-1} - x_0\|)$. Since $\varrho > (1 - \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$, $\|z - y_{2n-1}\| < \|y_{2n-1} - x_{2n-1}\| + \frac{\alpha_0}{2}\|x_{2n-1} - x_0\|$.

On the other hand, since $B(z, \varrho) \subset B(x_0, R_{2n-1})$ and $\varrho > (1-\frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$, we obtain $\|z - x_0\| < (1 + \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$. Therefore, $z \in \Phi(x_0, x_{2n-1}, y_{2n-1}, \alpha_0) \subset B(x_{2n-1}, (1+\frac{c}{8})\|x_{2n-1} - x_0\|)$. It follows that $B(z, \varrho) \cap B(x_{2n-1}, \eta_{2n-1})$ contains a ball disjoint from $A \cup (X \setminus U)$ with radius at least $\frac{c}{8}\|x_{2n-1} - x_0\|$, a contradictions. Therefore any ball contained in $B(x_0, R_{2n-1})$ and disjoint from $A \cup (X \setminus U)$ has a radius less than $(1 - \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$. Thus

$$\underline{p}(A \cup (X \setminus U), x_0) \leq \liminf_{n\to\infty} \frac{2(1 - \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|}{R_n} = 1 - \tfrac{\alpha_0}{2} < 1.$$

Hence, $X \setminus U$ is not lower strongly superporous at x_0 and $U \notin \underline{s}(X, \| \, \|)$. The proof is completed. □

Definition 2.23. We say that a normed space $(X, \| \, \|)$ satisfies condition (B) if for every $\varepsilon > 0$ there exists $\alpha \in (0, 1)$ such that for every collinear $x, y, z \in X$, where y lies between x and z, we have

$$\Phi(x, y, z, \alpha) \subset B\left(y, \varepsilon\|x - y\|\right).$$

Remark 2.24. Obviously, if a normed space $(X, \| \ \|)$ satisfies condition (B) then $(X, \| \ \|)$ satisfies condition (A).

Theorem 2.25. *If a normed space* $(X, \| \ \|)$ *satisfies condition* (B) *then* $\underline{s}(X, \| \ \|) \subset s(X, \| \ \|)$.

Proof. Let us take any $U \notin s(X, \| \ \|)$. Then $X \setminus U$ is not strongly superporous at some $x_0 \in U$. Hence there is $V \subset X$ satisfying $p(V, x_0) = 1$ and $p(V \cup (X \setminus U), x_0) = 1 - c < 1$. Since $p(V, x_0) = 1$, there exists a sequence of pairwise disjoint open balls $(B(x_n, \eta_n))_{n \geq 1}$ such that $\lim_{n\to\infty} x_n = x_0$, $B(x_n, \eta_n) \cap V = \emptyset$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{2\eta_n}{\|x_0 - x_n\| + \eta_n} = p(V, x_0) = 1$. Moreover, since $p(V \cup (X \setminus U), x_0) = 1 - c < 1$, $\lim_{n\to\infty} \frac{2\gamma(x_0, \|x_0 - x_n\| + \eta_n, V \cup (X\setminus U))}{\|x_0 - x_n\| + \eta_n} \leq 1 - c < 1$. Without loss of generality we may assume that $\eta_n > (1 - \frac{c}{8})\|x_n - x_0\|$ and $\frac{2\gamma(x_0, \|x_0 - x_n\| + \eta_n, V \cup (X\setminus U))}{\|x_0 - x_n\| + \eta_n} < 1 - \frac{7}{8}c$ for every $n \geq 1$.

Since $(X, \| \ \|)$ satisfies condition (B), we can find $\alpha_0 \in (0, \frac{c}{8})$ such that for every collinear $x, y, z \in X$, where y lies between x and z, we have $\Phi(x, y, z, \alpha) \subset B\left(y, \frac{c}{8}\|x - y\|\right)$ for every $\alpha < \alpha_0$.

If need be taking a subsequence, we may assume that

$$\|x_{n+1} - x_0\| + \eta_{n+1} < \tfrac{c}{4}\|x_n - x_0\| \qquad \text{for every } n. \tag{2.3}$$

Obviously, $p(X \setminus \bigcup_{n=1}^{\infty} B(x_n, \eta_n), x_0) = 1$.

By Lemma 2.18, there exists a sequence $(B(y_n, \delta_n))_{n\geq 1} \subset X \setminus \{x_0\}$ of balls such that $\underline{p}\,(X \setminus \bigcup_{n=1}^{\infty} B(y_n, \delta_n), x_0) = 1$, x_0, x_n, y_n are collinear, x_n lies between x_0 and y_n, $\|y_n - x_0\| - \delta_n = \|x_n - x_0\| - \eta_n$ and

$$\|y_{n+1} - x_0\| + \delta_{n+1} = \tfrac{c}{4}\|x_n - x_0\| \quad \text{for every } n \geq 1. \tag{2.4}$$

Let $z_n = x_0 - (y_{n-1} - x_0)\frac{\|y_n - x_0\|}{\|y_{n-1} - x_0\|}$ for $n > 1$. Then x_0, z_n, y_{n-1} are collinear, x_0 lies between z_n and y_{n-1} and $\|x_0 - z_n\| = \|x_0 - y_n\|$. Let $A = X \setminus \bigcup_{n=1}^{\infty}(B(y_{2n-1}, \delta_{2n-1}) \cup B(z_{2n}, \delta_{2n}))$. Obviously, $\underline{p}(A, x_0) = 1$. Since x_0, z_{2n}, y_{2n-1} are collinear, $B(y_{2n-1}, \delta_{2n-1}) \cap B(z_{2n}, \delta_{2n}) = \emptyset$. Moreover, by (2.3) and (2.4) if any ball is contained in $(X \setminus A) \cap B(x_0, \|x_{2n-1} - x_0\| + \eta_{2n-1})$ then it is contained either in $B(y_{2n-1}, \delta_{2n-1})$ or in $B(x_0, \frac{c}{4}\|x_{2n-1} - x_0\|)$.

We claim that $\underline{p}(A \cup (X \setminus U), x_0) < 1$. Let $R_n = \|x_n - x_0\| + \eta_n$ for $n \geq 1$. Obviously, $\lim_{n\to\infty} \frac{R_n}{2\|x_n - x_0\|} = 1$. Fix $n \geq 1$. Assume that there exists a ball $B(z, \varrho)$ contained in $B(x_0, R_{2n-1})$, disjoint from $A \cup (X \setminus U)$ and such that $\varrho > \|x_{2n-1} - x_0\| - \frac{\alpha_0}{2}\|x_{2n-1} - x_0\|$. Then $B(z, \varrho) \not\subset B(x_0, \frac{c}{4}\|x_{2n-1} - x_0\|)$ and therefore, $B(z, \varrho) \subset B(y_{2n-1}, \delta_{2n-1}) \subset B(y_{2n-1}, \|y_{2n-1} - x_0\|)$. Since $\varrho > (1 - \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$, we obtain $\|z - y_{2n-1}\| < \|y_{2n-1} - x_{2n-1}\| + \frac{\alpha_0}{2}\|x_{2n-1} - x_0\|$.

On the other hand, since $B(z, \varrho) \subset B(x_0, R_{2n-1})$ and $\varrho > (1 - \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$, we obtain $\|z - x_0\| < (1 + \frac{\alpha_0}{2})\|x_{2n-1} - x_0\|$. Therefore, $z \in \Phi(x_0, x_{2n-1}, y_{2n-1}, \alpha_0) \subset B(x_{2n-1}, \frac{c}{8}\|x_{2n-1} - x_0\|)$. It follows that $B(z, \varrho) \cap B(x_{2n-1}, \eta_{2n-1})$ contains a ball disjoint from $A \cup (X \setminus U)$ with radius at least $(1 - \frac{c}{4})\|x_{2n-1} - x_0\|$,

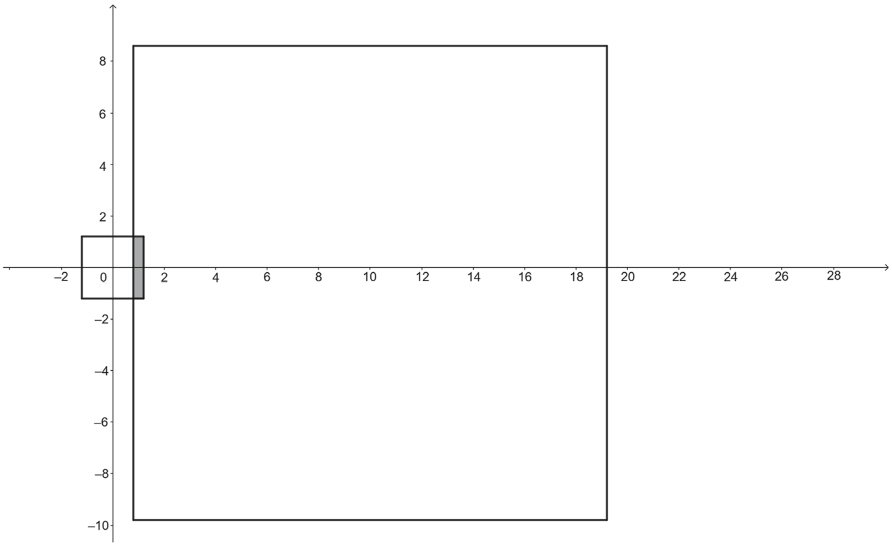

FIGURE 5. $\Phi((0,0),(1,0),(10,0),\alpha)$ in $(\mathbb{R}^2,\|\ \|_{\max})$

a contradictions. Therefore any ball contained in $B(x_0,R_{2n-1})$ and disjoint from $A\cup(X\setminus U)$ has a radius less than $(1-\frac{\alpha_0}{2})\|x_{2n-1}-x_0\|$. Thus

$$\underline{p}(A\cup(X\setminus U),x_0)\le\liminf_{n\to\infty}\frac{2(1-\frac{\alpha_0}{2})\|x_{2n-1}-x_0\|}{R_n}=1-\frac{\alpha_0}{2}<1.$$

Hence, $X\setminus U$ is not lower strongly superporous at x_0 and $U\notin\underline{s}(X,\|\ \|)$. The proof is completed. □

Example 2.26. Let $\mathcal{T}_{\max}$ be a topology in $\mathbb{R}^n$ generated by the norm $\|\ \|_{\max}$. Let $x=0_n$, $y=(1,0,\ldots,0)$ and $z=(10,0,\ldots,0)$. Then

$$\begin{aligned}\Phi(x,y,z,\alpha)&=[1-\alpha,1+\alpha]\times[-1-\alpha,1+\alpha]^{n-1}\\&\supset[1-\alpha,1+\alpha]\times[-1,1]^{n-1}\end{aligned}$$

and $B(y,\frac{1}{2}\|x-y\|_{\max})=B((1,0,\ldots,0),\frac{1}{2})=[\frac{1}{2},\frac{3}{2}]\times[-\frac{1}{2},\frac{1}{2}]^{n-1}\not\supset\Phi(x,y,z,\alpha)$ for any $\alpha>0$, (see Fig. 5).

Therefore $(\mathbb{R}^n,\|\ \|_{\max})$ does not satisfies condition (B) (and satisfies condition (A)).

Remark 2.27. Repeating arguments from the proof of Example 2.26 one can prove that $(l_\infty,\|\ \|_{\sup})$ does not satisfies condition (B) and satisfies condition (A).

Theorem 2.28. *Let $(X,\|\ \|)$ be either $(\mathbb{R}^n,\|\ \|_{\max})$ or $(l_\infty,\|\ \|_{\sup})$. Then $\underline{s}(X,\|\ \|)\subset s(X,\|\ \|)$, although neither $(\mathbb{R}^n,\|\ \|_{\max})$ nor $(l_\infty,\|\ \|_{\sup})$ satisfies condition (B).*

Proof. Take any $U \notin s(X, \| \ \|)$. Then $X \setminus U$ is not strongly superporous at some $x_0 \in U$. We may assume $x_0 = 0_X$. Hence we can find $A \subset X$ such that $p(A, 0_X) = 1$ and $p(A \cup (X \setminus U), 0_X) < 1$. Since $p(A, 0_X) = 1$, there exists a sequence of pairwise disjoint balls $(B(x_k, \eta_k))_{k\geq 1}$ such that $(\eta_k)_{k\geq 1}$ and $(\|x_k - x_0\|)_{k\geq 1}$ are decreasing and tend to 0, $A \cap \bigcup_{k=1}^{\infty} B(x_k, \eta_k) = \emptyset$ and $\lim_{k\to\infty} \frac{2\eta_k}{\|x_k - x_0\| + \eta_k} = 1$. Let $R_k = \|x_k - x_0\| + \eta_k$ for $k \geq 1$. By Lemma 1.1, there are sequences $(y_k)_{k\geq 1} \subset X$ and $(\zeta_k)_{k\geq 1} \subset \{-1, 1\}^k$ or $(\zeta_k)_{k\geq 1} \subset \{-1, 1\}^{\omega}$ such that

$$B(y_k, \eta_k - \tfrac{R_k}{4}) \subset B(x_k, \eta_k) \cap B(0_X, \tfrac{R_k}{2}) \cap H^{\zeta_k}$$

for every k. Let $\delta_k = \eta_k - \frac{R_k}{4}$. Then

$$\lim_{k\to\infty} \frac{2\delta_k}{\delta_k + \|y_k\|} \geq \lim_{k\to\infty} \frac{2\eta_k - \frac{R_k}{2}}{\frac{R_k}{2}} = \lim_{k\to\infty} (\tfrac{4\eta_k}{R_k} - 1) = 1 \tag{2.5}$$

and $p(X \setminus \bigcup_{k=1}^{\infty} B(y_k, \delta_k), 0_X) = 1$. Since

$$p\Big((X \setminus U) \cup \Big(X \setminus \bigcup_{k=1}^{\infty} B(y_k, \delta_k)\Big), 0_X\Big) \leq p((X \setminus U) \cup A, 0_X) < 1, \tag{2.6}$$

$\limsup_{k\to\infty} \frac{\gamma(0_X, \|y_k\| + \delta_k, (X\setminus U)\cup(X\setminus\bigcup_{k=1}^{\infty} B(y_k,\delta_k))}{\|y_k\|+\delta_k} < 1$.

For any ball $B(x, \nu)$ let $\varphi(B(x, \nu)) = (x_1 - \nu \operatorname{sgn}(x_1), x_2 - \nu \operatorname{sgn}(x_2), \ldots)$. Observe that if $B(x, \nu) \subset H^{\zeta}$, then $\varphi(B(x, \nu)) \in H^{\zeta}$ too and a point $\varphi(B(x, \nu))$ minimizes the distance between 0_X and $\overline{B}(x, \nu)$. Moreover, for any $B(x, \nu) \subset H^{\zeta}$ and $\varrho > 0$ there exists a unique $y \in X$ such that $\varphi(B(x, \nu)) = \varphi(B(y, \varrho))$ and $B(y, \varrho) \subset H^{\zeta}$ too (see Fig. 6).

For $k > 1$ let $\sigma_k = \frac{1}{2}\left(\|y_{k-1}\| - (\|y_k\| - \delta_k)\right)$ and z_k be such that $\varphi(B(y_k, \delta_k)) = \varphi(B(z_k, \sigma_k))$. Then $B(z_k, \sigma_k) \subset H^{\zeta_k}$ and $\|z_k\| + \sigma_k = \frac{1}{2}\|y_{k-1}\|$. Let $B = X \setminus \bigcup_{k=1}^{\infty} B(z_k, \sigma_k)$ and $r_k = \|y_k\| + \delta_k$ for $k \geq 1$. By Lemma 2.17, $\underline{p}(B, 0_X) = 1$.

On the other hand,

$$\gamma(0_X, r_k, (X \setminus U) \cup B) \leq \min\left\{\tfrac{1}{4}\|y_k\|, \gamma(0_X, r_k, H^{\zeta_n} \cap ((X \setminus U) \cup B))\right\}$$

and

$$\gamma\big(0_X, r_k, H^{\zeta_k} \cap ((X \setminus U) \cup B)\big) \leq \gamma(0_X, r_k, (X \setminus U) \cup A) + \operatorname{diam}((H^{\zeta_k} \cap B(0_X, r_k)) \setminus B(y_k, \delta_k)).$$

By (2.5) and (2.6), we obtain

$$\limsup_{k\to\infty} \frac{\gamma(0_X, r_k, H^{\zeta_n} \cap ((X \setminus U) \cup B))}{r_k} < 1.$$

Therefore $\underline{p}((X \setminus U) \cup B, 0_X) < 1$, $X \setminus U$ is not lower strongly superporous at 0_X and $U \notin \underline{s}(X, \| \ \|)$. The proof is completed. □

Theorem 2.29. $s(l_\infty, \| \ \|_{\sup}) \not\subset \underline{p}(l_\infty, \| \ \|_{\sup})$.

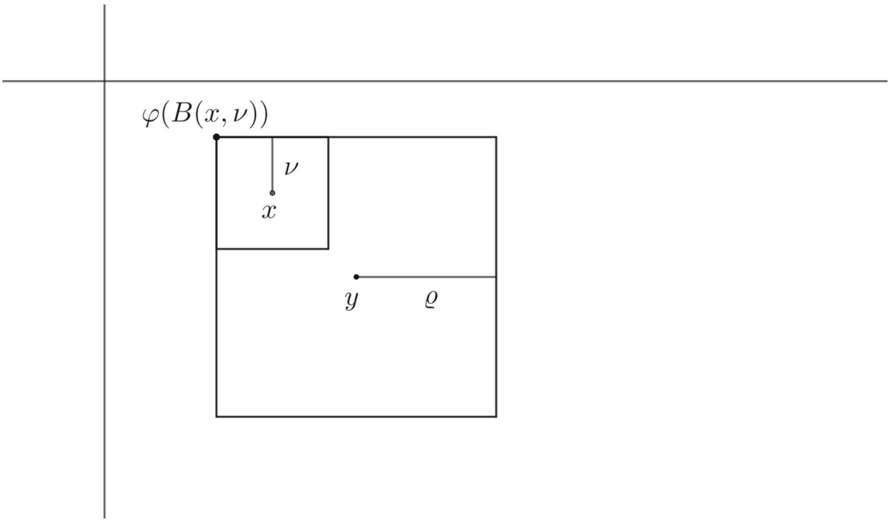

FIGURE 6. $\varphi(B(x,\nu))$ and $B(y,\varrho)$ in $(\mathbb{R}^2, \| \|_{\max})$

Proof. Define $b_n = \left(\frac{3}{4}\right)^{n-1}$, $a_n = \frac{b_n}{2} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$, $c_n = \frac{a_n+b_n}{2} = \left(\frac{3}{4}\right)^n$ for $n \geq 1$. Let $0_\infty \in l_\infty$, $0_\infty = (0,0,\ldots)$ and $x_n \in l_\infty$, $x_n = (-\frac{a_n}{2}, -\frac{a_n}{2}, \ldots, -\frac{a_n}{2}, \underbrace{c_n}_{n-th\ term}, -\frac{a_n}{2}, \ldots)$ for $n \geq 1$. Finally, let $U = l_\infty \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, \frac{a_n}{2})$. Observe that $\overline{B}(x_n, \frac{a_n}{2}) \subset H^{\zeta_n}$, where $\zeta_n = (-1,-1,\ldots,-1, \underbrace{1}_{n-th\ term}, -1, \ldots)$.

Since $\|x_n\|_{\sup} - \frac{a_n}{2} = c_n - \frac{a_n}{2} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$ and $\|x_{n+1}\|_{\sup} + \frac{a_{n+1}}{2} = c_{n+1} + \frac{a_{n+1}}{2} = b_{n+1} = \left(\frac{3}{4}\right)^n$, we obtain

$$\limsup_{n\to\infty} \frac{\|x_n\|_{\sup} - \frac{a_n}{2}}{\|x_{n+1}\|_{\sup} + \frac{a_{n+1}}{2}} = \limsup_{n\to\infty} \frac{\frac{1}{2}\left(\frac{3}{4}\right)^{n-1}}{\left(\frac{3}{4}\right)^n} = \frac{2}{3} < 1.$$

Therefore, by Lemma 2.17, $\underline{p}(U, 0_\infty) > 0$. Moreover, $\underline{p}(U \cup (l_\infty \setminus U), 0_\infty) = \underline{p}(l_\infty, 0_\infty) = 0$ and U is not lower superporous at $0_\infty \in E$. Therefore, $U \notin \underline{p}(l_\infty, \| \|_{\sup})$.

We claim that $U \in s(l_\infty, \| \|_{\sup})$. Clearly, $l_\infty \setminus U$ is strongly superporous at every $x \in U \setminus \{0_\infty\}$. Take any $A \subset l_\infty$ such that $p(A, 0_\infty) = 1$. There is a sequence of pairwise disjoint balls $(B(y_k, \eta_k))_{n\geq 1} \subset l_\infty \setminus (A \cup \{0_\infty\})$ such that $\lim_{n\to\infty} y_k = 0_\infty$ and $\lim_{n\to\infty} \frac{2\eta_k}{\|y_k\|_{\sup}+\eta_k} = 1$. By Lemma 1.1, there exist sequences $(B(z_k, \mu_k))_{k\geq 1} \subset l_\infty$ and $(\varsigma_k)_{k\geq 1} \subset \{-1,1\}^\omega$ such that $B(z_k, \mu_k) \subset B(y_k, \eta_k) \cap H^{\varsigma_k} \cap B(0_\infty, \frac{1}{2}(\|y_k\|_{\sup} + \eta_k))$ and $\mu_k > \eta_k - \frac{1}{4}(\|y_k\|_{\sup} + \eta_k)$. Then

$$\lim_{k\to\infty} \frac{2\mu_k}{\|z_k\|_{\sup} + \mu_k} \geq \lim_{k\to\infty} \frac{2\eta_k - \frac{1}{2}(\|y_k\|_{\sup} + \eta_k)}{\frac{1}{2}(\|y_k\|_{\sup} + \eta_k)} = 1$$

and $A \cap \bigcup_{k=1}^{\infty} B(z_k, \mu_k) = \emptyset$.

We construct a new sequence of open balls $(B(v_k, \nu_k))_{k\geq 1}$ such that $B(v_k, \nu_k) \subset B(z_k, \mu_k) \cap U$ and $p(l_\infty \setminus \bigcup_{k=1}^{\infty} B(v_k, \nu_k), 0_\infty) = 1$. Fix $k \geq 1$. If $\varsigma_k \neq \zeta_n$ for $n \geq 1$, i.e. $B(z_k, \mu_k) \subset U$ then we take $B(v_k, \nu_k) = B(z_k, \mu_k)$. Let us consider the case, where $\varsigma_k = \zeta_{n_k}$ for some $n_k \geq 1$ (obviously, $\varsigma_k \neq \zeta_n$ for $n \neq n_k$). Let $B(z_k, \mu_k) = (d_1^k, d_1^k + 2\mu_k) \times (d_2^k, d_2^k + 2\mu_k) \times \cdots$, where $d_{n_k}^k \geq 0$ and $d_n^k + 2\mu_k \leq 0$ for $n \neq n_k$. Since

$$\lim_{k\to\infty} \frac{2\mu_k}{\|z_k\|_{\sup} + \mu_k} \geq \lim_{k\to\infty} \frac{2\eta_k - \frac{1}{2}(\|y_k\|_{\sup} + \eta_k)}{\frac{1}{2}(\|y_k\|_{\sup} + \eta_k)} = 1,$$

we have $\lim_{n\to\infty} \frac{\sup\{|d_n^k|:\, n\geq 1\}}{\mu_k} = 0$. By Lemma 1.5, we can find $(d, e) \subset (d_{n_k}^k, d_{n_k}^k + 2\mu_k)$ such that $(d, e) \cap [a_{n_k}, b_{n_k}] = \emptyset$ and $\frac{e-d}{e} > 1 - \sqrt{\frac{3|d_{n_k}^k|}{|d_{n_k}^k| + 2\mu_k}}$. Then

$$\begin{aligned}(d_1^k, d_1^k + 2\mu_k) \ &\times \cdots \times (d_{n_k-1}^k, d_{n_k-1}^k + 2\mu_k) \times (d, e) \\ &\times (d_{n_k+1}^k, d_{n_k+1}^k + 2\mu_k) \times \cdots \subset U\end{aligned}$$

and we can find $B(v_k, \frac{e-d}{2}) \subset U$, i.e., we have $\nu_k = \frac{e-d}{2}$. Since

$$\lim_{k\to\infty} \left(1 - \sqrt{\frac{\sup\{3|d_n^k|:\, n \geq 1\}}{\inf\{|d_n^k| + 2\mu_k :\, n \geq 1\}}}\right) = 1,$$

we finally obtain $p(l_\infty \setminus \bigcup_{k=1}^{\infty} B(v_k, \nu_k), 0_\infty) = 1$. Therefore $l_\infty \setminus U$ is strongly superporous at 0_∞. Hence, $U \in s(l_\infty, \| \|_{\sup})$, which completed the proof. $\square$

Question 2.30. Does there exist a normed space $(X, \| \|)$ such that $s(X, \| \|) \subset \underline{p}(X, \| \|)$?

We may present relationships between considered topologies in the following diagram.

$$\begin{array}{ccccc} \mathcal{T}_{\| \|} \subseteqq \underline{s}(X, \| \|) & \overset{(A)}{\subsetneqq} & p(X, \| \|) & \subsetneqq & \underline{p}(X, \| \|) \\ \overset{(B)}{\subsetneqq} \\ s(X, \| \|) \end{array}$$

Inclusion $\mathcal{T}_{\| \|} \subset \underline{s}(X, \| \|)$ is just equality in some normed spaces, inclusion $\underline{s}(X, \| \|) \subsetneqq p(X, \| \|)$ holds under condition (A) and inclusion $\underline{s}(X, \| \|) \subsetneqq s(X, \| \|)$ holds under condition (B). No other inclusion, in general, holds, although we know only one example of a normed space in which $s(X, \| \|) \not\subset \underline{p}(X, \| \|)$.

3. Lower porouscontinuity

In [1] J. Borsík and J. Holos defined families of porouscontinuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Some properties of porouscontinuity can be found in [1,2,10]. Applying their ideas and replacing standard porosity in $\mathbb{R}$ by the lower porosity in X we transfer this concept for real functions defined on $(X, \|\ \|)$.

Definition 3.1. Let $(X, \|\ \|)$ be a normed space, $f\colon X \to R$ and $x \in X$.

Let $r \in [0, 1)$.The function f will be called:

- $\underline{\mathcal{P}_r}$-continuous at x if there exists a set $U \subset X$ such that $x \in U$, $\underline{p}(X \setminus U, x) > r$ and $f_{\restriction U}$ is continuous at x;
- $\underline{\mathcal{S}_r}$-continuous at x if for each $\varepsilon > 0$ there exists a set $U \subset X$ such that $x \in U$, $\underline{p}(X \setminus U, x) > r$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$;

Let $r \in (0, 1]$.The function f will be called:

- $\underline{\mathcal{M}_r}$-continuous at x if there exists a set $U \subset X$ such that $x \in U$, $\underline{p}(X \setminus U, x) \geq r$ and $f_{\restriction U}$ is continuous at x;
- $\underline{\mathcal{N}_r}$-continuous at x if for each $\varepsilon > 0$ there exists a set $U \subset X$ such that $x \in U$, $\underline{p}(X \setminus U, x) \geq r$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$;

By $\underline{\mathcal{P}_r}(f)$, $\underline{\mathcal{S}_r}(f)$, $\underline{\mathcal{M}_r}(f)$ and $\underline{\mathcal{N}_r}(f)$ we denote the sets of points at which f is $\underline{\mathcal{P}_r}$-continuous, $\underline{\mathcal{S}_r}$-continuous, $\underline{\mathcal{M}_r}$-continuous and $\underline{\mathcal{N}_r}$-continuous, respectively.

Proposition 3.2. *Let $f\colon X \to \mathbb{R}$ and $x \in X$. Then*

(1) $x \in \underline{\mathcal{S}_r}(f)$ if and only if $\underline{p}(X \setminus \{t\colon |f(t) - f(x)| < \varepsilon\}, x) > r$ for every $\varepsilon > 0$;
(2) $x \in \underline{\mathcal{N}_r}(f)$ if and only if $\underline{p}(X \setminus \{t\colon |f(t) - f(x)| < \varepsilon\}, x) \geq r$ for every $\varepsilon > 0$.

for corresponding r.

Similarly as in [1], we can easily check that f is $\underline{\mathcal{M}_r}$-continuous at x if and only if it is $\underline{\mathcal{N}_r}$-continuous at x.

If f is $\underline{\mathcal{P}_r}$-continuous, $\underline{\mathcal{S}_r}$-continuous, $\underline{\mathcal{M}_r}$-continuous at every point of X for some corresponding r then we say that f is $\underline{\mathcal{P}_r}$-continuous, $\underline{\mathcal{S}_r}$-continuous, $\underline{\mathcal{M}_r}$-continuous, respectively. All of these functions are called lower porouscontinuous functions.

Obviously, if f is continuous at some x then f is lower porouscontinuous (in each sense) at x. We introduce for corresponding r the following notations:

- $\underline{\mathcal{M}_r} = \underline{\mathcal{N}_r} = \{f\colon \underline{\mathcal{M}_r}(f) = X\}$;
- $\underline{\mathcal{P}_r} = \{f\colon \underline{\mathcal{P}_r}(f) = X\}$;
- $\underline{\mathcal{S}_r} = \{f\colon \underline{\mathcal{S}_r}(f) = X\}$.

In the paper we focus on $\underline{\mathcal{M}_1}$ and $\underline{\mathcal{S}_0}$. In [11] some properties of lower porouscontinuous functions $\underline{\mathcal{P}_r}$, $\underline{\mathcal{S}_r}$, $\underline{\mathcal{M}_r}$ for $r \in (0, 1)$ defined on $\mathbb{R}^2$ are presented.

Lemma 3.3. *Let* $(X, \| \ \|)$ *be a normed space. Then*

$$\operatorname{int}_{\underline{p}(X,\| \ \|)} E = \{x \in X : X \setminus E \text{ is lower superporous at } x\}$$

and

$$\operatorname{int}_{\underline{s}(X,\| \ \|)} E = \{x \in X : X \setminus E \text{ is lower strongly superporous at } x\}$$

for every $E \subset X$*.*

Proof. Denote $V = \operatorname{int}_{\underline{p}(X,\| \ \|)} E$. Take $x_0 \in V$. Then $V \in \underline{p}(X, \| \ \|)$ and $X \setminus V$ is lower superporous at x_0. Since $X \setminus E \subset X \setminus V$, the set $X \setminus E$ is lower superporous at x_0.

Now, let $x_0 \in \{x \in X : X \setminus E$ is lower superporous at $x\}$. Denote $V = \operatorname{int}_{\mathcal{T}_{\| \ \|}} E \cup \{x_0\}$. Obviously, $X \setminus V$ is lower superporous at each point $x \in V \setminus \{x_0\}$. Moreover, for each set $A \subset X$ we obtain $\underline{p}((X \setminus V) \cup A, x_0) = \underline{p}((X \setminus E) \cup A, x_0)$, because every open ball disjoint with $X \setminus E$ is contained in V. Since $X \setminus E$ is lower superporous at x_0, the set $X \setminus V$ is lower superporous at x_0, too. Therefore $V \in \underline{p}(X, \| \ \|)$. Finally $x_0 \in \operatorname{int}_{\underline{p}(X,\| \ \|)} E$, because $V \subset E$.

The proof of the second statement is very similar and we omit it. □

Lemma 3.4. *Let* A *be a closed subset of a normed space* $(X, \| \ \|)$ *and* $x_0 \in A$*. Then there exists* $E \subset X \setminus A$ *such that*

- $\operatorname{cl}_{\mathcal{T}_{\| \ \|}} E \subset E \cup \{x_0\}$*;*
- E *is discrete;*
- *for each* $B \subset X$*, if* $E \subset B$ *then* $\underline{p}(B, x_0) = \underline{p}(B \cup (X \setminus A), x_0)$*.*

Proof. Let $U_n = \overline{B}(x_0, \frac{1}{n}) \setminus B(x_0, \frac{1}{n+1})$ for $n \geq 1$. By the Zorn Lemma, for every n we can choose a discrete set $E_n \subset U_n \setminus A$ such that $U_n \setminus A \subset \bigcup_{x \in E_n} B\left(x, \frac{1}{(n+1)^2}\right)$ and $\|x_1 - x_2\| \geq \frac{1}{(n+1)^2}$ for $x_1, x_2 \in E_n$, $x_1 \neq x_2$. Let

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Then E is discrete, $E \cap A = \emptyset$ and $\operatorname{cl}_{\mathcal{T}_{\| \ \|}} E \subset E \cup \{x_0\}$. Take any $B \subset X$ such that $E \subset B$. The inequality $\underline{p}(B, x_0) \geq \underline{p}(B \cup (X \setminus A), x_0)$ is obvious. If $\underline{p}(B, x_0) = 0$ then certainly $\underline{p}(B \cup (X \setminus A), x_0) = 0$. Let $\underline{p}(B, x_0) = \alpha > 0$. Choose β, β_1 such that $0 < \beta < \beta_1 < \alpha$. We can find $n_0 > 1$ such that $\frac{1}{n_0} < \min\left\{\frac{\beta_1 - \beta}{4}, \frac{\varepsilon}{8}, \frac{\beta_1}{8}\right\}$. Since $\underline{p}(B, x_0) = \alpha > \beta_1$, we can find $R_0 \in (0, \frac{1}{4n_0})$ such that $\frac{2\gamma(x_0, R, B)}{R} > \beta_1$ for $R \in (0, R_0)$. Choose any $R \in (0, R_0)$. There exists $B(y, \eta)$ such that $\frac{2\eta}{\eta + \|x_0 - y\|} \geq \frac{2\eta}{R} > \beta_1$ and $B(y, \eta) \cap B = \emptyset$.

Suppose that $B(y, \eta) \not\subset A$ and take any $z \in B(y, \eta) \setminus A$. There exists n_1 such that $\frac{1}{n_1 + 1} < \|z - x_0\| \leq \frac{1}{n_1}$, i.e. $z \in U_n$. Since

$$\|z - x_0\| \leq \|z - y\| + \|y - x_0\| < \eta + \|y - x_0\| < 2\|y - x_0\| < \frac{1}{2n_0},$$

we obtain $\frac{1}{n_1+1} < \frac{1}{2n_0}$ and $n_1 > n_0$. By construction of E, there exists $v \in E_{n_1}$ such that $\|z - v\| \leq \frac{1}{(n_1+1)^2}$. Observe that $v \in E \subset B$ and $B \cap B(y, \eta) = \emptyset$. Therefore $v \notin B(y, \eta)$, i.e. $\|v - y\| \geq \eta$. Thus

$$\|z - y| \geq \|y - v\| - \|v - z\| \geq \eta - \frac{1}{(n_1+1)^2}.$$

This means that $B\Big(y, \eta - \frac{1}{(n_1+1)^2}\Big) \cap (B \cup (X \setminus A)) = \emptyset$. By inequality $\frac{2\eta}{\eta + \|x_0 - y\|} > \beta_1$, we obtain $2\eta > \beta_1 \|x_0 - y\|$. Hence

$$\eta > \frac{\beta_1 \|x_0 - y\|}{2} > \frac{\frac{1}{2}\beta_1 \|x_0 - z\|}{2} > \frac{\beta_1}{4(n_1+1)} > \frac{8}{n_0} \frac{1}{4(n_1+1)} > \frac{1}{n_1^2}$$

and $\eta - \frac{1}{(n_1+1)^2} > 0$. Moreover,

$$\begin{aligned}
&\frac{2\left(\eta - \frac{1}{(n_1+1)^2}\right)}{\eta - \frac{1}{(n_1+1)^2} + \|x_0 - y\|} > \frac{2\eta - \frac{2}{(n_1+1)^2}}{\eta + \|x_0 - y\|} \\
&\quad = \frac{2\eta}{\eta + \|x_0 - y\|} - \frac{2}{(n_1+1)^2 (\eta + \|x_0 - y\|)} > \beta_1 - \frac{2}{(n_1+1)^2 \|x_0 - z\|} \\
&\quad > \beta_1 - \frac{2}{(n_1+1)^2 \frac{1}{n_1+1}} = \beta_1 - \frac{2}{n_1+1} > \beta_1 - \frac{4}{n_0} > \beta_1 - (\beta_1 - \beta) = \beta.
\end{aligned}$$

Since $\beta \in (0, \alpha)$ was chosen arbitrary, $\underline{p}(B \cup (X \setminus A), x_0) \geq \alpha$, which completed the proof. $\square$

Lemma 3.5. *Let $(X, \|\ \|)$ be a normed space, $f\colon X \to \mathbb{R}$, $x_0 \in X$ and $\varrho > 0$. If f restricted to $\overline{B}(x_0, \varrho)$ is continuous then f is $\underline{p}(X, \|\ \|)$ and $\underline{s}(X, \|\ \|)$-continuous at every $x \in S(x_0, \varrho)$.*

Proof. For every $x \in S(x_0, \varrho)$ and $\varepsilon > 0$ there exists R_0 such that

$$B\left(x + \frac{R}{2}\frac{x_0 - x}{\|x_0 - x\|}, \frac{R}{2}\right) \subset B(x_0, \varrho) \cap B(x, R) \cap \{t \in X\colon |f(x) - f(x_0)| < \varepsilon\}$$

for every $R < R_0$, which completed the proof. $\square$

It is easily seen that result of addition and multiplication of functions from discussed classes of functions, in general, need not belong to these classes. Therefore we studied the following notion.

Definition 3.6 [3]. Let $\mathcal{F}$ be a family of real functions defined on $(X, \|\ \|)$. A set $\mathfrak{M}_a(\mathcal{F}) = \{g\colon X \to \mathbb{R}\colon \forall_{f \in \mathcal{F}} (f + g \in \mathcal{F})\}$ is called the maximal additive class for $\mathcal{F}$.

Remark 3.7. Let $f\colon X \to \mathbb{R}$, $f(x) = 0$ for $x \in X$ be a constant function. Clearly, if $f \in \mathcal{F}$ then $\mathfrak{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Let $\mathcal{C}_{\mathcal{T}}$ denote the class of continuous functions $f\colon (X, \mathcal{T}) \to \mathbb{R}$.

Theorem 3.8. $\mathfrak{M}_a(\underline{\mathcal{S}_0}) = \mathcal{C}_{\underline{p}(X, \|\ \|)}$. *Moreover,* $\mathfrak{M}_a(\underline{\mathcal{P}_0}) \subset \mathcal{C}_{\underline{p}(X, \|\ \|)}$.

Proof. Let $f \in \mathcal{C}_{\underline{p}(X,\| \|)}$. Take $g \in \underline{\mathcal{S}_0}$, $x_0 \in X$, $\varepsilon > 0$.

Denote $E_\varepsilon = \left\{x \in X\colon |g(x) - g(x_0)| < \frac{\varepsilon}{2}\right\}$. Then $\underline{p}(X \setminus E_\varepsilon, x_0) > 0$. Since $f \in \mathcal{C}_{\underline{p}(X,\| \|)}$, there exists a set U such that $x_0 \in U$, $U \in \underline{p}(X, \| \|)$ and $U \subset \left\{x \in X\colon |f(x) - f(x_0)| < \frac{\varepsilon}{2}\right\}$. By the definition of topology $\underline{p}(X, \| \|)$ the set $(X \setminus U) \cup (X \setminus E_\varepsilon)$ is lower porous. Moreover, $(X \setminus U) \cup (X \setminus E_\varepsilon) = X \setminus (U \cap E_\varepsilon)$. Thus $\underline{p}(X \setminus (U \cap E_\varepsilon), x_0) > 0$ and $|f(x) + g(x) - f(x_0) - g(x_0)| < \varepsilon$ for each $x \in U \cap E_\varepsilon$. Therefore $f + g$ is $\underline{\mathcal{S}_0}$-continuous at x_0. Hence $f \in \mathfrak{M}_a(\underline{\mathcal{S}_0})$.

Suppose that $f \notin \mathcal{C}_{\underline{p}(X,\| \|)}$. Then there exist $x_0 \in X$ and $\varepsilon > 0$ such that $x_0 \notin \operatorname{int}_{\underline{p}(X,\| \|)} E_\varepsilon$, where $E_\varepsilon = \{x \in X\colon |f(x) - f(x_0)| < \varepsilon\}$. The set E_ε does not contain any $\underline{p}$-neighbourhood of point x_0. By Lemma 3.3, the set $X \setminus E_\varepsilon$ is not lower superporous at x_0. Therefore there exists a set $F \subset X$ such that $\underline{p}(F, x_0) > 0$ and $\underline{p}((X \setminus E_\varepsilon) \cup F, x_0) = 0$. There exists a sequence of closed balls $\left(\overline{B}(x_n, \delta_n)\right)_{n \geq 1}$ such that $\bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n) \subset X \setminus F$, $x_0 \notin \bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n)$, $\lim_{n\to\infty} x_n = x_0$ and $\underline{p}(F, x_0) = \underline{p}\left(X \setminus \bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n), x_0\right) > 0$. Let $A = \{x_0\} \cup \bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n)$. By Lemma 3.4, we can find $E \subset X \setminus A$ such that for every $B \subset X$ if $E \subset B$ then $\underline{p}(B, x_0) = \underline{p}(B \cup (X \setminus A), x_0)$. Define $\widetilde{g}\colon (A \setminus \{x_0\}) \cup E \to \mathbb{R}$ by $\widetilde{g}(x) = 0$ for $x \in A \setminus \{x_0\}$ and $\widetilde{g}(x) = -f(x) + f(x_0) + \varepsilon$ for $x \in E$. Since $(A \setminus \{x_0\}) \cup E$ is a closed subset of $X \setminus \{x_0\}$ and $\widetilde{g}$ is continuous, by the Tietze Theorem, there exists a continuous extension $\widehat{g}\colon X \setminus \{x_0\} \to \mathbb{R}$ of $\widetilde{g}$. Finally, let $g\colon X \to \mathbb{R}$ be defined by $g(x) = \widehat{g}(x)$ for $x \neq x_0$ and $g(x_0) = 0$.

Since g is continuous at every point except x_0, $g(x) = g(x_0)$ for $x \in A$ and $\underline{p}(X \setminus A, x_0) = \underline{p}(F, x_0) > 0$, we have $g \in \underline{\mathcal{P}_0}$. On the other hand, $E \subset \{x \in X\colon |(f + g)(x) - (f + g)(x_0)| \geq \varepsilon\}$ and

$$\begin{aligned}
&\underline{p}(X \setminus \{x \in X\colon |(f + g)(x) - (f + g)(x_0)| < \varepsilon\}, x_0)\\
&\quad = \underline{p}((X \setminus A) \cup (X \setminus \{x \in X\colon |(f + g)(x) - (f + g)(x_0)| < \varepsilon\}), x_0)\\
&\quad = \underline{p}(X \setminus \{x \in A\colon |(f + g)(x) - (f + g)(x_0)| < \varepsilon\}, x_0)\\
&\quad = \underline{p}(X \setminus (E_\varepsilon \cap A), x_0) = \underline{p}((X \setminus E_\varepsilon) \cup (X \setminus A), x_0) \leq \underline{p}((X \setminus E_\varepsilon) \cup F, x_0) = 0.
\end{aligned}$$

Therefore $X \setminus \{x \in X\colon |(f + g)(x) - (f + g)(x_0)| < \varepsilon\}$ is not lower superporous at x_0 and $f + g \notin \underline{\mathcal{S}_0}$. It implies $\mathfrak{M}_a(\underline{\mathcal{S}_0}) \subset \mathcal{C}_{\underline{p}(X,\| \|)}$ and $\mathfrak{M}_a(\underline{\mathcal{P}_0}) \subset \mathcal{C}_{\underline{p}(X,\| \|)}$. The proof is completed. □

Theorem 3.9. $\mathfrak{M}_a(\underline{\mathcal{M}_1}) = \mathcal{C}_{\underline{s}(X,\| \|)}$.

Proof. The proof of inclusion $\mathcal{C}_{\underline{s}(X,\| \|)} \subset \mathfrak{M}_a(\underline{\mathcal{M}_1})$ is very similar to the proof of inclusion $\mathcal{C}_{\underline{p}(X,\| \|)} \subset \mathfrak{M}_a(\underline{\mathcal{S}_0})$ in the proof of Theorem 3.8 and we omit it.

Take any $f \notin \mathcal{C}_{\underline{s}(X,\| \|)}$. Then there exist $x_0 \in X$ and $\varepsilon > 0$ such that $x_0 \notin \operatorname{int}_{\underline{s}(X,\| \|)} E_\varepsilon$, where $E_\varepsilon = \{x \in X\colon |f(x) - f(x_0)| < \varepsilon\}$. By Lemma 3.3, the set $X \setminus E_\varepsilon$ is not lower strongly superporous at x_0. Similarly as in the proof of Theorem 3.8, we can find $F \subset X$ and $A = \{x_0\} \cup \bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n)$ such that $\underline{p}(F, x_0) = 1$, $\underline{p}((X \setminus E_\varepsilon) \cup F, x_0) < 1$, $A \subset X \setminus F$, $x_0 \notin \bigcup_{n=1}^\infty \overline{B}(x_n, \delta_n)$, $\lim_{n\to\infty} x_n = x_0$ and $\underline{p}(F, x_0) = \underline{p}\,(X \setminus A, x_0) > 0$. By Lemma 3.4, we can find $E \subset X \setminus A$ such that for every $B \subset X$ if $E \subset B$ then $\underline{p}(B, x_0) = \underline{p}(B \cup (X \setminus A), x_0)$. Define $\widetilde{g}\colon (A \setminus \{x_0\}) \cup E \to \mathbb{R}$ by $\widetilde{g}(x) = 0$ for $x \in A \setminus \{x_0\}$ and $\widetilde{g}(x) = -f(x) + f(x_0) + \varepsilon$ for $x \in E$. Again, similarly as in the proof of Theorem 3.8, we can define $g\colon X \to \mathbb{R}$ such that g is continuous at every point except x_0, $g(x) = 0$ for $x \in A$ and $g(x) = f(x_0) - f(x) + \varepsilon$ for $x \in E$. Since g is continuous at every point except x_0 and $\underline{p}(X \setminus A, x_0) = \underline{p}(F, x_0) = 1$, we have $g \in \underline{\mathcal{M}_1}$. On the other hand, $E \subset \{x \in X\colon |(f+g)(x) - (f+g)(x_0)| \geq \varepsilon\}$ and

$$\begin{aligned} &\underline{p}(X \setminus \{x \in X\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\}, x_0) \\ &\quad = \underline{p}((X \setminus A) \cup (X \setminus \{x \in X\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\}), x_0) \\ &\quad = \underline{p}(X \setminus \{x \in A\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\}, x_0) \\ &\quad = \underline{p}(X \setminus (E_\varepsilon \cap A), x_0) = \underline{p}((X \setminus E_\varepsilon) \cup (X \setminus A), x_0) \leq \underline{p}((X \setminus E_\varepsilon) \cup F, x_0) < 1. \end{aligned}$$

Therefore, $X \setminus \{x \in X\colon |(f+g)(x) - (f+g)(x_0)| < \varepsilon\}$ is not lower strongly superporous at x_0 and $f + g \notin \underline{\mathcal{M}_1}$. It implies $\mathfrak{M}_a(\underline{\mathcal{M}_1}) \subset \mathcal{C}_{\underline{s}(X,\|\ \|)}$. The proof is completed. □

Remark 3.10. In a similar way, applying $\underline{p}(X, \|\ \|)$ and $\underline{s}(X, \|\ \|)$, we can describe maximal multiplicative classes for $\underline{\mathcal{S}_0}$ and $\underline{\mathcal{M}_1}$. But in this case we need a notion of topology extended by a set, see [8,9].

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