Results Math (2022) 77:166

1422-6383/22/040001-13
published online June 29, 2022
https://doi.org/10.1007/s00025-022-01680-x

Results in Mathematics

Asymptotic and Non-asymptotic Results in the Approximation by Bernstein Polynomials

José A. Adell and Daniel Cárdenas-Morales

Abstract. This paper deals with the approximation of functions by the classical Bernstein polynomials in terms of the Ditzian–Totik modulus of smoothness. Asymptotic and non-asymptotic results are respectively stated for continuous and twice continuously differentiable functions. By using a probabilistic approach, known results are either completed or strengthened.

Mathematics Subject Classification. 41A17, 41A35.

Keywords. Bernstein polynomials, Ditzian–Totik modulus of smoothness, Steklov means, binomial random variable.

1. Introduction and Statements of the Main Results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As usual, $C[0,1]$ denotes the space of all real continuous functions defined on $[0,1]$, and $C^m[0,1]$, $m \in \mathbb{N}_0$, denotes the subspace of all m-times continuously differentiable functions, with the obvious understanding that $C^0[0,1] = C[0,1]$. For $m \in \mathbb{N}$, we denote by $\mathscr{C}^m[0,1] \supset C^m[0,1]$ the set of functions $f \in C^{m-1}[0,1]$ such that $f^{(m-1)}$ is absolutely continuous, i. e.,

$$f^{(m-1)}(y) - f^{(m-1)}(x) = \int_x^y g(u)du, \quad x, y \in [0,1],$$

for some bounded measurable function g, which can be denoted by $g = f^{(m)}$.

This work is partially supported by Research Project PGC2018-097621-B-I00. The second author is also supported by Junta de Andalucía Research Group FQM-0178.

The indicator function of a set A is denoted by 1_A, and $\mathbb{E}$ stands for mathematical expectation.

Let $f \in C[0,1]$. The sup-norm of f is simply denoted by $\|f\|$, although, more generally, we use the notation $\|f\|_A = \sup\{|f(x)| : x \in A\}$, $A \subseteq [0,1]$.

The second order central difference of f is defined by

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \quad h \geq 0,$$

whenever $x \pm h \in [0,1]$. The Ditzian–Totik modulus of smoothness of f with weight function $\varphi(x) = \sqrt{x(1-x)}$ is defined by

$$\omega_2^\varphi(f;\delta) = \sup\left\{\left|\Delta_{h\varphi(x)}^2 f(x)\right| : \ 0 \leq h \leq \delta, \ x \pm h\varphi(x) \in [0,1]\right\}, \quad \delta \geq 0.$$

The classical first order modulus of continuity is simply denoted by $\omega(f;\delta)$.

In this paper, we will make use of the following important inequality proved by Bustamante [2]:

$$\omega_2^\varphi(f;\lambda\delta) \leq (2+3\lambda^2)\omega_2^\varphi(f;\delta), \qquad \lambda, \delta \geq 0, \quad \lambda\delta \in [0,1). \tag{1}$$

Finally, the nth Bernstein polynomial of f is defined as

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,1,\ldots,n.$$

We have the probabilistic representation

$$B_n f(x) = \mathbb{E} f\left(\frac{S_n(x)}{n}\right), \tag{2}$$

where $S_n(x)$ is a random variable having the binomial law with parameters n and x, that is to say,

$$P(S_n(x) = k) = p_{n,k}(x), \quad k = 0,1,\ldots,n. \tag{3}$$

Throughout this paper, whenever we write f, n, x, and y, we are assuming that $f \in C[0,1]$, $n \in \mathbb{N}$, and $x, y \in [0,1]$.

Following the works by Ditzian and Ivanov [4] and Totik [9], the rates of uniform convergence for the Bernstein polynomials are characterized by

$$K_1 \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right) \leq \|B_n f - f\| \leq K_2 \omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right), \tag{4}$$

for some absolute constants K_1 and K_2. Whereas no specific values for K_1 have been provided yet, different authors completed statement (4) by showing specific values for the constant K_2. In this regard, Adell and Sangüesa [1] gave $K_2 = 4$, Gavrea et al. [5] and Bustamante [2] provided $K_2 = 3$, and finally, Păltănea [7] proved the validity of $K_2 = 2.5$, this being the best result up to date and up to our knowledge.

This notwithstanding, if additional smoothness conditions on f are added, then the second inequality in (4) may be valid for values of K_2

smaller than 2.5. In this respect, Bustamante and Quesada [3] and Păltănea [8] obtained the following asymptotic result

$$\lim_{n\to\infty} \frac{\|B_n f - f\|}{\omega_2^{\varphi}(f; 1/\sqrt{n})} = \frac{1}{2}, \quad f \in C^2[0,1], \tag{5}$$

provided that f is not an affine function.

The contribution of this paper is twofold. In first place, we strength statement (5) by giving a non-asymptotic version of it. In fact, we prove the following result.

Theorem 1. *If $f \in C^2[0,1]$, then*

$$\left| \|B_n f - f\| - \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) \right| \le \frac{1}{4n}\left(\omega\left(f''; \frac{1}{3\sqrt{n}}\right) + \frac{1}{4}\omega_2^{\varphi}\left(f''; \frac{1}{\sqrt{n}}\right)\right). \tag{6}$$

As a consequence, statement (5) *holds true.*

In second place, we complete statement (4) in the following asymptotic form.

Theorem 2. *Let $(\tau_n)_{n\ge 1}$ be a sequence of positive real numbers such that*

$$\tau_n \longrightarrow \infty, \quad \frac{\tau_n}{n} \longrightarrow 0, \qquad n \to \infty.$$

If $f \in C[0,1]$ is not an affine function, then

$$\limsup_{n\to\infty} \frac{1}{\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right)} \|B_n f - f\|_{[\tau_n/n, 1-\tau_n/n]} \le \frac{3}{2}. \tag{7}$$

Moreover, we have in (4),

$$K_2 \ge 1. \tag{8}$$

This result is based upon Theorem 3 in Sect. 3, which gives estimates of the form

$$|B_n f(x) - f(x)| \le K_2(n,x)\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right),$$

for some explicit constants $K_2(n,x)$ depending on n and x.

The paper is organized as follows. The proof of Theorem 1 is given in Sect. 2. We show Theorem 2 in Sect. 3 with the aid of two kinds of auxiliary results. On the one hand, we define certain smooth approximants $Q_h^a f$ of the function $f \in C[0,1]$, by antisymmetrizing in an appropriate way the classical Steklov means of f. On the other hand, we estimate the tail probabilities and the truncated variance of the random variable $S_n(x)$ appearing in the probabilistic representation of $B_n f$ given in (2).

2. Proof of Theorem 1

2.1. Preliminaries

The Taylor's formula of order $m \in \mathbb{N}$ for $f \in \mathscr{C}^m[0,1]$, with remainder in integral form can be written as

$$\begin{aligned} f(y) - \sum_{j=0}^{m-1} \frac{f^{(j)}(x)}{j!}(y-x)^j &= \frac{(y-x)^m}{(m-1)!}\int_0^1 (1-\theta)^{m-1} f^{(m)}(x+(y-x)\theta)d\theta \\ &= \frac{(y-x)^m}{m!}\mathbb{E}f^{(m)}(x+(y-x)\beta_m), \end{aligned} \tag{9}$$

where β_m is a random variable with the beta density $\rho_m(\theta) = m(1-\theta)^{m-1}$, $0 \le \theta \le 1$.

Lemma 1. *If $f \in C^2[0,1]$ and $\delta \ge 0$, then*

$$\left|\omega_2^\varphi(f;\delta) - \delta^2 \left\|\varphi^2 f''\right\|\right| \le \frac{\delta^2}{8}\omega_2^\varphi\left(f'';\delta\right).$$

Proof. Let $h \ge 0$ with $x \pm h \in [0,1]$. Using (9) with $m = 2$, we get

$$\begin{aligned} f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 \\ +\frac{h^2}{2}\mathbb{E}\left(f''(x-h\beta_2) - f''(x)\right), \end{aligned}$$

as well as

$$\begin{aligned} f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 \\ +\frac{h^2}{2}\mathbb{E}\left(f''(x+h\beta_2) - f''(x)\right). \end{aligned}$$

Adding these two identities, we obtain

$$\begin{aligned} \Delta_h^2 f(x) = f''(x)h^2 \\ +\frac{h^2}{2}\mathbb{E}\left(f''(x+h\beta_2) - 2f''(x) + f''(x-h\beta_2)\right). \end{aligned} \tag{10}$$

Replacing in (10) h by $h\varphi(x)$ and applying the reverse triangular inequality, we have

$$\begin{aligned} \left|\omega_2^\varphi(f;\delta) - \delta^2 \left\|\varphi^2 f''\right\|\right| &\le \frac{\delta^2}{2}\left\|\varphi^2\right\|\omega_2^\varphi\left(f'';\delta\right) \\ &= \frac{\delta^2}{8}\omega_2^\varphi(f'';\delta), \end{aligned}$$

thus completing the proof. □

Gonska et al. [6] showed that

$$\left\| B_n f - f - \frac{1}{2n}\varphi^2 f'' \right\| \leq \frac{1}{4n}\omega\left(f''; \frac{1}{3\sqrt{n}}\right). \tag{11}$$

2.2. Proof of Theorem 1

Statement (6) is an inmediate consequence of (11), Lemma 1 with $\delta = 1/\sqrt{n}$, and the reverse and direct triangular inequalities. On the other hand, we have from Lemma 1

$$\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) = \frac{1}{n}\left\|\varphi^2 f''\right\| + o\left(\frac{1}{n}\right),$$

since $f \in C^2[0,1]$. Thus, statement (5) readily follows from (6), and completes the proof.

3. Proof of Theorem 2

3.1. Auxiliary Results

Let $0 < h \leq 1/3$. We consider the Steklov means of f defined as

$$\begin{aligned} P_h f(y) &= \int_{-1}^{1}\int_{-1}^{1} f\left(y + \frac{h}{2}(v_1+v_2)\right) dv_1 dv_2 \\ &= \int_{-1}^{1} f(y+hv)\rho(v)dv, \quad h \leq y \leq 1-h, \end{aligned}$$

where

$$\rho(v) = (1+v)1_{[-1,0]} + (1-v)1_{(0,1]}, \quad -1 \leq v \leq 1.$$

In probabilistic terms, the Steklov means of f can be written as follows. Let V_1 and V_2 be independent identically distributed random variables having the uniform distribution on $[-1,1]$ and set $V = (V_1+V_2)/2$. Since $\rho(v)$ is the probability density of V, we can write

$$P_h f(y) = \mathbb{E} f(y+hV), \quad h \leq y \leq 1-h. \tag{12}$$

Lemma 2. *Let $0 < h \leq 1/3$ and let $P_h f(y)$ be as in (12). Then,*

(a)

$$|P_h f(y) - f(y)| \leq \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(y)}\right).$$

(b)

$$|(P_h f)''(y)| \leq \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(y)}\right).$$

Proof. Since V takes values in $[-1,1]$ and is symmetric (i. e., V and $-V$ have the same law), we see that

$$|P_h f(y) - f(y)| = \frac{1}{2}\left|\mathbb{E}(f(y+hV) + f(y-hV) - 2f(y))\right| \leq \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(y)}\right),$$

thus showing *(a)*. On the other hand, it can be checked that

$$P_h f(y) = \frac{1}{h^2}\left(f_{(2)}(y+h) + f_{(2)}(y-h) - 2f_{(2)}(y)\right),$$

where $f_{(2)}$ is a second antiderivative of f. This readily implies part *(b)* and completes the proof. □

We will make use of the approximant $P_h f$, whose domain is the interval $[h, 1-h]$, to define a further one whose domain is the whole interval $[0,1]$, keeping at the same time analogous properties to those given in Lemma 2. To this end, we assume that

$$n \geq 3, \qquad 0 < a < \frac{\varphi(a/2)}{\sqrt{n}} + a \leq 1. \tag{13}$$

and take

$$h = \frac{\varphi(ax)}{\sqrt{n}}, \qquad \frac{1}{a(n+1)} \leq x \leq \frac{1}{2}. \tag{14}$$

It turns out that

$$h \leq \min(ax, 1/3). \tag{15}$$

Now, we define the approximant $Q_h^a f(y)$ by antisymmetrizing $P_h f(y)$ around the axes $y = ax$ and $y = 1 - ax$ as follows

$$Q_h^a f(y) = \begin{cases} 2P_h f(ax) - P_h f(2ax - y), & y \in [0, ax); \\ P_h f(y), & y \in [ax, 1-ax]; \\ 2P_h f(1-ax) - P_h f(2(1-ax) - y), & y \in (1-ax, 1]. \end{cases} \tag{16}$$

The fact that $Q_h^a f$ is well defined readily follows from (13) and (14). Also, note that $Q_h^a f$ is twice differentiable except at the points ax and $1 - ax$. In these two points, $Q_h^a f$ only has sided second derivatives. This implies that $Q_h^a f \in \mathscr{C}^2[0,1]$.

Lemma 3. *Let $R_a = [ax, 1-ax]$. Under assumptions* (13) *and* (14)*, we have*

(a) If $y \in R_a$, then

$$|Q_h^a f(y) - f(y)| \leq \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right), \quad |(Q_h^a f)''(y)| \leq \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right).$$

(b) If $y \notin R_a$, then

$$|Q_h^a f(y) - f(y)| \leq \left(\frac{7}{2} + \frac{3\sqrt{anx}}{(1-a)^{3/2}}\right)\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right),$$

and

$$|(Q_h^a f)''(y)| \le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right).$$

Proof. *(a)* If $y \in R_a$, then

$$\frac{h}{\varphi(y)} = \frac{\varphi(ax)}{\varphi(y)}\frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{n}}. \tag{17}$$

Thus, the first inequality in part *(a)* follows from Lemma 2*(a)* and definition (16), whereas the second one follows from Lemma 2*(b)*.

(b) Suppose that $y \in [0, ax)$. By (16), we can write

$$\begin{aligned} Q_h^a f(y) - f(y) = 2\left(P_h f(ax) - f(ax)\right) & \\ - \left(P_h f(2ax-y) - f(2ax-y)\right) & \\ - \left(f(2ax-y) + f(y) - 2f(ax)\right). & \end{aligned} \tag{18}$$

Since $h \le ax \le 2ax - y \le 1 - h$, we see that

$$\varphi(ax) \ge \varphi(h), \quad \varphi(2ax-y) \ge \varphi(h). \tag{19}$$

We therefore have from Lemma 2*(a)*

$$\begin{aligned} |Q_h^a f(y) - f(y)| \le \frac{3}{2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right) & \\ + \omega_2^{\varphi}\left(f; \frac{ax}{\varphi(ax)}\right). & \end{aligned} \tag{20}$$

Applying (1) with $\lambda = ax\varphi(h)/(h\varphi(ax))$ and $\delta = h/\varphi(h)$, we obtain

$$\begin{aligned} \omega_2^{\varphi}\left(f; \frac{ax}{\varphi(ax)}\right) &\le \left(2 + \frac{3(ax)^2\varphi^2(h)}{\varphi^2(ax)h^2}\right)\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right) \\ &\le \left(2 + \frac{3\sqrt{anx}}{(1-a)^{3/2}}\right)\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right), \end{aligned} \tag{21}$$

as follows from (14) and some simple computations. Hence, the first inequality follows from (20) and (21).

On the other hand, we have from (16), (19), and Lemma 2*(b)*

$$\begin{aligned} |(Q_h^a f)''(y)| = |(P_h f)''(2ax-y)| &\le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(2ax-y)}\right) \\ &\le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right). \end{aligned}$$

If $y \in (1 - ax, 1]$, the proof es similar. □

The following estimates concerning the random variable $S_n(x)/n$ will be needed.

Lemma 4. *In the setting of Lemma 3, denote by $r = 1 - a$. Then,*

(a)

$$P\left(\frac{S_n(x)}{n} \notin R_a\right) \le e^{-nxr^2/2} + 3e^{-nxr^2/(2e)} =: \epsilon_n(x).$$

(b)

$$\frac{1}{h^2}\mathbb{E}\left(\frac{S_n(x)}{n} - x\right)^2 1_{\left\{\frac{S_n(x)}{n}\notin R_a\right\}} \le \frac{nx}{a(1-ax)}\left(e^{-nxr^2/2} + 6e^{-(n-2)xr^2/(2e)}\right) =: \delta_n(x).$$

Proof. (a) As follows from (3), we have

$$\mathbb{E}e^{\theta S_n(x)} = \left(1 + x(e^\theta - 1)\right)^n, \quad \theta \in \mathbb{R}. \tag{22}$$

Let $\theta \ge 0$. By (22) and Chebyshev's inequality, we have

$$\begin{aligned} P\left(S_n(x) < anx\right) &= P\left(e^{-\theta S_n(x)} > e^{-\theta anx}\right) \le \mathbb{E}e^{-\theta S_n(x) + \theta anx} \\ &= e^{-n\left(-\log\left(1 - x\left(1 - e^{-\theta}\right)\right) - \theta ax\right)} \le e^{-nx\left(\left(1 - e^{-\theta}\right) - a\theta\right)} \le e^{-nx\left(r\theta - \theta^2/2\right)}, \end{aligned} \tag{23}$$

where we have used the inequalities

$$-\log(1-u) \ge u, \quad u \ge 0, \qquad 1 - e^{-\theta} \ge \theta - \frac{\theta^2}{2}, \quad \theta \ge 0.$$

Choosing $\theta = r$ in (23) (the value minimizing the exponent), we get

$$P\left(S_n(x) < anx\right) \le e^{-nxr^2/2}. \tag{24}$$

On the other hand, we claim that

$$P\left(S_n(x) > n(1-ax)\right) \le P\left(S_n(x) > n(1-ax) - 1\right) \le 3e^{-nxr^2/(2e)}. \tag{25}$$

Indeed, let $0 \le \theta \le 1$. Using the inequalities

$$\log(1+u) \le u, \quad u \ge 0, \qquad e^\theta - 1 \le \theta + \frac{e\theta^2}{2}, \quad 0 \le \theta \le 1,$$

we have, as in the proof of (24),

$$\begin{aligned} P\left(S_n(x) > n(1-ax) - 1\right) &= P\left(e^{\theta S_n(x)} > e^{\theta n(1-ax) - \theta}\right) \\ &\le \mathbb{E}e^{\theta S_n(x) - n\theta(1-ax) + \theta} \le 3\mathbb{E}e^{\theta S_n(x) - n\theta(1-ax)} \\ &= 3e^{n\left(\log\left(1 + x\left(e^\theta - 1\right)\right) - \theta(1-ax)\right)} \le 3e^{n\left(x\theta + ex\theta^2/2 - \theta(1-ax)\right)} \\ &= 3e^{n\theta(2x-1)}e^{nx(e\theta^2/2 - r\theta)} \le 3e^{nx\left(e\theta^2/2 - r\theta\right)}, \end{aligned} \tag{26}$$

since $x \le 1/2$. Thus, claim (25) follows by choosing $\theta = r/e$ in (26). Hence, part *(a)* follows from (24) and (25).

(b) From (24), we see that

$$\mathbb{E}\left(\frac{S_n(x)}{n}-x\right)^2 1_{\{S_n(x)<anx\}}$$
$$\leq x^2 P\left(S_n(x)<anx\right) \leq x^2 e^{-nxr^2/2}. \tag{27}$$

On the other hand, since $1-ax \geq 1/2$, we have

$$\frac{k}{k-1} \leq \frac{n/2}{n/2-1} = \frac{n}{n-2}, \quad k > n(1-ax).$$

We therefore have

$$\begin{aligned}
&\mathbb{E}\left(\frac{S_n(x)}{n}-x\right)^2 1_{\{S_n(x)>n(1-ax)\}} \leq \frac{1}{n^2}\mathbb{E}S_n(x)^2 1_{\{S_n(x)>n(1-ax)\}} \\
&= \frac{1}{n^2}\sum_{k>n(1-ax)} \binom{n}{k} k^2 x^k (1-x)^{n-k} \\
&= \frac{n-1}{n}x^2 \sum_{k>n(1-ax)} \binom{n-2}{k-2}\frac{k}{k-1} x^{k-2}(1-x)^{n-k} \\
&\leq \frac{n-1}{n-2}x^2 P\left(S_{n-2}(x) > n(1-ax)-2\right) \\
&\leq 2x^2 P\left(S_{n-2}(x) > n(1-ax)-2\right),
\end{aligned} \tag{28}$$

since $n \geq 3$. Observe that

$$n(1-ax)-2 = (n-2)(1-ax)-2ax \geq (n-2)(1-ax)-1,$$

as follows from assumptions (13) and (14). By (25), the right-hand side in (28) can be bounded above by

$$2x^2 P\left(S_{n-2}(x) > (n-2)(1-ax)-1\right) \leq 6x^2 e^{-(n-2)xr^2/(2e)}.$$

This, together with (27) and (28), shows part *(b)* and completes the proof. □

We are in a position to give the following local estimate.

Theorem 3. *In the setting of Lemma 4, we have*

$$|B_n f(x) - f(x)| \leq \left(1+\frac{1}{2}\frac{\varphi^2(x)}{\varphi^2(ax)}\right)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{n}}\right) + \nu_n(x)\omega_2^{\varphi}\left(f;\frac{h}{\varphi(h)}\right),$$

where

$$\nu_n(x) = \left(\frac{7}{2}+\frac{3\sqrt{anx}}{(1-a)^{3/2}}\right)\epsilon_n(x) + \frac{1}{2}\delta_n(x). \tag{29}$$

Proof. We use the notation $Qf(y) = Q_h^a f(y)$ and write

$$\begin{aligned} B_n f(x) - f(x) &= (Qf(x) - f(x)) + (B_n f(x) - B_n(Qf)(x) \\ &\quad + (B_n(Qf)(x) - Qf(x)) \\ &=: I + II + III. \end{aligned} \tag{30}$$

By Lemma 3*(a)*, we have

$$|I| \le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right). \tag{31}$$

By (2) and Lemma 3*(a)* and *(b)*, we see that

$$\begin{aligned} |II| &= \left| \mathbb{E}Qf\left(\frac{S_n(x)}{n}\right) - \mathbb{E}f\left(\frac{S_n(x)}{n}\right) \right| \\ &\le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) \\ &\quad + \left(\frac{7}{2} + \frac{3\sqrt{anx}}{(1-a)^{3/2}}\right) P\left(\frac{S_n(x)}{n} \notin R_a\right) \omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right). \end{aligned} \tag{32}$$

Finally, denote by $\xi_n(x) = x + (S_n(x)/n - x)\beta_2$. Applying (9) with $m = 2$ and Lemma 3, we get

$$\begin{aligned} |III| &= \frac{1}{2}\left| \mathbb{E}(Qf)''(\xi_n(x)) \left(\frac{S_n(x)}{n} - x\right)^2 \right| \\ &\le \frac{1}{2h^2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) \mathbb{E}\left(\frac{S_n(x)}{n} - x\right)^2 1_{\{S_n(x)/n \in R_a\}} \\ &\quad + \frac{1}{2h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right) \mathbb{E}\left(\frac{S_n(x)}{n} - x\right)^2 1_{\{S_n(x)/n \notin R_a\}} \\ &\le \frac{1}{2}\frac{\varphi^2(x)}{\varphi^2(ax)}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) \\ &\quad + \frac{1}{2h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(h)}\right) \mathbb{E}\left(\frac{S_n(x)}{n} - x\right)^2 1_{\{S_n(x)/n \notin R_a\}}, \end{aligned} \tag{33}$$

where we have used (14), the inequality $1/\sqrt{n} \le h/\varphi(h)$, and the well known fact that

$$\mathbb{E}\left(\frac{S_n(x)}{n} - x\right)^2 = \frac{\varphi^2(x)}{n}.$$

The result follows from (30)–(33) and Lemma 4. □

3.2. Proof of Theorem 2

Since the random variables $S_n(x)$ and $n - S_n(1-x)$ have the same law, we have

$$B_n f(1-x) - f(1-x) = \mathbb{E}f\left(1 - \frac{S_n(x)}{n}\right) - f(1-x).$$

On the other hand, if $g(y) = f(1-y)$, we obviously have

$$\omega_2^\varphi(g;\delta) = \omega_2^\varphi(f;\delta), \quad \delta \geq 0.$$

Thus, without loss of generality, we can assume that $0 < x \leq 1/2$.

In the setting of Lemma 4, we claim that

$$\omega_2^\varphi\left(f;\frac{h}{\varphi(h)}\right) \leq \left(2+3\sqrt{\frac{anx}{1-a}}\right)\omega_2^\varphi\left(f;\frac{1}{\sqrt{n}}\right). \tag{34}$$

Actually, choose $\lambda = h\sqrt{n}/\varphi(h)$ and $\delta = 1/\sqrt{n}$. By definition (14) and the fact that $h \leq ax$, we see that

$$\lambda^2 = \frac{h^2 n}{\varphi^2(h)} = \frac{\varphi^2(ax)}{\varphi^2(h)} = \frac{ax(1-ax)}{h(1-h)} \leq \frac{ax}{h} = \frac{ax\sqrt{n}}{\varphi(ax)} = \frac{\sqrt{anx}}{\sqrt{1-ax}} \leq \sqrt{\frac{anx}{1-a}}.$$

This, in conjunction with (1), shows claim (34).

From Theorem 3 and (34), we have

$$\frac{|B_n f(x) - f(x)|}{\omega_2^\varphi(f;1/\sqrt{n})} \leq 1 + \frac{1}{2}\frac{\varphi^2(x)}{\varphi^2(ax)} + \left(2+3\sqrt{\frac{anx}{1-a}}\right)\nu_n(x), \tag{35}$$

where $\nu_n(x)$ is defined in (29). Recalling the definitions of $\epsilon_n(x)$ and $\delta_n(x)$ given in Lemma 4, we see that

$$\left(2+3\sqrt{\frac{anx}{1-a}}\right)\nu_n(x) \leq P_3(\sqrt{nx})e^{-cnx}, \tag{36}$$

where $P_3(\cdot)$ is a polynomial of degree three and c is a positive constant not depending on n and x. Observe that, whenever $t_n \to \infty$, as $n \to \infty$, we have

$$\lim_{n\to\infty} \sup_{u \geq t_n} P_3\left(\sqrt{u}\right)e^{-cu} = 0. \tag{37}$$

Let τ_n be as in Theorem 2. From (35) and (36), we get

$$\frac{1}{\omega_2^\varphi\left(f;1/\sqrt{n}\right)}\|B_n f - f\|_{[\tau_n/n,1/2]} \leq 1 + \frac{1}{2a} + \sup_{u\geq\tau_n} P_3\left(\sqrt{u}\right)e^{-cu}.$$

By (37) and the fact that $\tau_n \to \infty$, as $n \to \infty$, this implies that

$$\limsup_{n\to\infty} \frac{1}{\omega_2^\varphi\left(f;1/\sqrt{n}\right)}\|B_n f - f\|_{[\tau_n/n,1/2]} \leq 1 + \frac{1}{2a},$$

which shows (7), since $0 < a < 1$ is arbitrary.

On the other hand, let $x \in (0, 1/n)$. Consider the function

$$f_x(y) = \left(1 - \frac{y}{x}\right)1_{[0,x]}(y).$$

Observe that $\omega_2^\varphi(f_x;1/\sqrt{n}) = 1$, as well as

$$B_n f_x(x) - f_x(x) = \mathbb{E}f_x\left(\frac{S_n(x)}{n}\right) = P(S_n(x) = 0) = (1-x)^n,$$

thus implying that $K_2 \geq (1-x)^n$. Therefore, letting $x \to 0$, we see that $K_2 \geq 1$. This shows (8) and completes the proof.

Acknowledgements

The authors would like to thank an anonymous referee for her/his careful reading of the manuscript and valuable suggestions that led to a corrected and improved final version.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Competing interests The authors declare that they have no competing interests.

References

[1] Adell, J.A., Sangüesa, C.: Upper estimates in direct inequalities for Bernstein-type operators. J. Approx. Theory **109**, 229–241 (2001)

[2] Bustamante, J.: Estimates of positive linear operators in terms of second-order moduli. J. Math. Anal. Appl. **345**, 203–212 (2008)

[3] Bustamante, J., Quesada, J.M.: A property of Ditzian–Totik second order moduli. Appl. Math. Lett. **23**, 576–580 (2010)

[4] Ditzian, Z., Ivanov, K.G.: Strong converse inequalities. J. Anal. Math. **61**, 61–111 (1993)

[5] Gavrea, I., Gonska, H.H., Păltănea, R., Tachev, G.: General estimates for the Ditzian–Totik modulus. East J. Approx. **9**(2), 175–194 (2003)

[6] Gonska, H.H., Piţul, P., Raşa, I.: On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators. In: Agratini, O., Blaga, P. (eds.) Numerical Analysis and Approximation Theory (Proceedings of the International Conference on Cluj-Napoca 2006), pp. 55–80. Casa Cartii de Stiinta, Cluj-Napoca (2006)

[7] Păltănea, R.: Approximation Theory Using Positive Linear Operators. Birkhäuser Boston, Boston (2004)

[8] Păltănea, R.: Asymptotic constant in approximation of twice differentiable functions by a class of positive linear operators. Results Math. **73**(2), paper no. 64 (2018)

[9] Totik, V.: Strong converse inequalities. J. Approx. Theory **76**(3), 369–375 (1994)

José A. Adell
Departamento de Métodos Estadísticos
Universidad de Zaragoza
50009 Zaragoza
Spain
e-mail: adell@unizar.es

Daniel Cárdenas-Morales
Departamento de Matemáticas
Universidad de Jaén
23071 Jaén
Spain
e-mail: cardenas@ujaen.es

Received: October 27, 2021.
Accepted: April 13, 2022.