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**Results in Mathematics**

# Generalized Classical Weighted Means, the Invariance, Complementarity and Convergence of Iterates of the Mean-Type Mappings

Dorota Głazowska and Janusz Matkowski

**Abstract.** Under some simple conditions on real function $f$ defined on an interval $I$, the bivariable functions given by the following formulas

$$A_f(x,y) := f(x) + y - f(y),$$
$$G_f(x,y) := \frac{f(x)}{f(y)}\, y,$$
$$\text{and} \quad H_f(x,y) := \frac{xy}{f(x) + y - f(y)},$$

for all $x, y \in I$, generalize, respectively, the classical weighted arithmetic, geometric and harmonic means. The invariance equations

$$A_f \circ (G_g, H_h) = A_f, \quad G_g \circ (A_f, H_h) = G_g \quad \text{and} \quad H_h \circ (A_f, G_g) = H_h,$$

where $f$, $g$, $h$ are the unknown functions are, in some special cases, solved. The convergence of iterates of the relevant mean-type mappings is considered. As an application the solutions of some functional equations are determined.

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## 1. Introduction

The classical Pythagorean harmony proportion involving the bivariable symmetric arithmetic mean $\mathcal{A}$, harmonic mean $\mathcal{H}$ and geometric mean $\mathcal{G}$, equivalent to the equality

$$\mathcal{G} \circ (\mathcal{A}, \mathcal{H}) = \mathcal{G},$$

as well as its extension for the weighted means

$$\mathcal{G} \circ (\mathcal{A}_t, \mathcal{H}_t) = \mathcal{G},$$

where $t \in (0,1)$, and

$$\mathcal{A}_t (x,y) = tx + (1-t) y, \qquad \mathcal{H}_t (x,y) = \frac{xy}{tx + (1-t) y}, \qquad \mathcal{G} (x,y) = \sqrt{xy},$$

referred to as the *invariance of the geometric mean with respect to the mean-type mappings* $(\mathcal{A}_t, \mathcal{H}_t)$, has well known important consequences. In particular it implies that for every $t \in (0,1)$ the sequence $((\mathcal{A}_t, \mathcal{H}_t)^n : n \in \mathbb{N})$ of the iterates of the mean-type mapping $(\mathcal{A}_t, \mathcal{H}_t)$ converges to $(\mathcal{G}, \mathcal{G})$ (uniformly on compact subsets of $(0,\infty)^2$) [12] (also, under stronger conditions, Borwein and Borwein [2]).

This is a special case of the following more general fact. If $M, N$ are continuous bivariable strict means in an interval $I$, then there is a unique mean $K$ invariant with respect to the mean-type mapping $(M,N)$, that is satisfying the identity $K \circ (M,N) = K$; moreover the sequence of iterates $((M,N)^n : n \in \mathbb{N})$ converges to $(K,K)$ (uniformly on compact subsets of $I^2$) (see [9,10,12]). At this stage the mean $N$ is called *complementary to* $M$ *with respect to* $K$ (briefly, a $K$*-complementary to* $M$) and vice versa.

There is a rich literature related to the invariance equation problems. We refer the interested in the results dealing with invariant means, a survey paper [7]. Let us mention that invariance of the arithmetic mean with respect to the quasi-arithmetic mean-type mappings as well as some related questions were considered among others in [1,4–6,9].

Motivated by these facts, we give necessary and sufficient conditions for a real function $f$ defined on an interval $I$, under which the functions $A_f$, $G_f$, $H_f$ given by the following formulas

$$A_f (x,y) := f(x) + y - f(y),$$
$$G_f (x,y) := \frac{f(x)}{f(y)} y,$$
$$H_f (x,y) := \frac{xy}{f(x) + y - f(y)}$$

for $x, y \in I$, are bivariable means in $I$, generalizing respectively, the weighted arithmetic, geometric and harmonic means. In fact these means are symmetric, if and only if they coincide with $\mathcal{A}, \mathcal{G}, \mathcal{H}$, respectively. The invariance identity

$$\mathcal{G} \circ (A_f, H_f) = \mathcal{G},$$

extending the Pythagorean harmony proportion and confirming the adequacy of the generalized means, allows to conclude the suitable complementariness of $A_f$ and $H_f$ with respect to $\mathcal{G}$, and determine the convergence of sequence of the iterates of the mean-type mapping $(A_f, H_f)$ to $(\mathcal{G}, \mathcal{G})$ (Sect. 2).

In Sect. 3 we consider three related functional equations

$$A_f \circ (G_g, H_h) = A_f, \qquad H_h \circ (A_f, G_g) = H_h, \qquad G_g \circ (A_f, H_h) = G_g,$$

where $f, g, h$ are the unknown functions. We solve the first equation in the case when $A_f = \mathcal{A}$, the second in the case when $H_h = \mathcal{H}$, and the third in the case when $G_g = \mathcal{G}$. Moreover, for each of the classical symmetric means $\mathcal{A}, \mathcal{H}, \mathcal{G}$ and for some of the above generalized means $A_f$, $G_f$, $H_f$ we prove the existence and uniqueness of the respective complementary mean, we give its explicit formula, as well as the limit of the sequence of iterates of the relevant mean-type mappings.

In the last section we establish the form of all functions which are invariant with respect to the corresponding mean-type mappings and continuous on the diagonal.

## 2. Basic Notions and Generalization of the Weighted Arithmetic, Geometric and Harmonic Means

Let $I \subset \mathbb{R}$ be an interval. A bivariable function $M : I^2 \to \mathbb{R}$ is called a *mean* in $I$, if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

A mean $M$ is called *strict* if for all $x, y \in I$, $x \neq y$, these inequalities are sharp, and it is called *symmetric*, if $M(x, y) = M(y, x)$ for all $x, y \in I$ (see [2,3]).

*Remark 1.* If $M : I^2 \to I$ is a mean, then $M(J \times J) = J$ for any subinterval $J \subset I$.

Let $K, M, N : I^2 \to I$ be means. If

$$K(M(x, y), N(x, y)) = K(x, y), \qquad x, y \in I,$$

we write briefly $K \circ (M, N) = K$ and we say that:

(a) *$K$ is invariant with respect to the mean-type mapping* $(M, N) : I^2 \to I^2$, briefly, *$K$ is* $(M, N)$*-invariant*;
(b) *$N$ is complementary to $M$ with respect to $K$*, briefly, *$N$ is a $K$-complementary to $M$*.

Let us quote the following (see Remark 1 in [9])

**Lemma 1.** *Let $I \subset \mathbb{R}$ be an interval and $K : I^2 \to I$ be a symmetric mean which is continuous and strictly increasing with respect to the first variable. Then for every mean $M : I^2 \to I$ there exists a unique $K$-complementary mean $N : I^2 \to I$.*

*Remark 2* ([11]). Let $I \subset \mathbb{R}$ be an interval and let $f, \varphi : I \to \mathbb{R}$. Then the function

$$M(x,y) = f(x) + \varphi(y), \qquad x, y \in I,$$

is a mean if and only if $\varphi = \mathrm{id}|_I - f$, i.e. $M = A_f$, where $A_f : I^2 \to \mathbb{R}$ is defined by

$$A_f(x,y) := f(x) + y - f(y), \qquad x, y \in I, \tag{1}$$

and the functions $f$ and $\mathrm{id}|_I - f$ are increasing. Moreover

(i) $A_f$ is a mean if and only if the function $f$ is increasing and non-expansive;

(ii) $A_f$ is a strict mean if and only if $f$ and $\mathrm{id}|_I - f$ are strictly increasing, or equivalently, if and only if $f$ is strictly increasing and strictly contractive, i.e.

$$|f(x) - f(y)| < |x - y|, \qquad x, y \in I, \quad x \neq y;$$

(iii) $A_f$ is a weighted arithmetic mean of the weight $t \in [0,1]$, i.e.

$$A_f(x,y) = tx + (1-t)y, \qquad x, y \in I,$$

if and only if the function $I \ni x \longmapsto f(x) - tx$ is constant;

(iv) $A_f$ is symmetric if and only if $A_f = \mathcal{A}$, or equivalently, if and only if the function $I \ni x \longmapsto f(x) - \frac{x}{2}$ is constant.

Under the assumptions of this remark, the mean $A_f$ given by formula (1) is called a *generalized weighted arithmetic mean* and the function $f$, called its *generator*, being Lipschitzian, is absolutely continuous; consequently, it is differentiable almost everywhere.

*Remark 3* ([8]). Let $I \subset (0,\infty)$ be an interval and let $f, \psi : I \to (0,\infty)$. Then the function

$$M(x,y) = f(x)\,\psi(y), \qquad x, y \in I,$$

is a mean if and only if $\psi = \frac{\mathrm{id}|_I}{f}$, i.e. $M = G_f$, where $G_f : I^2 \to (0,\infty)$ is defined by

$$G_f(x,y) := \frac{f(x)}{f(y)}\,y, \qquad x, y \in I, \tag{2}$$

and the functions $f$ and $\frac{\mathrm{id}|_I}{f}$ are increasing. Moreover

(i) $G_f$ is a mean if and only if the function $f$ is increasing and

$$1 \leq \frac{f(y)}{f(x)} \leq \frac{y}{x}, \qquad x, y \in I,\ x < y;$$

(ii) $G_f$ is a strict mean if and only if the functions $f$ and $\frac{\mathrm{id}|_I}{f}$ are strictly increasing, or equivalently, if and only if

$$1 < \frac{f(y)}{f(x)} < \frac{y}{x}, \qquad x, y \in I,\ x < y;$$

(iii) $G_f$ is a weighted geometric mean of the weight $t \in [0,1]$, i.e.

$$G_f(x,y) = x^t y^{1-t}, \qquad x, y \in I,$$

if and only if the function $I \ni x \longmapsto \frac{f(x)}{x^t}$ is a constant;

(iv) $G_f$ is symmetric if and only if $G_f = \mathcal{G}$, or equivalently, if and only if the function $I \ni x \longmapsto \frac{f(x)}{\sqrt{x}}$ is constant.

Under the assumptions of this remark, the mean $G_f$ given by formula (2) is called a *generalized weighted geometric mean* and the function $f$, called its *generator*, is absolutely continuous; consequently, differentiable almost everywhere.

Note the following easy to prove

*Remark 4.* Let $I \subset (0,\infty)$ be an interval and let $f : I \to \mathbb{R}$. Then the function $H_f : I^2 \to \mathbb{R}$,

$$H_f(x,y) = \frac{xy}{f(x) + y - f(y)}, \qquad x, y \in I, \tag{3}$$

is a correctly defined (strict) mean in $I$ if and only if the functions $f$ and $\mathrm{id}|_I - f$ are (strictly) increasing. Moreover

(i) $H_f$ is a mean if and only if $A_f$ is a mean;

(ii) $H_f$ is a weighted harmonic mean of the weight $t \in [0,1]$, i.e.

$$H_f(x,y) = \frac{xy}{tx + (1-t)y}, \qquad x, y \in I,$$

if and only if the function $I \ni x \longmapsto f(x) - tx$ is constant;

(iii) $H_f$ is symmetric if and only if $H_f = \mathcal{H}$, or equivalently, if and only if the function $I \ni x \longmapsto f(x) - \frac{x}{2}$ is constant.

Under the assumptions of this remark, the mean $H_f$ given by formula (3) is called a *generalized weighted harmonic mean* and the function $f$, called its *generator*, is absolutely continuous; consequently, is differentiable almost everywhere.

The following result shows that, in particular, the above proposed definitions of generalizations of the classical means are natural.

**Theorem 1.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ be such that $f$, $h$, $\mathrm{id}|_I - f$, $\mathrm{id}|_I - h$ are increasing. Then the following conditions are pairwise equivalent*

(i) *the means $A_f$ and $H_h$ are mutually complementary with respect to the geometric mean $\mathcal{G}$ (see [9]);*

(ii) *the geometric mean* $\mathcal{G}$ *is invariant with respect to the mean-type mapping* $(A_f, H_h) : I^2 \to I^2$, *i.e.*

$$\mathcal{G} \circ (A_f, H_h) = \mathcal{G}; \tag{4}$$

(iii) *the function* $h - f$ *is constant, and*

$$H_h = H_f.$$

*Moreover, if the functions* $f$ , $h$, $\mathrm{id}|_I - f$, $\mathrm{id}|_I - h$ *are strictly increasing and* $h - f$ *is constant, then the sequence of iterates* $((A_f, H_h)^n : n \in \mathbb{N})$ *of the mean-type mapping* $(A_f, H_h)$ *converges, uniformly on compact subsets of* $I^2$, *to the mean-type mapping* $(\mathcal{G}, \mathcal{G})$.

*Proof.* Conditions (i) and (ii) are equivalent (see [9]).

Assume (ii). From the definitions of $\mathcal{G}$, $A_f$ and $H_h$ (see (1) and (3)), equality (4) holds, if and only if, for arbitrary $x, y \in I$,

$$\sqrt{(f(x) + y - f(y)) \; \frac{xy}{h(x) + y - h(y)}} = \mathcal{G} \circ (A_f, H_h)(x, y) = \mathcal{G}(x, y) = \sqrt{xy},$$

which (after simple calculations) can be written equivalently in the form

$$h(y) - f(y) = h(x) - f(x), \qquad x, y \in I.$$

The above equality holds if and only if the function $h - f$ is a constant, i.e. if and only if there is $c \in \mathbb{R}$ such that

$$h(x) = f(x) + c, \qquad x \in I.$$

From (3) it follows that $H_h = H_f$.

To prove the "moreover" result note that, in view of Remark 2 (ii), the function $f$ is continuous and, consequently, the mean-type mapping $(A_f, H_f)$ is continuous. Since the coordinate means are strict, the result follows from the main result of [10] (see also [12]). □

## 3. Invariant Means and Some Open Problems

In this section we consider some invariance equations involving the introduced generalized weighted means $A_f$, $G_f$ and $H_f$.

We begin with

**Problem 1.** Let $I \subset (0, \infty)$ be an interval. Find all functions $f, g, h : I \to (0, \infty)$ satisfying the equation

$$A_f \circ (G_g, H_h) = A_f,$$

assuming that $f, g, h$ are, respectively, the generators of generalized weighted arithmetic, geometric and harmonic means.

In the case when $A_f$ is symmetric, i.e. if $A_f = \mathcal{A}$, we prove the following

**Theorem 2.** *Let $I \subset (0,\infty)$ be an interval. Assume that $g : I \to (0,\infty)$ is strictly increasing and the functions $\frac{\mathrm{id}|_I}{g}$, $h : I \to (0,\infty)$ and $\mathrm{id}|_I - h$ are increasing. If*

$$\mathcal{A} \circ (G_g, H_h) = \mathcal{A}, \tag{5}$$

*then there exist $a \in (0,\infty)$ and $b \in \mathbb{R}$ such that*

$$g(x) = ax, \quad h(x) = x + b, \qquad x \in I,$$

*and*

$$G_g(x,y) = x, \quad H_h(x,y) = y, \qquad x, y \in I.$$

*Proof.* According to the definitions of the involved means (see (2) and (3)), Eq. (5) reduces to

$$\frac{g(x)}{g(y)}\, y + \frac{xy}{h(x) + y - h(y)} = x + y, \qquad x, y \in I,$$

which, after simple calculations, implies that

$$y^2 (g(x) - g(y)) = ((x+y)\, g(y) - yg(x))\,(h(x) - h(y)), \qquad x, y \in I. \tag{6}$$

The assumptions of $g$ and $h$ imply that they are absolutely continuous. Let $x \in I$ be a differentiability point of $g$ and $h$. Dividing both sides of the Eq. (6) by $x - y$, we get

$$y^2\, \frac{g(x) - g(y)}{x - y} = ((x+y)\, g(y) - yg(x))\, \frac{h(x) - h(y)}{x - y}, \qquad y \in I, y \neq x.$$

It follows that $g$ and $h$ are differentiable at the point $x$ and, letting $y \to x$, gives

$$x^2 g'(x) = xg(x)\, h'(x), \qquad x \in I,$$

whence

$$h'(x) = \frac{g'(x)}{g(x)}\, x, \qquad x \in I. \tag{7}$$

Differentiating both sides of (6) with respect to the variable $x$ at the point $x$, we obtain

$$\begin{aligned} y^2 g'(x) = {} & (g(y) - yg'(x))\,(h(x) - h(y)) \\ & + ((x+y)\, g(y) - yg(x))\, h'(x), \quad y \in I. \end{aligned}$$

On the other hand, differentiating both sides of this equality with respect to $y$ (at the points of differentiability of $g$ and $h$), we get

$$\begin{aligned} 2y\, g'(x) = {} & (g'(y) - g'(x))\,(h(x) - h(y)) - (g(y) - y\, g'(x))\, h'(y) \\ & + (g(y) + (x+y)\, g'(y) - g(x))\, h'(x), \end{aligned}$$

and this equality holds true for almost all $x$ and almost all $y$ in $I$. Setting here $y = x$ we have

$$2xg'(x) = (3xg'(x) - g(x))\, h'(x), \qquad \text{a.e. in } I,$$

whence, in view of (7), we obtain

$$2xg'(x) = (3xg'(x) - g(x)) \frac{xg'(x)}{g(x)}, \qquad \text{a.e. in } I,$$

which simplifies to

$$g'(x)(g(x) - xg'(x)) = 0, \qquad \text{a.e. in } I.$$

The absolute continuity and strict monotonicity of $g$ imply that $g'(x) > 0$ a.e. in $I$, so

$$g'(x) = \frac{g(x)}{x} \qquad \text{a.e. in } I.$$

Since the derivative of the absolutely continuous function $g$ coincides a.e. in $I$ with the continuous function $x \longmapsto \frac{g(x)}{x}$, it follows that $g$ must be continuously differentiable in $I$. Thus $g$ satisfies the differential equation

$$g'(x) = \frac{g(x)}{x}, \qquad x \in I.$$

Solving this equation we get

$$g(x) = ax, \qquad x \in I,$$

for some $a > 0$. Hence, in view of (7) we have

$$h'(x) = \frac{xg'(x)}{g(x)} = 1, \qquad x \in I,$$

so

$$h(x) = x + b, \qquad x \in I,$$

for some real $b$. Hence, by the definitions of $G_g$ and $H_h$, we get

$$G_g(x, y) = x, \quad H_h(x, y) = y, \qquad x, y \in I.$$

□

Let us note that Lemma 1 and main results of [10] (also [12]) allow to conclude:

*Remark 5.* Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\frac{\mathrm{id}|_I}{f}$ be increasing functions. Then

(i) there exists a unique function $M : I^2 \to \mathbb{R}$ such that

$$\mathcal{A} \circ (G_f, M) = \mathcal{A},$$

moreover $M$ is an $\mathcal{A}$-complementary mean for $G_f$, and

$$M(x, y) = x + y - \frac{f(x)}{f(y)} y, \qquad x, y \in I;$$

(ii) if the functions $f$ and $\frac{\mathrm{id}|_I}{f}$ are strictly increasing, then the sequence of iterates $((G_f, M)^n : n \in \mathbb{N})$ converges to $(\mathcal{A}, \mathcal{A})$ (uniformly on compact subsets of $I^2$).

We can also consider the invariance of the arithmetic mean $\mathcal{A}$ with respect to the mean-type mappings involving at least one of the introduced means $A_f$ or $H_f$. Similarly as in the above remark we will get the explicit formulas for the respective mean-type mappings ensuring the invariance, the limit of sequence of its iterates, as well as the complementary means, but we omit statements of these results.

**Corollary 1.** *Let $I \subset (0,\infty)$ be an interval. Assume that $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ are strictly increasing. The $\mathcal{A}$-complementary mean $M : I^2 \to I$ for $G_f$ obtained in Remark 5 is not a generalized weighted harmonic mean.*

Similarly one can raise

**Problem 2.** Let $I \subset (0,\infty)$ be an interval. Find all functions $f, g, h : I \to (0,\infty)$ satisfying the equation

$$H_h \circ (A_f, G_g) = H_h,$$

assuming that $f, g, h$ are, respectively, the generators of generalized weighted arithmetic, geometric and harmonic means.

In the case when $H_h$ is symmetric, i.e. if $H_h = \mathcal{H}$, we prove the following

**Theorem 3.** *Let $I \subset (0,\infty)$ be an interval. Assume that $f : I \to (0,\infty)$, $\mathrm{id}|_I - f$, $g : I \to (0,\infty)$, $\frac{\mathrm{id}|_I}{g}$ are increasing. If*

$$\mathcal{H} \circ (A_f, G_g) = \mathcal{H}, \tag{8}$$

*then there exist $a \in (0,\infty)$ and $b \in \mathbb{R}$ such that either*

$$f(x) = b, \quad g(x) = ax, \qquad x \in I,$$

*and*

$$A_f(x,y) = y, \quad G_g(x,y) = x, \qquad x, y \in I,$$

*or*

$$f(x) = x + b, \quad g(x) = a, \qquad x \in I,$$

*and*

$$A_f(x,y) = x, \quad G_g(x,y) = y, \qquad x, y \in I.$$

*Proof.* For the same reason as in the previous proof, the functions $f$ and $g$ are differentiable in $I$. By the definitions of the $\mathcal{H}$, $A_f$ and $G_g$ (see (1) and (2)), Eq. (8) can be written in the form

$$\frac{2\left(f(x) + y - f(y)\right)\frac{g(x)}{g(y)}\,y}{f(x) + y - f(y) + \frac{g(x)}{g(y)}\,y} = \frac{2xy}{x+y}, \qquad x, y \in I,$$

which, after simple calculations, reduces to

$$((x+y)\,g(x) - xg(y))\,(f(x) - f(y)) = xy^2\left(\frac{g(y)}{y} - \frac{g(x)}{x}\right), \quad x, y \in I. \tag{9}$$

Dividing both sides of this equation by $x - y$ we have

$$((x+y)\,g(x) - xg(y))\,\frac{f(x) - f(y)}{x-y} = -xy^2\,\frac{\frac{g(x)}{x} - \frac{g(y)}{y}}{x-y}, \qquad x, y \in I, x \neq y,$$

and letting $y$ tend to $x$ we get

$$xg(x)\,f'(x) = -x^3\,\frac{g'(x)\,x - g(x)}{x^2}, \qquad x \in I,$$

whence

$$f'(x) = \frac{g(x) - g'(x)\,x}{g(x)}, \qquad x \in I. \tag{10}$$

Differentiating both sides of (9) in $x$, (after a simplification) we obtain, for all $x, y \in I$,

$$(g(x) + (x+y)\,g'(x) - g(y))\,(f(x) - f(y)) + ((x+y)\,g(x) - xg(y))\,f'(x)$$
$$= yg(y) - y^2 g'(x).$$

Now, differentiating both sides of this equality in $y$, we get, for all $x, y \in I$,

$$(g'(x) - g'(y))\,(f(x) - f(y)) + (g(x) - (x+y)\,g'(x) - g(y))\,f'(y)$$
$$+\,(g(x) - xg'(y))\,f'(x) = g(y) + yg'(y) - 2yg'(x).$$

Taking here $y = x$ we obtain

$$(g(x) - 3xg'(x))\,f'(x) = g(x) - xg'(x), \qquad x \in I.$$

Note that

$$g(x) - 3xg'(x) \neq 0, \qquad x \in I,$$

as, if $g(x) - 3xg'(x) = 0$ then also $g(x) - xg'(x) = 0$, and we would have $g(x) = 0$, contradicting the assumption. Thus

$$f'(x) = \frac{g(x) - xg'(x)}{g(x) - 3xg'(x)}, \qquad x \in I.$$

Hence, making use of (10), we get

$$\frac{g(x) - g'(x)\,x}{g(x)} = \frac{g(x) - xg'(x)}{g(x) - 3xg'(x)}, \qquad x \in I,$$

which implies that for every $x \in I$,

$$\text{either} \quad g(x) - xg'(x) = 0 \quad \text{or} \quad g(x) = g(x) - 3xg'(x),$$

that is, for every $x \in I$,

$$\text{either} \quad g(x) - xg'(x) = 0 \quad \text{or} \quad g'(x) = 0.$$

It implies that either there is a constant $a > 0$ such that

$$g(x) = ax, \qquad x \in I,$$

or there is $a > 0$ such that

$$g(x) = a, \qquad x \in I.$$

From (10), in the first case we get $f' = 0$ in $I$, so there exists $b \in \mathbb{R}$ such that

$$f(x) = b, \qquad x \in I,$$

and in the second case, $f' = 1$ in $I$, so, for some real $b$,

$$f(x) = x + b, \qquad x \in I.$$

Consequently, in the first case we get

$$A_f(x, y) = y, \quad G_g(x, y) = x, \qquad x, y \in I,$$

and in the second case,

$$A_f(x, y) = x, \quad G_g(x, y) = y, \qquad x, y \in I.$$

□

Let us mention here also a result, related to the invariance of the harmonic mean, that gives us the explicit formula for an $\mathcal{H}$-complementary mean to generalized weighted arithmetic mean $A_f$. The result reads as follows

*Remark 6.* Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ be increasing functions. Then

(i) there exists a unique function $M : I^2 \to \mathbb{R}$ such that

$$\mathcal{H} \circ (A_f, M) = \mathcal{H},$$

moreover $M$ is an $\mathcal{H}$-complementary mean for $A_f$, and

$$M(x, y) = \frac{xy(f(x) + y - f(y))}{(x + y)(f(x) - f(y)) + y^2}, \qquad x, y \in I;$$

(ii) if the functions $f$ and $\mathrm{id}|_I - f$ are strictly increasing, then the sequence of iterates $((A_f, M)^n : n \in \mathbb{N})$ converges to $(\mathcal{H}, \mathcal{H})$ (uniformly on compact subsets of $I^2$).

Similarly, considering the invariance of the harmonic mean $\mathcal{H}$ with respect to the mean-type mappings involving at least one of the introduced means $G_f$ or $H_f$ one can determine the explicit formulas for the relevant mean-type mappings ensuring the invariance, the limit of sequence of its iterates, as well as the complementary means.

**Corollary 2.** *Let $I \subset (0, \infty)$ be an interval. Assume that $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ are strictly increasing. The $\mathcal{H}$-complementary to $A_f$ mean $M : I^2 \to I$ obtained in Remark 6 is not a generalized weighted geometric mean.*

Finally we formulate the following

**Problem 3.** Let $I \subset (0, \infty)$ be an interval. Find all functions $f, g, h : I \to (0, \infty)$ satisfying the equation

$$G_g \circ (A_f, H_h) = G_g,$$

assuming that $f, g, h$ are, respectively, the generators of generalized weighted arithmetic, geometric and harmonic means.

The solution of this problem in the case when $G_g$ is symmetric, i.e. if $G_g = \mathcal{G}$, is given in Theorem 1.

## 4. Applications: Invariant Functions

In this section we deal with a more general question: when a bivariable function (not necessarily a mean) is invariant with respect to the considered mean-type mappings.

Applying the results of the previous section we determine the form of all functions which are invariant with respect to the relevant mean-type mappings and continuous on the diagonal $\Delta := \{(x, x) : x \in I\}$, where $I \subset \mathbb{R}$ is an interval.

**Proposition 1.** *Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(\frac{f(x)}{f(y)}\, y, x + y - \frac{f(x)}{f(y)}\, y\right) = \Phi(x, y), \quad x, y \in I, \tag{11}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi(x, y) = \varphi\left(\frac{x + y}{2}\right), \qquad x, y \in I. \tag{12}$$

*Proof.* Assume that a continuous on the diagonal $\Delta$ function $\Phi : I^2 \to \mathbb{R}$ satisfies Eq. (11), and define the function $\varphi : I \to \mathbb{R}$ by

$$\varphi(u) := \Phi(u, u), \qquad u \in I. \tag{13}$$

Notice that Eq. (11) can be written in the form

$$\Phi(G_f(x, y), M(x, y)) = \Phi(x, y), \qquad x, y \in I, \tag{14}$$

where

$$G_f(x, y) = \frac{f(x)}{f(y)}\, y \quad \text{and} \quad M(x, y) = x + y - \frac{f(x)}{f(y)}\, y, \qquad x, y \in I.$$

From (14), by induction, we obtain

$$\Phi(x, y) = (\Phi \circ (G_f, M)^n)(x, y), \qquad x, y \in I,\ n \in \mathbb{N}, \tag{15}$$

where $(G_f, M)^n$ denotes the $n$-th iterates of the mean-type mapping $(G_f, M)$. In view of Remark 5 we get

$$\lim_{n\to\infty} (G_f, M)^n (x,y) = (\mathcal{A}(x,y), \mathcal{A}(x,y)) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right), \qquad x, y \in I.$$

Since the function $\Phi$ is continuous on the diagonal $\Delta$ and

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \in \Delta, \qquad x, y \in I,$$

it follows from (15) and (13) that, for all $x, y \in I$,

$$\begin{aligned}\Phi(x,y) &= \lim_{n\to\infty} (\Phi \circ (G_f, M)^n)(x,y) = \Phi(\mathcal{A}(x,y), \mathcal{A}(x,y)) \\ &= \Phi\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = \varphi\left(\frac{x+y}{2}\right),\end{aligned}$$

which proves (12).

Now, assume that there is a continuous function $\varphi : I \to \mathbb{R}$ such that (13) holds. Then by Remark 5 the arithmetic mean $\mathcal{A}$ is invariant with respect to the mean-type mapping $(G_f, M)$, and, for all $x, y \in I$,

$$\begin{aligned}\Phi(x,y) &= \varphi\left(\frac{x+y}{2}\right) = \varphi \circ \mathcal{A}(x,y) = \varphi \circ (\mathcal{A} \circ (G_f, M))(x,y) \\ &= \Phi(G_f(x,y), M(x,y)) = \Phi\left(\frac{f(x)}{f(y)}\, y, x+y-\frac{f(x)}{f(y)}\, y\right),\end{aligned}$$

which proves that the function $\Phi$ satisfies Eq. (11). □

**Corollary 3.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq. (11), then $\Phi = \mathcal{A}$.*

*Proof.* Assume that $\Phi : I^2 \to \mathbb{R}$ is a mean satisfying Eq. (11). Since every mean is continuous on the diagonal (see [12]), in view of Proposition 1 there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that

$$\Phi(x,y) = \varphi\left(\frac{x+y}{2}\right), \qquad x, y \in I.$$

Setting here $y = x$, by the reflexivity of every mean, we get

$$\varphi(x) = \varphi\left(\frac{x+x}{2}\right) = \Phi(x,x) = x, \qquad x \in I,$$

whence

$$\Phi(x,y) = \frac{x+y}{2} = \mathcal{A}(x,y), \qquad x, y \in I.$$

□

Since each of the results given below can be proved similarly as Proposition 1 or Corollary 3 we omit their proofs.

**Proposition 2.** *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ and $\mathrm{id}|_I - f$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(f(x) + y - f(y), f(y) + x - f(x)\right) = \Phi\left(x, y\right), \qquad x, y \in I, \tag{16}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{A}.$$

**Corollary 4.** *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq. (16), then $\Phi = \mathcal{A}$.*

**Proposition 3.** *Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(\frac{xy}{f(x) + y - f(y)}, x + y - \frac{xy}{f(x) + y - f(y)}\right) = \Phi\left(x, y\right), \quad x, y \in I, \tag{17}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{A}.$$

**Corollary 5.** *Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq. (17), then $\Phi = \mathcal{A}$.*

**Proposition 4.** *Let $I \subset (0, \infty)$ be an interval and let $f : I \to (0, \infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(f(x) + y - f(y), \frac{xy}{f(x) + y - f(y)}\right) = \Phi\left(x, y\right), \quad x, y \in I, \tag{18}$$

*or*

$$\Phi\left(\frac{xy}{f(x) + y - f(y)}, f(x) + y - f(y)\right) = \Phi\left(x, y\right), \qquad x, y \in I, \tag{19}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{G}.$$

**Corollary 6.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq.* (18) *or* (19)*, then $\Phi = \mathcal{G}$.*

**Proposition 5.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(\frac{f(x)}{f(y)}\,y, \frac{f(y)}{f(x)}\,x\right) = \Phi(x,y), \qquad x,y \in I, \tag{20}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{G}.$$

**Corollary 7.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq.* (20)*, then $\Phi = \mathcal{G}$.*

**Proposition 6.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(f(x)+y-f(y),\ \frac{xy\,(f(x)+y-f(y))}{(x+y)\,(f(x)-f(y))+y^2}\right) = \Phi(x,y), \quad x,y \in I, \tag{21}$$

*or*

$$\Phi\left(\frac{xy}{f(x)+y-f(y)}, \frac{yx}{f(y)+x-f(x)}\right) = \Phi(x,y), \qquad x,y \in I, \tag{22}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{H}.$$

**Corollary 8.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\mathrm{id}|_I - f$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq.* (21) *or* (22)*, then $\Phi = \mathcal{H}$.*

**Proposition 7.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. A function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\Delta$, satisfies the functional equation*

$$\Phi\left(\frac{f(x)}{f(y)}\,y, \frac{xyf(y)}{(x+y)f(x)-xf(y)}\right) = \Phi(x,y), \qquad x,y \in I, \tag{23}$$

*if and only if there is a single-variable continuous function $\varphi : I \to \mathbb{R}$ such that*

$$\Phi = \varphi \circ \mathcal{H}.$$

**Corollary 9.** *Let $I \subset (0,\infty)$ be an interval and let $f : I \to (0,\infty)$ and $\frac{\mathrm{id}|_I}{f}$ be strictly increasing functions. If a function $\Phi : I^2 \to \mathbb{R}$ is a mean and satisfies Eq.* (23)*, then $\Phi = \mathcal{H}$.*

We finish this section with the following

*Remark 7.* The condition of the continuity of $\Phi$ on the diagonal $\Delta$ in Propositions 1–7 cannot be omitted. Indeed, for example, the function $\Phi : I^2 \to \mathbb{R}$ defined by

$$\Phi(x,y) := \begin{cases} 0, \text{ if } (x,y) \in I^2 \setminus \Delta, \\ 1, \text{ if } (x,y) \in \Delta, \end{cases}$$

satisfies Eqs. (11), (16) and (17), is discontinuous at every point of $\Delta$ and, of course, it is not of the form $\varphi \circ \mathcal{A}$ where $\varphi$ is a single variable function defined on $I$.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.



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Dorota Głazowska and Janusz Matkowski
Institute of Mathematics
University of Zielona Góra
Szafrana 4A
65-516 Zielona Góra
Poland
e-mail: D.Glazowska@wmie.uz.zgora.pl;
j.matkowski@wmie.uz.zgora.pl

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