Parallel Packing Squares into a Triangle

Janusz Januszewski, Xi Liu, Zhanjun Su, and Łukasz Zielonka

Abstract. Assume that T_h is a triangle with the interior angles at the base of the measure not greater than 90° , with the base length 1 and the height h . Let S be a square with a side parallel to the base of T_h and let $\{S_n\}$ be a collection of the homothetic copies of S . A tight upper bound of the sum of the areas of squares from $\{S_n\}$ that can be parallel packed into a triangle T_h is determined.

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1. Introduction

Let P be a polygon and let S_i be a square for $i = 1, 2, \dots$. One side of P is called *the base* of P . A collection $S_1, S_2, \dots$ is said to be *packed* into P if their union is contained in P and if these squares have pairwise disjoint interiors. A packing is called *parallel* if a side of each packed square is parallel to the base of P .

The goal is to pack the squares into P with high density. The area of a polygon R is denoted by $|R|$. Let $\varrho(P)$ be the greatest number such that any collection of squares of the total area not greater than $\varrho(P) \cdot |P|$ can be parallel packed into P .

Assume that S is a square and that T_h is a triangle with the interior angles at the base of the measure not greater than 90° , with the base length 1 and the height h . There are many results concerning packing squares or rectangles.

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FIGURE 1. One square in T_h

Moon and Moser [11] showed that $\varrho(S) = 1/2$; in this case the most effective usual packing is the parallel packing. In many publications in which squares are placed into a rectangle, when authors write “packing” they mean “parallel packing” (see for example [1, 2, 5, 8–10, 12]). Let us add that for packing squares or rectangles of the side lengths not greater than 1 into a large square the most effective packings are not parallel (see [3, 7]).

The following results concerning parallel packing squares into a triangle were known:

- $\varrho(T_{\sqrt{3}/2}) = 2\sqrt{3} - 3$ for an equilateral triangle $T_{\sqrt{3}/2}$ (see [6]),
- $\varrho(T_{\sqrt{2}}) \geq (16 - 6\sqrt{2})/23$ for a right triangle $T_{\sqrt{2}}$ (see [4]),
- $\varrho(T_{\sqrt{2}/2}) = 6\sqrt{2} - 8$ and $\varrho(T_{\sqrt{2}/3}) = 6\sqrt{2}/(11 + 6\sqrt{2})$ (see [13]).

The aim of this note is to show that

$$\varrho(T_h) = \min \left\{ \frac{2h}{(h+1)^2}, \max \left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right] \right\}.$$

Without loss of generality one can assume that $\alpha \geq \beta$ (see Fig. 1).

2. Upper Bounds

Claim 1. *A square of the side length not greater than $h/(h+1)$ can be parallel packed into T_h while one square of the side length greater than $h/(h+1)$ cannot.*

Proof. Let a be the greatest number such that the square of the side length equal to a can be parallel packed into T_h (see Fig. 1). By Thales’s intercept theorem, $\frac{1}{a} = \frac{h}{h-a}$, i.e., $a = \frac{h}{h+1}$. $\square$

Corollary 1. *Any square of the area not greater than $2h/(h+1)^2 \cdot |T_h|$ can be parallel packed into T_h . Moreover, $\varrho(T_h) \leq 2h/(h+1)^2$.*

Proof. If the area of a square equals $2h/(h+1)^2 \cdot |T_h| = h^2/(h+1)^2$, then its side length is equal to $h/(h+1)$. $\square$

FIGURE 2. Two congruent squares in T_h for $\sqrt{2}/2 < h \leq 1$ FIGURE 3. Two congruent squares in T_h for $h > 1$

Claim 2. *Two congruent squares of the side length smaller than or equal to $\max[h/(2h+1), h/(h+2)]$ can be parallel packed into T_h while two squares of the side length greater than $\max[h/(2h+1), h/(h+2)]$ cannot.*

Proof. Let b be the greatest number such that two squares of the side length equal to b can be parallel packed into T_h along the base of T_h (see Fig. 2). By Thales's intercept theorem, $\frac{1}{2b} = \frac{h}{h-b}$, i.e., $b = \frac{h}{2h+1}$. Moreover, let d be the greatest number such that two squares of the side length equal to d can be parallel packed into T_h "vertically" as on Fig. 3. By Thales's intercept theorem, $\frac{1}{d} = \frac{h}{h-2d}$, i.e., $d = \frac{h}{h+2}$. $\square$

Corollary 2. *Any two congruent squares of the sum of the areas not greater than $\max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right] \cdot |T_h|$ can be parallel packed into T_h . Moreover, $\varrho(T_h) \leq \max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right]$.*

Proof. If the area of a square equals

$$\frac{1}{2} \cdot \max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right] \cdot |T_h| = \max\left[\frac{h^2}{(2h+1)^2}, \frac{h^2}{(h+2)^2}\right],$$

then its side length is equal to $\max\left(\frac{h}{2h+1}, \frac{h}{h+2}\right)$. $\square$

It can be checked that $4h/(2h+1)^2 \geq 4h/(h+2)^2$ provided $h \leq 1$. Moreover, $2h/(h+1)^2 \leq 4h/(2h+1)^2$ for $h \leq \sqrt{2}/2$ and $2h/(h+1)^2 \leq 4h/(h+2)^2$ for $h \geq \sqrt{2}$. Let

$$\begin{aligned} \varrho_h &= \min \left\{ \frac{2h}{(h+1)^2}, \max \left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right] \right\} \\ &= \begin{cases} \frac{2h}{(h+1)^2} & \text{for } 0 < h \leq \frac{1}{2}\sqrt{2} \\ \frac{4h}{(2h+1)^2} & \text{for } \frac{1}{2}\sqrt{2} < h \leq 1 \\ \frac{4h}{(h+2)^2} & \text{for } 1 < h \leq \sqrt{2} \\ \frac{2h}{(h+1)^2} & \text{for } h > \sqrt{2} \end{cases}. \end{aligned}$$

It is easy to verify that $\varrho_h \leq 1/2$.

Corollary 3. For any $h > 0$, $\varrho(T_h) \leq \varrho_h$.

Proof. If either $h \leq \frac{1}{2}\sqrt{2}$ or $h > \sqrt{2}$, then $\varrho_h = \frac{2h}{(h+1)^2}$. By Corollary 1 we get $\varrho(T_h) \leq \frac{2h}{(h+1)^2} = \varrho_h$. Note that the worst case appears when one square is packed: one square of the area greater than $\varrho_h|T_h|$ cannot be packed into T_h .

If $\frac{1}{2}\sqrt{2} < h \leq 1$, then $\varrho_h = \frac{4h}{(2h+1)^2} = \max \left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right]$. By Corollary 2 we obtain $\varrho(T_h) \leq \varrho_h$. Now the worst case appears when two squares are packed: two congruent squares of the total area greater than $\varrho_h|T_h|$ cannot be packed either along the base of T_h or vertically. However, it is possible to pack two congruent squares of the total area equal to $\varrho_h|T_h|$ horizontally (see Fig. 2).

If $1 < h \leq \sqrt{2}$, then $\varrho_h = \frac{4h}{(h+2)^2} = \max \left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right]$. By Corollary 2 we get $\varrho(T_h) \leq \varrho_h$. In that case the worst case appears when two squares are packed: two congruent squares of the total area greater than $\varrho_h|T_h|$ cannot be packed either along the base of T_h or vertically, but it is possible to pack two congruent squares of the total area equal to $\varrho_h|T_h|$ vertically (see Fig. 3). $\square$

In Sect. 5 we will show that $\varrho(T_h) \geq \varrho_h$.

3. Lemmas

Claim 3. Any two squares of the sum of the areas not greater than $\varrho_h \cdot |T_h|$ can be parallel packed into T_h .

Proof. The conditions $\alpha \geq \beta$ and $h \cot \alpha + h \cot \beta = 1$ (see Fig. 3) imply that

$$2 \cot \alpha \leq \cot \alpha + \cot \beta = \frac{1}{h}.$$

FIGURE 4. Claim 3

Let S_1 and S_2 be the squares of the side lengths a_1 , a_2 , respectively and let

$$a_1^2 + a_2^2 \leq \varrho_h \cdot |T_h| \leq \max \left[\frac{2h^2}{(2h+1)^2}, \frac{2h^2}{(h+2)^2} \right].$$

First we show that if S_1 and S_2 cannot be packed along the base of T_h , then $a_1^2 + a_2^2 > \frac{2h^2}{(2h+1)^2}$. If S_2 cannot be parallel packed into T_h "horizontally" as on Fig. 4, left, then $a_1 \cot \alpha + a_1 + a_2 + a_2 \cot \beta > 1$. Since

$$\begin{aligned} 1 &< a_1 \cot \alpha + a_1 + a_2 + a_2 \cot \beta = a_1 \cot \alpha + a_1 + a_2 + a_2 \left(\frac{1}{h} - \cot \alpha \right) \\ &= (a_1 - a_2) \cot \alpha + a_1 + a_2 + a_2 \cdot \frac{1}{h} \leq \frac{1}{2h} (a_1 - a_2) + a_1 + a_2 + a_2 \cdot \frac{1}{h} \\ &= (a_1 + a_2) \left(1 + \frac{1}{2h} \right) = \frac{2h+1}{2h} (a_1 + a_2), \end{aligned}$$

it follows that $a_1 + a_2 > \frac{2h}{2h+1}$, i.e., $a_2 > \frac{2h}{2h+1} - a_1$. Thus

$$\begin{aligned} a_1^2 + a_2^2 &> a_1^2 + \left(\frac{2h}{2h+1} - a_1 \right)^2 = 2 \left(a_1^2 - \frac{2ha_1}{2h+1} + \frac{2h^2}{(2h+1)^2} \right) \\ &\geq 2 \left(\frac{h^2}{(2h+1)^2} - \frac{2h^2}{(2h+1)^2} + \frac{2h^2}{(2h+1)^2} \right) = \frac{2h^2}{(2h+1)^2}. \end{aligned}$$

This means that $a_1^2 + a_2^2 \leq \frac{2h^2}{(h+2)^2}$. In that case we show that S_1 and S_2 can be packed into T_h "vertically" as on Fig. 4, right. The assumption $a_1^2 \leq \varrho_h \cdot |T_h|$, by Corollary 1 implies that S_1 can be packed into T_h . Let $p = a_1^2 + a_2^2$. Hence $a_1 + a_2 = a_1 + \sqrt{p - a_1^2}$. The function $f(a_1) = a_1 + \sqrt{p - a_1^2}$ has a local maximum at $a_1 = \sqrt{p/2}$. This means that $a_1 + a_2 \leq \sqrt{p/2} + \sqrt{p/2} = \sqrt{2p}$. Since $a_1 + a_2 = \sqrt{2p}$ for $a_1 = a_2$, it follows that if two congruent squares of the total area p can be parallel packed into T_h , then any two squares of the total area p can be packed into T_h . By Corollary 2 two congruent squares of the total area greater than $2h^2/(2h+1)^2$ and not greater than $2h^2/(h+2)^2$ can be parallel packed into T_h "vertically". $\square$

Let sT_h be the image of T_h after a homothety with the ratio s . Clearly, sT_h is a triangle of the base length s and the height sh . Since affine transformations preserve relative areas, by Claim 1 and 3 we get the following result.

Corollary 4. *Any square of the area not greater than $\varrho_h \cdot |sT_h|$ can be parallel packed into sT_h . Moreover, two squares of the sum of the areas not greater than $\varrho_h \cdot |sT_h|$ can be parallel packed into sT_h .*

The following two inequalities will be used in the proof of Lemma 3.

Lemma 1. *Let $f(x, h) = (1 + x^2)/(2h + 4hx + x)$. If $h > 0$ and if $x \leq 1$, then $f(x, h) > \varrho_h/(2h)$.*

Proof. Assume that $h > 0$ and that $x \leq 1$. We will show that

$$f(x, h) > \frac{\varrho_h}{2h} = \begin{cases} \frac{1}{(h+1)^2} & \text{for } h \in (0, \frac{1}{2}\sqrt{2}] \cup (\sqrt{2}, +\infty) \\ \frac{2}{(2h+1)^2} & \text{for } h \in (\frac{1}{2}\sqrt{2}, 1] \\ \frac{2}{(h+2)^2} & \text{for } h \in (1, \sqrt{2}] \end{cases}.$$

Observe that

$$\begin{aligned} f(x, h) - \frac{1}{(h+1)^2} &= \frac{x^2h^2 + 2x^2h + x^2 - 4hx - x + h^2 + 1}{(2h + 4hx + x)(h+1)^2} \\ &= \frac{(hx-1)^2(1-x) + (x-h)^2 + x^3h^2}{(2h + 4hx + x)(h+1)^2} > 0 \end{aligned}$$

for $h > 0$. Moreover, if $h \geq \sqrt{2}/2$, then $1/(h+1)^2 \geq 2/(2h+1)^2$ and, consequently, $f(x, h) > \frac{2}{(2h+1)^2}$. Finally observe that

$$\begin{aligned} f(x, h) - \frac{2}{(h+2)^2} &= \frac{x^2h^2 + 4x^2h + 4x^2 - 8hx - 2x + h^2 + 4}{(2h + 4xh + x)(h+2)^2} \\ &= \frac{(hx-2)^2(1-x) + (2x-h)^2 + x^3h^2 + 2x}{(2h + 4xh + x)(h+2)^2} > 0. \end{aligned}$$

□

Lemma 2. *Let $g(y, h) = (2 + y)/(6h + 4hy + 1)$. If $h > 0$ and if $y > 0$, then $g(y, h) > \varrho_h/(2h)$.*

Proof. We proceed as in the proof of Lemma 1. Observe that

$$g(y, h) - \frac{1}{(h+1)^2} = \frac{h^2 + (y+1)(h-1)^2}{(6h + 4hy + 1)(h+1)^2} > 0$$

and

$$g(y, h) - \frac{2}{(h+2)^2} = \frac{4 + 2(h-1)^2 + y(h-2)^2}{(6h + 4hy + 1)(h+2)^2} > 0.$$

□

FIGURE 5. Trapezoids L and L_1 FIGURE 6. Division of L_1

4. Packing Squares into Trapezoids

In the main packing method squares will be packed in trapezoid-shape layers. In this section an algorithm for packing squares into trapezoids is described.

Assume that S_i is a square of the side length a_i , where $a_1 \geq a_2 \geq \dots$. Let L be a trapezoid of the base length $b(L)$, with the height $h(L)$ and with the base angles measuring α and β , where $90^\circ \geq \alpha \geq \beta$ (see Fig. 5, left). Moreover assume that $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq b(L)$ and that $h(L) = a_1$ (see Fig. 5, right).

Description of the L -method for packing $S_1, S_2, \dots$ into L .

- The first square S_1 is packed into L as far to the left as possible. Let L_1 be the part of L lying to the right of S_1 , i.e., the trapezoid of the height $h(L) = a_1$, of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha)$ and with the base angles measuring 90° and β (see Fig. 5, right).
- If $a_2 > \frac{1}{2}a_1$ (see Fig. 5), then S_2 is packed into L_1 in the left-bottom corner; then $L_{2,1}$ is the part of L_1 lying to the right of S_2 , i.e., either $L_{2,1}$ is the trapezoid of the height $h(L) = a_1$, of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha) - a_2$ and with the base angles measuring 90° and β or $L_{2,1}$ is the triangle of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha) - a_2$, with the base angles measuring 90° and β , and of the height not greater than $h(L) = a_1$. If $a_2 \leq \frac{1}{2}a_1$, then L_1 is partitioned into smaller trapezoids. Let m_2 be an integer such that $2^{-m_2-1}a_1 < a_2 \leq 2^{-m_2}a_1$. Then L_1 is divided into 2^{m_2} trapezoids $L_{1,1}^+, \dots, L_{1,2^{m_2}}^+$ of the height $h_2 = 2^{-m_2}a_1$ (see Fig. 6). The square S_2 is packed into $L_{1,1}^+$ as far to the left as possible, i.e., in the left-bottom corner. New trapezoids are defined as follows:

- $L_{2,1}$ is the part of $L_{1,1}^+$ lying to the right of S_2 and $L_{2,i} = L_{1,i}^+$ for $i = 2, 3, \dots, 2^{m_2}$ (it is possible that $L_{2,1}$ is a triangle).
- Assume that squares $S_1, \dots, S_{k-1}$ were packed and that trapezoids $L_{k-1,i}$ for $i = 1, 2, \dots, 2^{m_{k-1}}$ (of the height h_{k-1}) were defined. Next square S_k is placed in the following way. If $a_k > \frac{1}{2}h_{k-1}$, then let j be the smallest integer such that $b(L_{k-1,j}) \geq a_k(1 + \cot \beta)$ (we find the lowest lying trapezoid into which S_k can be packed). Then S_k is packed into $L_{k-1,j}$ in the left-bottom corner. New trapezoids are defined: $L_{k,j}$ is the part of $L_{k-1,j}$ lying to the right of S_k and $L_{k,i} = L_{k-1,i}$ for $i \neq j$ (it is possible that some trapezoids are triangles). For example, squares S_2, S_3 and S_4 are packed on Fig. 8 (left) by this rule. Moreover, squares $S_4, \dots, S_{10}$ are packed by this rule on Fig. 9, left. If $a_k \leq \frac{1}{2}h_{k-1}$, then each trapezoid $L_{k-1,i}$ is partitioned into smaller trapezoids. Let n_k be an integer such that $2^{-n_k-1}h_{k-1} < a_k \leq 2^{-n_k}h_{k-1}$. Then $L_{k-1,i}$ (for $i = 1, \dots, 2^{m_{k-1}}$) is divided into 2^{n_k} trapezoids $L_{k-1,(i-1)2^{n_k}+1}^+, \dots, L_{k-1,i2^{n_k}}^+$ of the height $2^{-n_k}h_{k-1}$ (see Fig. 6, right). Let j be the smallest integer such that $b(L_{k-1,j}^+) \geq a_k(1 + \cot \beta)$ (we find the lowest lying trapezoid into which S_k can be packed). The square S_k is packed into $L_{k-1,j}^+$ in the left-bottom corner. New trapezoids are defined: $L_{k,j}$ is the part of $L_{k-1,j}^+$ lying to the right of S_k and $L_{k,i} = L_{k-1,i}^+$ for $i \neq j$ and $i \in \{1, 2, \dots, 2^{m_k}\}$, where $m_k = m_{k-1}n_k$ (it is possible that some trapezoids are triangles). For example, squares S_3 and S_9 are packed on Fig. 9 (right) by this rule.

Lemma 3. Assume that S_i is a square of the side length a_i for $i = 1, 2, \dots$, where $a_1 \geq a_2 \geq \dots$. Let L be a trapezoid of the base length $b(L)$, with the height $h(L) = a_1$ and with the base angles measuring α and β (see Fig. 5, left). Moreover assume that $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq b(L)$. If S_z is the first square that cannot be packed into L by the L -method, then $|S_1| + \dots + |S_{z-1}| > \varrho_h \cdot |L|$.

Proof. Since any homothety preserves relative areas, without loss of generality one can assume that $b(L) = 1$.

Consider four cases.

Case 1: $z = 3$, $a_z \geq a_1/2$. (see Fig. 7)

Let $x = a_z/a_1$. Since a_z cannot be packed into L , it follows that $a_1 \cot \alpha + a_1 + a_2 + a_z + a_z \cot \beta > 1$. Recall that $\cot \alpha + \cot \beta = 1/h$ and $\cot \alpha \leq \frac{1}{2h}$. By

$$\begin{aligned} 1 &< a_1 \cot \alpha + a_1 + a_2 + a_z + a_z \cot \beta = a_1 \cot \alpha + a_1 + 2a_z + a_z \left(\frac{1}{h} - \cot \alpha \right) \\ &= (a_1 - a_z) \cot \alpha + a_1 + 2a_z + a_z \cdot \frac{1}{h} \leq \frac{1}{2h} (a_1 - a_1 x) + a_1 + 2a_1 x + a_1 x \cdot \frac{1}{h} \\ &= a_1 \left(\frac{1+x}{2h} + 1 + 2x \right) = a_1 \cdot \frac{1+x+4hx+2h}{2h} \end{aligned}$$

FIGURE 7. $z = 3$, $a_z \geq a_1/2$

we get

$$a_1 > \frac{2h}{1 + 2h + 4hx + x}.$$

Denote by $\varrho(L)$ the ratio of the sum of the areas of squares packed into L to the area of L . Only two squares are packed in L , therefore

$$\varrho(L) = \frac{a_1^2 + a_2^2}{a_1 - \frac{1}{2}a_1^2(\cot \alpha + \cot \beta)} = \frac{a_1^2 + (xa_1)^2}{a_1 - \frac{a_1^2}{2h}} = \frac{a_1(1 + x^2) \cdot 2h}{2h - a_1}.$$

Since the function $f(a_1) = 2ha_1(1 + x^2)/(2h - a_1)$ is increasing, it follows that

$$\varrho(L) > \frac{2h(1 + x^2) \cdot 2h}{1 + 2h + 4hx + x} \cdot \frac{1 + 2h + 4hx + x}{2h \cdot (2h + 4hx + x)} = 2h \frac{(1 + x^2)}{2h + 4hx + x}.$$

By Lemma 1, $\varrho(L) > 2h \cdot \varrho_h/(2h) = \varrho_h$.

Case 2: $z > 3$, $a_z \geq a_1/2$.

Let L^- be the trapezoid with $b(L^-) = 1 - a_3 - \dots - a_{z-1}$, with the height a_1 and the base angles with the same measures as the base angles of L (see Fig. 8, where $z = 5$). Moreover, let sL^- be the image of L^- after a homothety with the ratio s such that the larger base of sL^- is equal to 1. By Case 1, the density of packing squares sS_1 and sS_2 into sL^- is greater than ϱ_h . This implies that the density of packing squares S_1 and S_2 into L^- is greater than ϱ_h . The density of packing squares $S_3, \dots, S_{z-1}$ into the rectangle $R_L = (a_3 + \dots + a_{z-1}) \times a_1$ is greater than $1/2 \geq \varrho_h$. Clearly, $|L^-| + |R_L| = |L|$. As a consequence, the density of packing squares $S_1, S_2, \dots, S_{z-1}$ into L is greater than ϱ_h .

Case 3: $z \geq 3$, $a_1/4 < a_z \leq a_1/2$.

Let k be the smallest integer such that $a_k < a_1/2$ (see Fig. 9, left). Obviously, $k \leq z$. Let $L_{k-1,1}$ be the part of L_1 lying to the right of S_{k-1} , i.e., either the trapezoid of the height $h_r = h(L_{k-1,1})$, of the base length $b_r = b(L_{k-1,i})$ and with the base angles measuring 90° and β or the triangle

FIGURE 8. $z = 5$, $a_z > a_1/2$

FIGURE 9. $k = 3$

of the base length $b_r = b(L_{k-1,1})$, with the base angles measuring 90° and β , and of the the height not greater than $h_r = h(L_{k-1,1})$,

Squares $S_k, S_{k+1}, \dots$ are packed into two rectangular trapezoids $L_{k-1,1}^+$ and $L_{k-1,2}^+$ with disjoint interiors, where $L_{k-1,1} = L_{k-1,1}^+ \cup L_{k-1,2}^+$ (it is possible that trapezoids are triangles).

Assume that S_v is the first square that cannot be packed into $L_{k-1,1}^+$. Clearly $a_v \leq h_r/2$ and $a_z \leq h_r/2$ (see Fig. 10, right).

Let $p_2 = |S_k| + \dots + |S_{z-1}|$, i.e., p_2 is equal to the sum of the areas of squares packed into $L_{k-1,1}$.

Subcase 3A: $b_r \geq a_v(1 + \cot \beta) + \frac{1}{2}h_r \cot \beta$. Observe that (see Fig. 10, right)

$$p_2 \geq [b_r - a_v(1 + \cot \beta)] a_v + [b_r - a_z(1 + \cot \beta) - \frac{1}{2}h_r \cot \beta] a_z.$$

Let

$$p_1 = [b_r - \frac{1}{2}h_r(1 + \cot \beta)] \frac{1}{2}h_r$$

FIGURE 10. Subcase 3A

and let $y = (2p_1)/(a_1 h_r)$. Obviously, p_1 is equal to the area of the gray rectangle and $y a_1$ is equal to the base length of the gray rectangle on Fig. 10, left.

Since

$$\begin{aligned}
 p_2 - p_1 &\geq b_r \left(a_v + a_z - \frac{1}{2} h_r \right) + (1 + \cot \beta) \left(\frac{1}{4} h_r^2 - a_v^2 - a_z^2 \right) - \frac{1}{2} h_r a_z \cot \beta \\
 &> \left[a_v (1 + \cot \beta) + \frac{1}{2} h_r \cot \beta \right] \left(a_v + a_z - \frac{1}{2} h_r \right) \\
 &\quad + (1 + \cot \beta) \left(\frac{1}{4} h_r^2 - a_v^2 - a_z^2 \right) - \frac{1}{2} h_r a_z \cot \beta \\
 &= a_z (1 + \cot \beta) (a_v - a_z) + \frac{1}{2} h_r \left(\frac{h_r}{2} - a_v \right)
 \end{aligned}$$

and $a_z \leq a_v \leq h_r/2$, we get $p_2 > p_1$. Hence, to show that $\sum_{i=1}^{z-1} |S_i| > \varrho_h |L|$, it suffices to verify that

$$a_1^2 + \dots + a_{k-1}^2 + p_1 > \varrho_h |L|.$$

First assume that $k = 2$. We argue as in Case 1. By conditions $a_1 \cot \alpha + a_1 + y a_1 + \frac{1}{2} a_1 + \frac{1}{2} a_1 \cot \beta = 1$, $\cot \alpha + \cot \beta = 1/h$ and $\cot \alpha \leq 1/(2h)$, we get

$$\begin{aligned}
 1 &= a_1 \cot \alpha + a_1 + y a_1 + \frac{1}{2} a_1 + \frac{1}{2} a_1 \cot \beta \\
 &= a_1 \cot \alpha + \frac{3}{2} a_1 + y a_1 + \frac{1}{2} a_1 \left(\frac{1}{h} - \cot \alpha \right) \\
 &= \frac{1}{2} a_1 \cot \alpha + \frac{3}{2} a_1 + y a_1 + \frac{a_1}{2h} \\
 &\leq \frac{1}{2} a_1 \cdot \frac{1}{2h} + \frac{3}{2} a_1 + y a_1 + \frac{a_1}{2h} = a_1 \cdot \frac{3h + 2yh + \frac{3}{2}}{2h}.
 \end{aligned}$$

Consequently,

$$a_1 \geq \frac{4h}{6h + 4yh + 3}.$$

FIGURE 11. Subcase 3C

Now we estimate the packing density:

$$\begin{aligned} \frac{a_1^2 + p_1}{|L|} &= \frac{a_1^2 + \frac{1}{2}a_1 \cdot ya_1}{a_1 - \frac{a_1^2}{2h}} = h(2+y) \cdot \frac{a_1}{2h - a_1} \\ &\geq h(2+y) \cdot \frac{\frac{4h}{6h+4yh+3}}{2h - \frac{4h}{6h+4yh+3}} = 2h \cdot \frac{2+y}{6h+4hy+1}. \end{aligned}$$

By Lemma 2, $a_1^2 + p_1 > |L| \cdot 2h \cdot \varrho_h / (2h) = \varrho_h \cdot |L|$.

If $k > 2$, then we argue as in Case 2.

Subcase 3B: $b_r < \frac{1}{2}h_r(1 + \cot \beta)$. This assumption implies that a square of the side length greater than $h_r/2$ cannot be packed in $L_{k-1,1}$. By Case 1 and Case 2, $a_1^2 + \dots + a_{k-1}^2 > \varrho_h |L|$.

Subcase 3C: $\frac{1}{2}h_r(1 + \cot \beta) \leq b_r \leq a_v(1 + \cot \beta) + \frac{1}{2}h_r \cot \beta$. The total area of squares packed in two trapezoids contained in $L_{k-1,1}$ (see Fig. 11, right) is not smaller than

$$p_2^- = [b_r - a_v(1 + \cot \beta)]a_v.$$

In this subcase, similarly as in Subcase 3a, it suffices to check that $p_2^-(L_{k-1,1}) \geq p_1^-(L_{k-1,1})$, where

$$p_1^- = [b_r - \frac{1}{2}h_r(1 + \cot \beta)] \cdot \frac{1}{2}h_r$$

(p_1^- is equal to the area of the gray rectangle on Fig. 11, left). Observe that

$$\begin{aligned} p_2^- - p_1^- &= b_r a_v - a_v^2(1 + \cot \beta) - \frac{1}{2}h_r b_r + \frac{1}{4}h_r^2(1 + \cot \beta) \\ &= \left(\frac{1}{4}h_r^2 - a_v^2\right)(1 + \cot \beta) + b_r a_v - \frac{1}{2}b_r h_r \\ &= \left(\frac{1}{2}h_r - a_v\right)\left(\frac{1}{2}h_r + a_v\right)(1 + \cot \beta) + \left(\frac{1}{2}h_r - a_v\right)b_r \\ &= \left(\frac{1}{2}h_r - a_v\right)\left[\frac{1}{2}h_r \cot \beta + a_v(1 + \cot \beta) - b_r + \frac{1}{2}h_r\right]. \end{aligned}$$

By $\frac{1}{2}h_r - a_v \geq 0$ and $\frac{1}{2}h_r \cot \beta + a_v(1 + \cot \beta) - b_r > 0$, we get $p_2^- - p_1^- > 0$.

Case 4: $z \geq 3$, $a_z \leq a_1/4$ (see Fig. 9, right).

We proceed in the same way as in Case 3. The sum of the areas of squares packed into two trapezoids $L_{l,i}^+, L_{l,i+1}^+$ contained in the trapezoid $L_{l,i}^+ \cup L_{l,i+1}^+$ is greater than the area of the corresponding "gray" rectangle of the height $h(L_{l,i}^+)$ and the base length $b(L_{l,i}^+) - h(L_{l,i}^+) \cdot (1 + \cot \beta)$ contained in $L_{l,i}^+ \cup L_{l,i+1}^+$. $\square$

5. Main Result

Theorem 1. *Any collection of squares of the total area not greater than $\varrho_h \cdot |T_h|$ can be parallel packed into T_h .*

Proof. Let $S_1, S_2, \dots$ be a collection of squares of the total area not greater than $\varrho_h \cdot |T_h|$. There is no loss of generality in assuming that $a_1 \geq a_2 \geq \dots$, where a_i denotes the side length of S_i .

Squares will be packed into trapezoid-shape layers in such a way that the packing density in each layer, i.e., the ratio of the sum of packed squares to the area of the layer, is greater than ϱ_h .

Since $a_1^2 + a_2^2 + \dots \leq \varrho_h \cdot |T_h|$, by Corollary 1 the first square can be packed into T_h . Moreover, if there are at least two squares in the collection (i.e., if $z > 1$), then, by Claim 3, squares S_1 and S_2 can be packed into T_h either horizontally or vertically, i.e., either $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq 1$ or $a_1 + a_2 \leq 2h/(h + 2)$ (see the proof of Claim 3).

If $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq 1$ (see Fig. 13), then let L^1 be the trapezoid contained in T_h with the base length 1, the height a_1 and the base angles measuring α and β . This trapezoid is called the *basic layer*. Squares $S_1, S_2, \dots$ are packed into L^1 by the L -method. Denote by S_{n_1} the first square that cannot be packed into L^1 ($n_1 = 3$ on Fig. 13). By Lemma 3, the total area of squares packed into L^1 is not smaller than $\varrho_h \cdot |L^1|$.

If $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) > 1$ and at the same time $a_1 + a_2 \leq \frac{2h}{2+h}$ (this case is possible for $h > 1$), then we pack S_1 and S_2 "vertically". The first condition implies that $1 < (a_1 + a_2)(2h + 1)/(2h)$ (see the proof of Claim 3). Let $\lambda = a_1 + a_2$. The so-called *double layer* L^1 of the height λ is created for packing only two squares, see Fig. 12. In that case $n_1 = 3$ (S_{n_1} is the first square that is not packed into L^1) and $\lambda > 2h/(2h + 1)$. It is easy to verify that $a_1^2 + a_2^2 \geq (\lambda/2)^2 + (\lambda/2)^2 = \lambda^2/2$. Recall that $\varrho_h \leq 1/2$ while the packing density in the double layer is not smaller than $1/2$:

$$\frac{a_1^2 + a_2^2}{1 \cdot \lambda - \frac{\lambda^2}{2h}} \geq \frac{\frac{1}{2}\lambda^2}{\lambda - \frac{\lambda^2}{2h}} = \frac{h\lambda}{2h - \lambda} > \frac{h \cdot \frac{2h}{2h+1}}{2h - \frac{2h}{2h+1}} = \frac{1}{2}.$$

Since the sum of the areas of the squares packed into L^1 is greater than $\varrho_h \cdot |L^1|$, it follows that the total area of the remaining squares is smaller than $\varrho_h \cdot |T_1|$, where $T_1 = T_h \setminus L^1$.

For packing squares $S_{n_1}, S_{n_1+1}, \dots$ the new layer L^2 of the base length $1 - a_1 \cot \alpha - a_1 \cot \beta$, with the base angles measuring α and β and with the

FIGURE 12. $a_1 + a_2 > 2h/(2h + 1)$

FIGURE 13. Packing method

height either a_{n_1} or $a_{n_1} + a_{n_1+1}$ is created directly above L^1 (see Fig. 13, where L^2 is a double layer). If $a_{n_1}(1 + \cot \alpha) + a_{n_1+1}(1 + \cot \beta) \leq b(L^2)$, then we place $S_{n_1}, S_{n_1+1}, \dots$ into the basic layer L^2 of the height a_{n_1} by the L -method (denote by S_{n_2} the first square that cannot be packed into L^2). Otherwise, we pack S_{n_1} and S_{n_1+1} into the double layer of the height $a_{n_1} + a_{n_1+1}$ and we take $n_2 = n_1 + 2$.

Assume that the layers $L^1, \dots, L^p$ were created and that S_{n_p} is the first square that cannot be packed into L^p . The total area of squares packed into each L^i ($i = 1, \dots, p$) is greater than $\varrho_p \cdot |L^i|$. This implies that $a_{n_p+1}^2 + a_{n_p+2}^2 + \dots$ is smaller than ϱ_h times the area of $T_p = T_h \setminus \bigcup_{i=1}^p L^i$. By Corollary 4, S_{n_p} can be packed into T_p . Moreover by Claim 3, S_{n_p} and S_{n_p+1} can be packed into T_p provided $z \geq n_p + 1$. There is an empty space in T_h to create either the basic or the double layer to pack S_{n_p} and S_{n_p+1} . This means that all the squares can be packed into T_h . $\square$

The following result is a consequence of Theorem 1 and Corollary 3.

Corollary 5. *For any $h > 0$, $\varrho(T_h) = \varrho_h$.*

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Janusz Januszewski and Łukasz Zielonka
Institute of Mathematics and Physics
Bydgoszcz University of Science and Technology
Al. Prof. S. Kaliskiego 7
85-789 Bydgoszcz
Poland
e-mail: januszew@pbs.edu.pl;
Lukasz.Zielonka@pbs.edu.pl

Xi Liu and Zhanjun Su
School of Mathematical Sciences
Hebei Normal University
Shijiazhuang 050024
China
e-mail: 1298789840@qq.com;
suzj888@163.com

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