Automorphisms of n -Ary Groups

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Abstract. Isotopies and autotopies of n -ary groups are described. As a consequence, we obtain various characterizations of the group of automorphisms of n -ary groups. We also determine the number of automorphisms of a given n -ary group.

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1. Introduction

Polyadic groups (called also n -ary groups) are a natural generalization of classical groups. The first important paper on n -ary groups was written (under the inspiration of Emmy Nöther) by Dörnte in 1928 (cf. [2]). Another important publication is the Post's article [16] which shows strong relationships with classical groups but also gives many significant differences between n -ary groups ($n > 2$) and ordinary groups. Today we know many such differences. For example, the intersection of two n -ary subgroups may be the empty set; there are n -ary groups without neutral elements (cf. [2]), but there are n -groups containing many neutral elements (cf. [4]). Also there are groups in which all elements are neutral.

Many authors have attempted to classify n -ary groups according to their isomorphism. Criteria were found to determine if the groups are isomorphic (for example, [8, 9]). However, isotopies between n -ary groups have not been studied so far. Also, little is known about the automorphisms and autotopies of n -ary groups (cf. [14]). The number of autotopies of a given n -ary group (quasigroup) is very large. An n -ary quasigroup of order four has at least $2^{\lfloor \frac{n}{2} \rfloor} + 2$ and not more than $6 \cdot 4^n$ autotopies (cf. [10]). For n -groups of greater order, the result is unknown. We only know the exact number of autotopies

and automorphisms for some n -ary quasigroups. For n -ary quasigroups linear over a commutative group it is given in [15].

In this article (inspired by [15]) we find a criterion for when two n -ary groups are isotopic. Then we characterize the autotopies of such groups and determine the number of all automorphisms of n -ary groups.

2. Preliminaries

Throughout the paper, we assume that $n > 2$. We will also use the abbreviated form of the notation. Instead of 'an n -ary group' we will write 'an n -group'. The sequence of elements $x_i, \dots, x_j$ we will be written as x_i^j , remembering that for $j < i$ it is an empty symbol.

The nonempty set G with one n -ary operation $f : G^n \rightarrow G$ is called an n -groupoid and is denoted by (G, f) . An n -groupoid is an n -quasigroup if for all $a_1^n, b \in G$ and each $i \in \overline{1, n} = \{1, 2, \dots, n\}$ there is exactly one $x_i \in G$ such that $f(a_1^{i-1}, x_i, a_{i+1}^n) = b$. If the operation f is associative, i.e.

$$f(f(a_1^n), a_{n+1}^{2n-1}) = f(a_1^j, f(a_{j+1}^{j+n}), a_{j+n+1}^{2n-1})$$

holds for all $a_1^{2n-1} \in G$ and all $1 \leq j < n$, then an n -groupoid (G, f) is called an n -semigroup. An n -quasigroup with an associative operation is called an n -group. For $n = 2$ we obtain quasigroup, semigroup, and group, respectively.

Note by the way that in the definition of the n -group (G, f) , assuming the associativity of the operation f , it is enough to postulate the existence of a solution of the equation $f(a_1^{i-1}, x_i, a_{i+1}^n) = b$ (cf. [16]).

In [7] the following characterization of n -groups is proved.

Theorem 2.1. *An n -semigroup (G, f) is an n -group if and only if for every $a \in G$ there exists an element $\bar{a} \in G$ such that*

$$f(\bar{a}, a, a, \dots, a, y) = f(y, a, a, \dots, a, \bar{a}) = y$$

for all $y \in G$.

Other characterizations of this type can be found in [5].

We say that an n -groupoid (G, f) has a *neutral element* if there exist $e \in G$ such that $f(e, \dots, e, x, e, \dots, e) = x$ no matter where x occurs. An n -quasigroup containing neutral elements (one of more) is called an n -loop. An element $a \in G$ with the property $f(a, \dots, a) = a$ is called *idempotent*. The set of all idempotents of an n -groupoid (G, f) is denoted by $E_{(G, f)}$. Generally, n -groups do not have such elements, but there are groups that have several neutral elements and groups that contain only neutral elements

Theorem 2.2. *The set of all neutral elements of an n -semigroup (G, f) is empty or is a commutative n -group contained in (G, f) .*

Proof. Let e be a neutral element of an n -semigroup (G, f) . Then $f(e, \dots, e) = e$. Moreover, e is the neutral element of a semigroup $(G, \cdot)$, where $xy = f(x, e, \dots, e, y)$. In this case $f(x_1^n) = x_1 x_2 \cdots x_n$. If a is another neutral element of (G, f) , then $x = f(x, a, \dots, a) = xa \cdots a = xa^{n-1}$ in $(G, \cdot)$. Thus $x = f(a, x, a, \dots, a) = axa^{n-2}$ implies $xa = ax$ for all $x \in G$.

Let $a_1, \dots, a_n$ be (not necessarily different) neutral elements of (G, f) . Then $f(f(a_1^n), \dots, f(a_1^n), x, f(a_1^n), \dots, f(a_1^n))$, calculated in $(G, \cdot)$, is equal to x . Hence $f(a_1^n)$ is a neutral element. Thus, the set E of all neutral elements of (G, f) is a commutative n -semigroup contained in (G, f) . Since the equation $f(x, a_2^n) = a_1$ is solved by $x = f_{(\cdot)}(a_1, \overset{(n-2)}{a_n}, \dots, \overset{(n-2)}{a_2}) \in E$, (E, f) is a commutative n -group. $\square$

A binary groupoid $(G, \cdot)$ with the operation $xy = f(x, a_2^{n-1}, y)$, where $a_2^{n-1} \in G$ are fixed, is called a *retract* of (G, f) and is denoted by $\text{ret}_{a_2^n}(G, f)$. The symbol $(G, \cdot) = \text{ret}_a(G, f)$ means that $xy = f(x, a, \dots, a, y)$.

It is clear that retracts of n -semigroups are semigroups, retracts of n -groups are groups. Using these retracts and their automorphisms, we can find a very useful characterization of n -ary groups known as Hosszú Theorem (cf. [13]). Since two years later Gluskin proved this theorem in a more general form (cf. [11]), this theorem is known also as Gluskin–Hosszú or Hosszú–Gluskin Theorem (cf. [6]).

Theorem 2.3. *An n -groupoid (G, f) is an n -group if and only if*

- (i) *on G one can define a binary operation $\cdot$ such that $(G, \cdot)$ is a group,*
- (ii) *there exist an automorphism φ of $(G, \cdot)$ and $b \in G$ such that $\varphi(b) = b$,*
- (iii) *$\varphi^{n-1}(x)b = bx$ for all $x \in G$,*
- (iv) *$f(x_1^n) = x_1 \varphi(x_2) \varphi^2(x_3) \cdots \varphi^{n-1}(x_n)b$ for all $x_1^n \in G$.*

In this case, we say that an n -group (G, f) is $\langle \varphi, b \rangle$ -derived from the group $(G, \cdot)$ and denote this fact by $(G, f) = \text{der}_{\varphi, b}(G, \cdot)$. All retracts of $(G, f) = \text{der}_{\varphi, b}(G, \cdot)$ are isomorphic to $(G, \cdot)$.

An n -groupoid (G, f) is *medial* if

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \dots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{2n}), \dots, f(x_{1n}^{nn}))$$

holds for all $x_{ij} \in G$ and $i, j \in \overline{1, n}$.

The following two characterizations of medial n -groups (proved in [3]) will be used later.

Lemma 2.4. *An n -group (G, f) is medial if and only if for some $a \in G$ the retract $\text{ret}_a(G, f)$ is commutative.*

Corollary 2.5. *An n -group (G, f) is medial if and only if it is $\langle \varphi, b \rangle$ -derived from an abelian group.*

3. Isotopies and Isomorphisms

Suppose (G, f) and (H, g) are two n -groupoids. If there are bijections $\alpha_1, \dots, \alpha_n, \delta$ from G to H , such that

$$\delta(f(x_1, x_2, \dots, x_n)) = g(\alpha_1(x_1), \alpha_2(x_2), \dots, \alpha_n(x_n)), \quad (1)$$

then we say that these two n -groupoids are *isotopic*. In this case, we also say that $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \delta)$ is an *isotopy* or an *isotopism*, or a *weak isomorphism* between (G, f) and (H, g) . Then we write $T : (G, f) \rightarrow (H, g)$ and say that (H, g) and (G, f) are isotopic or that (H, g) is an *isotope* of (G, f) . An isotopism with $\alpha_1 = \alpha_2 = \dots = \alpha_n = \delta$ is an isomorphism.

An isomorphism saves all properties of n -groupoids. Unfortunately, an isotopism does not do that. For example, an n -groupoid isotopic to a commutative n -groupoid may not be commutative; an n -groupoid isotopic to an idempotent n -groupoid may not be idempotent. An n -groupoid isotopic to an n -group may not be an n -group, but it will always be an n -quasigroup. To confirm the above statements, it is enough to consider two 5-groupoids: $(\mathbb{Z}_5, f) = \text{der}_0(\mathbb{Z}_5, +)$ and $(\mathbb{Z}_5, g)$ with $g(x_1^5) = (x_1 - x_2 + x_3 + x_4 + x_5) \pmod{5}$.

The isotopy $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \varepsilon)$, where ε is the identity mapping, is called a *principal isotopism*.

Proposition 3.1. *Every isotope of n -groupoid (G, f) is isomorphic to a certain principal isotope of (G, f) .*

Proof. Let (H, g) be an isotope of an n -groupoid (G, f) . Then there are bijections $\alpha_i, \delta : G \rightarrow H$ satisfying (1). Consider the n -ary operation

$$h(x_1^n) = f(\alpha_1^{-1}\delta(x_1), \alpha_2^{-1}\delta(x_2), \dots, \alpha_n^{-1}\delta(x_n)).$$

Since all $\alpha_i^{-1}\delta$ are bijections from G onto G , the n -groupoid (G, h) is a principal isotope of (G, f) .

Also,

$$\begin{aligned} \delta(h(x_1, \dots, x_n)) &= \delta f(\alpha_1^{-1}\delta(x_1), \alpha_2^{-1}\delta(x_2), \dots, \alpha_n^{-1}\delta(x_n)) \\ &= g(\delta(x_1), \delta(x_2), \dots, \delta(x_n)). \end{aligned}$$

Thus, (H, g) is isomorphic to the principal isotope (G, h) . □

In a binary case ($n = 2$), the isotopic groups are isomorphic (cf. [17]). Unfortunately, for n -groups ($n > 2$) this is not true. These two $(n + 1)$ -groups $(\mathbb{Z}_n, f) = \text{der}_0(\mathbb{Z}_n, +)$ and $(\mathbb{Z}_n, g) = \text{der}_1(\mathbb{Z}_n, +)$ are isotopic but not isomorphic. The first is idempotent, the second has not idempotents. However, for n -groups we have the following result.

Theorem 3.2. *Two n -groups are isotopic if and only if their retracts are isomorphic.*

Proof. Let n -groups (G, f) and (H, g) be isotopic. Then, by (1), for retracts $(G, \circ) = \text{ret}_{a_2^{n-1}}(G, f)$ and $(H, \diamond) = \text{ret}_{b_2^{n-1}}(H, g)$, with $b_i = \alpha_i(a_i)$, $1 < i < n$, we have

$$\delta(x_1 \circ x_n) = \delta(f(x_1, a_2^{n-1}, x_n)) = g(\alpha_1(x_1), b_2^{n-1}, \alpha_n(x_n)) = \alpha_1(x_1) \diamond \alpha_n(x_n).$$

So, $(G, \circ)$ and $(H, \diamond)$ are isotopic groups. Hence they are isomorphic.

Conversely, let the retracts of n -groups (G, f) and (H, g) be isomorphic. By Hosszú Theorem, there are groups and their automorphisms such that $(G, f) = \text{der}_{\varphi, a}(G, *)$ and $(H, g) = \text{der}_{\psi, b}(H, \cdot)$. All retracts of (G, f) are isomorphic to $(G, *)$, all retracts of (H, g) to $(H, \cdot)$. So, there is an isomorphism $\delta : (G, *) \rightarrow (H, \cdot)$. Thus, the mappings $\alpha_1 = \delta$, $\alpha_2 = \psi^{-1}\delta\varphi$, $\alpha_3 = \psi^{-2}\delta\varphi^2, \dots, \alpha_{n-1} = \psi^{2-n}\delta\varphi^{n-2}$ and $\alpha_n = R_b^{-1}\psi^{1-n}\delta\varphi^{n-1}P_a$, where $R_b(x) = x \cdot b$, $P_a(x) = x * a$, are bijections. Also,

$$\begin{aligned} \delta(f(x_1^n)) &= \delta(x_1 * \varphi(x_2) * \varphi^2(x_3) * \dots * \varphi^{n-2}(x_{n-1}) * \varphi^{n-1}(x_n * a)) \\ &= \delta(x_1) \cdot \delta\varphi(x_2) \cdot \delta\varphi^2(x_3) \cdot \dots \cdot \delta\varphi^{n-2}(x_{n-1}) \cdot \delta\varphi^{n-1}P_a(x_n) \\ &= \alpha_1(x_1) \cdot \psi\alpha_2(x_2) \cdot \dots \cdot \psi^{n-2}\alpha_{n-1}(x_{n-1}) \cdot \psi^{n-1}R_b\alpha_n(x_n) \\ &= g(\alpha_1(x_1), \alpha_2(x_2), \dots, \alpha_{n-1}(x_{n-1}), \alpha_n(x_n)). \end{aligned}$$

This means that $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \delta)$ is the isotopy between (G, f) and (H, g) . This completes the proof. $\square$

As a consequence of Theorem 3.2 and Corollary 2.5 we obtain

Corollary 3.3. *An n -group isotopic to a medial n -group also is medial.*

Corollary 3.4. *All n -groups of the same prime order are isotopic but not necessarily isomorphic.*

All n -groups (for fixed n and order of G) can be divided into disjoint classes containing isotopic n -groups. Each class can be represented by an n -group $\text{der}(G, \cdot)$. The number of such classes coincides with the number of non-isomorphic groups of a given order. For order $n \leq 30$ there is

$$1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4$$

such classes, but for $|G| = 64$ we have 267 such classes (for each $n > 2$).

Using Theorem 3.2 we can prove

Proposition 3.5. *Any isotopism $T : (G, f) \rightarrow (H, g)$ of n -groups $(G, f) = \text{der}_{\varphi, a}(G, \star)$ and $(H, g) = \text{der}_{\psi, b}(H, \cdot)$ can be expressed in the form*

$$T = (h, \psi^{-1}h\varphi, \psi^{-2}h\varphi^2, \dots, \psi^{-(n-2)}h\varphi^{n-2}, R_{b^{-1}}\psi^{-(n-1)}h\varphi^{n-1}R_a^*, h),$$

where $h : (G, \star) \rightarrow (H, \cdot)$ is a group isomorphism, R_c^* and R_c —right translations in $(G, \star)$ and $(H, \cdot)$, respectively.

An isomorphism of n -groups can be characterized by the following proposition.

Proposition 3.6. *Let $(G, \star)$ and $(H, \cdot)$ be groups with neutral elements $e^\star$ and e , respectively. Then $h : G \rightarrow H$ such that $h(e^\star) = e$ is an isomorphism of n -groups $(G, f) = \text{der}_{\varphi, a}(G, \star)$ and $(H, g) = \text{der}_{\psi, b}(H, \cdot)$ if and only if it is an isomorphism of groups $(G, \star)$ and $(H, \cdot)$, $h(a) = b$ and $h\varphi = \psi h$.*

Proof. Any isomorphism is an isotopism, so, by Proposition 3.5, we get $h = \alpha_2$. This gives $\psi h = h\varphi$. Now $h = \alpha_n$ implies $h(a) = b$. Therefore,

$$\begin{aligned} e &= h(e^\star) = h(a^{-1} \star \varphi(e^\star) \star \cdots \star \varphi^{n-1}(e^\star) \star a) = hf(a^{-1}, e^\star, \dots, e^\star) \\ &= g(h(a^{-1}), h(e^\star), \dots, h(e^\star)) = g(h(a^{-1}), e, \dots, e) = h(a^{-1}) \cdot b. \end{aligned}$$

Thus $h(a^{-1}) = b^{-1}$ in $(H, \cdot)$.

Since, by Hosszú Theorem, $\varphi(a^{-1}) = a^{-1}$ and $\varphi^{n-1}(x) \star a = a \star x$, we obtain

$$\begin{aligned} h(x \star y) &= h(x \star \varphi(a^{-1}) \star \varphi^2(e^\star) \star \cdots \star \varphi^{n-2}(e^\star) \star a \star y) \\ &= hf(x, a^{-1}, e^\star, \dots, e^\star, y) = g(h(x), h(a^{-1}), h(e^\star), \dots, h(e^\star), h(y)) \\ &= g(h(x), b^{-1}, e, \dots, e, h(y)) \\ &= h(x) \cdot \psi(b^{-1}) \cdot \psi^2(e) \cdots \psi^{n-2}(e) \cdot \psi^{n-1}h(y) \cdot b \\ &= h(x) \cdot b^{-1} \cdot \psi^{n-1}h(y) \cdot b = h(x) \cdot b^{-1} \cdot b \cdot h(y) = h(x) \cdot h(y). \end{aligned}$$

This shows that h is a group isomorphism.

The converse statement is obvious. $\square$

Theorem 3.7. *Every isotopism $T : G \rightarrow H$ of n -groups $(G, f) = \text{der}(G, \star)$ and $(H, g) = \text{der}(H, \cdot)$ can be uniquely presented in the form*

$$T = (L_{a_1}I_{c_1}\beta, L_{a_2}I_{c_2}\beta, \dots, L_{a_n}I_{c_n}\beta, R_{c_n}\beta), \quad (2)$$

where $a_1, \dots, a_n \in H$, $c_i = a_1a_2 \cdots a_i$, $I_{c_i}(x) = c_i^{-1}xc_i$ and $\beta : G \rightarrow H$ is an isomorphism of groups $(G, \star)$ and $(H, \cdot)$.

Proof. Direct calculations show that such defined T is an isotopism of these n -groups.

To prove the converse statement, let $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \delta)$ be an isotopism of n -groups $(G, f) = \text{der}(G, \star)$ and $(H, g) = \text{der}(H, \cdot)$, i.e. let

$$\delta f(x_1^n) = \delta(x_1 \star x_2 \star \cdots \star x_n) = \alpha_1(x_1)\alpha_2(x_2) \cdots \alpha_n(x_n), \quad (3)$$

for bijections $\alpha_1, \dots, \alpha_n, \delta : G \rightarrow H$.

For the sake of clarity, let $e^\star$ be the identity of $(G, \star)$, $a_i = \alpha_i(e^\star)$ and $c_i = a_1a_2 \cdots a_i$, $i \in \overline{1, n}$.

From (3) we obtain $\delta(e^\star) = a_1a_2 \cdots a_n$ and

$$\delta(x) = a_1a_2 \cdots a_{i-1}\alpha_i(x)a_{i+1} \cdots a_n = R_{a_{i+1} \cdots a_n}L_{a_1 \cdots a_{i-1}}\alpha_i(x)$$

for every $x \in G$. Thus

$$\alpha_i(x) = L_{a_1 \cdots a_{i-1}}^{-1}R_{a_{i+1} \cdots a_n}^{-1}\delta(x) = L_{(a_1 \cdots a_{i-1})^{-1}}R_{(a_{i+1} \cdots a_n)^{-1}}\delta(x).$$

Hence, (3) can be rewritten in the form

$$\begin{aligned} \delta f(x_1^n) &= R_{(a_2 \cdots a_n)^{-1}} \delta(x_1) L_{a_1^{-1}} R_{(a_3 \cdots a_n)^{-1}} \delta(x_3) \\ &\quad \cdots L_{(a_1 \cdots a_{n-2})^{-1}} R_{a_{n-1}^{-1}} \delta(x_{n-1}) L_{(a_1 \cdots a_n)^{-1}} \delta(x_n) \\ &= \delta(x_1) a_n^{-1} \cdots a_2^{-1} a_1^{-1} \delta(x_2) a_n^{-1} \cdots a_3^{-1} a_2^{-1} a_1^{-1} \delta(x_3) a_n^{-1} \cdots a_1^{-1} \\ &\quad \cdots a_n^{-1} \cdots a_1^{-1} \delta(x_{n-1}) a_n^{-1} \cdots a_1^{-1} \delta_n \\ &= \delta(x_1) d \cdot \delta(x_2) d \cdot \delta(x_3) d \cdots \delta(x_{n-1}) d \cdot \delta(x_n), \end{aligned}$$

where $d = a_n^{-1} \cdots a_2^{-1} a_1^{-1}$. From this, multiplying by d , we get

$$\delta f(x_1^n) d = \delta(x_1) d \cdot \delta(x_2) d \cdot \delta(x_3) d \cdots \delta(x_{n-1}) d \cdot \delta(x_n) d,$$

which shows that $\beta = R_d \delta : G \rightarrow H$ is an isomorphism of n -groups (G, f) and (H, g) . Since $\beta(e^*) = \delta(e^*) d = a_1 a_2 \cdots a_n d = e$, where e is the identity of $(H, \cdot)$, β is an isomorphism of groups $\text{ret}_{e^*}(G, f) = (G, \star)$ and $\text{ret}_e(H, g) = (H, \cdot)$ too.

From $\beta = R_d \delta$ we also have $\delta = R_d^{-1} \beta = R_{d^{-1}} \beta = R_{a_1 a_2 \cdots a_n} \beta$. Thus

$$\begin{aligned} \alpha_i &= L_{(a_1 \cdots a_{i-1})^{-1}} R_{(a_{i+1} \cdots a_n)^{-1}} \delta = L_{(a_1 \cdots a_{i-1})^{-1}} R_{(a_{i+1} \cdots a_n)^{-1}} R_{a_1 a_2 \cdots a_n} \beta \\ &= L_{(a_1 \cdots a_{i-1})^{-1}} R_{a_i \cdots a_n} \beta = L_{a_i} L_{a_i^{-1}} L_{(a_1 \cdots a_{i-1})^{-1}} R_{a_i \cdots a_n} \beta \\ &= L_{a_i} L_{(a_1 \cdots a_i)^{-1}} R_{a_i \cdots a_n} \beta = L_{a_i} I_{a_1 \cdots a_i} \beta, \end{aligned}$$

where I_a is the inner automorphism of $(H, \cdot)$.

Therefore,

$$T = (L_{a_1} I_{c_1} \beta, L_{a_2} I_{c_2} \beta, \dots, L_{a_n} I_{c_n} \beta, R_{c_n} \beta),$$

where $c_i = a_1 a_2 \cdots a_i$, is an isotopy of n -groups (G, f) and (H, g) .

This representation is unique. Indeed, if also

$$T = (L_{b_1} I_{d_1} \beta, L_{b_2} I_{d_2} \beta, \dots, L_{b_n} I_{d_n} \beta, R_{d_n} \sigma),$$

for some $b_1, \dots, b_n \in H$, $d_i = b_1 b_2 \cdots b_i$ and an automorphism σ of $(H, \cdot)$, then $R_{c_n} \beta = R_{d_n} \sigma$ and $L_{a_i} I_{c_i} \beta = L_{b_i} I_{d_i} \sigma$ for all $i \in \overline{1, n}$. Thus $L_{a_1} I_{c_1} \beta(e) = L_{b_1} I_{d_1} \sigma(e)$, which gives $a_1 = b_1$ and $\beta = \sigma$. Consequently, $L_{a_2} I_{c_2} = L_{b_2} I_{d_2}$ implies $a_2 = b_2$, and so on. $\square$

Corollary 3.8. *Every isotopism $T : G \rightarrow H$ of n -groups $(G, f) = \text{der}(G, \star)$ and $(H, g) = \text{der}(H, \cdot)$, where $(H, \cdot)$ is an abelian group, can be uniquely presented in the form*

$$T = (L_{a_1} \beta, L_{a_2} \beta, \dots, L_{a_n} \beta, L_{c_n} \beta), \quad (4)$$

where $a_1, a_2, \dots, a_n \in H$, $c_n = a_1 a_2 \cdots a_n$ and $\beta : G \rightarrow H$ is an isomorphism of groups $(G, \star)$ and $(H, \cdot)$.

Proof. In an abelian group $R_c = L_c$ and $I_c = \varepsilon$. $\square$

Lemma 3.9. *Let $(G, f) = \text{der}(G, \star)$ and $(H, g) = \text{der}(H, \cdot)$ be two fixed n -groups. Mapping $h : G \rightarrow H$ is an isomorphism of these n -groups if and only if there is an isomorphism $\beta : (G, \star) \rightarrow (H, \cdot)$ such that $h = R_{h(e^*)} \beta$, where e^* is the identity of $(G, \star)$, and $h(e^*)^n = h(e^*)$ belongs to the center of $(H, \cdot)$.*

Proof. An isomorphism h can be considered as an isotopism (2) of which all components are equal. Thus $L_{a_1}I_{a_1}\beta(e^*) = L_{a_2}I_{a_1a_2}\beta(e^*)$ implies $a_1 = a_2$. Analogously, $L_{a_2}I_{a_1a_2}\beta(e) = L_{a_3}I_{a_1a_2a_3}\beta(e^*)$ implies $a_2=a_3$, and so on. Hence, $h(x)=\beta(x)a$. Consequently, $h(e^*) = a$. Therefore, $h = R_{h(e^*)}\beta$ and $h(e^*) = hf(e^*, \dots, e^*) = h(e^*)^n$. Since $hf(x, e^*, \dots, e^*) = h(x) = hf(e^*, x, e^*, \dots, e^*)$, $h(e^*)$ must be in the center of $(H, \cdot)$.

The converse statement is obvious. $\square$

Theorem 3.10. *A mapping $h : G \rightarrow H$ is an isomorphism of n -groups $(G, f) = \text{der}_{\varphi, a}(G, *)$ and $(H, g) = \text{der}_{\psi, b}(H, \cdot)$ if and only if it can be expressed in the form $h = R_c\beta$, where $c = c^n$ belongs to the center of $(H, \cdot)$, $\beta : (G, *) \rightarrow (H, \cdot)$ is an isomorphism, $\beta\varphi = \psi\beta$, $\beta(a) = b$ and $\psi(c) = c$.*

Proof. Any isomorphism $h : (G, f) \rightarrow (H, g)$ can be considered as an isotopism $P = (h, h, \dots, h) : (G, f) \rightarrow (H, g)$ such that $P = S \circ A \circ T$, where $T = (\varepsilon, \varphi, \varphi^2, \dots, \varphi^{n-2}, R_a^*\varphi^{n-1}, \varepsilon)$ is an isotopism between (G, f) and $\text{der}(G, *)$, $A = (\theta, \theta, \dots, \theta)$, $\theta : \text{der}(G, *) \rightarrow \text{der}(H, \cdot)$ is an isomorphism of n -groups $\text{der}(G, *)$ and $\text{der}(H, \cdot)$, $S = (\varepsilon, \psi^{-1}, \psi^{-2}, \dots, \psi^{-(n-2)}, R_{b^{-1}}\psi^{-(n-1)}, \varepsilon)$ is an isotopism between $\text{der}(H, \cdot)$ and (H, g) .

$$\begin{array}{ccc} (G, f) & \xrightarrow{T} & \text{der}(G, *) \\ P \downarrow & & \downarrow A \\ (H, g) & \xleftarrow{S} & \text{der}(H, \cdot) \end{array}$$

By Lemma 3.9, $\theta = R_c\beta$, $\beta : (G, *) \rightarrow (H, \cdot)$ is an isomorphism, $c = \theta(e^*)$, where e^* is the neutral element of $(G, *)$. Then

$$P = (R_c\beta, \psi^{-1}R_c\beta\varphi, \dots, \psi^{2-n}R_c\beta\varphi^{n-2}, R_{b^{-1}}\psi^{1-n}R_c\beta R_a^*\varphi^{n-1}, R_c\beta).$$

Since P is an isomorphism, all components of P should be equal. In particular, $R_c\beta = \psi^{-1}R_c\beta\varphi = R_{b^{-1}}\psi^{1-n}R_c\beta R_a^*\varphi^{n-1}$. From the first equality, we obtain $R_c\beta = \psi^{-k}R_c\beta\varphi^k$ for all $1 \leq k < n$. Thus, the first $n-1$ components are equal. From $R_c\beta = \psi^{-1}R_c\beta\varphi$, for $x = e^*$, we also have $c = \psi^{-1}(c)$. Hence, $\psi(c) = c$ and $\beta(x)c = \psi^{-1}\beta\varphi(x)c$. So, $\psi\beta = \beta\varphi$.

The equality $R_c\beta = R_{b^{-1}}\psi^{1-n}R_c\beta R_a^*\varphi^{n-1}$, implies $cb = \psi^{1-n}(\beta(a)c)$. Thus $\psi^{n-1}(bc) = \beta(a)c$, which, by Hosszú Theorem, gives $bc = \beta(a)c$. So, $\beta(a) = b$.

Conversely, let $h = R_c\beta$, where c and β are as in theorem. Then h is a bijection $G \rightarrow H$ and

$$\begin{aligned}
hf(x_1^n) &= R_c\beta(x_1 * \varphi(x_2) * \varphi^2(x_3) * \cdots * \varphi^{n-1}(x_n) * a) \\
&= \beta(x_1) \cdot \beta\varphi(x_2) \cdot \beta\varphi^2(x_3) \cdots \beta\varphi^{n-1}(x_n) \cdot \beta(a) \cdot c \\
&= \beta(x_1) \cdot \psi\beta(x_2) \cdot \psi^2\beta(x_3) \cdots \psi^{n-1}\beta(x_n) \cdot b \cdot c^n \\
&= \beta(x_1)c \cdot \psi\beta(x_2)c \cdot \psi^2\beta(x_3)c \cdots \psi^{n-1}\beta(x_n)c \cdot b \\
&= R_c\beta(x_1) \cdot \psi R_c\beta(x_2) \cdot \psi^2 R_c\beta(x_3) \cdots \psi^{n-1} R_c\beta(x_n) \cdot b \\
&= g(R_c\beta(x_1), R_c\beta(x_2), R_c\beta(x_3), \dots, R_c\beta(x_n)) \\
&= g(h(x_1), h(x_2), h(x_3), \dots, h(x_n)).
\end{aligned}$$

This shows that h is an isomorphism between (G, f) and (H, g) . $\square$

Corollary 3.11. *A mapping $h : G \rightarrow H$ is an isomorphism of n -groups $(G, f) = \text{der}_{\varphi, a}(G, *)$ and $(H, g) = \text{der}_{\psi, b}(H, \cdot)$ if and only if it can be expressed in the form $h = R_c\beta$, where $\beta : (G, *) \rightarrow (H, \cdot)$ is an isomorphism, $\beta\varphi = \psi\beta$ and $c \in Z(H, \cdot)$ is such that $\beta(a) = \psi(c)\psi^2(c) \cdots \psi^{n-1}(c)\alpha$.*

4. Autotopies

If for an n -groupoid (G, f) there are bijections $\alpha_1, \dots, \alpha_n, \delta : G \rightarrow G$ such that

$$\delta f(x_1, x_2, \dots, x_n) = f(\alpha_1(x_1), \alpha_2(x_2), \dots, \alpha_n(x_n)), \quad (5)$$

then $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \delta)$ is called an *autotopy* or a *weak automorphism* of (G, f) . The set of all autotopies of (G, f) is denoted by $\mathcal{A}(G, f)$. It is clear that $\mathcal{A}(G, f)$ is a group with respect to the composition of autotopies:

$$(\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \circ (\beta_1, \beta_2, \dots, \beta_{n+1}) = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_{n+1}\beta_{n+1}).$$

The group $\text{Aut}(G, f)$ of all automorphisms of (G, f) , considered as an $(n+1)$ -tuples $(\varphi, \dots, \varphi)$, is a normal subgroup of the group $\mathcal{A}(G, f)$.

We start with the following theorem proved by Belousov (cf. [1]) and give a short proof of this theorem.

Theorem 4.1. *Groups of autotopies of isotopic n -groups are isomorphic.*

Proof. Let $A = (\alpha_1, \alpha_2, \dots, \alpha_n, \alpha)$ and $B = (\beta_1, \beta_2, \dots, \beta_n, \beta)$ be autotopies of n -groups (G, f) and (H, g) , respectively.

Consider the mapping $\beta : \mathcal{A}(G, f) \rightarrow \mathcal{A}(H, g)$ defined by $\beta(A) = T \circ A \circ T^{-1}$, where $T = (\delta_1, \delta_2, \dots, \delta_n, \delta)$ is an isotopy from (G, f) to (H, g) . Then

$$\begin{aligned}
T^{-1} : \delta^{-1}g(x_1^n) &= f(\delta_1^{-1}(x_1), \dots, \delta_n^{-1}(x_n)) \\
A : \alpha\delta^{-1}g(x_1^n) &= \alpha f(\delta_1^{-1}(x_1), \dots, \delta_n^{-1}(x_n)) = f(\alpha_1\delta_1^{-1}(x_1), \dots, \alpha_n\delta_n^{-1}(x_n)) \\
T : \delta\alpha\delta^{-1}g(x_1^n) &= \delta f(\alpha_1\delta_1^{-1}(x_1), \dots, \alpha_n\delta_n^{-1}(x_n)) \\
&= g(\delta_1\alpha_1\delta_1^{-1}(x_1), \dots, \delta_n\alpha_n\delta_n^{-1}(x_n)).
\end{aligned}$$

Hence, $\beta(A) = (\delta_1\alpha_1\delta_1^{-1}, \dots, \delta_n\alpha_n\delta_n^{-1}, \delta\alpha\delta^{-1}) \in \mathcal{A}(H, g)$.

In a similar way, we can show that $\Phi(B) = T^{-1} \circ B \circ T \in \mathcal{A}(G, f)$. Thus β is one-to-one and onto. Obviously $\beta(A_1 \circ A_2) = \beta(A_1) \circ \beta(A_2)$. So, β is an isomorphism. $\square$

Corollary 4.2. *For any n -group (G, f) and any $a \in G$ the groups $\mathcal{A}(G, f)$ and $\mathcal{A}(\text{der}(\text{ret}_a(G, f)))$ are isomorphic. In particular,*

$$\mathcal{A}(\text{der}_{\varphi, b}(G, \cdot)) \cong \mathcal{A}(\text{der}(G, \cdot)).$$

As a consequence of Theorem 3.7 and Corollary 3.8 we obtain

Corollary 4.3. *Every autotopy T of an n -group $(G, f) = \text{der}(G, \cdot)$ can be uniquely presented in the form (2), where $a_1, \dots, a_n \in G$, $c_i = a_1 a_2 \cdots a_i$, $I_c(x) = c^{-1} x c$ and $\beta \in \text{Aut}(G, \cdot)$.*

Corollary 4.4. *Every autotopy T of a medial n -group $(G, f) = \text{der}(G, \cdot)$ can be uniquely presented in the form (4), where $a_1, \dots, a_n \in G$, $c_n = a_1 a_2 \cdots a_n$ and $\beta \in \text{Aut}(G, \cdot)$.*

Theorem 4.5. *The number of autotopies of an n -group (G, f) of order $|G|$ is equal to $|G|^n \cdot |\text{Aut}(G, \cdot)|$.*

Proof. By Hosszú Theorem, any n -group (G, f) is $\langle \varphi, b \rangle$ -derived from certain group $(G, \cdot)$, i.e. $f(x_1^n) = x_1 \varphi(x_2) \varphi^2(x_2) \cdots \varphi^{n-1}(x_n) b$, where $b \in G$ is fixed and φ is an automorphism of $(G, \cdot)$. This means that (G, f) is isotopic to the n -group $(G, g) = \text{der}(G, \cdot)$. The isotopy has the form $T = (\varepsilon, \varphi, \varphi^2, \dots, \varphi^{n-2}, R_b \varphi^{n-1}, \varepsilon)$. Hence, these two n -groups have the same number of autotopies (Theorem 4.1). i-central We will calculate the number of autotopies of the n -group $(G, g) = \text{der}(G, \cdot)$. According to Corollary 4.3 each autotopy of (G, g) can be written in the form

$$T = T_1 \circ T_2 = (L_{a_1} I_{c_1}, L_{a_2} I_{c_2}, \dots, L_{a_n} I_{c_n}, R_{c_n}) \circ (\beta, \beta, \dots, \beta).$$

Since the autotopy T_1 is uniquely determined by $a_1, a_2, \dots, a_n \in G$, there are exactly $|G|^n$ autotopies. If the autotopy P belongs to the common part of $\{(L_{a_1} I_{c_1}, L_{a_2} I_{c_2}, \dots, L_{a_n} I_{c_n}, R_{c_n}) \mid a_1, \dots, a_n \in G\}$ and $\{(\beta, \dots, \beta) \mid \beta \in \text{Aut}(G, \cdot)\}$, then all its components are equal to some automorphism $\beta \in \text{Aut}(G, \cdot)$. In particular, $L_{a_1} I_{c_1}(x) = \beta(x)$, which for $x = e$ gives $a_1 = e$. Consequently, $\beta = \varepsilon$. This means that all components of the autotopy P are equal. Therefore $|\mathcal{A}(G, g)| = |G|^n \cdot |\text{Aut}(G, \cdot)|$. $\square$

The number of autotopies of a given n -group is very large. For example, for n -groups with p elements (p prime) it is equal to $p^n(p-1)$.

In the case when an n -group (G, f) has p^m elements (p prime) we can determine the number of autotopies using the formula for the number of automorphisms of abelian p -groups (cf. [12]). Using this formula and Theorem 4.5 we obtain

Theorem 4.6. *The number of autotopies of an n -group $\langle \varphi, b \rangle$ -derived from the group $\mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_m}}$ is equal to*

$$p^{n(\alpha_1 + \cdots + \alpha_m)} \prod_{k=1}^m (p^{d_k} - p^{k-1}) \prod_{j=1}^m (p^{\alpha_j})^{m-d_j} \prod_{i=1}^m (p^{\alpha_i-1})^{m+1-c_i},$$

where $d_k = \max\{l : \alpha_l = \alpha_k\}$ and $c_k = \min\{l : \alpha_l = \alpha_k\}$.

For medial n -groups we have a stronger result.

Theorem 4.7. *If an n -group (G, f) is medial, then*

$$\mathcal{A}(G, f) \cong \bigoplus_{i=1}^n (G, \cdot)_i \ltimes \text{Aut}(G, \cdot),$$

where $\bigoplus_{i=1}^n (G, \cdot)_i$ is the direct sum of n copies of the group $(G, \cdot) = \text{ret}_a(G, f)$.

Proof. An n -group (G, f) is $\langle \varphi, b \rangle$ -derived from the group $(G, \cdot) = \text{ret}_a(G, f)$. All retracts of this n -group are isomorphic to $(G, \cdot)$. So, it is sufficient to prove our theorem for $(G, f) = \text{der}(G, \cdot)$ (Theorem 4.1). Since (G, f) is medial, $(G, \cdot)$ is an abelian group. Hence, by Corollary 4.4, any autotopy T of $(G, f) = \text{der}(G, \cdot)$ can be uniquely presented in the form

$$T = T_1 \circ T_2 = (L_{a_1}, L_{a_2}, \dots, L_{a_n}, R_{a_1 \dots a_n}) \circ (\beta, \beta, \dots, \beta),$$

where $a_1, a_2, \dots, a_n \in G$, $\beta \in \text{Aut}(G, \cdot)$.

The set $\mathcal{T}_1 = \{(L_{a_1}, L_{a_2}, \dots, L_{a_n}, R_{a_1 \dots a_n}) \mid a_1, \dots, a_n \in G\}$ forms a subgroup in $\mathcal{A}(G, f)$. Since for any autotopy $T = (L_{b_1} \beta, \dots, L_{b_n} \beta, R_{b_1 \dots b_n} \beta)$ its inverse has the form $T^{-1} = (L_{d_1} \beta^{-1}, \dots, L_{d_n} \beta^{-1}, R_d \beta^{-1})$, where $d_i = \beta^{-1}(b_i^{-1})$ and $d = (b_1 \cdots b_n)^{-1}$. Then for all $T_1 \in \mathcal{T}_1$ and $T \in \mathcal{A}(G, f)$

$$\begin{aligned} T^{-1} \circ T_1 \circ T &= (\dots, L_{d_i} \beta^{-1} L_{a_i} L_{b_i} \beta, \dots, R_d \beta^{-1} R_{a_1 \dots a_n} R_{b_1 \dots b_n} \beta) \\ &= (\dots, L_{\beta^{-1}(a_i)}, \dots, R_{b_1 \dots b_n a_1 \dots a_n (b_1 \dots b_n)^{-1}}) \in \mathcal{T}_1. \end{aligned}$$

Thus $\mathcal{T}_1$ is a normal subgroup of $\mathcal{A}(G, f)$.

Clearly, the autotopy $T_1 = (L_{a_1}, L_{a_2}, \dots, L_{a_n}, R_{a_1 \dots a_n})$ is uniquely determined by $a_1, a_2, \dots, a_n \in G$. So, $\mathcal{T}_1$ can be identified with the direct sum of n copies of the group $(G, \cdot)$.

Elements of the set $\mathcal{T}_2 = \{(\beta, \beta, \dots, \beta) \mid \beta \in \text{Aut}(G, f)\}$ can be identified with β . Consequently, $\mathcal{T}_2$ can be identified with $\text{Aut}(G, \cdot)$.

Since $\mathcal{T}_1 \cap \mathcal{T}_2 = \{(\varepsilon, \varepsilon, \dots, \varepsilon)\}$, $\mathcal{A}(G, f)$ is isomorphic to the semidirect product of $\bigoplus_{i=1}^n (G, \cdot)_i$ and $\text{Aut}(G, \cdot)$. $\square$

5. Automorphisms

From the results obtained in the previous sections we deduce the following characterizations of automorphisms of n -groups.

From Proposition 3.6 we can deduce

Corollary 5.1. *If $(G, \star)$ is a group with a neutral elements $e^\star$, then a mapping $h : G \rightarrow G$ such that $h(e^\star) = e^\star$ is an automorphism of an n -group $(G, f) = \text{der}_{\varphi, a}(G, \star)$ if and only if $h \in \text{Aut}(G, \star)$, $h(a) = a$ and $h\varphi = \varphi h$.*

Lemma 3.9 gives

Corollary 5.2. *A mapping $h : G \rightarrow G$ is an automorphism of an n -group $(G, f) = \text{der}(G, \cdot)$ if and only if there is an automorphism of the group $(G, \cdot)$ such that $h = R_{h(e)}\varphi$ and $h(e)^n = h(e)$, where e is the identity of $(G, \cdot)$, belongs to the center of $(G, \cdot)$.*

As a consequence of Theorem 3.10 and Corollary 3.11 we obtain

Proposition 5.3. *A mapping $h : G \rightarrow G$ is an automorphism of an n -group $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ if and only if $h = R_c\beta$, where β is an automorphism of $(G, \cdot)$ such that $\beta\varphi = \varphi\beta$*

- (1) $\varphi(c) = c = c^n$, $c \in Z(G, \cdot)$ and $\beta(a) = a$, or equivalently
- (2) $c \in Z(G, \cdot)$ and $\beta(a) = \varphi(c)\varphi^2(c) \cdots \varphi^{n-1}(c)a$.

Corollary 5.4. *Let $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$, $c \in G$ and $\beta \in \text{Aut}(G, \cdot)$ be fixed. Then $R_c\beta \in \text{Aut}(G, f)$ if and only if $R_c \in \text{Aut}(G, f)$ and $\beta \in \text{Aut}(G, f)$.*

Proof. Indeed, if $h = R_c\beta$ is an automorphism of $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$, then $\beta\varphi = \varphi\beta$ and $\beta(a) = a$. Therefore,

$$\begin{aligned} \beta f(x_1^n) &= \beta(x_1\varphi(x_2)\varphi^2(x_3) \cdots \varphi^{n-1}(x_n)a) \\ &= \beta(x_1)\varphi\beta(x_2)\varphi^2\beta(x_3) \cdots \varphi^{n-1}\beta(x_n)a \\ &= f(\beta(x_1), \beta(x_2), \dots, \beta(x_n)). \end{aligned}$$

Hence $\beta \in \text{Aut}(G, f)$. Consequently, also $R_c = h\beta^{-1} \in \text{Aut}(G, f)$.

The converse statement is obvious. □

Theorem 5.5. *If $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$, then*

$$\text{Aut}(G, f) \cong \mathcal{R}_\varphi(G, f) \ltimes \text{Aut}_{\varphi, a}(G, \cdot)$$

where

$$\begin{aligned} \mathcal{R}_\varphi(G, f) &= \{R_c \mid \varphi(c) = c = c^n, \ c \in Z(G, \cdot)\} \text{ and} \\ \text{Aut}_{\varphi, a}(G, \cdot) &= \{\beta \in \text{Aut}(G, \cdot) \mid \beta\varphi = \varphi\beta, \ \beta(a) = a\}. \end{aligned}$$

Proof. $\mathcal{R}_\varphi$ and $\text{Aut}_{\varphi, a}(G, \cdot)$ are subgroups of $\text{Aut}(G, f)$ and $\text{Aut}(G, \cdot)$, respectively.

By Proposition 5.3, for every $h = R_d\delta \in \text{Aut}(G, f)$ and every $R_c \in \mathcal{R}_\varphi$ we have $h^{-1}R_ch = \delta^{-1}R_{d^{-1}}R_cR_d\delta = R_{\delta^{-1}(c)} \in \mathcal{R}_\varphi$. Hence $\mathcal{R}_\varphi$ is a normal subgroup of $\text{Aut}(G, f)$. Moreover, if $\psi \in \mathcal{R}_\varphi(G, f) \cap \text{Aut}_{\varphi, a}(G, \cdot)$, then $\psi = R_c = \beta$. Thus, $R_c(a) = \beta(a) = a$, which implies $c = e$. Therefore, $\mathcal{R}_\varphi(G, f) \cap \text{Aut}_{\varphi, a}(G, \cdot) = \{\varepsilon\}$. Therefore $\text{Aut}(G, f) \cong \mathcal{R}_\varphi(G, f) \ltimes \text{Aut}_{\varphi, a}(G, \cdot)$. □

An idempotent of an n -group $(G, f) = \text{der}_{\varphi,a}(G, \cdot)$ is called *quasi-central* if it belongs to the center of the group $(G, \cdot)$. Quasi-central idempotents exist in certain n -groups. For example, in ternary group (S_3, f) with the operation $f(x, y, z) = xyza$, where $a \in S_3$ is a fixed element of order 3, there are no such idempotents. The set of all quasi-central idempotents of an n -group is denoted by $CE_{(G,f)}$.

Proposition 5.6. *Quasi-central elements of an n -group $(G, g) = \text{der}_{\varphi}(G, \cdot)$ form a normal subgroup of $(G, \cdot)$ and a commutative n -subgroup of (G, g) .*

Proof. Let $(G, g) = \text{der}_{\varphi}(G, \cdot)$. Then the neutral element e of the group $(G, \cdot)$ is a quasi-central idempotent of (G, g) . Thus, the set $CE_{(G,g)}$ is nonempty.

Furthermore, for any $a \in CE_{(G,g)}$ we have $a = a\varphi(a)\varphi^2(a) \cdots \varphi^{n-1}(a)$ which implies $\varphi(a)\varphi^2(a) \cdots \varphi^{n-1}(a) = e = \varphi(a^{-1})\varphi^2(a^{-1}) \cdots \varphi^{n-1}(a^{-1})$. Thus $a \in CE_{(G,g)}$ if and only if $a^{-1} \in CE_{(G,g)}$. Also $ac \in CE_{(G,g)}$ for $a, c \in CE_{(G,g)}$. Hence $CE_{(G,g)}$ is a normal subgroup of $(G, \cdot)$.

Consequently, for $a_1, \dots, a_n \in CE_{(G,g)}$ we have

$$\begin{aligned} g(a_1^n) &= a_1\varphi(a_2)\varphi^2(a_3) \cdots \varphi^{n-1}(a_n) \\ &= a_1\varphi(a_1)\varphi^2(a_1) \cdots \varphi^{n-1}(a_1) \cdot \varphi\left(a_2\varphi(a_2)\varphi^2(a_2) \cdots \varphi^{n-1}(a_2)\right) \\ &\quad \cdot \varphi^2\left(a_3\varphi(a_3)\varphi^2(a_3) \cdots \varphi^{n-1}(a_3)\right) \cdots \varphi^{n-1}\left(a_n\varphi(a_n)\varphi^2(a_n) \cdots \varphi^{n-1}(a_n)\right) \\ &= a_1\varphi(a_2)\varphi^2(a_3) \cdots \varphi^{n-1}(a_n) \cdot \varphi\left(a_1\varphi(a_2)\varphi^2(a_3) \cdots \varphi^{n-1}(a_n)\right) \\ &\quad \cdot \varphi^2\left(a_1\varphi(a_2)\varphi^2(a_3) \cdots \varphi^{n-1}(a_n)\right) \cdots \varphi^{n-1}\left(a_1\varphi(a_2)\varphi^2(a_3) \cdots \varphi^{n-1}(a_n)\right) \\ &= g(g(a_1^n), g(a_1^n), \dots, g(a_1^n)). \end{aligned}$$

This shows that $(CE_{(G,g)}, g)$ is an idempotent n -semigroup contained in (G, g) . Since in (G, g) we have $g(x, a, \dots, a) = g(a, \dots, a, x) = x$ for all $a, x \in CE_{(G,g)}$, $(CE_{(G,g)}, g)$ is an n -group (Theorem 2.1) contained in (G, g) . $\square$

Theorem 5.7. *If an n -group $(G, f) = \text{der}_{\varphi,a}(G, \cdot)$ has at least one quasi-central idempotent, then*

$$\text{Aut}(G, g) \cong \mathcal{R}_{E(G,g)} \ltimes Z(\varphi)$$

where $\mathcal{R}_{E(G,f)} = \{R_c \mid c \in CE_{(G,f)}\}$ and $Z(\varphi) = \{\beta \in \text{Aut}(G, \cdot) \mid \varphi\beta = \beta\varphi\}$.

Proof. Let $(G, f) = \text{der}_{\varphi,a}(G, \cdot)$. We will show first that (G, f) is isomorphic to $(G, g) = \text{der}_{\varphi}(G, \cdot)$.

Let $c \in CE_{(G,f)}$. Then $c = f(c, c, \dots, c) = c\varphi(c)\varphi^2(c) \cdots \varphi^{n-1}(c)a$. Thus, $ac^{-1} = c^{-1}\varphi(c^{-1})\varphi^2(c^{-1}) \cdots \varphi^{n-1}(c^{-1})$. Hence

$$\begin{aligned} R_{c^{-1}}f(x_1^n) &= x_1\varphi(x_2)\varphi^2(x_3) \cdots \varphi^{n-1}(x_n)ac^{-1} \\ &= x_1\varphi(x_2)\varphi^2(x_3) \cdots \varphi^{n-1}(x_n)c^{-1}\varphi(c^{-1})\varphi^2(c^{-1}) \cdots \varphi^{n-1}(c^{-1}) \\ &= x_1c^{-1}\varphi(x_2)\varphi(c^{-1})\varphi^2(x_3)\varphi^2(c^{-1}) \cdots \varphi^{n-1}(x_n)\varphi^{n-1}(c^{-1}) \\ &= x_1c^{-1}\varphi(x_2c^{-1})\varphi^2(x_3c^{-1}) \cdots \varphi^{n-1}(x_nc^{-1}) \\ &= R_{c^{-1}}(x_1)\varphi R_{c^{-1}}(x_2)\varphi^2 R_{c^{-1}} \cdots \varphi^{n-1} R_{c^{-1}}(x_n) \\ &= g(R_{c^{-1}}(x_1), R_{c^{-1}}(x_2), \dots, R_{c^{-1}}(x_n)). \end{aligned}$$

Therefore $R_{c^{-1}} : (G, f) \rightarrow (G, g)$ is a homomorphism. Since it is one-to-one and onto, $(G, f) \cong (G, g)$. Then also $\text{Aut}(G, f) \cong \text{Aut}(G, g)$ and $\mathcal{R}_{E(G, f)} \cong \mathcal{R}_{E(G, g)}$. So it is sufficient to prove our theorem for (G, g) .

The neutral element of $(G, \cdot)$ is a quasi-central idempotent of (G, g) . Thus the set $\mathcal{R}_{E(G, g)}$ is non-empty. Because $R_d R_c = R_{cd}$ and $cd \in CE_{(G, g)}$ for all $c, d \in CE_{(G, g)}$, $\mathcal{R}_{E(G, g)}$ is a subgroup of $\text{Aut}(G, g)$. Moreover, $(R_b \beta)^{-1} \circ R_c \circ R_b \beta = R_{\beta^{-1}(c)}$ for any $R_b \beta \in \text{Aut}(G, g)$ and $c \in CE_{(G, g)}$.

Since, by Corollary 5.4, β is an automorphism of (G, g) , we have $\beta^{-1}(c) = \beta^{-1}g(c, c, \dots, c) = g(\beta^{-1}(c), \beta^{-1}(c), \dots, \beta^{-1}(c))$. Thus $\beta^{-1}(c) \in CE_{(G, g)}$. Consequently, $R_{\beta^{-1}(c)} \in \mathcal{R}_{E(G, g)}$. Therefore, $\mathcal{R}_{E(G, g)}$ is a normal subgroup of $\text{Aut}(G, g)$.

Clearly $Z(\varphi)$ also is a subgroup of $\text{Aut}(G, g)$ and $\mathcal{R}_{E(G, g)} \cap Z(\varphi) = \{\varepsilon\}$. This means that $\text{Aut}(G, g) \cong \mathcal{R}_{E(G, g)} \ltimes Z(\varphi)$. $\square$

Corollary 5.8. *An n -group $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ containing at least one quasi-central idempotent has $|CE_{(G, f)}| \cdot |Z(\varphi)|$ automorphisms.*

Corollary 5.9. *If an n -group $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ has only one quasi-central idempotent, then*

$$\text{Aut}(G, f) \cong Z(\varphi).$$

Corollary 5.10. *If $m|n$, then $\text{Aut}(\mathbb{Z}_m, f) \cong \text{Aut}(\mathbb{Z}_m, +)$ for an n -group $(G, f) = \text{der}_{\varphi}(\mathbb{Z}_m, +)$.*

Corollary 5.11. *If a medial n -group $(G, f) = \text{der}_a(G, \cdot)$ has only one idempotent, then*

$$\text{Aut}(G, f) \cong \text{Aut}(G, \cdot).$$

Corollary 5.12. *A medial n -group $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ containing at least one idempotent has only the identity automorphism if and only if $(G, \cdot)$ is the direct product of cyclic groups of order 2.*

Proof. In abelian group $\alpha(x) = x^{-1}$ is an automorphism commuting with all other automorphisms. If $|\text{Aut}(G, f)| = 1$, then also $|Z(\varphi)| = 1$. Therefore must be $\alpha = \varepsilon$. Consequently, $x = \alpha(x) = x^{-1}$ for all $x \in G$. Hence $(G, \cdot)$, as a Boolean group, is the direct product of copies of the group $\mathbb{Z}_2$. $\square$

To calculate the number of automorphisms of a given n -group (G, f) let's assume that $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ and $\delta(x) = \varphi(x)\varphi^2(x) \cdots \varphi^{n-1}(x)$. Then δ is an endomorphism of $(G, \cdot)$. Moreover let $K = \text{Ker } \delta$, $P = \{\beta(a)a^{-1} \mid \beta \in Z(\varphi)\} \cap \delta(G)$, $D = \{(c, \beta) \in Z(G, \cdot) \times Z(\varphi) \mid \beta(a) = a\delta(c)\}$ and $S_a = \{\beta \in Z(\varphi) \mid \beta(a) = a\}$.

By Proposition 5.3, the mapping $h : G \rightarrow G$ is an automorphism of an n -group $(G, f) = \text{der}_{\varphi, a}(G, \cdot)$ if and only if there are $c \in Z(G, \cdot)$ and $\beta \in Z(\varphi)$ such that $h = R_c \beta$ and $\beta(a) = \delta(c)a$.

Thus, the enumeration of all automorphisms of an n -group $(G, f) = \text{der}_{\varphi,a}(G, \cdot)$ can be reduced to the enumeration of all pairs $(c, \beta) \in Z(G, \cdot) \times Z(\varphi)$ such that $\beta(a) = \delta(c)a$. Denote the set of such pairs by D .

Note that $(c, \beta) \in D$ implies $(cd, \beta) \in D$ for every $d \in K = \text{Ker } \delta$. On the other hand, from $(c, \beta) = (b, \beta)$ it follows $cb^{-1} \in K$. Thus each automorphism $\beta \in Z(\varphi)$ determines exactly $|K|$ pairs $(c, \beta) \in D$, i.e. exactly $|K|$ elements in G . Thus $|D| = |K| \cdot t$, where t is the number of possible automorphisms $\beta \in Z(\varphi)$ for which there is a pair $(c, \beta) \in D$.

To determine the number of β corresponding to the fixed $c \in P$, note first that for each $c \in P$ there is at least one $g_c \in G$ such that $c = \beta(a)a^{-1} = \delta(g_c)$. If also $\delta(g) = c$, then $\delta(gg_c^{-1}) = e$, i.e. $gg_c^{-1} \in K = \text{Ker } \delta$. Therefore, $g = g_cd$ for some $d \in K$. This means that each element $c \in P$ corresponds to exactly $|K|$ elements $g_c \in G$ such that $\delta(g_c) = c$. This element c corresponds to $|S_a|$ elements β_c of the group $Z(\varphi)$ such that $\beta_c(a)a^{-1} = c$. If $c, d \in P$, $c \neq d$, then $\delta(g_c) \neq \delta(g_d)$ and $\beta_c \neq \beta_d$. Therefore $|\text{Aut}(Q, f)| = |K| \cdot |P| \cdot |S_a|$.

In this way we obtain

Theorem 5.13. *If $(G, f) = \text{der}_{\varphi,a}(G, \cdot)$, then $|\text{Aut}(Q, f)| = |P| \cdot |K| \cdot |S_a|$.*

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