Semi-smooth Points in Some Classical Function Spaces

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Abstract. The investigations of the smooth points in the spaces of continuous function were started by Banach in 1932 considering function space $\mathcal{C}(\Omega)$. Singer and Sundaresan extended the result of Banach to the space of vector valued continuous functions $\mathcal{C}(\mathcal{T}, E)$, where $\mathcal{T}$ is a compact metric space. The aim of this paper is to present a description of semi-smooth points in spaces of continuous functions $\mathcal{C}_0(\mathcal{T}, E)$ (instead of smooth points). Moreover, we also find necessary and sufficient condition for semi-smoothness in the general case.

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1. Introduction

Smoothness in Banach space has been intensively studied for decades from many different points of view. We refer to the monographs [2, 6] and the papers [10, 11] for motivations, history, various aspects and problems connected with the subject.

In this paper, motivated by the results published by Banach [2], Singer [10], Sundaresan [11] and Milićić [8, 9], we study the notion of semi-smooth points in Banach spaces of continuous functions.

It is necessary to mention a classical result: if Ω is a compact Hausdorff space, then

$$\begin{aligned} \text{smooth } (\mathcal{C}(\Omega)) &= \{f \in \mathcal{C}(\Omega) : \text{there exists } t_o \text{ such that } \|f\| \\ &= |f(t_o)| > |f(t)| \text{ for } t \neq t_o\} \end{aligned}$$

(cf. Banach classical monograph [2, p. 168]). Singer [10] characterized the smooth points of the unit ball of $\mathcal{C}([0, 1], E)$, the space of continuous functions with values in the Banach space E . This was extended by Sundaresan [11] to the space of vector valued continuous functions $\mathcal{C}(\mathcal{T}, E)$, where $\mathcal{T}$ is a compact metric space.

The main object of this paper is to study the semi-smooth points of the unit ball of $\mathcal{C}_0(\mathcal{T}, E)$, when $\mathcal{T}$ is a locally compact topological space. Moreover, in this paper, we find necessary and sufficient condition for semi-smoothness.

It is worth mentioning that in some cases the notion of semi-smoothness may be more convenient than the notion of smoothness. Indeed, recently the concept of semi-smoothness played a significant role in the paper [12], where the notion of semi-smooth points was helpful to solve the open problem posed in 2010.

2. Preliminaries

The paper is organized as follows. In Sect. 3 of the paper, we discuss semi-smoothness in the space $\mathcal{C}_0(\mathcal{T}, E)$. In the Sect. 4 we study semi-smooth points and relate them to the set of supporting functionals.

Before stating the results, we recall some definitions and introduce our notation.

2.1. Norm Derivatives

Let $(X, \|\cdot\|)$ be a real normed space. We define *norm derivatives*:

$$\rho'_\pm(x, y) := \lim_{\lambda \rightarrow 0^\pm} \frac{\|x + \lambda y\|^2 - \|x\|^2}{2\lambda} = \|x\| \cdot \lim_{\lambda \rightarrow 0^\pm} \frac{\|x + \lambda y\| - \|x\|}{\lambda}, \quad x, y \in X.$$

Convexity of the norm yields that the above definition is meaningful. The mappings ρ'_+ and ρ'_- are sometimes called the *superior* and *inferior semi inner products* and their following properties can be found, e.g., in [1, 7]:

- (nd1) $\forall x, y \in X \quad \forall \alpha \in \mathbb{R} \quad \rho'_\pm(x, \alpha x + y) = \alpha \|x\|^2 + \rho'_\pm(x, y);$
- (nd2) $\forall x, y \in X \quad \forall \alpha \geq 0 \quad \rho'_\pm(\alpha x, y) = \alpha \rho'_\pm(x, y) = \rho'_\pm(x, \alpha y);$
- (nd3) $\forall x, y \in X \quad \forall \alpha < 0 \quad \rho'_\pm(\alpha x, y) = \alpha \rho'_\mp(x, y) = \rho'_\pm(x, \alpha y);$
- (nd4) $\forall x \in X \quad \rho'_\pm(x, x) = \|x\|^2;$
- (nd5) $\forall x, y \in X \quad |\rho'_\pm(x, y)| \leq \|x\| \cdot \|y\|;$
- (nd6) $\forall x, y \in X \quad \rho'_-(x, y) \leq \rho'_+(x, y).$

Moreover, mappings ρ'_+, ρ'_- are continuous with respect to the second variable, but not necessarily with respect to the first one. Note that if $(X, \langle \cdot | \cdot \rangle)$ is an inner product space, then $\langle y | x \rangle = \rho'_+(x, y) = \rho'_-(x, y)$ for arbitrary $x, y \in X$.

Miličić introduced (see [8]) the following concept. Let us define $\rho' : X \times X \rightarrow \mathbb{R}$ by

$$\rho'(x, y) := \frac{1}{2} (\rho'_-(x, y) + \rho'_+(x, y)), \quad x, y \in X.$$

The above functional ρ' is also denoted by $\langle y|x \rangle_g$ and called an *M-semi inner product*—cf. Miličić [9] and also [7]. From the above properties of the mappings $\rho'_\pm$ we get:

$$(\text{Msip1}) \quad \forall_{x,y \in X} \quad \forall_{\alpha \in \mathbb{R}} \quad \rho'(x, \alpha x + y) = \alpha \|x\|^2 + \rho'(x, y);$$

$$(\text{Msip2}) \quad \forall_{x,y \in X} \quad \forall_{\alpha \in \mathbb{R}} \quad \rho'(\alpha x, y) = \alpha \rho'(x, y) = \rho'(x, \alpha y);$$

$$(\text{Msip3}) \quad \forall_{x \in X} \quad \rho'(x, x) = \|x\|^2;$$

$$(\text{Msip4}) \quad \forall_{x,y \in X} \quad |\rho'(x, y)| \leq \|x\| \cdot \|y\|.$$

As a consequence of (nd6) we have

$$\rho'_-(x, y) \leq \rho'(x, y) \leq \rho'_+(x, y), \quad x, y \in X. \quad (2.1)$$

Moreover, ρ' is continuous with respect to the second variable.

It is worth mentioning that *M-semi inner product* ρ' was recently very helpful to extended results concerning Birkhoff-James orthogonally additive mappings; in particular, see [3, Theorem 3.3].

The unit sphere of X is denoted by $S(X)$. Fix $x \in X \setminus \{0\}$. We consider the set $J(x)$ defined as follows

$$J(x) := \{x^* \in X^* : \|x^*\| = 1, \|x^*(x)\| = \|x\|\}.$$

It is easy to check that the set $J(x)$ is convex, closed and $J(x) \subseteq S(X^*)$. By the Hahn-Banach theorem we get $J(x) \neq \emptyset$ for all $x \in X \setminus \{0\}$. Moreover, it is easy to see that $J(\alpha x) = J(x)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$, $x \in X \setminus \{0\}$.

Let us recall the following result containing a representation of the norm derivatives $\rho'_\pm$ in terms of supporting functionals.

Theorem 2.1. ([7, p. 36]). *Let X be a real normed space. Then one has the representation:*

$$\begin{aligned} \rho'_-(x, y) &= \|x\| \cdot \inf \{x^*(y) : x^* \in J(x)\} \text{ and} \\ \rho'_+(x, y) &= \|x\| \cdot \sup \{x^*(y) : x^* \in J(x)\} \text{ for all } x, y \in X. \end{aligned}$$

So, in particular,

$$\forall_{x^* \in J(x)} \quad \rho'_-(x, y) \leq \|x\| \cdot x^*(y) \leq \rho'_+(x, y). \quad (2.2)$$

2.2. Smoothness and Semi-smoothness

From now on we assume that the considered normed spaces are real and their dimensions are not less than 2. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$. We say that X is *smooth at point* x_o if $J(x)$ contains a unique element in X^* . Now, we consider a set

$$\mathcal{N}_{sm}(X) := \{x \in X : X \text{ is smooth at } x\} \cup \{0\}.$$

It will be helpful to reformulate the definition of smoothness as follows:

$$x \in \mathcal{N}_{sm}(X) \Leftrightarrow \text{card}J(x) = 1. \quad (2.3)$$

If X is a separable real Banach space, then $\mathcal{N}_{sm}(X)$ is dense. We say that a normed space X is *smooth* if $X = \mathcal{N}_{sm}(X)$.

Now we will give a characterization of smoothness at a point in terms of the norm derivatives (see [1, 7]).

Theorem 2.2. ([1, 7]). *Let X be a real normed space and let $x_o \in X \setminus \{0\}$. Then the following statements are equivalent:*

- (a) X is smooth at x_o , i.e., $x_o \in \mathcal{N}_{sm}(X)$;
- (b) $\rho'_-(x_o, \cdot)$ is linear;
- (c) $\rho'_+(x_o, \cdot)$ is linear;
- (d) $\rho'_-(x_o, \cdot) \in X^*$;
- (e) $\rho'_+(x_o, \cdot) \in X^*$;
- (f) $\forall_{x,y \in X} \quad \rho'_-(x_o, y) = \rho'_+(x_o, y)$.

It is clear that if X is smooth, then the M -semi-inner product in the sense of Miličić $\rho'(\diamond, \cdot)$ is linear in the second argument. However, there also exists non-smooth spaces from which the mapping $\rho'(\diamond, \cdot)$ is linear in the second variable too ([8, Example 8.1], [7, p. 51]).

A real normed space X is called *semi-smooth* if ρ' is additive (or, equivalently, linear; see (Msip2)) with respect to the second variable, i.e.,

$$\rho'(x, y + z) = \rho'(x, y) + \rho'(x, z), \quad x, y, z \in X.$$

Each smooth space is semi-smooth in the above sense but not conversely (l^1 is a suitable example; see [8, Example 8.1] and [7, p. 51]).

The notion of semi-smooth spaces motivates this paper. We say that a normed space $(X, \|\cdot\|)$ is *semi-smooth at the point* $x_o \in X$ if it satisfies

$$\forall_{y,z \in X} \quad \rho'(x_o, y + z) = \rho'(x_o, y) + \rho'(x_o, z),$$

or, equivalently, if the mapping $\rho'(x_o, \cdot): X \rightarrow \mathbb{R}$ is linear. As was mentioned, semi-smooth points played a crucial role in [12].

Similarly as above, we define

$$\mathcal{N}_{sm}^s(X) := \{x \in X : X \text{ is semi-smooth at } x\}.$$

Thus we can write $X = \mathcal{N}_{sm}^s(X)$ if and only if X is semi-smooth. We observe that

$$\mathcal{N}_{sm}(X) \subseteq \mathcal{N}_{sm}^s(X). \quad (2.4)$$

Indeed, let $x \in \mathcal{N}_{sm}(X)$. Then for all $y \in X$ we have $\rho'_-(x, y) = \rho'_+(x, y)$. Applying (2.1) and (2.2) we get $\rho'(x, \cdot) = \rho'_-(x, \cdot) = \rho'_+(x, \cdot) = \|x\| \cdot x^*(\cdot)$ for all $x^* \in J(x)$, hence $\rho'(x, \cdot)$ is linear, and therefore $x \in \mathcal{N}_{sm}^s(X)$. As we know already from [8, Example 8.1], [7, p. 51], we have $\mathcal{N}_{sm}(X)(l^1) \subsetneq \mathcal{N}_{sm}^s(X)(l^1)$.

We summarize our observations in the following simple result.

Theorem 2.3. *Let X be a real normed space and let $x_o \in X \setminus \{0\}$. Then the following statements are equivalent:*

- (a) X is semi-smooth at x_o , i.e., $x_o \in \mathcal{N}_{sm}^s(X)$;
- (b) $\rho'(x_o, \cdot)$ is additive;
- (c) $\rho'(x_o, \cdot)$ is linear;
- (d) $\rho'(x_o, \cdot) \in X^*$;
- (e) X is semi-smooth at αx_o for all $\alpha \in \mathbb{R} \setminus \{0\}$.

In the papers [4, 5], ρ and $\rho_{\pm}$ -orthogonalities were considered as well as other characterizations of semi-smooth spaces in terms of approximately $\rho, \rho_{\pm}$ -orthogonalities.

3. Semi-smoothness in Spaces of Continuous Functions

In this section we discuss the notion of semi-smoothness in continuous function spaces.

If $\mathcal{T}$ is a locally compact Hausdorff space and E a real normed linear space, we let $\mathcal{C}_0(\mathcal{T}, E)$ denote the space of all continuous functions f from $\mathcal{T}$ to E which vanish at infinity (namely, a function f *vanishes at infinity* if for every $\varepsilon > 0$ there is a compact set $K \subseteq \mathcal{T}$ such that $\|f(t)\|_E < \varepsilon$ if $t \in \mathcal{T} \setminus K$) with $\|f\|_{\infty} := \sup\{\|f(t)\|_E : t \in \mathcal{T}\}$. In this section we want to focus our attention on $\mathcal{C}_0(\mathcal{T}, E)$. We denote $\mathcal{M}(f) := \{t \in \mathcal{T} : \|f(t)\|_E = \|f\|_{\infty}\}$.

Lemma 3.1. *For any $f \in \mathcal{C}_0(\mathcal{T}, E)$ the set $\mathcal{M}(f)$ is not empty and compact.*

Proof. Take $f \neq 0$ and consider $\varepsilon = \frac{\|f\|_{\infty}}{2}$. Take a compact set $K \subseteq \mathcal{T}$ such that $\|f(t)\|_E < \varepsilon$ if $t \in \mathcal{T} \setminus K$. We know that the mapping $K \ni t \mapsto \|f(t)\|_E \in [0, \|f\|_{\infty}] \subseteq \mathbb{R}$ is continuous and K is compact. Therefore this mapping attains its maximum, i.e., there is $t_0 \in K$ such that $\|f\|_{\infty} = \|f(t_0)\|_E$. Thus the set $\mathcal{M}(f)$ is not empty. Moreover, it is easy to check that the set $\mathcal{M}(f)$ is closed. Therefore it is compact. $\square$

The next theorem will be useful for the study of semi-smooth points. The same symbol, ρ'_+ , will be used to denote the norm derivatives on E and on $\mathcal{C}_0(\mathcal{T}, E)$.

Theorem 3.2. ([13, Theorem 20.8, p. 462]). *Let $\mathcal{T}$ be a locally compact Hausdorff. Let E be a real normed space. Assume that $f, g \in \mathcal{C}_0(\mathcal{T}, E) \setminus \{0\}$. Then*

$$\begin{aligned} \rho'_+(f, g) &= \max \{ \rho'_+(f(t), g(t)) : t \in \mathcal{M}(f) \}, \\ \rho'_-(f, g) &= \min \{ \rho'_-(f(t), g(t)) : t \in \mathcal{M}(f) \}. \end{aligned} \quad (3.1)$$

Now we prove the first main result of this section.

Theorem 3.3. *Suppose that $\mathcal{T}$ is a locally compact space. Let E be a real normed space. Assume that $f \in \mathcal{C}_0(\mathcal{T}, E)$, $f \neq 0$ and let $\text{card} \mathcal{M}(f) = 1$, i.e., $\mathcal{M}(f) = \{t_1\}$. The following statements are equivalent:*

- (A) f is a semi-smooth point, i.e., $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$,
- (B) $f(t_1)$ is a semi-smooth point, i.e., $f(t_1) \in \mathcal{N}_{sm}^s(E)$.

Proof. By (3.1) we have

$$\begin{aligned}\rho'(f, g) &= \frac{1}{2} (\rho'_-(f, g) + \rho'_+(f, g)) = \\ &= \frac{1}{2} (\rho'_-(f(t_1), g(t_1)) + \rho'_+(f(t_1), g(t_1))) = \\ &= \rho'(f(t_1), g(t_1)).\end{aligned}$$

Therefore

$$\forall_{g \in \mathcal{C}_0(\mathcal{T}, E)} \quad \rho'(f, g) = \rho'(f(t_1), g(t_1)). \quad (3.2)$$

We prove (A) $\Rightarrow$ (B). Suppose $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$. By Theorem 2.3, we may assume that $\|f\|_\infty = 1$, and then $\|f(t_1)\|_E = 1$. Fix x, y in $E \setminus \{0\}$. The functions defined by $g(t) := \|f(t)\|_E \cdot x$, $h(t) := \|f(t)\|_E \cdot y$ are continuous and $g, h \in \mathcal{C}_0(\mathcal{T}, E)$. Then, using (3.2) again we deduce

$$\begin{aligned}\rho'(f(t_1), x+y) &= \rho'(f(t_1), g(t_1)+h(t_1)) \stackrel{(3.2)}{=} \rho'(f, g+h) \stackrel{(A)}{=} \\ &= \rho'(f, g) + \rho'(f, h) \stackrel{(3.2)}{=} \rho'(f(t_1), g(t_1)) + \rho'(f(t_1), h(t_1)) = \\ &= \rho'(f(t_1), y) + \rho'(f(t_1), y),\end{aligned}$$

which means that $f(t_1) \in \mathcal{N}_{sm}^s(E)$. On the other hand, from (3.2) it is very easy to prove that (B) $\Rightarrow$ (A). $\square$

A topological space is called *perfectly normal* (or T_6) if it is normal and every closed subset is a G_δ subset. The next result is from topology. It may be known to the reader, but it is presented here for the convenience.

Theorem 3.4. (Vedenisov). *In a perfectly normal (i.e., T_6) topological space (X, τ) there is for each pair F, H of disjoint closed sets a continuous function $f: X \rightarrow [0, 1]$ that is 0 on F and 1 on H and $0 < f(t) < 1$ for $t \in X \setminus (F \cup H)$.*

Of more immediate interest to us are the facts that metric spaces are perfectly normal spaces. Now we prove the second main result of this section.

Theorem 3.5. *Suppose that $\mathcal{T}$ is a locally compact perfectly normal space. Let E be a real normed space. Assume that $f \in \mathcal{C}_0(\mathcal{T}, E)$, $f \neq 0$ and let $\text{card} \mathcal{M}(f) \geq 3$. Then f is not a semi-smooth point, i.e., $f \notin \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$.*

Proof. There exist $t_1, t_2, t_3 \in \mathcal{M}(f)$ such that $t_1 \neq t_2 \neq t_3 \neq t_1$. Using the Vedenisov Extension Theorem (Theorem 3.4) twice, we are able to find a function $\gamma \in \mathcal{C}(\mathcal{T}, \mathbb{R})$ such that $\gamma(t_1) = 1$, $\gamma(t_2) = -1$, $\gamma(t_3) = 0$, and $\gamma(t_2) < \gamma(t) < \gamma(t_1)$ for all $t \in \mathcal{T} \setminus \{t_1, t_2\}$.

Using again the Vedenisov Extension Theorem, we are able to find a function $\eta \in \mathcal{C}(\mathcal{T}, \mathbb{R})$ such that $\eta(t_1) = 0$, $\eta(t_2) = -1$, $\eta(t_3) = 1$, and $\eta(t_2) < \eta(t) < \eta(t_3)$ for all $t \in \mathcal{T} \setminus \{t_2, t_3\}$.

Let us define $g, h: \mathcal{T} \rightarrow E$ by $g := \gamma \cdot f$, $h := \eta \cdot f$. Since $\gamma, \eta \in \mathcal{C}_0(\mathcal{T}, \mathbb{R})$, $|\gamma(t)| \leq 1$, $|\eta(t)| \leq 1$ (for all $t \in \mathcal{T}$) and $f \in \mathcal{C}_0(\mathcal{T}, E)$, we have $g, h \in \mathcal{C}_0(\mathcal{T}; E)$.

Thus we obtain

$$\begin{aligned}
 \rho'_+(f, g) &\stackrel{(3.1)}{=} \max \{ \rho'_+(f(t), g(t)) : t \in \mathcal{M}(f) \} \\
 &= \max \{ \rho'_+(f(t), \gamma(t) \cdot f(t)) : t \in \mathcal{M}(f) \} \\
 &\stackrel{(\text{nd2}, \text{nd3})}{=} \max \{ \gamma(t) \cdot \rho'_\pm(f(t), f(t)) : t \in \mathcal{M}(f) \} \\
 &\stackrel{(\text{nd4})}{=} \max \{ \gamma(t) \cdot \|f(t)\|_E^2 : t \in \mathcal{M}(f) \} \\
 &= \max \{ \gamma(t) \cdot \|f\|_\infty^2 : t \in \mathcal{M}(f) \} = \gamma(t_1) \cdot \|f\|_\infty^2 = \|f\|_\infty^2.
 \end{aligned}$$

Similarly, using (3.1) and (nd2-4) one gets $\rho'_-(f, g) = -\|f\|_\infty^2$ and consequently,

$$\rho'(f, g) = \frac{1}{2}(\rho'_+(f, g) + \rho'_-(f, g)) = \frac{1}{2}(\|f\|_\infty^2 - \|f\|_\infty^2) = 0. \quad (3.3)$$

In a similar way one can prove: $\rho'_+(f, h) = \|f\|_\infty^2$, $\rho'_-(f, h) = -\|f\|_\infty^2$. Hence

$$\rho'(f, h) = 0. \quad (3.4)$$

Since $g + h = (\gamma + \eta) \cdot f$, it is easy to verify that

$$g(t_1) + h(t_1) = f(t_1), \quad g(t_2) + h(t_2) = -2f(t_2), \quad g(t_3) + h(t_3) = f(t_3).$$

In the same manner, one can prove $\rho'_-(f, g+h) = -2\|f\|_\infty^2$. It is easy to verify that $\gamma(t) + \eta(t) < 2$ for all $t \in \mathcal{M}(f)$. Since $\mathcal{M}(f)$ is compact (see Lemma 3.1), there is $\alpha \in (0, 2)$ such that $\gamma(t) + \eta(t) \leq \alpha$ for all $t \in \mathcal{M}(f)$. Since $g + h = (\gamma + \eta) \cdot f$, it follows that

$$\begin{aligned}
 \rho'_+(f, g+h) &\stackrel{(3.1)}{=} \max \{ \rho'_+(f(t), (\gamma(t) + \eta(t)) \cdot f(t)) : t \in \mathcal{M}(f) \} \\
 &\stackrel{(\text{nd2}, \text{nd3})}{=} \max \{ (\gamma(t) + \eta(t)) \cdot \rho'_\pm(f(t), f(t)) : t \in \mathcal{M}(f) \} \\
 &\stackrel{(\text{nd4})}{=} \|f\|_\infty^2 \cdot \max \{ (\gamma(t) + \eta(t)) : t \in \mathcal{M}(f) \} \leq \|f\|_\infty^2 \cdot \alpha
 \end{aligned}$$

and

$$\begin{aligned}
 \rho'(f, g+h) &= \frac{1}{2}(\rho'_-(f, g+h) + \rho'_+(f, g+h)) = \frac{1}{2}(-2\|f\|_\infty^2 + \rho'_+(f, g+h)) \\
 &\leq \frac{1}{2}(-2\|f\|_\infty^2 + \alpha \cdot \|f\|_\infty^2) < \frac{1}{2}(-2\|f\|_\infty^2 + 2\|f\|_\infty^2) = 0.
 \end{aligned} \quad (3.5)$$

Thus by (3.3), (3.4) and (3.5), we have $\rho'(f, g+h) \neq \rho'(f, g) + \rho'(f, h)$. This means that $f \notin \mathcal{N}_{sm}^s(\mathcal{C}(\mathcal{T}; E))$. $\square$

We are going to present the third main results of this section. We will need a simple lemma.

Lemma 3.6. *Let X be a real normed space and let $x_o \in X \setminus \{0\}$. Then the following statements are equivalent:*

- (i) X is smooth at x_o , i.e., $x_o \in \mathcal{N}_{sm}(X)$;
- (ii) $\forall y, z \in X \quad \|y\| \leq 1, \|z\| \leq 1 \Rightarrow \rho'_-(x_o, y+z) = \rho'_-(x_o, y) + \rho'_-(x_o, z)$;
- (iii) $\forall y, z \in X \quad \|y\| = \|z\| \Rightarrow \rho'_-(x_o, y+z) = \rho'_-(x_o, y) + \rho'_-(x_o, z)$.

Proof. Assume condition (i). It follows from Theorem 2.2 ((a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d)) that (ii) and (iii) holds.

For the proof of (iii) $\Rightarrow$ (i) take any $y \neq 0$. It follows from (iii) that

$$0 = \rho'_-(x_o, y + (-y)) = \rho'_-(x_o, y) + \rho'_-(x_o, -y). \quad (3.6)$$

Using (3.6) and (nd3) we have

$$\rho'_-(x_o, y) = -\rho'_-(x_o, -y) = \rho'_+(x_o, y) \quad (3.7)$$

and thus $\rho'_-(x_o, \cdot) = \rho'_+(x_o, \cdot)$. By Theorem 2.2 we have $x_o \in \mathcal{N}_{sm}(X)$.

The proof of (ii) $\Rightarrow$ (i) is similar: we fix y with $\|y\| \leq 1$ and follow the argument to obtain equation (3.7). Then by (nd2) we get that $\rho'_-(x_o, y) = \rho'_+(x_o, y)$ for all $y \in Y$. Thus $x_o \in \mathcal{N}_{sm}(X)$. $\square$

We are now in a position to prove the third main results of this section.

Theorem 3.7. *Suppose that $\mathcal{T}$ is a locally compact space. Let E be a real normed space. Assume that $f \in \mathcal{C}_0(\mathcal{T}, E)$, $f \neq 0$ and let $\text{card } \mathcal{M}(f) = 2$, i.e., $\mathcal{M}(f) = \{t_1, t_2\}$. The following statements are equivalent:*

- (A) *f is a semi-smooth point, i.e., $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$,*
- (B) *$f(t_1), f(t_2)$ are smooth points, i.e., $f(t_1), f(t_2) \in \mathcal{N}_{sm}(E)$.*

Proof. We start with proving (B) $\Rightarrow$ (A). We assume that $\mathcal{M}(f) = \{t_1, t_2\}$. Fix arbitrarily $g \in \mathcal{C}_0(\mathcal{T}, E)$ such that $g \neq 0$. From Theorem 3.2 we have

$$\rho'_+(f, g) = \rho'_+(f(t_1), g(t_1)) \quad \text{and} \quad \rho'_-(f, g) = \rho'_-(f(t_2), g(t_2)), \quad (3.8)$$

or

$$\rho'_+(f, g) = \rho'_+(f(t_2), g(t_2)) \quad \text{and} \quad \rho'_-(f, g) = \rho'_-(f(t_1), g(t_1)). \quad (3.9)$$

Since both $f(t_1)$ and $f(t_2)$ are smooth points, we have

$$\begin{aligned} \rho'_-(f(t_1), g(t_1)) &= \rho'_+(f(t_1), g(t_1)) \quad \text{and} \\ \rho'_-(f(t_2), g(t_2)) &= \rho'_+(f(t_2), g(t_2)). \end{aligned} \quad (3.10)$$

From (3.8), (3.9), (3.10) we have

$$\begin{aligned} \rho'(f, g) &= \frac{1}{2} (\rho'_+(f, g) + \rho'_-(f, g)) = \\ &= \frac{1}{2} (\rho'_+(f(t_1), g(t_1)) + \rho'_+(f(t_2), g(t_2))) \end{aligned} \quad (3.11)$$

for all $g \in \mathcal{C}_0(\mathcal{T}, E)$.

Fix arbitrarily two functions $h, k \in \mathcal{C}_0(\mathcal{T}, E)$. Then we have

$$\begin{aligned}
 \rho'(f, h+k) &\stackrel{(3.11)}{=} \frac{1}{2} (\rho'_+(f(t_1), h(t_1)+k(t_1)) + \rho'_+(f(t_2), h(t_2)+k(t_2))) \\
 &= \left(\begin{smallmatrix} f(t_1), f(t_2) \\ \in \mathcal{N}_{sm}(E) \end{smallmatrix} \right) \frac{1}{2} (\rho'_+(f(t_1), h(t_1)) + \rho'_+(f(t_1), k(t_1)) + \\
 &\quad + \rho'_+(f(t_2), h(t_2)) + \rho'_+(f(t_2), k(t_2))) \\
 &= \frac{1}{2} (\rho'_+(f(t_1), h(t_1)) + \rho'_+(f(t_2), h(t_2))) + \\
 &\quad + \frac{1}{2} (\rho'_+(f(t_1), k(t_1)) + \rho'_+(f(t_2), k(t_2))) \\
 &\stackrel{(3.11)}{=} \rho'(f, h) + \rho'(f, k).
 \end{aligned}$$

which means that $f \in \mathcal{N}_{sm}^s(\mathcal{C}(\mathcal{T}, E))$.

For the proof of (A) $\Rightarrow$ (B) fix arbitrarily two vectors $y, z \in E$ such that $\|y\| \leq 1$, $\|z\| \leq 1$. In order to show that $f(t_1) \in \mathcal{N}_{sm}(E)$, we use Lemma 3.6. By Urysohn's Lemma there is a continuous function $\varphi: \mathcal{T} \rightarrow [0, 1]$ such that $\varphi(t_1) = 0$, $\varphi(t_2) = 1$.

Let us introduce the functions g, h from $\mathcal{T}$ into E defined by

$$\begin{aligned}
 g(t) &:= \|f(t)\|_E \cdot (1 - \varphi(t)) \cdot y + \varphi(t) \cdot f(t), \\
 h(t) &:= \|f(t)\|_E \cdot (1 - \varphi(t)) \cdot z + \varphi(t) \cdot f(t).
 \end{aligned}$$

It is a straightforward verification to show that $g, h \in \mathcal{C}(\mathcal{T}, E)$ and $\|g\|_\infty \leq \|f\|_\infty$, $\|h\|_\infty \leq \|f\|_\infty$. Moreover,

$$g(t_1) = \|f\|_\infty y, \quad g(t_2) = f(t_2), \quad h(t_1) = \|f\|_\infty z, \quad h(t_2) = f(t_2). \quad (3.12)$$

We have

$$\begin{aligned}
 \rho'_+(f(t_1), \|f\|_\infty y) &= \rho'_+(f(t_1), g(t_1)) =: \alpha, & \rho'_+(f(t_2), g(t_2)) &= \|f\|_\infty^2, \\
 \rho'_-(f(t_1), \|f\|_\infty y) &= \rho'_-(f(t_1), g(t_1)) =: \beta, & \rho'_-(f(t_2), g(t_2)) &= \|f\|_\infty^2
 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned}
 \rho'_+(f(t_1), \|f\|_\infty z) &= \rho'_+(f(t_1), h(t_1)) =: \gamma, & \rho'_+(f(t_2), h(t_2)) &= \|f\|_\infty^2, \\
 \rho'_-(f(t_1), \|f\|_\infty z) &= \rho'_-(f(t_1), h(t_1)) =: \delta, & \rho'_-(f(t_2), h(t_2)) &= \|f\|_\infty^2.
 \end{aligned} \quad (3.14)$$

Combining (nd5) and (3.13), (3.14), we immediately get

$$|\alpha| \leq \|f\|_\infty^2, \quad |\beta| \leq \|f\|_\infty^2, \quad |\gamma| \leq \|f\|_\infty^2, \quad |\delta| \leq \|f\|_\infty^2. \quad (3.15)$$

Using (3.1), (3.13), (3.14), (3.15) and the equality $M(f) = \{t_1, t_2\}$, we obtain

$$\rho'_+(f, g) = \|f\|_\infty^2, \quad \rho'_-(f, g) = \beta \quad \text{and} \quad \rho'_+(f, h) = \|f\|_\infty^2, \quad \rho'_-(f, h) = \delta,$$

and hence

$$\rho'(f, g) + \rho'(f, h) = \frac{1}{2} (\|f\|_\infty^2 + \beta) + \frac{1}{2} (\|f\|_\infty^2 + \delta) = \|f\|_\infty^2 + \frac{1}{2} (\beta + \delta). \quad (3.16)$$

On the other hand, we also consider

$$\begin{aligned}\rho'_+(f(t_1), g(t_1) + h(t_1)) &=: \varepsilon, & \rho'_+(f(t_2), g(t_2) + h(t_2)) &= 2\|f\|_\infty^2, \\ \rho'_-(f(t_1), g(t_1) + h(t_1)) &=: \eta, & \rho'_-(f(t_2), g(t_2) + h(t_2)) &= 2\|f\|_\infty^2.\end{aligned}\quad (3.17)$$

By (3.17) and (nd5) we have

$$\begin{aligned}|\varepsilon| &= |\rho'_+(f(t_1), g(t_1) + h(t_1))| \leq \|f(t_1)\| \cdot \|g(t_1) + h(t_1)\| \leq 2\|f\|_\infty^2, \\ |\eta| &= |\rho'_-(f(t_1), g(t_1) + h(t_1))| \leq \|f(t_1)\| \cdot \|g(t_1) + h(t_1)\| \leq 2\|f\|_\infty^2.\end{aligned}\quad (3.18)$$

Combining (3.1), (3.17) and (3.18) and the equality $M(f) = \{t_1, t_2\}$, we immediately get $\rho'_+(f, g+h) = 2\|f\|_\infty^2$, $\rho'_-(f, g+h) = \eta$, and therefore

$$\rho'(f, g+h) = \frac{1}{2}(2\|f\|_\infty^2 + \eta) = \|f\|_\infty^2 + \frac{1}{2}\eta. \quad (3.19)$$

Then, by (3.16), (3.19) and the assumption $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$ we have

$$\|f\|_\infty^2 + \frac{1}{2}(\beta + \delta) = \rho'(f, g) + \rho'(f, h) = \rho'(f, g+h) = \|f\|_\infty^2 + \frac{1}{2}\eta,$$

whence $\eta = \beta + \delta$. Using (3.12), (3.13), (3.14), (3.17) and (nd2) we have

$$\begin{aligned}\rho'_-(f(t_1), y+z) &\stackrel{(3.12)}{=} \rho'_-\left(f(t_1), \frac{1}{\|f\|_\infty}g(t_1) + \frac{1}{\|f\|_\infty}h(t_1)\right) \\ &\stackrel{(3.17)}{=} \frac{1}{\|f\|_\infty}\eta = \frac{1}{\|f\|_\infty}\beta + \frac{1}{\|f\|_\infty}\delta \\ &\stackrel{(3.13), (3.14)}{=} \rho'_-\left(f(t_1), \frac{1}{\|f\|_\infty}g(t_1)\right) + \rho'_-\left(f(t_1), \frac{1}{\|f\|_\infty}h(t_1)\right) \\ &\stackrel{(3.12)}{=} \rho'_-(f(t_1), y) + \rho'_-(f(t_1), z).\end{aligned}$$

It follows from Lemma 3.6 that $f(t_1) \in \mathcal{N}_{sm}(E)$. In a similar way, one checks that $f(t_2) \in \mathcal{N}_{sm}(E)$. Therefore, we may say that the proof of Theorem 3.7 is complete. $\square$

Combining Theorems 3.3, 3.5 and 3.7, we immediately get the following characterizations.

Theorem 3.8. *Suppose that $\mathcal{T}$ is a locally compact metric space and let E be a normed space. Assume that $f \in \mathcal{C}_0(\mathcal{T}, E)$, $f \neq 0$, and $f(t)$ is a smooth point for every $t \in \mathcal{M}(f)$. The following statements are equivalent:*

- (A) f is a semi-smooth point, i.e., $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$,
- (B) $\text{card}\mathcal{M}(f) \leq 2$.

Theorem 3.9. *Suppose that $\mathcal{T}$ is a locally compact metric space. Let E be a smooth normed space. Assume that $f \in \mathcal{C}_0(\mathcal{T}, E)$, $f \neq 0$. The following statements are equivalent:*

- (A) f is a semi-smooth point, i.e., $f \in \mathcal{N}_{sm}^s(\mathcal{C}_0(\mathcal{T}, E))$,
- (B) $\text{card}\mathcal{M}(f) \leq 2$.

In this part of the section, we present another application of our main results in the classical sequence spaces, i.e., c_0 and c .

The sequence $x = (x_1, x_2, \dots) \in c_0$ can be identified with a function $f_x : \mathbb{N} \rightarrow \mathbb{R}$ by $f_x(n) := x_n$. Clearly, $f_x \in \mathcal{C}_0(\mathbb{N}, \mathbb{R})$ and such an identification is a linear and topological isomorphism, which is also an isometry. In the same manner we can see that the space c is identified with $\mathcal{C}(\mathbb{N} \cup \{+\infty\}, \mathbb{R})$, and in particular $\mathcal{C}(\mathbb{N} \cup \{+\infty\}, \mathbb{R}) = \mathcal{C}_0(\mathbb{N} \cup \{+\infty\}, \mathbb{R})$.

Theorem 3.10. *Suppose that $x = (x_1, x_2, \dots) \in c_0$ and $x \neq 0$. Then x is a semi-smooth point, i.e., $x \in \mathcal{N}_{sm}^s(c_0)$, if and only if one of the following conditions holds:*

- (a) $\exists k, \ell \in \mathbb{N} \forall n \in \mathbb{N} \setminus \{k, \ell\} : |x_n| < |x_k| = |x_\ell| = \|x\|_\infty$,
- (b) $\exists! k \in \mathbb{N} \forall n \neq k : |x_n| < |x_k| = \|x\|_\infty$.

Proof. We can make the identification $f_x \in \mathcal{C}_0(\mathbb{N}, \mathbb{R})$ by $f_x(n) := x_n$. Theorem 3.9 implies f_x is semi-smooth if and only if $\text{card} \mathcal{M}(f_x) \leq 2$, which translates to conditions (a) and (b). $\square$

Theorem 3.11. *Suppose that $x = (x_1, x_2, \dots) \in c$ and $x \neq 0$. Then x is a semi-smooth point, i.e., $x \in \mathcal{N}_{sm}^s(c)$, if and only if one of the following conditions holds:*

- (a) $\exists k, l \in \mathbb{N} \forall n \in \mathbb{N} \setminus \{k, l\} : |x_n| < |x_k| = |x_l| = \|x\|_\infty$ and $\lim_{n \rightarrow +\infty} x_n \in (-\|x\|_\infty, \|x\|_\infty)$,
- (b) $\forall n \in \mathbb{N} : |x_n| < \|x\|_\infty$ and $\lim_{n \rightarrow +\infty} x_n = \|x\|_\infty$ or $-\|x\|_\infty$,
- (c) $\exists! k \in \mathbb{N} \forall n \neq k : |x_n| < |x_k| = \|x\|_\infty$ and $\lim_{n \rightarrow +\infty} x_n \in (-\|x\|_\infty, \|x\|_\infty)$.

Proof. Applying Theorem 3.9 again, this time to the identification $c = \mathcal{C}(\mathbb{N} \cup \{+\infty\}, \mathbb{R})$, we obtain the desired assertion. $\square$

Example 3.12. Consider the set $\mathcal{T} := \Sigma_2 = \{-1, 1\}^{\mathbb{N}}$. By Tychonoff theorem, Σ_2 is a compact metric space with product topology given by

$$d(x, y) := \sum_{j=0}^{+\infty} \frac{|x_j - y_j|}{2^j},$$

where $x = (x_j)_{j \in \mathbb{N}}$ and $y = (y_j)_{j \in \mathbb{N}}$ are in Σ_2 . Denote $1^\infty = (1, 1, 1, \dots)$ and $-1^\infty = (-1, -1, -1, \dots)$.

Fix a number $\alpha > 0$ and take $f_\alpha : \Sigma_2 \rightarrow \mathbb{R}$ given by the formula

$$f_\alpha(x) = f_\alpha(x_0, x_1, \dots) := \sum_{j=0}^{+\infty} \frac{x_j}{(j+1)^\alpha}.$$

It is easy to verify that $f_\alpha \in \mathcal{C}_0(\Sigma_2, \mathbb{R})$. Furthermore, $M(f_\alpha) = \{-1^\infty, 1^\infty\}$ and thus by Theorem 3.9 we conclude that f_α is a semi-smooth point in the space $\mathcal{C}_0(\Sigma_2, \mathbb{R})$.

Clearly, one can take any $f \in \mathcal{C}_0(\Sigma_2, \mathbb{R})$ with the following representation: $f(x) = \sum_{j=0}^{+\infty} a_j x_j$ for some suitable sequence $(a_j)_{j \in \mathbb{N}}$ with $a_j \neq 0$ for all $j \in \mathbb{N}$ such that $f(x)$ converges for any $x \in \Sigma_2$. Then $\|f\|_\infty = \|f((\operatorname{sgn}(a_j))_{j \in \mathbb{N}})\| = \|f(-(\operatorname{sgn}(a_j))_{j \in \mathbb{N}})\|$ and $\|f(x)\| < \|f\|_\infty$ for $x \in \Sigma_2 \setminus \{-(\operatorname{sgn}(a_j))_{j \in \mathbb{N}}, (\operatorname{sgn}(a_j))_{j \in \mathbb{N}}\}$. Thus such an f is a semi-smooth point.

The above considerations can be rewritten as the following result.

Theorem 3.13. *Every unconditionally convergent series is a semi-smooth point in $\mathcal{C}_0(\Sigma_2, \mathbb{R})$.*

Proof. The series $\sum_{j=0}^{+\infty} a_j$ is identified with $f(x_0, x_1, \dots) = \sum_{j=0}^{+\infty} a_j x_j$. The rest is clear. $\square$

Let X be a real normed linear space. Let $X \oplus_\infty \dots \oplus_\infty X$ denote the direct sum of X with itself n times, and in a natural way we introduce the norm $\|(x_1, \dots, x_n)\|_\infty := \max\{\|x_1\|, \dots, \|x_n\|\}$. Note that $X \oplus_\infty X$ is identified with $\mathcal{C}_0(\{1, 2\}, X)$. As an immediate consequence of Theorems 3.7 and 3.9, we have the following.

Corollary 3.14. *X is a smooth space if and only if $X \oplus_\infty X$ is a semi-smooth space.*

Proof. Assume that X is a smooth space. Take any non-zero $f \in \mathcal{C}_0(\{1, 2\}, X)$. Since $X \oplus_\infty X = \{(g(1), g(2)) : g \in \mathcal{C}_0(\{1, 2\}, X)\}$ and $\|(f(1), f(2))\|_\infty = \max\{\|f(1)\|, \|f(2)\|\}$, we have $\operatorname{card} \mathcal{M}(f) \leq 2$. Thus, by Theorem 3.9 we obtain that f is a semi-smooth point.

Conversely, let $X \oplus_\infty X$ be semi-smooth. Fix $x \in X \setminus \{0\}$. We can find a non-zero $f \in \mathcal{C}_0(\{1, 2\}, X)$ such that $x = f(1) = f(2)$. Then $\operatorname{card} \mathcal{M}(f) = 2$ and by Theorem 3.7, we obtain that $x = f(1)$ is a smooth point. $\square$

We may identify $X \oplus_\infty X \oplus_\infty X$ with a space $\mathcal{C}_0(\{1, 2, 3\}, X)$, and applying again Theorem 3.5 and using a similar argument as in the proof of the above corollary, we obtain

Corollary 3.15. *Let X be any nontrivial normed space. Then $X \oplus_\infty X \oplus_\infty X$ is not a semi-smooth space.*

Bearing in mind that the space l^1 is semi-smooth, we have the following result.

Proposition 3.16. *Let X be any nontrivial normed space. Then, there is no linear isometry μ from $X \oplus_\infty X \oplus_\infty X$ into l^1 .*

4. Semi-smooth Points in Normed Spaces and Supporting Functionals

The condition (2.3) motivates this section. When x_o is not a smooth point, consider the set $J(x_o)$. An interesting question in analysis is to study points for which $J(x_o)$ is a “large set”. The question is: What about spaces which are not smooth? Motivated by these, in this section we prove some characterization of semi-smoothness (instead of smoothness).

Let us fix $D \subseteq X$ and $a \in D$. We say that D is *symmetric (with respect to the point a)* if for all $x \in D$ we have $2a - x \in D$ or, equivalently, $D = 2a - D$.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a real normed linear space and $x_o \in X \setminus \{0\}$. Then the following statements are equivalent:*

(A) *X is semi-smooth in x_o ;*

(B) *$J(x_o)$ is symmetric with respect to some point m^* .*

Moreover, $\rho'(x_o, y) = m^*(y)$ for $y \in X$.

Proof. Without loss of generality, we can assume that $\|x_o\| = 1$ (see Theorem 2.3, (a) $\Leftrightarrow$ (e)). We start with proving (B) $\Rightarrow$ (A). Fix $y \in X$. By (2.2) we obtain

$$x^*(y) \leq \rho'_+(x_o, y) \text{ for all } x^* \in J(x_o).$$

It yields

$$\forall_{x^* \in J(x_o)} \quad 2m^*(y) - \rho'_+(x_o, y) \leq 2m^*(y) - x^*(y). \quad (4.1)$$

It follows from (B) that $2m^* - x^* \in J(x_o)$ and $J(x_o) = 2m^* - J(x_o)$. Now the condition (4.1) becomes (putting $z^* := 2m^* - x^*$ in place of $2m^* - x^*$ in the above inequality)

$$\forall_{z^* \in J(x)} \quad 2m^*(y) - \rho'_+(x_o, y) \leq z^*(y).$$

Passing to the infimum over $z^* \in J(x)$ (and applying Theorem 2.1) we get

$$2m^*(y) - \rho'_+(x_o, y) \leq \rho'_-(x_o, y).$$

It follows from the above inequality that $m^*(y) \leq \frac{1}{2}\rho'_-(x_o, y) + \frac{1}{2}\rho'_+(x_o, y) = \rho'(x_o, y)$. Thus we have

$$\forall_{y \in X} \quad m^*(y) \leq \rho'(x_o, y). \quad (4.2)$$

Putting $-y$ in place of y in the above inequality we get $m^*(-y) \leq \rho'(x_o, -y)$. Using (Msip2), we get $m^*(y) \geq \rho'(x_o, y)$, which means

$$\forall_{y \in X} \quad m^*(y) \geq \rho'(x_o, y). \quad (4.3)$$

Inequalities (4.2) and (4.3) yield $\forall_{y \in X} \quad m^*(y) = \rho'(x_o, y)$. Thus finally we get $\rho'(x_o, \cdot) \in X^*$. That means $x_o \in \mathcal{N}_{sm}^s(X)$ (see Theorem 2.3).

We prove the converse implication. If we assume (A), then $\rho'(x_o, \cdot) \in X^*$. By (Msip3) and (Msip4) we get $\rho'(x_o, \cdot) \in J(x_o)$. Let $m^* \in X^*$ be defined by $m^*(\cdot) := \rho'(x_o, \cdot)$. We show that $J(x_o)$ is symmetric. Fix $z^* \in J(x_o)$. Therefore,

$2m^*(x_o) - z^*(x_o) = 2\rho'(x_o, x_o) - \|x_o\| = 2\|x_o\|^2 - \|x_o\| = 2 \cdot 1 - 1 = \|x_o\|$. Next we prove that $\|2m^* - z^*\| = 1$. Fix $y \in S(X)$ and notice that we have

$$\begin{aligned} 2m^*(y) - z^*(y) &= 2\rho'(x_o, y) - z^*(y) = \rho'_-(x_o, y) + \rho'_+(x_o, y) - z^*(y) \\ &\stackrel{(2.2)}{\geq} \rho'_-(x_o, y) + \rho'_+(x_o, y) - \rho'_+(x_o, y) = \rho'_-(x_o, y) \stackrel{(\text{nd5})}{\geq} -1 \end{aligned}$$

and

$$\begin{aligned} 2m^*(y) - z^*(y) &= 2\rho'(x_o, y) - z^*(y) = \rho'_-(x_o, y) + \rho'_+(x_o, y) - z^*(y) \\ &\stackrel{(2.2)}{\leq} \rho'_-(x_o, y) + \rho'_+(x_o, y) - \rho'_-(x_o, y) = \rho'_+(x_o, y) \stackrel{(\text{nd5})}{\leq} 1. \end{aligned}$$

It follows from the above inequalities that $|2m^*(y) - z^*(y)| \leq 1$ for all $y \in S(X)$. We know that $2m^*(x_o) - z^*(x_o) = 1$ and $\|x_o\| = 1$. Therefore $\|2m^* - z^*\| = 1$, and hence $2m^* - z^* \in J(x_o)$. $\square$

As a consequence of the above theorem, we get the following result.

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a real normed linear space and $x_o \in X \setminus \{0\}$. If $J(x_o)$ is a segment, i.e., $J(x_o) = \text{conv}\{x^*, y^*\}$, then X is semi-smooth at $x_o \in S_{X^*}$. Moreover, $\rho'(x_o, \cdot) = \frac{1}{2}x^*(\cdot) + \frac{1}{2}y^*(\cdot)$.*

Fix a nonzero $x \in X$. If $\dim X = 2$, then $\dim X^* = 2$. Since $J(x)$ is a convex and $J(x) \subseteq S(X^*)$, the equality $\dim X^* = 2$ shows that $\text{card} J(x) = 1$ or $J(x) = \text{conv}\{x^*, y^*\}$. Hence $x \in \mathcal{N}_{sm}(X)$ or $x \in \mathcal{N}_{sm}^s(X)$. As an immediate consequence of this reasoning, we have the following.

Corollary 4.3. *Every two-dimensional normed space is semi-smooth.*

5. Concluding Remarks

If X is a Banach space that is separable, then $\mathcal{N}_{sm}(X)$ and $\mathcal{N}_{sm}^s(X)$ must be dense. We know that $\mathcal{N}_{sm}(X) \subseteq \mathcal{N}_{sm}^s(X)$. What does this say about $\mathcal{N}_{sm}^s(X) \setminus \mathcal{N}_{sm}(X)$? We have already seen that

$$\mathcal{N}_{sm}^s(\mathcal{C}[0, 1]) \setminus \mathcal{N}_{sm}(\mathcal{C}[0, 1]) = \{f \in \mathcal{C}[0, 1] : \text{card} \mathcal{M}(f) = 2\}. \quad (5.1)$$

Now, let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{K}(\mathcal{H})$ be a space of compact linear operators. The following result was obtained in [14]:

$$\mathcal{N}_{sm}^s(\mathcal{K}(\mathcal{H})) \setminus \mathcal{N}_{sm}(\mathcal{K}(\mathcal{H})) = \{A \in \mathcal{K}(\mathcal{H}) : \dim \text{span} M(A) = 2\}, \quad (5.2)$$

where $M(A) := \{x \in S_{\mathcal{H}} : \|Ax\| = \|A\|\}$. It is easy to check that the above two sets, i.e., (5.1) and (5.2), are dense and nowhere dense, respectively.

It is known that

$$x \in \mathcal{N}_{sm}(c_0) \Leftrightarrow \exists!_{k \in \mathbb{N}} \forall_{n \neq k} : |x_n| < |x_k| = \|x\|_{\infty}.$$

Therefore, $\mathcal{N}_{sm}(c_0)$ is easily seen to be open. But since $\mathcal{N}_{sm}(c_0)$ is dense, $\mathcal{N}_{sm}^s(c_0) \setminus \mathcal{N}_{sm}(c_0)$ is nowhere dense. This is summarized as follows.

X	$\mathcal{N}_{sm}(X)$	$\mathcal{N}_{sm}^s(X)$	$\mathcal{N}_{sm}^s(X) \setminus \mathcal{N}_{sm}(X)$
$\mathcal{K}(\mathcal{H})$	dense	dense	nowhere dense
$\mathcal{C}[0, 1]$	dense	dense	dense
c_0	dense and open	dense	nowhere dense
$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$	$\emptyset$

It is amazing that the sets $\mathcal{N}_{sm}^s$, $\mathcal{N}_{sm}$ are always huge but the set $\mathcal{N}_{sm}^s \setminus \mathcal{N}_{sm}$ may be large or small.

We end this work with an inquisitive question. Is it possible to find a normed linear space X such that $\dim X \geq 3$ and $\mathcal{N}_{sm}(X) = \mathcal{N}_{sm}^s(X) \neq X$?

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