The Mixed Scalar Curvature of Almost-Product Metric-Affine Manifolds, II

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Abstract. We continue our study of the mixed Einstein–Hilbert action as a functional of a pseudo-Riemannian metric and a linear connection. Its geometrical part is the total mixed scalar curvature on a smooth manifold endowed with a distribution or a foliation. We develop variational formulas for quantities of extrinsic geometry of a distribution on a metric-affine space and use them to derive Euler–Lagrange equations (which in the case of space-time are analogous to those in Einstein–Cartan theory) and to characterize critical points of this action on vacuum space-time. Together with arbitrary variations of metric and connection, we consider also variations that partially preserve the metric, e.g., along the distribution, and also variations among distinguished classes of connections (e.g., statistical and metric compatible, and this is expressed in terms of restrictions on contorsion tensor). One of Euler–Lagrange equations of the mixed Einstein–Hilbert action is an analog of the Cartan spin connection equation, and the other can be presented in the form similar to the Einstein equation, with Ricci curvature replaced by the new Ricci type tensor. This tensor generally has a complicated form, but is given in the paper explicitly for variations among semi-symmetric connections.

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1. Introduction

We study the mixed Einstein–Hilbert action as a functional of two variables: a pseudo-Riemannian metric and a linear connection. Its geometrical part is the total mixed scalar curvature on a smooth manifold endowed with a distribution

or a foliation. Our goals are to obtain the Euler–Lagrange equations of the action, present them in the classical form of Einstein equations and find their solutions for the vacuum case under various geometric assumptions.

1.1. State of the Art

The Metric-Affine Geometry (founded by E. Cartan) generalizes pseudo-Riemannian Geometry: it uses a linear connection $\bar{\nabla}$ with torsion, instead of the Levi-Civita connection ∇ of a pseudo-Riemannian metric $g = \langle \cdot, \cdot \rangle$ on a manifold M, e.g., [16], and appears in such context as almost Hermitian and Finsler manifolds and theory of gravity. To describe geometric properties of $\bar{\nabla}$, we use the difference $\mathfrak{T} = \bar{\nabla} - \nabla$ (called the *contorsion tensor*) and also auxiliary (1,2)-tensors $\mathfrak{T}^*$ and $\mathfrak{T}^\wedge$ defined by

$$\langle \mathfrak{T}^*_X Y, Z \rangle = \langle \mathfrak{T}_X Z, Y \rangle, \quad \mathfrak{T}^\wedge_X Y = \mathfrak{T}_Y X, \quad X, Y, Z \in \mathfrak{X}_M.$$

The following distinguished classes of metric-affine manifolds $(M, g, \bar{\nabla})$ are considered important.

     • *Riemann–Cartan manifolds*, where the $\bar{\nabla}$-parallel transport along the curves preserves the metric, i.e., $\bar{\nabla} g = 0$, e.g., [14,28], This condition is equivalent to $\mathfrak{T}^* = -\mathfrak{T}$ and $\bar{\nabla}$ is then called a metric compatible (or: metric) connection, e.g., [9], where the torsion tensor is involved in the Cartan spin connection equation, see (3b). More specific types of metric connections (e.g., the *semisymmetric connections* [11,33] and *adapted metric connections* [5]) also find applications in geometry and theoretical physics.

     • *Statistical manifolds*, where the tensor $\bar{\nabla} g$ is symmetric in all its entries and connection $\bar{\nabla}$ is torsion-free, e.g., [10,18,19]. These conditions are equivalent to $\mathfrak{T}^\wedge = \mathfrak{T}$ and $\mathfrak{T}^* = \mathfrak{T}$. The theory of affine hypersurfaces in $\mathbb{R}^{n+1}$ is a natural source of such manifolds; they also find applications in theory of probability and statistics.

     The above classes of connections admit a natural definition of the sectional curvature: in case of metric connections by the same formula as for the Levi-Civita connection, and for statistical connections by the analogue introduced in [18]. For the curvature tensor $\bar{R}_{X,Y} = [\bar{\nabla}_Y, \bar{\nabla}_X] + \bar{\nabla}_{[X,Y]}$ of an affine connection $\bar{\nabla}$, we have

$$\bar{R}_{X,Y} - R_{X,Y} = (\nabla_Y \mathfrak{T})_X - (\nabla_X \mathfrak{T})_Y + [\mathfrak{T}_Y, \mathfrak{T}_X], \tag{1}$$

where $R_{X,Y} = [\nabla_Y, \nabla_X] + \nabla_{[X,Y]}$ is the Riemann curvature tensor of ∇. Similarly as in Riemannian geometry, one can also consider the scalar curvature $\bar{S}$ of $\bar{R}$.

     Many notable examples of pseudo-Riemannian metrics come (as critical points) from variational problems, a particularly famous of which is the *Einstein-Hilbert action*, e.g., [7]. Its Einstein–Cartan generalization in the framework of metric-affine geometry, given (on a smooth manifold M) by

$$\bar{J} : (g, \mathfrak{T}) \to \int_M \left\{ \frac{1}{2\mathfrak{a}} (\bar{S} - 2\Lambda) + \mathcal{L} \right\} \mathrm{dvol}_g, \tag{2}$$

extends the original formulation of general relativity and provides interesting examples of metrics as well as connections, e.g., [1, Chapter 17]. Here, Λ is a constant (the "cosmological constant"), $\mathcal{L}$ is Lagrangian describing the matter contents, and $\mathfrak{a} > 0$ is the coupling constant. To deal also with non-compact manifolds, it is assumed that the integral above is taken over M if it converges; otherwise, one integrates over arbitrarily large, relatively compact domain $\Omega \subset M$, which also contains supports of variations of g and $\mathfrak{T}$. The Euler–Lagrange equation for (2) when g varies is

$$\overline{\mathrm{Ric}} - (1/2)\,\overline{\mathrm{S}} \cdot g + \Lambda\,g = \mathfrak{a}\,\varXi \tag{3a}$$

(called the Einstein equation) with the non-symmetric Ricci curvature $\overline{\mathrm{Ric}}$ and the asymmetric energy-momentum tensor $\varXi$ (generalizing the stress tensor of Newtonian physics), given in coordinates by $\varXi_{\mu\nu} = -2\,\partial\mathcal{L}/\partial g^{\mu\nu} + g_{\mu\nu}\mathcal{L}$. The Euler–Lagrange equation for (2) when $\mathfrak{T}$ varies is an algebraic constraint with the torsion tensor $\mathcal{S}$ of $\overline{\nabla}$ and the spin tensor $s^c_{\mu\nu} = 2\,\partial\mathcal{L}/\partial\mathfrak{T}^c_{\mu\nu}$ (used to describe the intrinsic angular momentum of particles in spacetime, e.g., [30]):

$$\mathcal{S}(X,Y) + \mathrm{Tr}(\mathcal{S}(\cdot,Y) - \mathcal{S}(X,\cdot)) = \mathfrak{a}\,s(X,Y), \quad X, Y \in \mathfrak{X}_M. \tag{3b}$$

Since $\mathcal{S}(X,Y) = \mathfrak{T}_X Y - \mathfrak{T}_Y X$, (3b) can be rewritten using the contorsion tensor. The solution of (3a,b) is a pair $(g, \mathfrak{T})$, satisfying this system, where the pair of tensors $(\varXi, s)$ (describing a specified type of matter) is given. In vacuum space-time, Einstein and Einstein–Cartan theories coincide. The classification of solutions of (3a,b) is a deep and largely unsolved problem [30], even for $\mathfrak{T} = 0$ [7].

1.2. Objectives

On a manifold equipped with an additional structure (e.g., almost product, complex or contact), one can consider an analogue of (2) adjusted to that structure. In pseudo-Riemannian geometry, it may mean restricting g to a certain class of metrics (e.g., conformal to a given one, in the Yamabe problem [7]) or even constructing a new, related action (e.g., the Futaki functional on a Kahler manifold [7], or several actions on contact manifolds [8]), to cite only few examples. The latter approach was taken in authors' previous papers, where the scalar curvature in the Einstein–Hilbert action on a pseudo-Riemannian manifold was replaced by the mixed scalar curvature of a given distribution or a foliation.

In this paper, a similar change in (2) will be considered on a connected smooth $(n + p)$-dimensional manifold M endowed with an affine connection and a smooth n-dimensional distribution $\widetilde{\mathcal{D}}$ (a subbundle of the tangent bundle TM). Distributions and foliations (that can be viewed as integrable distributions) on manifolds appear in various situations, e.g., [5,20]. When a pseudo-Riemannian metric g on M is non-degenerate along $\widetilde{\mathcal{D}}$, it defines the orthogonal p-dimensional distribution $\mathcal{D}$ such that both distributions span the

tangent bundle: $TM = \widetilde{\mathcal{D}} \oplus \mathcal{D}$ and define a Riemannian almost-product structure on (M, g), e.g., [15]. From a mathematical point of view, a *space-time* of general relativity is a $(n + 1)$-dimensional time-oriented (i.e., with a given timelike vector field) Lorentzian manifold, see [4]. A space-time admits a global *time function* (i.e., increasing function along each future directed nonspacelike curve) if and only if it is stable causal; in particular, a globally hyperbolic spacetime is naturally endowed with a codimension-one foliation (the level hypersurfaces of a given time-function), see [6,12].

The *mixed Einstein–Hilbert action* on $(M, \widetilde{\mathcal{D}})$,

$$\bar{J}_{\widetilde{\mathcal{D}}} : (g, \mathfrak{T}) \mapsto \int_M \left\{ \frac{1}{2\mathfrak{a}} \left(\overline{\mathrm{S}}_{\mathrm{mix}} - 2\,\Lambda \right) + \mathcal{L} \right\} \mathrm{d\,vol}_g, \tag{4}$$

is an analog of (2), where $\overline{\mathrm{S}}$ is replaced by the mixed scalar curvature $\overline{\mathrm{S}}_{\mathrm{mix}}$, see (9), for the affine connection $\overline{\nabla} = \nabla + \mathfrak{T}$. The physical meaning of (4) is discussed in [2] for the case of $\mathfrak{T} = 0$. Our action (4) can be useful for the multi-time Geometric Dynamics, e.g., [17] and survey [31]. This was introduced like Multi-time World Force Law involving field potentials, gravitational potentials (components of the two Riemannian metrics), and the Yang–Mills potentials (components of the Riemannian connections and the nonlinear connection).

In view of the formula $\overline{\mathrm{S}} = 2\overline{\mathrm{S}}_{\mathrm{mix}} + \overline{\mathrm{S}}^\top + \overline{\mathrm{S}}^\perp$, where $\overline{\mathrm{S}}^\top$ and $\overline{\mathrm{S}}^\perp$ are the scalar curvatures along the distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, one can combine the actions (2) and (4) to obtain the new *perturbed Einstein–Hilbert action* on $(M, \widetilde{\mathcal{D}})$: $\bar{J}_\epsilon : (g, \mathfrak{T}) \mapsto \int_M \left\{ \frac{1}{2\mathfrak{a}} \left(\overline{\mathrm{S}} + \epsilon\,\overline{\mathrm{S}}_{\mathrm{mix}} - 2\,\Lambda \right) + \mathcal{L} \right\} \mathrm{d\,vol}_g$ with $\epsilon \in \mathbb{R}$, whose critical points may describe geometry of the space-time in an extended theory of gravity.

The mixed scalar curvature (being an averaged mixed sectional curvature) is one of the simplest curvature invariants of a pseudo-Riemannian almost-product structure. If a distribution is spanned by a unit vector field N, i.e., $\langle N, N \rangle = \epsilon_N \in \{-1, 1\}$, then $\overline{\mathrm{S}}_{\mathrm{mix}} = \epsilon_N \overline{\mathrm{Ric}}_{N,N}$, where $\overline{\mathrm{Ric}}_{N,N}$ is the Ricci curvature in the N-direction. If $\dim M = 2$ and $\dim \widetilde{\mathcal{D}} = 1$, then obviously $2\overline{\mathrm{S}}_{\mathrm{mix}} = \overline{\mathrm{S}}$. If $\mathfrak{T} = 0$ then $\overline{\mathrm{S}}_{\mathrm{mix}}$ reduces to the mixed scalar curvature $\mathrm{S}_{\mathrm{mix}}$ of ∇, see (10), which can be defined as a sum of sectional curvatures of planes that non-trivially intersect with both of the distributions. Investigation of $\mathrm{S}_{\mathrm{mix}}$ led to multiple results regarding the existence of foliations and submersions with interesting geometry, e.g., integral formulas and splitting results, curvature prescribing and variational problems, see survey [23]. The trace of the partial Ricci curvature (rank 2) tensor $r_{\mathcal{D}}$ is $\mathrm{S}_{\mathrm{mix}}$, see Sect. 2. The understanding of the mixed curvature, especially, $r_{\mathcal{D}}$ and $\mathrm{S}_{\mathrm{mix}}$, is a fundamental problem of extrinsic geometry of foliations, see [20].

Varying (4) with fixed $\mathfrak{T} = 0$, as a functional of g only, we obtain the Euler–Lagrange equations in the form similar to (3a), see [2] for space-times, and for $\widetilde{\mathcal{D}}$ of any dimension, see [25,26], i.e.,

$$\mathrm{Ric}_{\,\mathcal{D}} - (1/2)\,\mathrm{S}_{\,\mathcal{D}} \cdot g + \Lambda\,g = \mathfrak{a}\,\Xi, \tag{5}$$

where the Ricci and scalar curvature are replaced by the *mixed Ricci curvature* $\mathrm{Ric}_{\mathcal{D}}$, see (26), and its trace $\mathrm{S}_{\mathcal{D}}$. In [27], we obtained the Euler–Lagrange equations for (4) with fixed g and variable $\mathfrak{T}$, see (30a–h), and examined critical contorsion tensors (and corresponding connections) in general and in distinguished classes of (1,2)-tensors. We have shown that $\mathfrak{T}$ is critical for (4) with fixed g if and only if $\mathfrak{T}$ obeys certain system of algebraic equations, however, unlike (3b), these equations heavily involve also the pseudo-Riemannian geometry of the distributions.

   In this article we generalize these results, considering variations of (4) with respect to both g and $\mathfrak{T}$, at their arbitrary values. As we are less inclined to discuss particular physical theories, we basically confine ourselves to studying the total mixed scalar curvature—the geometric part of the mixed Einstein–Hilbert action, i.e., we set $\Lambda = \mathcal{L} = 0$ in (4), which in physics correspond to vacuum space-time and no "cosmological constant":

$$\bar{J}_{\mathrm{mix}} : (g, \mathfrak{T}) \mapsto \int_M \overline{\mathrm{S}}_{\mathrm{mix}} \, \mathrm{d\,vol}_g \,. \tag{6}$$

Considering variations of the metric that preserve the volume of the manifold, we can also obtain the Euler–Lagrange equations for (6), that coincide with those for (4) with $\mathcal{L} = 0$ and $\Lambda \neq 0$.

   The terms of $\overline{\mathrm{S}}_{\mathrm{mix}}$ without covariant derivatives of $\mathfrak{T}$ make up the *mixed scalar $\mathfrak{T}$-curvature*, see Sect. 2, which we find interesting on its own. In particular, $\mathrm{S}_{\mathfrak{T}}$ can be viewed as the Riemannian mixed scalar curvature of a distribution with all sectional curvatures of planes replaced by their $\mathfrak{T}$-curvatures (see [18]), and for statistical connections we have $\overline{\mathrm{S}}_{\mathrm{mix}} = \mathrm{S}_{\mathrm{mix}} + \mathrm{S}_{\mathfrak{T}}$. Thus, we also study (in Sect. 3.1) the following, closely related to (6), action on $(M, \widetilde{\mathcal{D}})$:

$$I : (g, \mathfrak{T}) \mapsto \int_M \mathrm{S}_{\mathfrak{T}} \, \mathrm{d\,vol}_g \,. \tag{7}$$

For each of the examined actions (6) and (7), we obtain the Euler–Lagrange equations and formulate results about existence and examples of their solutions, that we describe in more detail further below. In particular, from [27] we know that if $\mathfrak{T}$ is critical for the action (6), then $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are totally umbilical with respect to ∇—and to express this together with other conditions, a pair of equations like (3a,b) is not sufficient. Due to this fact, only in the special case of semi-symmetric connections we present the Euler–Lagrange equation in the form, which directly generalizes (5):

$$\overline{\mathrm{Ric}}_{\mathcal{D}} - (1/2) \overline{\mathrm{S}}_{\mathcal{D}} \cdot g + \Lambda g = \mathfrak{a} \, \Xi \tag{8}$$

and a separate condition (61), similar to (3b), for the vector field parameterizing this type of connection. In general case, instead of a single equation like (3b), we obtain a system of equations (30a–h), which we then use to write the analogue of (3a) explicitly in terms of extrinsic geometry of distributions.

1.3. Structure of the Paper

The article has the Introduction and three other Sections.

Section 2 contains background definitions and necessary results from [3, 25–27], among them the notions of the mixed scalar curvature and the mixed and the partial Ricci tensors are central.

Section 3 contains the main results, described in detail below.

Section 4 contains auxiliary lemmas with necessary, but lengthy computations, and the References include 32 items.

In Sect. 3, we derive the Euler–Lagrange equations for (6) and (7) and find some of their solutions–critical pairs $(g, \mathfrak{T})$ for different kinds of variations of metric and connection. Apart from varying among all metrics that are non-degenerate on $\widetilde{\mathcal{D}}$, we also restrict to the case when metric remains fixed on the distribution, and the complementary case when metric varies only on the distribution–preserving its orthogonal complement and the metric on it. This approach (first, applied in [24] for codimension one foliations) can be used to finding an optimal extension of a metric given only on the distribution–which is the problem of the relationship between sub-Riemannian and Riemannian geometries. Moreover, in analogy to the Einstein–Hilbert action, all variations are considered in two kinds: with and without preserving the volume of the manifold, see [7]. In addition, together with arbitrary variations of connection, we consider variations among such distinguished classes as statistical and metric connections and express this in terms of constraints on $\mathfrak{T}$.

Section 3 is divided into four subsections, according to additional conditions we impose on connections (e.g., metric, adapted and statistical) or actions we consider (defined by the mixed scalar curvature $\overline{\mathrm{S}}_{\mathrm{mix}}$ and the algebraic curvature-type invariant of a contorsion tensor $\mathrm{S}_{\mathfrak{T}}$).

In Sect. 3.1, we vary functional (7) with respect to metric g. Compared to its variation with fixed g, which was considered in [27], we obtain additional conditions for general and metric connections. On the other hand, restricting (7) to pairs of metrics and statistical connections also does not give any new Euler–Lagrange equations than those obtained in [27]. Similarly, a metric-affine doubly twisted product is critical for (7) if and only if it is critical for the action with fixed g.

In Sect. 3.2, for arbitrary variations of $(g, \mathfrak{T})$ we show that statistical connections critical for (6) on a closed M are exactly those that are critical for (6) with fixed g, and for $n + p > 2$ these exist only on metric products. On the other hand, for every g critical for (6) with fixed $\mathfrak{T} = 0$, there exist statistical connections, satisfying algebraic conditions (37a,b), such that $(g, \mathfrak{T})$ is critical for (6) restricted to all metrics, but only statistical connections. Note that (37b) is equivalent to $\mathfrak{T}$ acting invariantly on each distribution, i.e., with only components $\mathfrak{T} : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ and $\mathfrak{T} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$. Equations (37a,b) imply also that the traces $\mathrm{Tr}^{\top} \mathfrak{T}$ and $\mathrm{Tr}^{\perp} \mathfrak{T}$ vanish, and these are the only restrictions for $\mathfrak{T}$ critical among statistical connections.

In Sect. 3.3 we show that for $n, p > 1$ the critical value of (6) attained by $(g, \mathfrak{T})$, where $\mathfrak{T}$ corresponds to a metric connection, depends only on g and is non-negative on a Riemannian manifold. In other words, pseudo-Riemannian geometry determines the mixed scalar curvature of any critical metric connection. For general metric connections, we consider only adapted variations of the metric (see Definition 2) due to complexity of the variational formulas. Compared to (6) with fixed g, we get a new condition (47a), involving the symmetric part of $\mathfrak{T}|_{\mathcal{D} \times \mathcal{D}}$ and of $\mathfrak{T}|_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}}$ in the dual equation. This condition is strong enough to prevent existence of critical points of (6) in some settings, e.g. for $\widetilde{\mathcal{D}}$ spanned by the Reeb field on a closed contact manifold with associated metric. Under some assumptions, trace of (47a) depends only on the pseudo-Riemannian geometry of $(M, g, \widetilde{\mathcal{D}})$ and thus gives a necessary condition for the metric to admit a critical point of (6) in a large class of connections (e.g., adapted), or for integrable distributions $\mathcal{D}$. On the other hand, in the case of adapted variations, antisymmetric parts of $(\mathfrak{T}|_{\mathcal{D} \times \mathcal{D}})^\perp$ and $(\mathfrak{T}|_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}})^\top$ remain free parameters of any critical metric connection, as they do not appear in Euler–Lagrange equations (note that these components define part of the critical connection's torsion). Thus, for a given metric g that admits critical points of (6), one can expect to have multiple critical metric connections, and examples in Sect. 3.3 confirm that.

Section 3.4 deals with a semi-symmetric connection (parameterized by a vector field), as a simple case of a metric connection. Although such connections are critical for the action (6) and arbitrary variations of connections only on metric-affine products, when we restrict variations of the mixed scalar curvature to semi-symmetric connections, we obtain meaningful Euler–Lagrange equations (in Theorem 6), which allow us to explicitly present the mixed Ricci tensor—analogous to the Ricci tensor in the Einstein equation.

2. Preliminaries

Here, we recall definitions of some functions and tensors, used also in [3,25–27,32], and introduce several new notions related to geometry of $(M, g, \bar{\nabla})$ endowed with a non-degenerate distribution.

2.1. The Mixed Scalar Curvature

Let $\mathrm{Sym}^2(M)$ be the space of symmetric $(0, 2)$-tensors tangent to a smooth connected manifold M. A *pseudo-Riemannian metric* $g = \langle \cdot, \cdot \rangle$ of index q on M is an element $g \in \mathrm{Sym}^2(M)$ such that each g_x ($x \in M$) is a non-degenerate bilinear form of index q on the tangent space $T_x M$. For $q = 0$ (i.e., g_x is positive definite) g is a Riemannian metric and for $q = 1$ it is called a Lorentz metric. Let $\mathrm{Riem}(M) \subset \mathrm{Sym}^2(M)$ be the subspace of pseudo-Riemannian metrics of a given signature.

A smooth subbundle $\widetilde{\mathcal{D}} \subset TM$ (that is, a regular distribution) is *non-degenerate*, if g_x is non-degenerate on $\widetilde{\mathcal{D}}_x \subset T_xM$ for $x \in M$; in this case, the orthogonal complement $\mathcal{D}$ of $\widetilde{\mathcal{D}}$ is also non-degenerate, and we have $\widetilde{\mathcal{D}}_x \cap \mathcal{D}_x = 0, \widetilde{\mathcal{D}}_x \oplus \mathcal{D}_x = T_xM$ for all $x \in M$. Let $\mathfrak{X}_M, \mathfrak{X}^{\perp}, \mathfrak{X}^{\top}$ be the modules over $C^{\infty}(M)$ of sections (vector fields) of $TM, \mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively.

Let $\mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D}) \subset \mathrm{Riem}(M)$ be the subspace of pseudo-Riemannian metrics making $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ (of ranks $\dim \widetilde{\mathcal{D}} = n \geq 1$ and $\dim \mathcal{D} = p \geq 1$) orthogonal and non-degenerate. Given $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$, a local adapted orthonormal frame $\{E_a, \mathcal{E}_i\}$, where $\{E_a\} \subset \widetilde{\mathcal{D}}$ and $\epsilon_i = \langle \mathcal{E}_i, \mathcal{E}_i \rangle \in \{-1, 1\}$, $\epsilon_a = \langle E_a, E_a \rangle \in \{-1, 1\}$, always exists on M. The following convention is adopted for the range of indices:

$$a, b, c \ldots \in \{1 \ldots n\}, \quad i, j, k \ldots \in \{1 \ldots p\}.$$

All the quantities defined below with the use of an adapted orthonormal frame do not depend on the choice of this frame. We have $X = \widetilde{X} + X^{\perp}$, where $\widetilde{X} \equiv X^{\top}$ is the $\widetilde{\mathcal{D}}$-component of $X \in \mathfrak{X}_M$ (respectively, $X^{\perp}$ is the $\mathcal{D}$-component of X) with respect to g. Set $\mathrm{id}^{\top}(X) = X^{\top}$ and $\mathrm{id}^{\perp}(X) = X^{\perp}$.

Definition 1. The function on $(M, g, \bar{\nabla})$ endowed with a non-degenerate distribution $\widetilde{\mathcal{D}}$,

$$\bar{\mathrm{S}}_{\mathrm{mix}} = \frac{1}{2} \sum\nolimits_{a,i} \epsilon_a \epsilon_i \big(\langle \bar{R}_{E_a, \mathcal{E}_i} E_a, \mathcal{E}_i \rangle + \langle \bar{R}_{\mathcal{E}_i, E_a} \mathcal{E}_i, E_a \rangle \big), \tag{9}$$

is called the *mixed scalar curvature with respect to connection* $\bar{\nabla}$. In particular case of the Levi-Civita connection ∇, the function on (M, g),

$$\mathrm{S}_{\mathrm{mix}} = \mathrm{Tr}_g\, r_{\mathcal{D}} = \sum\nolimits_{a,i} \epsilon_a \epsilon_i\, \langle R_{E_a, \mathcal{E}_i} E_a,\, \mathcal{E}_i \rangle \tag{10}$$

is called the *mixed scalar curvature* (with respect to ∇). The symmetric $(0, 2)$-tensor

$$r_{\mathcal{D}}(X, Y) = \sum\nolimits_a \epsilon_a\, \langle R_{E_a,\, X^{\perp}} E_a,\, Y^{\perp} \rangle, \quad X, Y \in \mathfrak{X}_M, \tag{11}$$

is called the *partial Ricci tensor* related to $\widetilde{\mathcal{D}}$.

Remark that on $(M, \widetilde{\mathcal{D}})$, the $\mathrm{S}_{\mathrm{mix}}$ and g-orthogonal complement to $\widetilde{\mathcal{D}}$ are determined by the choice of metric g. In particular, if $\dim \widetilde{\mathcal{D}} = 1$ then $r_{\mathcal{D}} = \epsilon_N R_N$, where $R_N = R_{N,\cdot}\, N$ is the Jacobi operator, and if $\dim \mathcal{D} = 1$ then $r_{\mathcal{D}} = \mathrm{Ric}_{N,N}\, g^{\perp}$, where the symmetric $(0,2)$-tensor $g^{\perp}$ is defined by $g^{\perp}(X, Y) = \langle X^{\perp}, Y^{\perp} \rangle$ for $X, Y \in \mathfrak{X}_M$.

We use the following convention for components of various $(1, 1)$-tensors in an adapted orthonormal frame $\{E_a, \mathcal{E}_i\}$: $\mathfrak{T}_a = \mathfrak{T}_{E_a}$, $\mathfrak{T}_i = \mathfrak{T}_{\mathcal{E}_i}$, etc. Following the notion of $\mathfrak{T}$-sectional curvature of a symmetric $(1, 2)$-tensor $\mathfrak{T}$ on a vector space endowed with a scalar product and a cubic form, see [18], we define the

mixed scalar $\mathfrak{T}$-curvature by (12), as a sum of $\mathfrak{T}$-sectional curvatures of planes that non-trivially intersect with both of the distributions,

$$\mathrm{S}_{\mathfrak{T}} = \sum\nolimits_{a,i} \epsilon_a \epsilon_i (\langle [\mathfrak{T}_i, \mathfrak{T}_a] E_a, \mathcal{E}_i \rangle + \langle [\mathfrak{T}_a, \mathfrak{T}_i] \mathcal{E}_i, E_a \rangle). \tag{12}$$

The definitions (12), (9)–(10) do not depend on the choice of an adapted local orthonormal frame. Thus, we can consider $\bar{\mathrm{S}}_{\mathrm{mix}}$ and $\mathrm{S}_{\mathfrak{T}}$ on $(M, \widetilde{\mathcal{D}})$ as functions of g and $\mathfrak{T}$. If $\mathfrak{T}$ is either symmetric or anti-symmetric then (12) reads as $\mathrm{S}_{\mathfrak{T}} = \sum_{a,i} \epsilon_a \epsilon_i \langle [\mathfrak{T}_i, \mathfrak{T}_a] E_a, \mathcal{E}_i \rangle$. As was mentioned in the Introduction, the *mixed scalar $\mathfrak{T}$-curvature* (for the contorsion tensor $\mathfrak{T}$) is a part of $\bar{\mathrm{S}}_{\mathrm{mix}}$, in fact we have [27, Eq. (6)]:

$$\bar{\mathrm{S}}_{\mathrm{mix}} = \mathrm{S}_{\mathrm{mix}} + \mathrm{S}_{\mathfrak{T}} + \bar{Q}/2, \tag{13}$$

where $\bar{Q}$ consists of all terms with covariant derivatives of $\mathfrak{T}$,

$$\bar{Q} = \sum\nolimits_{a,i} \epsilon_a \epsilon_i \big(\langle (\nabla_i \mathfrak{T})_a E_a, \mathcal{E}_i \rangle - \langle (\nabla_a \mathfrak{T})_i E_a, \mathcal{E}_i \rangle$$
$$+ \langle (\nabla_a \mathfrak{T})_i \mathcal{E}_i, E_a \rangle - \langle (\nabla_i \mathfrak{T})_a \mathcal{E}_i, E_a \rangle \big).$$

The formulas for the mixed scalar curvature in the next two propositions are essential in our calculations. The propositions use tensors defined in [25], which are briefly recalled below.

Proposition 1. *The following presentation of the partial Ricci tensor in* (11) *is valid, see* [3,25]:

$$r_{\mathcal{D}} = \mathrm{div}\, \tilde{h} + \langle \tilde{h}, \tilde{H} \rangle - \widetilde{\mathcal{A}}^{\flat} - \widetilde{\mathcal{T}}^{\flat} - \Psi + \mathrm{Def}_{\mathcal{D}} H. \tag{14}$$

Tracing (14), *we have, see* [32],

$$\mathrm{S}_{\mathrm{mix}} = \langle H, H \rangle + \langle \tilde{H}, \tilde{H} \rangle - \langle h, h \rangle - \langle \tilde{h}, \tilde{h} \rangle + \langle T, T \rangle + \langle \tilde{T}, \tilde{T} \rangle + \mathrm{div}(H + \tilde{H}). \tag{15}$$

For totally umbilical distributions, i.e., $h = \frac{1}{n} H\, g^{\top}$ *and* $\tilde{h} = \frac{1}{p}\, \tilde{H}\, g^{\perp}$, (15) *reads as*

$$\mathrm{S}_{\mathrm{mix}} = \frac{n-1}{n} \langle H, H \rangle + \frac{p-1}{p} \langle \tilde{H}, \tilde{H} \rangle + \langle T, T \rangle + \langle \tilde{T}, \tilde{T} \rangle + \mathrm{div}(H + \tilde{H}), \tag{16}$$

Denote by $\langle B, C \rangle_{|V}$ the inner product of tensors B, C restricted to $V = (\tilde{\mathcal{D}} \times \mathcal{D}) \cup (\mathcal{D} \times \tilde{\mathcal{D}})$.

Proposition 2. (see [21]) *We have using* (1),

$$2\,(\bar{\mathrm{S}}_{\mathrm{mix}} - \mathrm{S}_{\mathrm{mix}}) = \mathrm{div}\, \big((\mathrm{Tr}^{\top}(\mathfrak{T} - \mathfrak{T}^*))^{\perp} + (\mathrm{Tr}^{\perp}(\mathfrak{T} - \mathfrak{T}^*))^{\top} \big) - Q, \tag{17}$$

where

$$Q = -\langle \mathrm{Tr}^{\perp} \mathfrak{T}, \mathrm{Tr}^{\top} \mathfrak{T}^* \rangle - \langle \mathrm{Tr}^{\top} \mathfrak{T}, \mathrm{Tr}^{\perp} \mathfrak{T}^* \rangle + \langle \mathfrak{T}^*, \mathfrak{T}^{\wedge} \rangle_{|V}$$
$$- \langle \mathrm{Tr}^{\top}(\mathfrak{T} - \mathfrak{T}^*) - \mathrm{Tr}^{\perp}(\mathfrak{T} - \mathfrak{T}^*),\, H - \tilde{H} \rangle$$
$$- \langle \mathfrak{T} - \mathfrak{T}^* + \mathfrak{T}^{\wedge} - \mathfrak{T}^{*\wedge},\, \tilde{A} - \tilde{T}^{\sharp} + A - T^{\sharp} \rangle. \tag{18}$$

and the partial traces of $\mathfrak{T}$ (similarly, for $\mathfrak{T}^*$, etc.) are given by

$$\mathrm{Tr}^\top \mathfrak{T} = \sum_a \epsilon_a \mathfrak{T}_a E_a, \quad \mathrm{Tr}^\perp \mathfrak{T} = \sum_i \epsilon_i \mathfrak{T}_i \mathcal{E}_i. \tag{19}$$

The tensors used in the above results (and other ones) are defined below for one of the distributions (say, $\mathcal{D}$; similar tensors for $\widetilde{\mathcal{D}}$ are denoted using $^\top$ or $^\sim$ notation).

The integrability tensor and the second fundamental form $T, h : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \mathcal{D}$ of $\widetilde{\mathcal{D}}$ are given by

$$T(X,Y) = (1/2)\,[X,\,Y]^\perp, \quad h(X,Y) = (1/2)\,(\nabla_X Y + \nabla_Y X)^\perp, \quad X, Y \in \mathfrak{X}^\top.$$

The mean curvature vector field of $\widetilde{\mathcal{D}}$ is given by $H = \mathrm{Tr}_g\, h = \sum_a \epsilon_a h(E_a, E_a)$. We call $\widetilde{\mathcal{D}}$ totally umbilical, minimal, or totally geodesic, if $h = \frac{1}{n} H\, g^\top$, $H = 0$, or $h = 0$, respectively.

The "musical" isomorphisms $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, if $\omega \in \Lambda^1(M)$ is a 1-form and $X, Y \in \mathfrak{X}_M$ then $\omega(Y) = \langle \omega^\sharp, Y \rangle$ and $X^\flat(Y) = \langle X, Y \rangle$. For arbitrary (0,2)-tensors A and B we also have $\langle A, B \rangle = \mathrm{Tr}_g(A^\sharp B^\sharp) = \langle A^\sharp, B^\sharp \rangle$.

The Weingarten operator A_Z of $\widetilde{\mathcal{D}}$ with $Z \in \mathfrak{X}^\perp$, and the operator $T_Z^\sharp$ are defined by

$$\langle A_Z(X), Y \rangle = \langle h(X,Y), Z \rangle, \quad \langle T_Z^\sharp(X), Y \rangle = \langle T(X,Y), Z \rangle, \quad X, Y \in \mathfrak{X}^\top.$$

The norms of tensors are obtained using

$$\langle h, h \rangle = \sum_{a,b} \epsilon_a \epsilon_b \, \langle h(E_a, E_b), h(E_a, E_b) \rangle,$$

$$\langle T, T \rangle = \sum_{a,b} \epsilon_a \epsilon_b \, \langle T(E_a, E_b), T(E_a, E_b) \rangle, \quad \text{etc.}$$

The *divergence* of a vector field $X \in \mathfrak{X}_M$ is given by

$$(\mathrm{div}\,X)\,\mathrm{d\,vol}_g = \mathcal{L}_X(\mathrm{d\,vol}_g), \tag{20}$$

where $\mathrm{d\,vol}_g$ is the volume form of g. One may show that

$$\mathrm{div}\,X = \sum_i \epsilon_i \, \langle \nabla_i X, \mathcal{E}_i \rangle + \sum_a \epsilon_a \, \langle \nabla_a X, E_a \rangle.$$

The $\mathcal{D}$-*divergence* of a vector field X is given by $\mathrm{div}^\perp X = \sum_i \epsilon_i \, \langle \nabla_i X, \mathcal{E}_i \rangle$. Thus, $\mathrm{div}\,X = \mathrm{Tr}(\nabla X) = \mathrm{div}^\perp X + \widetilde{\mathrm{div}}\,X$. Observe that for $X \in \mathfrak{X}^\perp$ we have

$$\mathrm{div}^\perp X = \mathrm{div}\,X + \langle X, H \rangle. \tag{21}$$

For a (1,2)-tensor P define a (0,2)-tensor $\mathrm{div}^\perp P$ by

$$(\mathrm{div}^\perp P)(X,Y) = \sum_i \epsilon_i \, \langle (\nabla_i P)(X,Y), \mathcal{E}_i \rangle, \quad X, Y \in \mathfrak{X}_M.$$

For a $\mathcal{D}$-valued (1,2)-tensor P, similarly to (21), we have

$$(\mathrm{div}^\top P)(X,Y) = \sum_a \epsilon_a \, \langle (\nabla_a P)(X,Y), E_a \rangle = -\langle P(X,Y), H \rangle,$$

$$\mathrm{div}^\perp P = \mathrm{div}\,P + \langle P, H \rangle,$$

where $\langle P, H \rangle$ is a $(0,2)$-tensor, $\langle P, H \rangle(X,Y) = \langle P(X,Y), H \rangle$. For example, $\operatorname{div}^\perp h = \operatorname{div} h + \langle h, H \rangle$. For a function f on M, we use the notation $\nabla^\perp f = (\nabla f)^\perp$ of the projection of ∇f onto $\mathcal{D}$.

The $\mathcal{D}$-*deformation tensor* $\operatorname{Def}_\mathcal{D} Z$ of $Z \in \mathfrak{X}_M$ is the symmetric part of ∇Z restricted to $\mathcal{D}$,

$$2 \operatorname{Def}_\mathcal{D} Z(X,Y) = \langle \nabla_X Z, Y \rangle + \langle \nabla_Y Z, X \rangle, \quad X, Y \in \mathfrak{X}^\perp.$$

The self-adjoint $(1,1)$-tensors: $\mathcal{A}$ (the *Casorati type operator*) and $\mathcal{T}$ and the symmetric $(0,2)$-tensor Ψ, see [3,25], are defined by

$$\mathcal{A} = \sum\nolimits_i \epsilon_i A_i^2, \quad \mathcal{T} = \sum\nolimits_i \epsilon_i (T_i^\sharp)^2,$$

$$\Psi(X,Y) = \operatorname{Tr}(A_Y A_X + T_Y^\sharp T_X^\sharp), \quad X, Y \in \mathfrak{X}^\perp.$$

For readers' convenience, we gather below also definitions of all other basic tensors that will be used in further parts of the paper. We define a self-adjoint $(1,1)$-tensor $\mathcal{K}$ by the formula

$$\mathcal{K} = \sum\nolimits_i \epsilon_i [T_i^\sharp, A_i] = \sum\nolimits_i \epsilon_i (T_i^\sharp A_i - A_i T_i^\sharp),$$

and the $(1,2)$-tensors α, θ and $\tilde{\delta}_Z$ (defined for a given vector field $Z \in \mathfrak{X}_M$) on $(M, \tilde{\mathcal{D}}, g)$:

$$\alpha(X,Y) = \frac{1}{2}(A_{X^\perp}(Y^\top) + A_{Y^\perp}(X^\top)), \quad \theta(X,Y) = \frac{1}{2}(T_{X^\perp}^\sharp(Y^\top) + T_{Y^\perp}^\sharp(X^\top)),$$

$$\tilde{\delta}_Z(X,Y) = \frac{1}{2}\left(\langle \nabla_{X^\top} Z, Y^\perp \rangle + \langle \nabla_{Y^\top} Z, X^\perp \rangle \right), \quad X, Y \in \mathfrak{X}_M.$$

For any $(1,2)$-tensors P, Q and a $(0,2)$-tensor S on TM, define the following $(0,2)$-tensor $\Upsilon_{P,Q}$:

$$\langle \Upsilon_{P,Q}, S \rangle = \sum\nolimits_{\lambda,\mu} \epsilon_\lambda \epsilon_\mu \left[S(P(e_\lambda, e_\mu), Q(e_\lambda, e_\mu)) + S(Q(e_\lambda, e_\mu), P(e_\lambda, e_\mu)) \right],$$

where on the left-hand side we have the inner product of $(0,2)$-tensors induced by g, $\{e_\lambda\}$ is a local orthonormal basis of TM and $\epsilon_\lambda = \langle e_\lambda, e_\lambda \rangle \in \{-1, 1\}$. Note that

$$\Upsilon_{P,Q} = \Upsilon_{Q,P}, \quad \Upsilon_{P, fQ_1 + Q_2} = f\Upsilon_{P,Q_1} + \Upsilon_{P,Q_2}.$$

Finally, for the contorsion tensor and $X \in TM$ we define $\mathfrak{T}_X^\top : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ by

$$\mathfrak{T}_X^\top Y = (\mathfrak{T}_X(Y^\top))^\top, \quad Y \in TM.$$

Remark 1. From now on, we shall omit factors ϵ_μ in all expressions with sums over an adapted frame (or its part), effectively identifying symbols $\sum_\mu$ with $\sum_\mu \epsilon_\mu$ etc. As we assume in this paper that g is non-degenerate on the distribution $\tilde{\mathcal{D}}$, the presence of factors ϵ_μ in the sums is the only difference in formulas with adapted frames for a Riemannian and a pseudo-Riemannian metric g. With the definitions given in this section, all tensor equations that

follow look exactly the same in both these cases. In more complicated formulas we shall also omit summation indices, assuming that every sum is taken over all indices that appear repeatedly after the summation sign and contains appropriate factors ϵ_μ.

2.2. The Mixed Ricci Curvature

Let (M, g) be a pseudo-Riemannian manifold endowed with a non-degenerate distribution $\widetilde{\mathcal{D}}$. We consider smooth 1-parameter variations $\{g_t \in \mathrm{Riem}(M) : |t| < \epsilon\}$ of the metric $g_0 = g$. Let the infinitesimal variations, represented by a symmetric $(0, 2)$-tensor

$$B_t \equiv \partial g_t / \partial t,$$

be supported in a relatively compact domain Ω in M, i.e., $g_t = g$ outside Ω for all $|t| < \epsilon$. We call a variation g_t *volume-preserving* if $\mathrm{Vol}(\Omega, g_t) = \mathrm{Vol}(\Omega, g)$ for all t. We adopt the notations $\partial_t \equiv \partial / \partial t$, $B \equiv \partial_t g_{t \,|\, t=0} = \dot{g}$, but we shall also write B instead of B_t to make formulas easier to read, wherever it does not lead to confusion. Since B is symmetric, then $\langle C, B \rangle = \langle \mathrm{Sym}(C), B \rangle$ for any $(0, 2)$-tensor C. We denote by $\otimes$ the product of tensors and use the symmetrization operator to define the symmetric product of tensors: $B \odot C = \mathrm{Sym}(B \otimes C) = \frac{1}{2} (B \otimes C + C \otimes B)$.

Definition 2. A family of metrics $\{g_t \in \mathrm{Riem}(M) : |t| < \epsilon\}$ such that $g_0 = g$ will be called

(i) $g^{\pitchfork}$-*variation* if $g_t(X, Y) = g_0(X, Y)$ for all $X, Y \in \mathfrak{X}^\top$ and $|t| < \epsilon$.

(ii) *adapted variation*, if the g_t-orthogonal complement $\mathcal{D}_t$ remain g_0-orthogonal to $\widetilde{\mathcal{D}}$ for all t.

(iii) $g^\top$-*variation*, if it is adapted and $g_t(X, Y) = g_0(X, Y)$ for all $X, Y \in \mathfrak{X}^\perp$ and $|t| < \epsilon$.

(iv) $g^\perp$-*variation*, if it is adapted $g^{\pitchfork}$-variation.

In other words, for $g^{\pitchfork}$-variations the metric on $\widetilde{\mathcal{D}}$ is preserved. For adapted variation we have $g_t \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ for all t. For $g^\top$-variations only the metric on $\widetilde{\mathcal{D}}$ changes, and for $g^\perp$-*variations* only the metric on $\mathcal{D}$ changes, and $\mathcal{D}$ remains to be g_t-orthogonal to $\widetilde{\mathcal{D}}$.

The symmetric tensor $B_t = \dot{g}_t$ (of any variation) can be decomposed into the sum of derivatives of $g^{\pitchfork}$- and $g^\top$-variations, see [26]. Namely, $B_t = B_t^{\pitchfork} + \tilde{B}_t$, where

$$B_t^{\pitchfork} = \begin{pmatrix} B_{t \,|\, \mathcal{D} \times \mathcal{D}} & B_{t \,|\, \mathcal{D} \times \widetilde{\mathcal{D}}} \\ B_{t \,|\, \widetilde{\mathcal{D}} \times \mathcal{D}} & 0 \end{pmatrix}, \quad \tilde{B}_t = \begin{pmatrix} 0 & 0 \\ 0 & B_{t \,|\, \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} \end{pmatrix}.$$

Thus, for $g^{\pitchfork}$-variations $B(X, Y) = 0$ for all $X, Y \in \mathfrak{X}^\top$. Denote by $^\top$ and $^\perp$ the g_t-orthogonal projections of vectors onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}(t)$ (the g_t-orthogonal complement of $\widetilde{\mathcal{D}}$), respectively.

Proposition 3. (see [26]) *Let g_t be a $g^{\pitchfork}$-variation of $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Let $\{E_a, \mathcal{E}_i\}$ be a local $(\widetilde{\mathcal{D}}, \mathcal{D})$-adapted and orthonormal for $t = 0$ frame, that evolves according to*

$$\partial_t E_a = 0, \qquad \partial_t \mathcal{E}_i = -(1/2)\,(B_t^{\sharp}(\mathcal{E}_i))^{\perp} - (B_t^{\sharp}(\mathcal{E}_i))^{\top}. \tag{22}$$

Then, for all t, $\{E_a(t), \mathcal{E}_i(t)\}$ is a g_t-orthonormal frame adapted to $(\widetilde{\mathcal{D}}, \mathcal{D}(t))$.

For any $g^{\pitchfork}$-variation of metric the evolution of $\mathcal{D}(t)$ gives rise to the evolution of both $\widetilde{\mathcal{D}}$- and $\mathcal{D}(t)$-components of any $X \in \mathfrak{X}_M$:

$$\partial_t(X^{\top}) = (\partial_t X)^{\top} + (B^{\sharp}(X^{\perp}))^{\top}, \quad \partial_t(X^{\perp}) = (\partial_t X)^{\perp} - (B^{\sharp}(X^{\perp}))^{\top}.$$

The Divergence Theorem (with $X \in \mathfrak{X}_M$) states that

$$\int_M (\mathrm{div}\, X)\, \mathrm{d\, vol}_g = 0, \tag{23}$$

when M is closed (compact and without boundary); this is also true if M is open and X is supported in a relatively compact domain $\Omega \subset M$. For any variation g_t of metric g on M with $B = \partial_t g$ we have

$$\partial_t \left(\mathrm{d\, vol}_g \right) = \frac{1}{2}\, (\mathrm{Tr}_g B)\, \mathrm{d\, vol}_g, \tag{24}$$

e.g., [29]. By Lemma 1 and (23)–(24),

$$\frac{d}{dt} \int_M (\mathrm{div}\, X)\, \mathrm{d\, vol}_g = \int_M \mathrm{div} \left(\partial_t X + \frac{1}{2}\,(\mathrm{Tr}_g B) X \right) \mathrm{d\, vol}_g = 0 \tag{25}$$

for any variation g_t of metric with $\mathrm{supp}\,(\partial_t g) \subset \Omega$, and t-dependent $X \in \mathfrak{X}_M$ with $\mathrm{supp}\,(\partial_t X) \subset \Omega$.

Let V be the linear subspace of $TM \times TM$ spanned by $(\mathcal{D} \times \widetilde{\mathcal{D}}) \cup (\widetilde{\mathcal{D}} \times \mathcal{D})$. Thus, the product $TM \times TM$ is the sum of three subbundles, $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$, $\mathcal{D} \times \mathcal{D}$ and V. Using this decomposition, we define the tensor in (5).

Definition 3. (see [22]) The symmetric $(0, 2)$-tensor $\mathrm{Ric}_{\mathcal{D}}$ in (5), defined by its restrictions on three complementary subbundles of $TM \times TM$, is referred to as the *mixed Ricci curvature*:

$$\begin{cases} \mathrm{Ric}_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} = r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \widetilde{\mathcal{A}}^{\flat} - \widetilde{\mathcal{T}}^{\flat} + \Psi - \mathrm{Def}_{\mathcal{D}}\, H + \widetilde{\mathcal{K}}^{\flat} \\ \quad + H^{\flat} \otimes H^{\flat} - \frac{1}{2}\, \Upsilon_{h,h} - \frac{1}{2}\, \Upsilon_{T,T} - \frac{n-1}{p+n-2}\, \mathrm{div}(\tilde{H} - H)\, g^{\perp}, \\ \mathrm{Ric}_{\mathcal{D}\,|\,V} = -4\langle \theta, \tilde{H} \rangle - 2(\mathrm{div}(\alpha - \tilde{\theta}))_{|V} - 2\langle \tilde{\theta} - \tilde{\alpha}, H \rangle \\ \quad - 2\,\mathrm{Sym}(H^{\flat} \otimes \tilde{H}^{\flat}) + 2\, \tilde{\delta}_H - 4\,\Upsilon_{\tilde{\alpha},\theta} - 2\,\Upsilon_{\alpha,\tilde{\alpha}} - 2\,\Upsilon_{\theta,\tilde{\theta}}, \\ \mathrm{Ric}_{\mathcal{D}\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} = r_{\widetilde{\mathcal{D}}} - \langle h, H \rangle + \mathcal{A}^{\flat} - \mathcal{T}^{\flat} + \widetilde{\Psi} - \mathrm{Def}_{\widetilde{\mathcal{D}}}\, \tilde{H} + \mathcal{K}^{\flat} \\ \quad + \tilde{H}^{\flat} \otimes \tilde{H}^{\flat} - \frac{1}{2}\, \Upsilon_{\tilde{h},\tilde{h}} - \frac{1}{2}\, \Upsilon_{\tilde{T},\tilde{T}} + \frac{p-1}{p+n-2}\, \mathrm{div}(\tilde{H} - H)\, g^{\top}. \end{cases} \tag{26}$$

Here $(26)_3$ is dual to $(26)_1$ with respect to interchanging distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, and their last terms vanish if $n = p = 1$. Also, $\mathrm{S}_{\mathcal{D}} := \mathrm{Tr}_g\, \mathrm{Ric}_{\mathcal{D}} = \mathrm{S}_{\mathrm{mix}} + \frac{p-n}{n+p-2}\, \mathrm{div}(H - \tilde{H})$.

The following theorem, which allows us to restore the partial Ricci curvature (26), is based on calculating the variations with respect to g of components in (15) and using (25) for divergence terms. According to this theorem and Definition 3 we conclude that a metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}})$ is critical for the action (6) with fixed $\mathfrak{T} = 0$ (i.e., considered as a functional of g only), with respect to volume-preserving variations of metric if and only if (5) holds.

Theorem 1. (see [26]) *A metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}})$ is critical for the action* (6) *with fixed $\mathfrak{T} = 0$, with respect to volume-preserving $g^{\pitchfork}$-variations if and only if*

$$r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \widetilde{\mathcal{A}}^{\flat} - \widetilde{\mathcal{T}}^{\flat} + \Psi - \mathrm{Def}_{\mathcal{D}} H + \widetilde{\mathcal{K}}^{\flat} + H^{\flat} \otimes H^{\flat} - \frac{1}{2}\Upsilon_{h,h} - \frac{1}{2}\Upsilon_{T,T}$$

$$-\frac{1}{2}\left(\mathrm{S}_{\mathrm{mix}} + \mathrm{div}(\tilde{H} - H) \right) g^{\perp} = \lambda\, g^{\perp}, \tag{27a}$$

$$-4\langle \theta, \tilde{H} \rangle - 2(\mathrm{div}(\alpha - \tilde{\theta}))_{|\mathrm{V}} - 2\langle \tilde{\theta} - \tilde{\alpha}, H \rangle - 2\, H^{\flat} \odot \tilde{H}^{\flat} + 2\, \tilde{\delta}_{H}$$

$$-4\Upsilon_{\tilde{\alpha},\theta} - 2\Upsilon_{\alpha,\tilde{\alpha}} - 2\Upsilon_{\theta,\tilde{\theta}} = 0 \tag{27b}$$

for some $\lambda \in \mathbb{R}$. The Euler–Lagrange equation for volume-preserving $g^{\top}$-variations is dual to (27a).

Example 1. For a *space-time* (M^{p+1}, g) endowed with $\widetilde{\mathcal{D}}$ spanned by a timelike unit vector field N, the tensor $\mathrm{Ric}_{\mathcal{D}}$, see (26) with $n = 1$, and its trace have the following particular form:

$$\begin{cases} \mathrm{Ric}_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} = \epsilon_{N}(R_{N} + (\widetilde{A}_{N})^{2} - (\widetilde{T}_{N}^{\sharp})^{2} + [\widetilde{T}_{N}^{\sharp},\, \widetilde{A}_{N}])^{\flat} \\ \quad + H^{\flat} \otimes H^{\flat} - \tilde{\tau}_{1}\, \tilde{h}_{sc} - \mathrm{Def}_{\mathcal{D}} H, \\ \mathrm{Ric}_{\mathcal{D}}(\cdot, N)_{|\,\mathcal{D}} = \mathrm{div}^{\perp}\widetilde{T}_{N}^{\sharp}|_{\mathcal{D}} + 2\,(\widetilde{T}_{N}^{\sharp}(H))^{\flat}, \\ \mathrm{Ric}_{\mathcal{D}}(N, N) = \epsilon_{N}\, \mathrm{Ric}_{N,N} - 2\,\|\widetilde{T}\|^{2} - \mathrm{div}\, H, \end{cases} \tag{28}$$

$$\mathrm{S}_{\mathcal{D}} = \epsilon_{N}\, \mathrm{Ric}_{N,N} + \mathrm{div}(\epsilon_{N}\, \tilde{\tau}_{1} N - H). \tag{29}$$

Here $\tilde{\tau}_{i} = \mathrm{Tr}((\widetilde{A}_{N})^{i})$, $\widetilde{A}_{N}$ is the shape operator, $\widetilde{T}$ is the integrability tensor and $\tilde{h}_{sc}$ is the scalar second fundamental form of $\mathcal{D}$. Note that the right-hand side of $(28)_{2}$ vanishes when $\mathcal{D}$ is integrable.

2.3. Variations with Respect to $\mathfrak{T}$

The next theorem is based on calculating the variations with respect to $\mathfrak{T}$ of components $\mathrm{S}_{\mathfrak{T}}$ and $\bar{Q}/2$ in (13) and using (25) for divergence terms. Here $\{e_{\lambda}\}$ are vectors of an adapted frame, without distinguishing distribution to which they belong.

Theorem 2. *The Euler–Lagrange equation for* (4) *with fixed g, considered as a functional of an arbitrary $(1,2)$-tensor $\mathfrak{T}$, for all variations of $\mathfrak{T}$, is the following algebraic system with spin tensor $s_{\mu\nu}^{c} = 2\,\partial\mathcal{L}/\partial\mathfrak{T}_{\mu\nu}^{c}$ (hence $s_{\alpha\beta}^{\gamma} = \langle s(e_{\alpha}, e_{\beta}), e_{\gamma} \rangle$):*

$$\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T}^* - \tilde{H}, Z\,\rangle\,\langle X, Y\rangle + \langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T} + \tilde{H}, Y\,\rangle\,\langle X, Z\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(X, Y), Z\rangle, \tag{30a}$$

$$\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T}^* - H, W\,\rangle\,\langle U, V\rangle + \langle\, \mathrm{Tr}^{\top}\, \mathfrak{T} + H, V\,\rangle\,\langle U, W\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(U, V), W\rangle, \tag{30b}$$

$$\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T}^* + H, U\,\rangle\,\langle X, Y\rangle - \langle (A_U - T_U^{\sharp} + \mathfrak{T}_U)X,\, Y\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(X, Y), U\rangle, \tag{30c}$$

$$\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T}^* + \tilde{H}, X\,\rangle\,\langle U, V\rangle - \langle (\tilde{A}_X - \tilde{T}_X^{\sharp} + \mathfrak{T}_X)U,\, V\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(U, V), X\rangle, \tag{30d}$$

$$\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T} - H, U\,\rangle\,\langle X, Y\rangle + \langle (A_U + T_U^{\sharp} - \mathfrak{T}_U)Y,\, X\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(X, U), Y\rangle, \tag{30e}$$

$$\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T} - \tilde{H}, X\,\rangle\,\langle U, V\rangle + \langle (\tilde{A}_X + \tilde{T}_X^{\sharp} - \mathfrak{T}_X)V,\, U\rangle$$
$$= -(\mathfrak{a}/2)\,\langle s(U, X), V\rangle, \tag{30f}$$

$$2\,\langle \tilde{T}_X^{\sharp} U,\, V\rangle + \langle \mathfrak{T}_U V + \mathfrak{T}_V^* U,\, X\rangle = (\mathfrak{a}/2)\,\langle s(X, U), V\rangle, \tag{30g}$$

$$2\,\langle T_U^{\sharp}X,\, Y\rangle + \langle \mathfrak{T}_X Y + \mathfrak{T}_Y^* X,\, U\rangle = (\mathfrak{a}/2)\,\langle s(U, X), Y\rangle, \tag{30h}$$

for all $X, Y, Z \in \widetilde{\mathcal{D}}$ and $U, V, W \in \mathcal{D}$, see [27, Eqs. (15a–h)], where variations of Lagrangian $\mathcal{L}$, i.e., spin tensor in (30a–h), are omitted. Here, (30b, d, f, h) are dual to (30a, c, e, g).

Proof. Set $S = \partial_t \mathfrak{T}^t{}_{|\,t=0}$ for a one-parameter family $\mathfrak{T}^t$ ($|t| < \varepsilon$) of $(1, 2)$-tensors. Using Proposition 2 and removing integrals of divergences of compactly supported (in a domain Ω) vector fields, we get

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_M \bar{\mathrm{S}}_{\mathrm{mix}}(\mathfrak{T}^t)\,\mathrm{d\,vol}_g\,|_{t=0} = \frac{1}{2} \int_M \sum \Big\{ \langle S_a E_b, E_c\rangle$$
$$\times \big(\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T}^* - \tilde{H}, E_c\,\rangle\,\langle E_a, E_b\rangle + \langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T} + \tilde{H}, E_b\,\rangle\,\langle E_a, E_c\rangle\big)$$
$$+ \langle S_a E_b, \mathcal{E}_i\rangle\big(\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T}^* + H, \mathcal{E}_i\,\rangle\,\langle E_a, E_b\rangle - \langle (A_i - T_i^{\sharp})E_a, E_b\rangle - \langle \mathfrak{T}_i E_a, E_b\rangle\big)$$
$$+ \langle S_a \mathcal{E}_i, E_b\rangle\big(\langle\, \mathrm{Tr}^{\perp}\, \mathfrak{T} - H, \mathcal{E}_i\,\rangle\,\langle E_a, E_b\rangle + \langle (A_i + T_i^{\sharp})E_b, E_a\rangle - \langle \mathfrak{T}_i E_b, E_a\rangle\big)$$
$$+ \langle S_a \mathcal{E}_i, \mathcal{E}_j\rangle\big(\langle (\tilde{A}_a - \tilde{T}_a^{\sharp})\mathcal{E}_i, \mathcal{E}_j\rangle - \langle (\tilde{A}_a + \tilde{T}_a^{\sharp})\mathcal{E}_i, \mathcal{E}_j\rangle - \langle \mathfrak{T}_i \mathcal{E}_j + \mathfrak{T}_j^* \mathcal{E}_i, E_a\rangle\big)$$
$$+ \langle S_i \mathcal{E}_j, \mathcal{E}_k\rangle\big(\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T}^* - H, \mathcal{E}_k\,\rangle\,\langle \mathcal{E}_i, \mathcal{E}_j\rangle + \langle\, \mathrm{Tr}^{\top}\, \mathfrak{T} + H, \mathcal{E}_j\,\rangle\,\langle \mathcal{E}_i, \mathcal{E}_k\rangle\big)$$
$$+ \langle S_i \mathcal{E}_j, E_a\rangle\big(\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T}^* + \tilde{H}, E_a\,\rangle\,\langle \mathcal{E}_i, \mathcal{E}_j\rangle - \langle (\tilde{A}_a + \tilde{T}_a^{\sharp})\mathcal{E}_j, \mathcal{E}_i\rangle - \langle \mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j\rangle\big)$$
$$+ \langle S_i E_a, \mathcal{E}_j\rangle\big(\langle\, \mathrm{Tr}^{\top}\, \mathfrak{T} - \tilde{H}, E_a\,\rangle\,\langle \mathcal{E}_i, \mathcal{E}_j\rangle + \langle (\tilde{A}_a + \tilde{T}_a^{\sharp})\mathcal{E}_j, \mathcal{E}_i\rangle - \langle \mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i\rangle\big)$$
$$+ \langle S_i E_a, E_b\rangle\big(\langle (A_i - T_i^{\sharp})E_a, E_b\rangle - \langle (A_i + T_i^{\sharp})E_a, E_b\rangle - \langle \mathfrak{T}_a E_b + \mathfrak{T}_b^* E_a, \mathcal{E}_i\rangle\big)\Big\}\,\mathrm{d\,vol}_g\,.$$

Since no further assumptions are made about S or $\mathfrak{T}$, all the components $\langle S_\mu e_\lambda, e_\rho\rangle$ are independent and the above formula gives rise to (30a–h), where $X, Y, Z \in \widetilde{\mathcal{D}}$ and $U, V, W \in \mathcal{D}$ are any vectors from an adapted frame. Observe that in every equation from (30a–h) each term contains the same set of those vectors and is trilinear in them, so all these equations hold in fact for all vectors

$X, Y, Z \in \widetilde{\mathcal{D}}$ and $U, V, W \in \mathcal{D}$. Further below, we obtain many other formulas from computations in adapted frames, in the same way. $\square$

Taking difference of symmetric (in X, Y) parts of (30c,e) with $s = 0$ yields that $\widetilde{\mathcal{D}}$ is totally umbilical—similar result for $\mathcal{D}$ follows from dual equations (e.g., [27]). For vacuum space-time ($\mathcal{L} = 0$), the (30a–h) are simplified to the following equations (31a–j).

Corollary 1. (see Theorem 1 in [27]) *Let a metric-affine manifold* $(M, g, \bar{\nabla} = \nabla + \mathfrak{T})$ *be endowed with a non-degenerate distribution* $\widetilde{\mathcal{D}}$. *Then* $\mathfrak{T}$ *is critical for the action* (6) *with fixed g for all variations of* $\mathfrak{T}$ *if and only if* $\widetilde{\mathcal{D}}$ *and* $\mathcal{D}$ *are totally umbilical and* $\mathfrak{T}$ *satisfies the following linear algebraic system for all* $X, Y \in \widetilde{\mathcal{D}}$ *and* $U, V \in \mathcal{D}$:

$$(\mathfrak{T}_U\, V + \mathfrak{T}_V^*\, U)^\top = -2\,\widetilde{T}(U,V), \tag{31a}$$

$$(\mathrm{Tr}^\perp \mathfrak{T}^*)^\top = \tilde{H} = -(\mathrm{Tr}^\perp \mathfrak{T})^\top \quad \text{for } n > 1, \tag{31b}$$

$$\mathfrak{T}_U^\top - \mathfrak{T}_U^{*\top} = 2\, T_U^\sharp, \tag{31c}$$

$$\mathfrak{T}_U^\top + \mathfrak{T}_U^{*\top} = \langle\, \mathrm{Tr}^\perp (\mathfrak{T} + \mathfrak{T}^*),\, U \rangle\, \mathrm{id}^\top, \tag{31d}$$

$$(\mathrm{Tr}^\perp (\mathfrak{T} - \mathfrak{T}^*))^\perp = (2 - 2/n)\, H, \tag{31e}$$

$$(\mathfrak{T}_X\, Y + \mathfrak{T}_Y^*\, X)^\perp = -2\, T(X,Y), \tag{31f}$$

$$(\mathrm{Tr}^\top \mathfrak{T}^*)^\perp = H = -(\mathrm{Tr}^\top \mathfrak{T})^\perp \quad \text{for } p > 1, \tag{31g}$$

$$\mathfrak{T}_X^\perp - \mathfrak{T}_X^{*\perp} = 2\, \tilde{T}_X^\sharp, \tag{31h}$$

$$\mathfrak{T}_X^\perp + \mathfrak{T}_X^{*\perp} = \langle\, \mathrm{Tr}^\top (\mathfrak{T} + \mathfrak{T}^*),\, X \rangle\, \mathrm{id}^\perp, \tag{31i}$$

$$(\mathrm{Tr}^\top (\mathfrak{T} - \mathfrak{T}^*))^\top = (2 - 2/p)\, \tilde{H}. \tag{31j}$$

Example 2. For $(M^{p+1}, g, \widetilde{\mathcal{D}})$ of Example 1, the system (30a–h) reduces to

$$\langle\, \mathrm{Tr}^\perp (\mathfrak{T}^* + \mathfrak{T}), N \rangle = -(\mathfrak{a}/2)\, \langle s(N,N), N \rangle,$$

$$\langle\, \mathrm{Tr}^\top \mathfrak{T}^* - H, W \rangle \langle U, V \rangle + \langle\, \mathrm{Tr}^\top \mathfrak{T} + H, V \rangle \langle U, W \rangle = -(\mathfrak{a}/2) \langle s(U,V), W \rangle,$$

$$\langle\, \mathrm{Tr}^\perp \mathfrak{T}^*,\, U \rangle - \langle \mathfrak{T}_U\, N, N \rangle = -(\mathfrak{a}/2)\, \langle s(N,N), U \rangle,$$

$$(\langle\, \mathrm{Tr}^\top \mathfrak{T}^*, N \rangle + \tilde{\tau}_1)\, \langle U, V \rangle - \langle (\tilde{A}_N - \tilde{T}_N^\sharp + \mathfrak{T}_N) U,\, V \rangle = -(\mathfrak{a}/2)\, \langle s(U,V), N \rangle,$$

$$\langle\, \mathrm{Tr}^\perp \mathfrak{T},\, U \rangle - \langle \mathfrak{T}_U\, N, N \rangle = -(\mathfrak{a}/2)\, \langle s(N,U), N \rangle,$$

$$(\langle\, \mathrm{Tr}^\top \mathfrak{T}, N \rangle - \tilde{\tau}_1)\, \langle U, V \rangle + \langle (\tilde{A}_N + \tilde{T}_N^\sharp - \mathfrak{T}_N) V,\, U \rangle = -(\mathfrak{a}/2)\, \langle s(U,N), V \rangle,$$

$$\langle\, 2\,\widetilde{T}(U,V) + \mathfrak{T}_U\, V + \mathfrak{T}_V^*\, U,\, N \rangle = (\mathfrak{a}/2)\, \langle s(N,U), V \rangle,$$

$$\langle (\mathfrak{T} + \mathfrak{T}^*)_N\, N,\, U \rangle = (\mathfrak{a}/2)\, \langle s(U,N), N \rangle,$$

where $U, V, W \in \mathcal{D}$.

3. Main Results

In Sect. 3.1 we consider the total mixed scalar curvature of contorsion tensor for general and particular connections, e.g., metric and statistical, and metric-affine doubly twisted products. In Sect. 3.2 we consider the total mixed scalar curvature of statistical manifolds endowed with a distribution. In Sect. 3.3 we consider the total mixed scalar curvature of Riemann–Cartan manifolds endowed with a distribution. In Sect. 3.4, we derive the Euler–Lagrange equations for semi-symmetric connections and present the mixed Ricci tensor explicitly in (64). Our aims are to find out which metrics admit critical points of examined functionals and which components of $\mathfrak{T}$ in these particular cases determine whether or not its mixed scalar curvature is critical in its class of connections. This might help to achieve better understanding of both mixed scalar curvature invariant and the role played by some components of contorsion tensor.

3.1. Variational Problem with Contorsion Tensor

By Proposition 2 and (12), we have the following decomposition [21] (note that these are terms of $-Q$ in the first line of (18)):

$$2\,\mathrm{S}_{\mathfrak{T}} = \langle\,\mathrm{Tr}^\top\mathfrak{T},\ \mathrm{Tr}^\perp\mathfrak{T}^*\rangle + \langle\,\mathrm{Tr}^\perp\mathfrak{T},\ \mathrm{Tr}^\top\mathfrak{T}^*\rangle - \langle\mathfrak{T}^\wedge,\mathfrak{T}^*\rangle_{|V}.$$

We consider arbitrary variations $\mathfrak{T}(t)$, $\mathfrak{T}(0) = \mathfrak{T}$, $|t| < \epsilon$, and variations corresponding to metric and statistical connections, while $\Omega \subset M$ contains supports of infinitesimal variations $\partial_t\mathfrak{T}(t)$. In such cases, the Divergence Theorem states that if $X \in \mathfrak{X}_M$ is supported in Ω then (23) holds.

Theorem 3. *A pair $(g, \mathfrak{T})$ is critical for the action (7) with respect to all variations of $\mathfrak{T}$ and g if and only if $\mathfrak{T}$ satisfies the following algebraic systems (for all $X, Y, Z \in \widetilde{\mathcal{D}}$ and $U, V, W \in \mathcal{D}$):*

$$\mathrm{Tr}^\top(\mathfrak{T}_V\mathfrak{T}_U^\wedge) - \frac{1}{2}\,\langle\mathfrak{T}_V\,U + \mathfrak{T}_U\,V,\ \mathrm{Tr}^\top\mathfrak{T}^*\rangle = 0, \tag{32a}$$

$$\langle\,\mathrm{Tr}^\perp\mathfrak{T} - \mathrm{Tr}^\top\mathfrak{T},\ \mathfrak{T}_Y^*\,U\rangle - \langle\mathfrak{T}_Y U + \mathfrak{T}_U\,Y,\ \mathrm{Tr}^\top\mathfrak{T}^*\rangle - \mathrm{Tr}^\perp(\mathfrak{T}_Y^*(\mathfrak{T}^*)_U^\wedge)$$
$$+\,\mathrm{Tr}^\top\left(\mathfrak{T}_Y^*(\mathfrak{T}^*)_U^\wedge + \mathfrak{T}_U\mathfrak{T}_Y^\wedge + \mathfrak{T}_Y\mathfrak{T}_U^\wedge\right) = 0 \tag{32b}$$

$$\mathrm{Tr}^\top(\mathfrak{T}_Y\mathfrak{T}_X^\wedge) - \frac{1}{2}\langle\mathfrak{T}_Y\,X + \mathfrak{T}_X\,Y,\ \mathrm{Tr}^\perp\mathfrak{T}^*\rangle = 0, \tag{32c}$$

and

$$(\mathfrak{T}_X^*\,Y + \mathfrak{T}_Y\,X)^\perp = 0, \tag{33a}$$

$$(\mathfrak{T}_U\,V + \mathfrak{T}_V^*\,U)^\top = 0, \tag{33b}$$

$$\langle X, Z\rangle\langle\,\mathrm{Tr}^\perp\mathfrak{T}, Y\rangle + \langle X, Y\rangle\langle\,\mathrm{Tr}^\perp\mathfrak{T}^*, Z\rangle = 0, \tag{33c}$$

$$\langle U, V\rangle\langle\,\mathrm{Tr}^\top\mathfrak{T}^*, W\rangle + \langle U, W\rangle\langle\,\mathrm{Tr}^\top\mathfrak{T}, V\rangle = 0, \tag{33d}$$

$$\mathfrak{T}_U^\top = \langle\,\mathrm{Tr}^\perp\mathfrak{T}, U\rangle\,\mathrm{id}^\top, \tag{33e}$$

$$\mathfrak{T}_X^\perp = \langle\,\mathrm{Tr}^\top\mathfrak{T}^*, X\rangle\,\mathrm{id}^\perp, \tag{33f}$$

$$(\mathrm{Tr}^\perp(\mathfrak{T} - \mathfrak{T}^*))^\perp = 0, \qquad (\mathrm{Tr}^\top(\mathfrak{T} - \mathfrak{T}^*))^\top = 0. \tag{33g}$$

Moreover, if $n > 1$ and $p > 1$ then (33c,d) read as

$$(\mathrm{Tr}^\perp \mathfrak{T})^\top = 0 = (\mathrm{Tr}^\perp \mathfrak{T}^*)^\top, \quad (\mathrm{Tr}^\top \mathfrak{T}^*)^\perp = 0 = (\mathrm{Tr}^\top \mathfrak{T})^\perp. \tag{34}$$

Proof. From Proposition 2 and Lemma 3, for a $g^\pitchfork$-variation g_t of metric g we obtain

$$
\begin{aligned}
2\,\partial_t \mathrm{S}_{\mathfrak{T}}(g_t) &= \partial_t \langle \mathrm{Tr}^\top \mathfrak{T}, \mathrm{Tr}^\perp \mathfrak{T}^* \rangle + \partial_t \langle \mathrm{Tr}^\perp \mathfrak{T}, \mathrm{Tr}^\top \mathfrak{T}^* \rangle - \partial_t \langle \mathfrak{T}^\wedge, \mathfrak{T}^* \rangle_{|V} \\
&= \frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_j)\big(\langle \mathrm{Tr}^\top \mathfrak{T}, \mathfrak{T}_i^* \mathcal{E}_j - \mathfrak{T}_j^* \mathcal{E}_i \rangle - \langle \mathfrak{T}_j \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_j, \mathrm{Tr}^\top \mathfrak{T}^* \rangle + 2 \langle \mathfrak{T}_j^* E_a, \mathfrak{T}_a \mathcal{E}_i \rangle \big) \\
&\quad + \sum B(\mathcal{E}_i, E_b)\big(\langle \mathrm{Tr}^\perp \mathfrak{T} - \mathrm{Tr}^\top \mathfrak{T}, \mathfrak{T}_b^* \mathcal{E}_i \rangle - \langle \mathfrak{T}_b \mathcal{E}_i + \mathfrak{T}_i E_b, \mathrm{Tr}^\top \mathfrak{T}^* \rangle \\
&\quad + \langle \mathfrak{T}_a^* \mathcal{E}_i, \mathfrak{T}_b E_a \rangle + \langle \mathfrak{T}_b^* E_a, \mathfrak{T}_a \mathcal{E}_i \rangle + \langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a E_b \rangle - \langle \mathfrak{T}_j^* \mathcal{E}_i, \mathfrak{T}_b \mathcal{E}_j \rangle \big).
\end{aligned}
\tag{35}
$$

Thus, $\partial_t \mathrm{S}_{\mathfrak{T}}(g_t) = 0$ if and only if the right hand side of (35) vanishes for all symmetric tensors $B = \partial_t g$. For the $(\mathcal{D} \times \mathcal{D})$-part of B we get

$$\sum B(\mathcal{E}_i, \mathcal{E}_j)\big(\frac{1}{2}\langle \mathrm{Tr}^\top \mathfrak{T}, \mathfrak{T}_i^* \mathcal{E}_j - \mathfrak{T}_j^* \mathcal{E}_i \rangle - \frac{1}{2}\langle \mathfrak{T}_j \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_j, \mathrm{Tr}^\top \mathfrak{T}^* \rangle + \mathrm{Tr}^\top (\mathfrak{T}_j \mathfrak{T}_i^\wedge) \big) = 0,$$

but since B is arbitrary and symmetric and $\mathfrak{T}_i^* \mathcal{E}_j - \mathfrak{T}_j^* \mathcal{E}_i$ is skew-symmetric, this can be written as (32a). For the mixed part of B (i.e., B restricted to the subspace V) we get the following Euler–Lagrange equation:

$$
\begin{aligned}
\sum B(\mathcal{E}_i, E_b)\big(\langle \mathrm{Tr}^\perp \mathfrak{T}, \mathfrak{T}_b^* \mathcal{E}_i \rangle - \langle \mathrm{Tr}^\top \mathfrak{T}, \mathfrak{T}_b^* \mathcal{E}_i \rangle - \langle \mathfrak{T}_b \mathcal{E}_i + \mathfrak{T}_i E_b, \mathrm{Tr}^\top \mathfrak{T}^* \rangle \\
+ \langle \mathfrak{T}_a^* \mathcal{E}_i, \mathfrak{T}_b E_a \rangle + \langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a E_b \rangle + \langle \mathfrak{T}_b^* E_a, \mathfrak{T}_a \mathcal{E}_i \rangle - \langle \mathfrak{T}_j^* \mathcal{E}_i, \mathfrak{T}_b \mathcal{E}_j \rangle \big) = 0.
\end{aligned}
$$

From this we obtain (32b). Taking dual equation to (32a) with respect to interchanging distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, we obtain (32c), which is the Euler–Lagrange equation for $g^\top$-variations. The proof of (33a–g), see [27], is based on calculation of variations with respect to $\mathfrak{T}$ of $\mathrm{S}_{\mathfrak{T}}$ and using (25). $\qquad\square$

Definition 4. (see Sect. 4 in [27]) The *doubly twisted product* $B \times_{(v,u)} F$ of metric-affine manifolds $(B, g_B, \mathfrak{T}_B)$ and $(F, g_F, \mathfrak{T}_F)$ (or the *metric-affine doubly twisted product*) is a manifold $M = B \times F$ with the metric $g = g^\top + g^\perp$ and the affine connection, whose contorsion tensor is $\mathfrak{T} = \mathfrak{T}^\top + \mathfrak{T}^\perp$, where

$$
\begin{aligned}
g^\top(X, Y) &= v^2 g_B(X^\top, Y^\top), \quad g^\perp(X, Y) = u^2 g_F(X^\perp, Y^\perp), \\
\mathfrak{T}_X^\top Y &= u^2 (\mathfrak{T}_B)_{X^\top} Y^\top, \quad \mathfrak{T}_X^\perp Y = v^2 (\mathfrak{T}_F)_{X^\perp} Y^\perp,
\end{aligned}
$$

and the warping functions $u, v \in C^\infty(M)$ are positive.

From Theorem 3 we obtain the following

Corollary 2. *A metric-affine doubly twisted product $B \times_{(v,u)} F$ with $\sum_a \epsilon_a \neq 0 \neq \sum_i \epsilon_i$ is critical for (7) with respect to all variations of $\mathfrak{T}$ and g if and only if*

$$\mathrm{Tr}\,\mathfrak{T}_B = 0 = \mathrm{Tr}\,\mathfrak{T}_F. \tag{36}$$

Proof. It was proven in [27, Corollary 13] that a metric-affine doubly twisted product $B \times_{(v,u)} F$ is critical for (7) with fixed g and for variations of $\mathfrak{T}$ if and only if (36) holds. It can be easily seen that for such doubly twisted product satisfying $\operatorname{Tr} \mathfrak{T}_B = 0 = \operatorname{Tr} \mathfrak{T}_F$ all terms in (32a–c) vanish. $\qquad\square$

Corollary 3. *A pair $(g, \mathfrak{T})$, where $\mathfrak{T}$ is the contorsion tensor of a statistical connection on (M, g), is critical for the action (7) with respect to all variations of metric, and variations of $\mathfrak{T}$ corresponding to statistical connections if and only if $\mathfrak{T}$ satisfies the following algebraic system:*

$$(\operatorname{Tr}^\top \mathfrak{T})^\top = 0 = (\operatorname{Tr}^\perp \mathfrak{T})^\perp, \tag{37a}$$

$$(\mathfrak{T}_X Y)^\perp = 0 = (\mathfrak{T}_U V)^\top, \quad X, Y \in \widetilde{\mathcal{D}}, \ \ U, V \in \mathcal{D}. \tag{37b}$$

Proof. By [27, Corollary 7], $\mathfrak{T}$ is critical for the action $\mathfrak{T} \mapsto \int_M S_{\mathfrak{T}} \, d\operatorname{vol}_g$, see (7), with respect to variations of $\mathfrak{T}$ corresponding to statistical connections if and only if the following equations hold:

$$(\operatorname{Tr}^\top \mathfrak{T})^\perp = 0 = (\operatorname{Tr}^\perp \mathfrak{T})^\top, \tag{38a}$$

$$(\mathfrak{T}_U V)^\top = \frac{1}{2} \langle U, V \rangle (\operatorname{Tr}^\top \mathfrak{T})^\top, \tag{38b}$$

$$(\mathfrak{T}_X Y)^\perp = \frac{1}{2} \langle X, Y \rangle (\operatorname{Tr}^\perp \mathfrak{T})^\perp, \tag{38c}$$

for all $X, Y \in \widetilde{\mathcal{D}}$ and $U, V \in \mathcal{D}$. If (37a,b) hold, then also (38a–c) hold, moreover if (37b) is satisfied and $\mathfrak{T}$ corresponds to a statistical connection, then all terms in equations (32a–c) vanish.

On the other hand, if (38a–c) hold, then (32a) becomes

$$\frac{n}{4} (\operatorname{Tr}^\perp \mathfrak{T})^{\perp\flat} \otimes (\operatorname{Tr}^\perp \mathfrak{T})^{\perp\flat} - \frac{3}{4} \langle (\operatorname{Tr}^\top \mathfrak{T})^\top, (\operatorname{Tr}^\top \mathfrak{T})^\top \rangle \, g^\perp = 0, \tag{39}$$

and (32c) becomes dual to the above. If $p > 1$ and $\langle (\operatorname{Tr}^\perp \mathfrak{T})^\perp, (\operatorname{Tr}^\perp \mathfrak{T})^\perp \rangle \neq 0$, then there is $W \in \mathcal{D}$ such that $\langle W, W \rangle \neq 0$ and $\langle W, (\operatorname{Tr}^\perp \mathfrak{T})^\perp \rangle = 0$, and evaluating (39) on $W \otimes W$ we obtain $(\operatorname{Tr}^\top \mathfrak{T})^\top = 0$ and then it also follows from (39) that $(\operatorname{Tr}^\perp \mathfrak{T})^\perp = 0$. If $p > 1$ and $(\operatorname{Tr}^\perp \mathfrak{T})^\perp = 0$, then we obtain $(\operatorname{Tr}^\top \mathfrak{T})^\top = 0$ from (32a), as $g^\perp$ is non-degenerate. If $p > 1$ and $\langle (\operatorname{Tr}^\perp \mathfrak{T})^\perp, (\operatorname{Tr}^\perp \mathfrak{T})^\perp \rangle = 0$ but $(\operatorname{Tr}^\perp \mathfrak{T})^\perp \neq 0$, then (32a) evaluated on $(\operatorname{Tr}^\perp \mathfrak{T})^\perp \otimes W$, where $W \in \mathcal{D}$, implies that

$$\langle (\operatorname{Tr}^\top \mathfrak{T})^\top, (\operatorname{Tr}^\top \mathfrak{T})^\top \rangle \langle (\operatorname{Tr}^\perp \mathfrak{T})^\perp, W \rangle = 0$$

and since W here is arbitrary, it follows that $\langle (\operatorname{Tr}^\top \mathfrak{T})^\top, (\operatorname{Tr}^\top \mathfrak{T})^\top \rangle = 0$, and then it also follows from (39) that $(\operatorname{Tr}^\perp \mathfrak{T})^\perp = 0$.

Equalities $(\operatorname{Tr}^\top \mathfrak{T})^\top = 0 = (\operatorname{Tr}^\perp \mathfrak{T})^\perp$ together with (38b,c) yield (37b).

If $n > 1$ we can similarly use (32c) for the same effect, and if $n = p = 1$ then (39) becomes

$$\langle (\operatorname{Tr}^\perp \mathfrak{T})^\perp, (\operatorname{Tr}^\perp \mathfrak{T})^\perp \rangle = 3 \langle (\operatorname{Tr}^\top \mathfrak{T})^\top, (\operatorname{Tr}^\top \mathfrak{T})^\top \rangle,$$

which together with its dual imply $(\mathrm{Tr}^\top \mathfrak{T})^\top = 0 = (\mathrm{Tr}^\perp \mathfrak{T})^\perp$, and again we obtain (37b) from (38b,c). □

Corollary 4. *A pair $(g, \mathfrak{T})$, where $\mathfrak{T}$ is the contorsion tensor of a metric connection on (M, g), is critical for (7) with respect to all variations of metric, and variations of $\mathfrak{T}$ corresponding to metric connections if and only if $\mathfrak{T}$ satisfies the following linear algebraic system (for all $X, Y \in \widetilde{\mathcal{D}}$ and $U, V \in \mathcal{D}$):*

$$(\mathfrak{T}_Y X + \mathfrak{T}_X^* Y)^\perp = 0 = (\mathfrak{T}_U V + \mathfrak{T}_V^* U)^\top, \tag{40a}$$

$$(\mathrm{Tr}^\top \mathfrak{T})^\top = 0 = (\mathrm{Tr}^\perp \mathfrak{T})^\perp, \tag{40b}$$

$$\mathfrak{T}_X^\perp = 0 = \mathfrak{T}_U^\top, \tag{40c}$$

$$(\mathrm{Tr}^\perp \mathfrak{T})^\top = 0 \quad for\ n > 1, \quad (\mathrm{Tr}^\top \mathfrak{T})^\perp = 0 \quad for\ p > 1. \tag{40d}$$

and for all $X \in \widetilde{\mathcal{D}}$ and $U \in \mathcal{D}$ we have

$$\mathrm{Tr}^\top ((\mathfrak{T}_U)^\perp (\mathfrak{T}_X^\wedge)^\perp + 2\,(\mathfrak{T}_U^\wedge)^\top (\mathfrak{T}_X)^\top) - \mathrm{Tr}^\perp ((\mathfrak{T}_U^\wedge)^\top (\mathfrak{T}_X)^\top)$$
$$+ \langle \mathrm{Tr}^\perp \mathfrak{T}, (\mathfrak{T}_X U)^\top \rangle = 0.$$

Proof. By [27, Corollary 8], $\mathfrak{T}$ is critical for the action $\mathfrak{T} \mapsto \int_M \mathrm{S}_{\mathfrak{T}}\,\mathrm{d\,vol}_g$, see (7), with respect to variations of $\mathfrak{T}$ corresponding to metric connections if and only if (40a–d) hold.

In (32a), by (40c) we have $\langle \mathfrak{T}_a \mathcal{E}_i, E_b \rangle = 0 = \langle \mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_k \rangle$, and by (40b) also $\langle \mathfrak{T}_a^* E_a, E_b \rangle = 0$. Hence, what remains in (32a) is

$$\langle (\mathfrak{T}_j \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_j)^\perp, \mathrm{Tr}^\top \mathfrak{T}^* \rangle = 0, \quad \forall\, i, j.$$

By (40d), this is identity if $p > 1$. On the other hand, for $p = 1$ it reduces to

$$2 \langle \mathfrak{T}_1 \mathcal{E}_1, \mathcal{E}_1 \rangle \langle \mathrm{Tr}^\top \mathfrak{T}^*, \mathcal{E}_1 \rangle = 0,$$

and by (40b), $\langle \mathfrak{T}_1 \mathcal{E}_1, \mathcal{E}_1 \rangle = 0$. Therefore, (32a) is satisfied if (40a–c) and the second equation in (40d) are satisfied. Using dual parts of (40a–d) we obtain analogous result for (32c). From (40a–d) we have for all b, c, i, k,

$$\sum \langle \mathfrak{T}_a E_a, E_c \rangle = 0, \quad \langle \mathfrak{T}_b^* \mathcal{E}_i, \mathcal{E}_k \rangle = 0, \quad \sum \langle \mathfrak{T}_a^* E_a, E_c \rangle = 0,$$
$$\langle \mathfrak{T}_b^* \mathcal{E}_i, \mathcal{E}_k \rangle = 0, \quad \langle \mathfrak{T}_i^* E_b, E_c \rangle = 0, \quad \langle \mathfrak{T}_b \mathcal{E}_i, \mathcal{E}_k \rangle = 0.$$

Thus, in (32b) we have only the following terms:

$$\sum \langle \mathfrak{T}_j \mathcal{E}_j, E_c \rangle \langle \mathfrak{T}_b^* \mathcal{E}_i, E_c \rangle + \sum \langle \mathfrak{T}_a^* \mathcal{E}_i, E_c \rangle \langle \mathfrak{T}_b E_a, E_c \rangle d + \sum \langle \mathfrak{T}_i^* E_a, \mathcal{E}_k \rangle \langle \mathfrak{T}_a E_b, \mathcal{E}_k \rangle$$
$$+ \sum \langle \mathfrak{T}_b^* E_a, E_c \rangle \langle \mathfrak{T}_a \mathcal{E}_i, E_c \rangle - \sum \langle \mathfrak{T}_j^* \mathcal{E}_i, E_c \rangle \langle \mathfrak{T}_b \mathcal{E}_j, E_c \rangle = 0$$

for all b, i. Using $\mathfrak{T}^* = -\mathfrak{T}$ (metric compatibility of $\mathfrak{T}$), we obtain that (32b) is equivalent to

$$\sum \langle \mathfrak{T}_j \mathcal{E}_j, E_c \rangle \langle \mathfrak{T}_b \mathcal{E}_i, E_c \rangle + 2 \sum \langle \mathfrak{T}_a \mathcal{E}_i, E_c \rangle \langle \mathfrak{T}_b E_a, E_c \rangle$$
$$+ \sum \langle \mathfrak{T}_i E_a, \mathcal{E}_j \rangle \langle \mathfrak{T}_a E_b, \mathcal{E}_j \rangle - \sum \langle \mathfrak{T}_j \mathcal{E}_i, E_c \rangle \langle \mathfrak{T}_b \mathcal{E}_j, E_c \rangle = 0$$

for all b, i. This completes the proof. $\square$

The results obtained when considering the action (7) on metric-affine doubly twisted products, allow us to determine which of these structures are critical for the action (6).

Proposition 4. *A metric-affine doubly twisted product $B \times_{(v,u)} F$ is critical for* (6) *with respect to all variations of g and $\mathfrak{T}$ if and only if* (36) *holds and*

$$\nabla^\top u = 0 = \nabla^\perp v. \tag{41}$$

Proof. It was proven in [27] that a metric-affine doubly twisted product $B \times_{(v,u)} F$ is critical for action (6) with fixed g, with respect to all variations of $\mathfrak{T}$, if and only if (41) and (36) hold. Note that (41) means that TB and TF as (integrable) distributions on $B \times_{(v,u)} F$ are totally geodesic. It can be easily seen that if (36) holds and the distributions are integrable and totally geodesic, then all terms in all variation formulas obtained in Lemma 3 vanish. $\square$

3.2. Statistical Connections

We define a new tensor $\Theta = \mathfrak{T} - \mathfrak{T}^* + \mathfrak{T}^\wedge - \mathfrak{T}^{*\wedge}$, composed of some terms appearing in (18). Note that $\langle \mathfrak{T}_X^{*\wedge} Y, Z \rangle = \langle \mathfrak{T}_Y^* X, Z \rangle = \langle \mathfrak{T}_Y Z, X \rangle$ for all $X, Y, Z \in \mathfrak{X}_M$.

Theorem 4. *Let $(g, \mathfrak{T})$ correspond to a statistical connection. Then $(g, \mathfrak{T})$ is critical for* (6) *with respect to volume-preserving variations of g and variations of $\mathfrak{T}$ among all $(1, 2)$-tensors if and only if the following conditions are satisfied:*

1. *$\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are both integrable,*
2. *$(\mathrm{Tr}^\top \mathfrak{T})^\top = 0 = (\mathrm{Tr}^\perp \mathfrak{T})^\perp$, see (37a, b),*
3. *$\mathfrak{T}_X : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ for all $X \in \widetilde{\mathcal{D}}$,*
4. *$\mathfrak{T}_U : \mathcal{D} \to \mathcal{D}$ for all $U \in \mathcal{D}$,*
5. *if $n > 1$ then $\tilde{H} = 0$,*
6. *if $p > 1$ then $H = 0$,*
7. *$\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are both totally umbilical,*

and the following equations (trivial when $n > 1$ and $p > 1$, see 5. and 6. above) hold for some $\lambda \in \mathbb{R}$:

$$\frac{n-1}{n} H^\flat \otimes H^\flat - \frac{1}{2}\Big(\frac{n-1}{n} \langle H, H \rangle + \frac{p-1}{p} \langle \tilde{H}, \tilde{H} \rangle + \frac{2(p-1)}{p} \operatorname{div} \tilde{H}\Big) g^\perp =$$
$$\lambda g^\perp, \tag{42a}$$

$$\frac{n-1}{n} \Big(\tilde{\delta}_H - \frac{p-1}{p} H^\flat \odot \tilde{H}^\flat\Big) = 0, \tag{42b}$$

$$\frac{p-1}{p} \tilde{H}^\flat \otimes \tilde{H}^\flat - \frac{1}{2}\Big(\frac{p-1}{p} \langle \tilde{H}, \tilde{H} \rangle + \frac{n-1}{n} \langle H, H \rangle + \frac{2(n-1)}{n} \operatorname{div} H\Big) g^\top =$$
$$\lambda g^\top. \tag{42c}$$

Proof. For any $\mathfrak{T}$ that corresponds to a statistical connection, we have $\mathfrak{T}^\wedge = \mathfrak{T}$ and $\mathfrak{T}^* = \mathfrak{T}$. Condition 1 follows from (31a,f) and $\mathfrak{T} = \mathfrak{T}^\wedge$. Then (31a,f), condition 1 and

$$\langle \mathfrak{T}_i \mathcal{E}_j, E_a \rangle = \langle \mathfrak{T}_j^* \mathcal{E}_i, E_a \rangle = \langle \mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j \rangle, \quad \forall i, j, a,$$

yield condition 3. We get condition 5 from $\mathfrak{T} = \mathfrak{T}^*$ and (31b). Conditions 4 and 6 are dual to conditions 3 and 5, and are obtained analogously. Condition 2 follows from $\mathfrak{T} = \mathfrak{T}^*$, condition 3 (and its dual condition 5) and (31c) (and its dual (31g)). Condition 7 follows from Corollary 1.

Let g_t be a $g^\pitchfork$-variation of g. Although for statistical manifolds, (17) reads as

$$\bar{\mathrm{S}}_{\mathrm{mix}} - \mathrm{S}_{\mathrm{mix}} = \mathrm{S}_{\mathfrak{T}} = \langle \mathrm{Tr}^\top \mathfrak{T}, \, \mathrm{Tr}^\perp \mathfrak{T} \rangle - \frac{1}{2} \langle \mathfrak{T}, \mathfrak{T} \rangle_{\,|\,V}, \tag{43}$$

we cannot vary this formula with respect to metric with fixed $\mathfrak{T}$, because when g changes, $\mathfrak{T}$ may no longer correspond to statistical connections (condition $\mathfrak{T} = \mathfrak{T}^*$ may not be preserved by the variation). Instead, we use Lemma 3 and derive from (67) for $\mathfrak{T}$ corresponding to a statistical connection (for which $\mathfrak{T} = \mathfrak{T}^* = \mathfrak{T}^\wedge$ and $\Theta = 0$) that

$$\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = \sum B(\mathcal{E}_i, E_b) \big(\langle \mathfrak{T}_j \mathcal{E}_i, \mathfrak{T}_b \mathcal{E}_j \rangle - 3 \langle \mathfrak{T}_a \mathcal{E}_i, \mathfrak{T}_b E_a \rangle \big)$$
$$- \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathfrak{T}_j E_a, \mathfrak{T}_a \mathcal{E}_i \rangle.$$

From conditions 3–4: $\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = 0$. From (68) with $\Theta = 0$ we have

$$\partial_t \langle \Theta, A \rangle = 2 \sum B(\mathcal{E}_j, E_b) \big(\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a \mathcal{E}_i, \mathcal{E}_j \rangle$$
$$- \langle h(E_a, E_c), \mathcal{E}_j \rangle \langle \mathfrak{T}_a E_b, E_c \rangle \big) - 2 \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a \mathcal{E}_j, E_b \rangle.$$

For totally umbilical distribution, the last equation further simplifies to

$$\partial_t \langle \Theta, A \rangle = \frac{2}{n} \sum B(\mathcal{E}_j, E_b) \big(\langle H, \mathcal{E}_i \rangle \langle \mathfrak{T}_b \mathcal{E}_i, \mathcal{E}_j \rangle - \langle H, \mathcal{E}_j \rangle \langle \mathfrak{T}_a E_b, E_a \rangle \big)$$
$$- \frac{2}{n} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle H, \mathcal{E}_i \rangle \langle \mathfrak{T}_a \mathcal{E}_j, E_a \rangle.$$

From conditions 2-4 we obtain in the above $\partial_t \langle \Theta, A \rangle = 0$. For integrable distributions, since $\Theta = 0$, we have

$$\partial_t \langle \Theta, T^\sharp \rangle = 0, \quad \partial_t \langle \Theta, \tilde{T}^\sharp \rangle = 0,$$

and from (71), with $\Theta = 0$ and totally umbilical distributions, we have

$$\partial_t \langle \Theta, \tilde{A} \rangle = \frac{2}{p} \sum B(\mathcal{E}_j, E_b) \big(\langle \tilde{H}, E_a \rangle \langle \mathfrak{T}_a \mathcal{E}_j, E_b \rangle - \langle \tilde{H}, E_b \rangle \langle \mathfrak{T}_j \mathcal{E}_i, \mathcal{E}_i \rangle \big)$$
$$+ \frac{2}{p} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \tilde{H}, E_a \rangle \langle \mathfrak{T}_a \mathcal{E}_j, \mathcal{E}_i \rangle.$$

From conditions 3–4 and 2 we get in the above

$$\partial_t \langle \Theta, \tilde{A} \rangle = -\frac{2}{p} \sum B(\mathcal{E}_j, E_b) \langle \tilde{H}, E_b \rangle \langle \mathfrak{T}_i \mathcal{E}_i, \mathcal{E}_j \rangle = 0.$$

From conditions 3–4, using (72) and (73), we get

$$\partial_t \langle \operatorname{Tr}^\top \mathfrak{T}, \operatorname{Tr}^\perp \mathfrak{T}^* \rangle = -\sum B(\mathcal{E}_i, E_b) \langle \operatorname{Tr}^\top \mathfrak{T}, E_c \rangle \langle E_c, \mathfrak{T}_b \mathcal{E}_i \rangle = 0,$$

$$\partial_t \langle \operatorname{Tr}^\top \mathfrak{T}^*, \operatorname{Tr}^\perp \mathfrak{T} \rangle = \sum B(\mathcal{E}_j, E_b) \langle \operatorname{Tr}^\perp \mathfrak{T} - 2 \operatorname{Tr}^\top \mathfrak{T}, \mathfrak{T}_b \mathcal{E}_j \rangle$$

$$- \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathfrak{T}_j \mathcal{E}_i, \operatorname{Tr}^\top \mathfrak{T} \rangle.$$

From conditions 3–4 and 2 we get $\partial_t \langle \operatorname{Tr}^\top \mathfrak{T}^*, \operatorname{Tr}^\perp \mathfrak{T} \rangle = 0$. From $\mathfrak{T}^* = \mathfrak{T}$, using (74), we obtain

$$\partial_t \langle \operatorname{Tr}^\top (\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \operatorname{Tr}^\top \mathfrak{T}, \mathcal{E}_j \rangle \langle \mathcal{E}_i, H \rangle$$

$$+ \sum B(\mathcal{E}_j, E_b) \big(\langle \mathfrak{T}_b \mathcal{E}_j, \tilde{H} - H \rangle + \langle \operatorname{Tr}^\top \mathfrak{T}, E_b \rangle \langle \mathcal{E}_j, H \rangle - \langle \operatorname{Tr}^\top \mathfrak{T}, \mathcal{E}_j \rangle \langle E_b, \tilde{H} \rangle \big).$$

From conditions 3–4 and 2 we get $\partial_t \langle \sum (\mathfrak{T}_a^* - \mathfrak{T}_a) E_a, \tilde{H} - H \rangle = 0$. Similarly, from (75) we obtain

$$\partial_t \langle \operatorname{Tr}^\perp (\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \big(\langle \mathfrak{T}_i \mathcal{E}_j, \tilde{H} - H \rangle + \langle \operatorname{Tr}^\perp \mathfrak{T}, \mathcal{E}_i \rangle \langle H, \mathcal{E}_j \rangle \big)$$

$$+ \sum B(\mathcal{E}_j, E_b) \big(\langle \operatorname{Tr}^\perp \mathfrak{T}, E_b \rangle \langle H, \mathcal{E}_j \rangle + \langle \mathfrak{T}_j E_b, \tilde{H} - H \rangle - \langle \operatorname{Tr}^\perp \mathfrak{T}, \mathcal{E}_j \rangle \langle \tilde{H}, E_b \rangle \big).$$

From conditions 3–4 and 2 we get in the above

$$\partial_t \langle \operatorname{Tr}^\perp (\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = -\sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathfrak{T}_j \mathcal{E}_i, H \rangle.$$

By condition 6 we have $H = 0$ if $p > 1$ and if $p = 1$ we only have $i = j = k = 1$ and by condition 2,

$$\langle \mathfrak{T}_j \mathcal{E}_i, \mathcal{E}_k \rangle = \langle \operatorname{Tr}^\perp \mathfrak{T}, \mathcal{E}_1 \rangle = 0.$$

Hence, for $\mathfrak{T}$ corresponding to a statistical connection satisfying the assumptions, any variation of $\overline{S}_{\mathrm{mix}}$ with respect to g is just a variation of S_{mix} with respect to g. Thus, remaining (42a–c) are equations of Theorem 1 written for both distributions integrable and umbilical. $\qquad\square$

Corollary 5. *Let M be a closed manifold. Then $(g, \mathfrak{T})$, where $\mathfrak{T}$ corresponds to a statistical connection on (M, g), is critical for the action (6) with respect to all variations of g and $\mathfrak{T}$ if and only if $(g, \mathfrak{T})$ satisfy conditions 1–7 of Theorem 4; furthermore, either $n = p = 1$ or $H = 0 = \tilde{H}$.*

Proof. Clearly, (42a–c) hold when $n = p = 1$. If $n, p > 1$ then conditions 5 and 6 imply $H = \tilde{H} = 0$. Suppose that $n > 1$, $p = 1$ and $H \neq 0$ and let $N \in \mathcal{D}$ be a local unit vector field. Then, evaluating (42a) on $N \otimes N$, we obtain

$$\frac{n-1}{2n} \langle H, H \rangle = \lambda. \tag{44}$$

For $p = 1$ we have $H = -(\operatorname{div} N)N$ and $\int_M \tau_1 \, \mathrm{d} \operatorname{vol}_g = 0$ for $\tau_1 = \langle H, N \rangle$, e.g., [24]. The integral formula shows that τ_1 vanishes somewhere on M. On the other hand, (44) yields that $\langle H, H \rangle = \tau_1^2$ is constant on M, hence $H = 0$. Since $n > 1$, condition 5 in Theorem 4 implies also $\tilde{H} = 0$.   □

Equation (43) and Corollary 3 imply the following

Corollary 6. *Let $(g, \mathfrak{T})$ correspond to a statistical connection. Then $(g, \mathfrak{T})$ is critical for the action (6) with respect to all variations of metric and variations of $\mathfrak{T}$ corresponding to statistical connections if and only if (37a, b) and equations of Theorem 1 hold.*

Note that conditions for a statistical connection to be critical for (6) with fixed g are actually those from Corollary 3 (instead of conditions in [27, Theorem 3], which do not consider all symmetries of $\partial_t \mathfrak{T}$ for variation among statistical connections). Indeed, for a family of statistical connections on (M, g) and $S = \partial_t \mathfrak{T}_t|_{t=0}$, we have for statistical connections $g(S_a E_b, \mathcal{E}_i) = g(S_a \mathcal{E}_i, E_b) = g(S_i E_a, E_b)$. Gathering together terms appearing by these components in [27, Eq. (14)], we obtain

$$\sum g(S_i E_a, E_b) \cdot \big(g(\operatorname{Tr}^\perp \mathfrak{T} + H^\perp, \mathcal{E}_i)\delta_{ab} - g((A_i - T_i^\sharp)E_a, E_b) - g(\mathfrak{T}_i E_a, E_b)$$
$$+ g(\operatorname{Tr}^\perp \mathfrak{T} - H^\perp, \mathcal{E}_i)\delta_{ab} + g((A_i + T_i^\sharp)E_b, E_a) - g(\mathfrak{T}_i E_b, E_a)$$
$$+ g((A_i - T_i^\sharp)E_a, E_b) - g((A_i + T_i^\sharp)E_a, E_b) - g(\mathfrak{T}_a E_b + \mathfrak{T}_b E_a, \mathcal{E}_i) \big) = 0.$$

However, we have $g(S_i E_a, E_b) = (S_i E_b, E_a)$, thus $\sum (S_i E_a, E_b) g(T_i^\sharp E_a, E_b) = 0$. Considering this, instead of [27, Eq. (28c)] we obtain the following Euler–Lagrange equation:

$$g(\mathfrak{T}_i E_a, E_b) = \frac{1}{2} g(\operatorname{Tr}^\perp \mathfrak{T}, \mathcal{E}_i)\delta_{ab},$$

which can be transformed into the third equation in [27, Cor. 7], and the second equation in [27, Cor. 7] is dual to it with respect to interchanging distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. Similarly, for terms appearing in [27, Eq. (14)] by $g(S_a E_b, E_c)$, we obtain

$$g(S_a E_b, E_c) \cdot \big(g(\operatorname{Tr}^\perp \mathfrak{T} - H, E_c)\delta_{ab} + g(\operatorname{Tr}^\perp \mathfrak{T} + H, E_b)\delta_{ac} \big) = 0,$$

for all S such that $g(S_a E_b, E_c) = g(S_b E_a, E_c) = g(S_a E_c, E_b)$ for all a, b, c. Considering this symmetry, we obtain the following Euler–Lagrange equation:

$$g(\operatorname{Tr}^\perp \mathfrak{T}, E_c)\delta_{ab} + g(\operatorname{Tr}^\perp \mathfrak{T}, E_b)\delta_{ac} + g(\operatorname{Tr}^\perp \mathfrak{T}, E_a)\delta_{bc} = 0,$$

and considering arbitrary $a = b = c$ we get $(\operatorname{Tr}^\perp \mathfrak{T})^\top = 0$, which - together with its dual - is the first equation in [27, Cor. 7]. Remaining components of S in [27, Eq. (14)] are dual to the ones we considered here, so they will not give new Euler–Lagrange equations. Therefore, distributions in [27, Thm. 3] do not need to be integrable and conditions given there should be those from [27, Cor. 7]. We note that [27, Cor. 3] remains true under additional assumption that $\sum_a \epsilon_a \neq 0 \neq \sum_i \epsilon_i$, as then we have $\operatorname{Tr}^\top \mathfrak{T} = 0 = \operatorname{Tr}^\perp \mathfrak{T}$.

3.3. Metric Connections

Here, we consider g and $\mathfrak{T}$ as independent variables in the action (6), hence for every pair $(g, \mathfrak{T})$ critical for (6) the contorsion tensor $\mathfrak{T}$ must be critical for (6) with fixed g, and thus satisfy Corollary 1. Using this fact, we characterize those critical values of (6), that are attained on the set of pairs $(g, \mathfrak{T})$, where $\mathfrak{T}$ is the contorsion tensor of a metric (in particular, adapted) connection for g. Note that by [27, Corollary 2], restricting variations of $\mathfrak{T}$ to tensors corresponding to a metric connection gives the same conditions as considering variations among arbitrary $\mathfrak{T}$.

Proposition 5. *Let the contorsion tensor $\mathfrak{T}$ of a metric connection $\bar{\bar{\nabla}}$ be critical for the action* (6) *with fixed g. Then $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are both totally umbilical and for Q given in* (18) *we have*

$$\frac{1}{2}\, Q = \frac{2n-1}{n}\, \langle\, \mathrm{Tr}^{\top} \mathfrak{T}, H \rangle + \frac{2p-1}{p}\, \langle\, \mathrm{Tr}^{\perp} \mathfrak{T}, \tilde{H} \rangle$$
$$+ \frac{p-1}{p}\, \langle \tilde{H}, \tilde{H} \rangle + \frac{n-1}{n}\, \langle H, H \rangle + \langle T, T \rangle + \langle \tilde{T}, \tilde{T} \rangle. \tag{45}$$

Proof. By Corollary 1, both distributions are totally umbilical. In this case, using (31a-j), we have

$$\langle\, \mathrm{Tr}^{\top}(\mathfrak{T} - \mathfrak{T}^*), H - \tilde{H} \rangle = 2\, \langle\, \mathrm{Tr}^{\top} \mathfrak{T}, H \rangle - 2\, \frac{p-1}{p}\, \langle \tilde{H}, \tilde{H} \rangle,$$

$$\langle\, \mathrm{Tr}^{\perp}(\mathfrak{T} - \mathfrak{T}^*),\, H - \tilde{H} \rangle = 2\, \frac{n-1}{n}\, \langle H, H \rangle - 2\, \langle\, \mathrm{Tr}^{\perp} \mathfrak{T}, \tilde{H} \rangle,$$

$$-\langle\, \mathrm{Tr}^{\top} \mathfrak{T},\, \mathrm{Tr}^{\perp} \mathfrak{T}^* \rangle = \frac{n-1}{n}\, \langle\, \mathrm{Tr}^{\top} \mathfrak{T}, H \rangle + \frac{p-1}{p}\, \langle\, \mathrm{Tr}^{\perp} \mathfrak{T}, \tilde{H} \rangle,$$

$$-\langle\, \mathrm{Tr}^{\perp} \mathfrak{T},\, \mathrm{Tr}^{\top} \mathfrak{T}^* \rangle = \frac{p-1}{p}\, \langle\, \mathrm{Tr}^{\perp} \mathfrak{T}, \tilde{H} \rangle + \frac{n-1}{n}\, \langle\, \mathrm{Tr}^{\top} \mathfrak{T}, H \rangle.$$

For totally umbilical distributions and critical metric connection, (31a-j) yield

$$-2\langle \mathfrak{T} + \mathfrak{T}^{\wedge}, A \rangle = 4\langle\, \mathrm{Tr}^{\top} \mathfrak{T}, H \rangle,$$
$$-2\langle \mathfrak{T} + \mathfrak{T}^{\wedge}, \tilde{A} \rangle = 4\langle\, \mathrm{Tr}^{\perp} \mathfrak{T}, \tilde{H} \rangle,$$
$$\langle \mathfrak{T} + \mathfrak{T}^{\wedge}, T^{\sharp} \rangle = 2 \sum \langle \mathfrak{T}_a \mathcal{E}_i + \mathfrak{T}_i E_a, T_i^{\sharp} E_a \rangle = 4\langle T, T \rangle,$$
$$\langle \mathfrak{T} + \mathfrak{T}^{\wedge}, \tilde{T}^{\sharp} \rangle = 4\langle \tilde{T}, \tilde{T} \rangle,$$
$$\langle \mathfrak{T}^*, \mathfrak{T}^{\wedge} \rangle_{\,|\,V} = -\langle \mathfrak{T}, \mathfrak{T}^{\wedge} \rangle_{\,|\,V} = -2 \sum \langle \mathfrak{T}_i E_a, \mathfrak{T}_a \mathcal{E}_i \rangle = -2\langle T, T \rangle - 2\langle \tilde{T}, \tilde{T} \rangle.$$

Using the above in (18), and simplifying the expression, completes the proof. $\square$

Remark 2. Let $n, p > 1$. By (34), for critical metric connection equation (45) becomes

$$\frac{1}{2}\, Q = -\langle H, H \rangle - \langle \tilde{H}, \tilde{H} \rangle + \langle T, T \rangle + \langle \tilde{T}, \tilde{T} \rangle.$$

By this and (17), for any critical metric connection on a closed manifold (M, g) we have

$$\int_M \overline{S}_{\mathrm{mix}} \, d \, \mathrm{vol}_g \overset{(15)}{=} \int_M \big(\frac{2n-1}{n} \langle H, H \rangle + \frac{2p-1}{p} \langle \tilde{H}, \tilde{H} \rangle \big) \, d \, \mathrm{vol}_g \, .$$

Thus, the right hand side of the above equation is the only critical value of the action (6) (with fixed g on a closed manifold M) restricted to metric connections for g. Notice that it does not depend on $\mathfrak{T}$, but only on the pseudo-Riemannian geometry of distributions on (M, g). Moreover, on a Riemannian manifold it is always non-negative.

Consider pairs $(g, \mathfrak{T})$, where $\mathfrak{T}$ corresponds to a metric connection, critical for (6) with respect to $g^\perp$-variations. We apply only adapted variations, as they will allow to obtain the Euler–Lagrange equations without explicit use of adapted frame or defining multiple new tensors. The case of general variations, mostly due to complicated form of tensor F defined by (66) that appears in variation formulas, is significantly more involved and beyond the scope of this paper. Set

$$\chi = \sum\nolimits_{a,j} (\mathfrak{T}_j E_a)^{\perp \flat} \odot (\tilde{T}_a^\sharp \mathcal{E}_j)^{\perp \flat}, \qquad \phi(X, Y) = (\mathfrak{T} + \mathfrak{T}^\wedge)_{X^\perp} Y^\perp. \quad (46)$$

Define also tensors $\phi^\top$ and $\phi^\perp$ by $\phi^\top(X, Y) = (\phi(X, Y))^\top$ and $\phi^\perp(X, Y) = (\phi(X, Y))^\perp$ for $X, Y \in \mathfrak{X}_M$.

Theorem 5. *A pair $(g, \mathfrak{T})$, where $\mathfrak{T}$ corresponds to a metric connection on M, is critical for (6) with respect to $g^\perp$-variations of metric and arbitrary variations of $\mathfrak{T}$ if and only if all the following conditions are satisfied: $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are totally umbilical, the following Euler–Lagrange equation holds:*

$$- \frac{5n-5}{n} H^\flat \otimes H^\flat - \frac{1}{2} \Upsilon_{T,T} + 2 \widetilde{T}^\flat$$

$$+ \big(\frac{3p-3}{p} \operatorname{div} \tilde{H} - \frac{2n-1}{n} \langle \operatorname{Tr}^\top \mathfrak{T}, H \rangle - \frac{2p-1}{p} \langle \operatorname{Tr}^\perp \mathfrak{T}, \tilde{H} \rangle - \operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) \big) g^\perp$$

$$- 2 \operatorname{div} \phi^\top + \langle \phi, \frac{3}{2} \tilde{H} - \frac{1}{2} H + \frac{1}{2} (\operatorname{Tr}^\top \mathfrak{T})^\perp \rangle + 7 \chi$$

$$+ \frac{3n+2}{n} H^\flat \odot (\operatorname{Tr}^\top \mathfrak{T})^{\perp \flat} = 0, \quad (47a)$$

$\mathfrak{T}$ satisfies the following linear algebraic system:

$$(\mathfrak{T}_V U - \mathfrak{T}_U V)^\top = 2 \tilde{T}(U, V), \quad (47b)$$

$$\mathfrak{T}_U^\top = T^\sharp, \quad (47c)$$

$$(\operatorname{Tr}^\perp \mathfrak{T})^\perp = \frac{n-1}{n} H, \quad (47d)$$

$$(\mathfrak{T}_Y X - \mathfrak{T}_X Y)^\perp = 2 T(X, Y), \quad (47e)$$

$$\mathfrak{T}_X^\perp = \tilde{T}_X^\sharp, \quad (47f)$$

$$(\operatorname{Tr}^\top \mathfrak{T})^\top = \frac{p-1}{p} \tilde{H}, \quad (47g)$$

for all $X, Y \in \widetilde{\mathcal{D}}$ *and* $U, V \in \mathcal{D}$; *and*

$$(\mathrm{Tr}^\top \mathfrak{T})^\perp = -H, \quad if \ p > 1, \qquad (\mathrm{Tr}^\perp \mathfrak{T})^\top = -\tilde{H}, \quad if \ n > 1. \quad (47\mathrm{h})$$

Proof. By Corollary 1, $\mathfrak{T}$ is critical for (6) (with fixed g) if and only if distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are totally umbilical and (47b–g) (together with (47h) if their respective assumptions on n and p hold) are satisfied. Let $\mathfrak{T}$ be critical for the action (6) with fixed g. We shall prove that a pair $(g, \mathfrak{T})$ is critical for the action (6) with respect to $g^\perp$-variations of metric if and only if (47a) holds.

By Proposition 2, for any variation g_t of metric such that $\mathrm{supp}(B) \subset \Omega$, where Ω is a compact set on M, and Q in (18), we have

$$\frac{d}{dt} \int_M \left(2(\bar{\mathrm{S}}_{\mathrm{mix}} - \mathrm{S}_{\mathrm{mix}}) + Q \right) \mathrm{d\,vol}_g = \frac{d}{dt} \int_M (\mathrm{div}\, X)\, \mathrm{d\,vol}_g,$$

where $X = (\mathrm{Tr}^\top(\mathfrak{T} - \mathfrak{T}^*))^\perp + (\mathrm{Tr}^\perp(\mathfrak{T} - \mathfrak{T}^*))^\top$. Although X is not necessarily zero on $\partial\Omega$, we have $\mathrm{supp}\,(\partial_t X) \subset \Omega$, thus, $\frac{d}{dt} \int_M (\mathrm{div}\, X)\, \mathrm{d\,vol}_g = 0$, see (25), and hence:

$$\frac{d}{dt} \int_M (\bar{\mathrm{S}}_{\mathrm{mix}} - \mathrm{S}_{\mathrm{mix}})\, \mathrm{d\,vol}_g = -\frac{1}{2} \int_M (\partial_t Q)\, \mathrm{d\,vol}_g - \frac{1}{4} \int_M Q \langle B, g \rangle\, \mathrm{d\,vol}_g,$$

where, up to divergence of a compactly supported vector field, $\partial_t Q$ is given in Lemma 5. For $g^\perp$-variations we get (see [26, Eq. (29)] for more general case of $g^\pitchfork$-variations),

$$\frac{d}{dt} \int_M \mathrm{S}_{\mathrm{mix}}\, \mathrm{d\,vol}_g = \int_M \left\langle -\mathrm{div}\,\tilde{h} - \widetilde{\mathcal{K}}^\flat - H^\flat \otimes H^\flat + \frac{1}{2}\Upsilon_{h,h} + \frac{1}{2}\Upsilon_{T,T} + 2\widetilde{\mathcal{T}}^\flat \right.$$
$$\left. + \frac{1}{2}\left(\mathrm{S}_{\mathrm{mix}} + \mathrm{div}(\tilde{H} - H) \right) g^\perp,\ B \right\rangle \mathrm{d\,vol}_g.$$

For totally umbilical distributions we have

$$\widetilde{\mathcal{K}}^\flat = 0, \quad \mathrm{div}\,\tilde{h} = \frac{1}{p}(\mathrm{div}\,\tilde{H})\, g^\perp, \quad \left\langle \frac{1}{2}\Upsilon_{h,h}, B \right\rangle = \left\langle \frac{1}{n} H^\flat \otimes H^\flat, B \right\rangle.$$

Hence,

$$\frac{d}{dt} \int_M \mathrm{S}_{\mathrm{mix}}\, \mathrm{d\,vol}_g = \int_M \left\langle \frac{1}{2}\Upsilon_{T,T} + 2\widetilde{\mathcal{T}}^\flat - \frac{n-1}{n} H^\flat \otimes H^\flat \right.$$
$$\left. + \frac{1}{2}\left(\mathrm{S}_{\mathrm{mix}} + \mathrm{div}\left(\frac{p-2}{p}\tilde{H} - H \right) - \frac{1}{2}Q \right) g^\perp + \frac{1}{2}\delta Q,\ B \right\rangle \mathrm{d\,vol}_g,$$

where δQ is defined by the equality $\langle \delta Q, B \rangle = -\partial_t Q$, see Lemma 5. Thus, the Euler–Lagrange equation for $g^\perp$-variations of metric and totally umbilical distributions is the following:

$$-\frac{2n-2}{n} H^\flat \otimes H^\flat + \Upsilon_{T,T} + 4\widetilde{\mathcal{T}}^\flat + \left(\mathrm{S}_{\mathrm{mix}} + \mathrm{div}\left(\frac{p-2}{p}\tilde{H} - H \right) - \frac{1}{2}Q \right) g^\perp$$
$$+ \delta Q = 0. \tag{48}$$

Using Lemma 5, Proposition 5 and (16) in (48), we obtain

$$
-\frac{5n-5}{n} H^\flat \otimes H^\flat - \frac{1}{2}\, \Upsilon_{T,T} + 2\, \widetilde{\mathcal{T}}^\flat + \left(\frac{3p-3}{p}\,\operatorname{div}\tilde{H} - \frac{2n-1}{n} \langle \operatorname{Tr}^\top \mathfrak{T},\, H \rangle \right.
$$

$$
\left. -\frac{2p-1}{p}\, \langle \operatorname{Tr}^\perp \mathfrak{T},\, \tilde{H} \rangle - \operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) \right) g^\perp - 2\operatorname{div}\phi^\top
$$

$$
+ \left\langle \phi,\, \frac{p+2}{p}\, \tilde{H} - \frac{1}{2} H + \frac{1}{2}\operatorname{Tr}^\top \mathfrak{T} \right\rangle + 7\chi + \frac{3n+2}{n}\, H^\flat \odot (\operatorname{Tr}^\top \mathfrak{T})^{\perp\flat} = 0.
$$

By (47g), from the above we get (47a). $\square$

Remark 3. Note that for volume-preserving variations, the right hand sides of (47a) and (48) should be $\lambda\, g^\perp$, with $\lambda \in \mathbb{R}$ being an arbitrary constant [26]. This obviously applies also to the special cases of the Euler–Lagrange equation (47a) discussed below.

If $p > 1$ and $n > 1$ then (47a) can be written as

$$
\frac{3-8n}{n} H^\flat \otimes H^\flat - \frac{1}{2}\,\Upsilon_{T,T} + 2\,\widetilde{\mathcal{T}}^\flat - 2\operatorname{div}\phi^\top + \left\langle \phi, \frac{3}{2}\tilde{H} - H \right\rangle + 7\chi
$$

$$
+ \left(\frac{4p-3}{p}\operatorname{div}\tilde{H} + \frac{2n-1}{n}\langle H, H \rangle + \frac{2p-1}{p}\langle \tilde{H}, \tilde{H} \rangle \right) g^\perp = 0. \qquad (49)
$$

Taking trace of (49) and using (47d, g–i) and equalities $\operatorname{Tr}_g \Upsilon_{T,T} = 2\langle T, T \rangle$ and $\operatorname{Tr}_g \widetilde{\mathcal{T}}^\flat = -\langle \tilde{T}, \tilde{T} \rangle$, we obtain the following result.

Corollary 7. *Let a pair $(g, \mathfrak{T})$, where g is a pseudo-Riemannian metric on M and $\mathfrak{T}$ corresponds to a metric connection, be critical for (6) with respect to $g^\perp$-variations of metric and arbitrary variations of $\mathfrak{T}$. Then for $n, p > 1$ we have*

$$
\frac{(2n-1)(p-5)}{n}\, \langle H, H \rangle - \langle T, T \rangle - 2\langle \tilde{T}, \tilde{T} \rangle + (4p-1)\operatorname{div}\tilde{H}
$$

$$
+ 2(p-2)\langle \tilde{H}, \tilde{H} \rangle + 7\operatorname{Tr}_g \chi = 0, \qquad (50)
$$

and for $n = 1$ and $p > 1$ we get

$$
(p-5)\langle H, H \rangle - 2\langle \tilde{T}, \tilde{T} \rangle + 3(p-1)\operatorname{div}\tilde{H}
$$

$$
- (p+4)\operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) + 2(2-p)\langle \operatorname{Tr}^\perp \mathfrak{T},\, \tilde{H} \rangle + 7\operatorname{Tr}_g \chi = 0.
$$

Corollary 8. *Let M be a closed manifold, $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ be both integrable with dimensions $n, p > 1$ and let a pair $(g, \mathfrak{T})$, where g is a pseudo-Riemannian metric on M, postive or negative definite on each distribution $\widetilde{\mathcal{D}}$, $\mathcal{D}$, and $\mathfrak{T}$ corresponds to a metric connection, be critical for (6) with respect to adapted variations of metric and arbitrary variations of $\mathfrak{T}$. Then:*

     1. if $n \neq 2$ or $p \neq 5$ then $\widetilde{\mathcal{D}}$ is totally geodesic,
     2. if $p \neq 2$ or $n \neq 5$ then $\mathcal{D}$ is totally geodesic.

Proof. By the assumptions, T, $\tilde{T}$ and χ all vanish in (50), which considered with its dual and integrated over M yields two homogeneous linear equations for $\int_M \langle H, H \rangle \, \mathrm{d}\operatorname{vol}_g$ and $\int_M \langle \tilde{H}, \tilde{H} \rangle \, \mathrm{d}\operatorname{vol}_g$. It follows that if $n \neq 2$ or $p \neq 5$ then $\int_M \langle H, H \rangle \, \mathrm{d}\operatorname{vol}_g = 0$. Since g is definite on $\mathcal{D}$, we obtain $H = 0$ and since $\widetilde{\mathcal{D}}$ is totally umbilical by Theorem 5, the first claim follows; the second is analogous. $\qquad\square$

Recall that an *adapted connection* to $(\mathcal{D}, \widetilde{\mathcal{D}})$, see e.g., [5], is defined by

$$\bar{\nabla}_Z X \in \mathfrak{X}^\perp, \quad \bar{\nabla}_Z Y \in \mathfrak{X}^\top, \quad X \in \mathfrak{X}^\perp, \ Y \in \mathfrak{X}^\top, \ Z \in \mathfrak{X}_M,$$

and an example is the Schouten-Van Kampen connection with contorsion tensor

$$\mathfrak{T}_X Y = -(\nabla_{X^\top} Y^\perp)^\top - (\nabla_{X^\top} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top - (\nabla_{X^\perp} Y^\top)^\perp, \quad X, Y \in \mathfrak{X}_M.$$

Proposition 6. *Let $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ both be totally umbilical. Then contorsion tensor $\mathfrak{T}$ corresponding to an adapted metric connection satisfies* (47a–i) *if and only if it satisfies the equations*

$$\mathfrak{T}_U^\top = T_U^\sharp, \tag{51a}$$

$$(\operatorname{Tr}^\perp \mathfrak{T})^\perp = \frac{n-1}{n} H, \tag{51b}$$

$$\mathfrak{T}_X^\perp = \tilde{T}_X^\sharp, \tag{51c}$$

$$(\operatorname{Tr}^\top \mathfrak{T})^\top = \frac{p-1}{p} \tilde{H}, \tag{51d}$$

$$\frac{3-8n}{n} H^\flat \otimes H^\flat - \frac{1}{2} \Upsilon_{T,T} - 5\, \widetilde{T}^\flat - \langle \phi, H \rangle$$
$$+ \Big(\frac{4p+1}{p} \operatorname{div} \tilde{H} + \frac{2p-4}{p} \langle \tilde{H}, \tilde{H} \rangle + \frac{2n-1}{n} \langle H, H \rangle \Big) g^\perp = 0, \tag{51e}$$

for all $X \in \widetilde{\mathcal{D}}$ and $U \in \mathcal{D}$.

Proof. For adapted connection and totally umbilical distribution $\mathcal{D}$ we have $\phi^\top = -2\tilde{h} = -\frac{2}{p} \tilde{H} g^\perp$, see [27, Section 2.5], and

$$\mathfrak{T}_X Y = -(\nabla_{X^\top} Y^\top)^\perp - (\nabla_{X^\perp} Y^\perp)^\top + (A_{Y^\perp} + T_{Y^\perp}^\sharp) X^\top$$
$$+ (\tilde{A}_{Y^\top} + \tilde{T}_{Y^\top}^\sharp) X^\perp + (\mathfrak{T}_X Y^\top)^\top + (\mathfrak{T}_X Y^\perp)^\perp. \tag{52}$$

Moreover, an adapted connection is critical for (6) with fixed g if and only if (51a–d) hold, see [27]. Note that for adapted connection from (52) we obtain $\chi = -\widetilde{T}^\flat$, as for $X, Y \in \mathcal{D}$ we have

$$2\chi(X,Y) = \sum \big(2\langle \tilde{T}_a^\sharp \mathcal{E}_j, X \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_j, Y \rangle + \langle \tilde{A}_a \mathcal{E}_j, X \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_j, Y \rangle$$
$$+ \langle \tilde{A}_a \mathcal{E}_j, Y \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_j, X \rangle \big) = -2 \sum \langle \tilde{T}_a^\sharp \tilde{T}_a^\sharp X, Y \rangle$$

for umbilical distributions. Also (47h) hold, in all dimensions n, p. Thus, for a critical adapted connection, (47a) simplifies to (51e). $\qquad\square$

If $p > 1$ then $\phi^\perp$ is not determined by $(\mathrm{Tr}^\perp \mathfrak{T})^\perp$ and by (52) in Proposition 6 can be set arbitrary for an adapted metric connection. Using this fact and taking trace of (51e) yield the following.

Corollary 9. *Let $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ both be totally umbilical. If a contorsion tensor $\mathfrak{T}$, corresponding to an adapted metric connection, satisfies (47a–i) then the metric g satisfies*

$$\frac{(2n-1)(p-5)}{n}\langle H, H\rangle - \langle T, T\rangle + 5\langle \tilde{T}, \tilde{T}\rangle + (4p+1)\,\mathrm{div}\,\tilde{H} + 2(p-2)\langle \tilde{H}, \tilde{H}\rangle = 0. \tag{53}$$

If $p > 1$ and at every point of M we have $H \neq 0$, then for a given (M, g) satisfying (53) there exists a metric adapted connection such that $(g, \mathfrak{T})$ is critical for the action (6) with respect to all variations of $\mathfrak{T}$ and $g^\perp$-variations of metric.

Example 3. In [13] it was proved that on a Sasaki manifold (M, g, ξ, η) (that is, M with a normal contact metric structure [8]) there exists a unique metric connection with a skew-symmetric, parallel torsion tensor, and its contorsion tensor is given by $\langle \mathfrak{T}_X Y, Z\rangle = \frac{1}{2}(\eta \wedge d\eta)(X, Y, Z)$, where $X, Y, Z \in \mathfrak{X}_M$ and η is the contact form on M. Let $\widetilde{\mathcal{D}}$ be the one-dimensional distribution spanned by the Reeb field ξ. It follows that for this connection we have $\phi = 0$ and for $X, Y \in \mathcal{D}$

$$\chi(X, Y) = -\frac{1}{4}\sum_i \Big[(\eta \wedge d\eta)(\xi, \mathcal{E}_i, X) \cdot \langle \tilde{T}_\xi^\sharp \mathcal{E}_i, Y\rangle$$
$$+ (\eta \wedge d\eta)(\xi, \mathcal{E}_i, Y) \cdot \langle \tilde{T}_\xi^\sharp \mathcal{E}_i, X\rangle \Big] = -\widetilde{\mathcal{T}}^\flat(X, Y),$$

see (46), as $d\eta(X, Y) = 2\langle X, \tilde{T}_\xi^\sharp Y\rangle$ (we use here the same convention for differential of forms as in [13], which is different than the one in [8]). Since g is a Sasaki metric, both distributions are totally geodesic, and for volume-preserving variations the Euler–Lagrange equation (47a) gets $\lambda g^\perp$ on the right-hand side (see Remark 3) and becomes

$$-5\,\widetilde{\mathcal{T}}^\flat = \lambda g^\perp. \tag{54}$$

As on a Sasakian manifolds we have $\widetilde{\mathcal{T}}^\flat = -\frac{1}{p}\langle \tilde{T}, \tilde{T}\rangle g^\perp$ and $\langle \tilde{T}, \tilde{T}\rangle = p$ (e.g., [26, Section 3.3]), we see that (54) holds in this case for $\lambda = 5$.

We can slightly modify this example to obtain a family of critical connections (although no longer with parallel torsion) on a contact manifold.

Proposition 7. *Let (M, η) be a contact manifold and let $\widetilde{\mathcal{D}}$ be the one-dimensional distribution spanned by the Reeb field ξ. Let g be an associated metric [8] on (M, η).*

1. *There exist metric connections $\nabla + \mathfrak{T}$ on M such that $(g, \mathfrak{T})$ is critical for (6) with respect to volume-preserving $g^\perp$-variations of metric and arbitrary variations of $\mathfrak{T}$.*

2. *If M is closed, then there exist no metric connections $\nabla + \mathfrak{T}$ on M such that $(g, \mathfrak{T})$ is critical for (6) with respect to adapted volume-preserving variations of metric and arbitrary variations of $\mathfrak{T}$.*

Proof. 1. Let $\mathfrak{T}_\xi \xi = 0$ and for all $X, Y \in \mathcal{D}$ let $(\mathfrak{T}_\xi X)^\top = 0$, $(\mathfrak{T}_X \xi)^\top = 0$ and

$$\langle \mathfrak{T}_X Y, \xi \rangle = \frac{1}{2}(\eta \wedge d\eta)(X, Y, \xi) = -\langle \mathfrak{T}_X \xi, Y \rangle, \quad \langle \mathfrak{T}_\xi X, Y \rangle = -\frac{1}{2}(\eta \wedge d\eta)(\xi, X, Y).$$

For all $X, Y, Z \in \mathcal{D}$, let $\langle \mathfrak{T}_X Y, Z \rangle = \omega(X, Y, Z)$, where ω is any 3-form. Then connection $\nabla + \mathfrak{T}$ will satisfy all Euler–Lagrange equations (47b–i) and (47a) with $\lambda g^\perp = 5 g^\perp$ on the right-hand side (see Remark 3).

2. Recall [26, Remark 4(ii)] that the Euler–Lagrange equations for volume-preserving adapted variations of the metric are (47a) with the right-hand side $\lambda g^\perp$ and its dual (with the same constant λ). Note that the tensor dual to ϕ is given by $\tilde{\phi}(X, Y) = \mathfrak{T}_{X^\top} Y^\top + \mathfrak{T}_{Y^\top} X^\top$.

For an an associated metric g, $(\tilde{T}_\xi^\sharp, \xi, \eta, g)$ is a contact metric structure [8], which implies [26]

$$(\tilde{T}_\xi^\sharp)^2 = -\operatorname{id}^\perp, \langle \tilde{T}, \tilde{T} \rangle = p, \tag{55}$$

Suppose that $(g, \mathfrak{T})$ satisfy (47b–i). By (47g), (47h), the left-hand side of the equation dual to (47a) reduces to $-\langle \tilde{T}, \tilde{T} \rangle$. Hence, by (47a) and $(55)_2$, a pair $(g, \mathfrak{T})$ is critical for the action (6) with respect to volume-preserving adapted variations if and only if

$$2\widetilde{\mathcal{T}}^\flat - \operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) g^\perp - 2 \operatorname{div} \phi^\top + 7\chi = -p g^\perp \tag{56}$$

Since for $X, Y \in \mathcal{D}$ we have $\mathfrak{T}_X Y = \frac{1}{2}\phi(X, Y) + \frac{1}{2}(\mathfrak{T}_X Y - \mathfrak{T}_Y X)$, by (47b) we have

$$
\begin{aligned}
\chi(X, Y) &= -\frac{1}{4}\langle \mathfrak{T}_j X - \mathfrak{T}_X \mathcal{E}_j, \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, Y \rangle - \frac{1}{4}\langle \mathfrak{T}_j X + \mathfrak{T}_X \mathcal{E}_j, \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, Y \rangle \\
&\quad - \frac{1}{4}\langle \mathfrak{T}_j Y - \mathfrak{T}_Y \mathcal{E}_j, \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, X \rangle - \frac{1}{4}\langle \mathfrak{T}_j Y + \mathfrak{T}_Y \mathcal{E}_j, \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, X \rangle \\
&= -\frac{1}{2}\langle \tilde{T}(X, \mathcal{E}_j), \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, Y \rangle - \frac{1}{2}\langle \tilde{T}(Y, \mathcal{E}_j), \xi \rangle \langle \tilde{T}_\xi^\sharp \mathcal{E}_j, X \rangle \\
&\quad + \frac{1}{4}\langle \phi(X, \mathcal{E}_j), \xi \rangle \langle \tilde{T}_\xi^\sharp Y, \mathcal{E}_j \rangle + \frac{1}{4}\langle \phi(Y, \mathcal{E}_j), \xi \rangle \langle \tilde{T}_\xi^\sharp X, \mathcal{E}_j \rangle \\
&= -\widetilde{\mathcal{T}}^\flat(X, Y) + \frac{1}{4}\langle \phi(\tilde{T}_\xi^\sharp Y, X), \xi \rangle + \frac{1}{4}\langle \phi(\tilde{T}_\xi^\sharp X, Y), \xi \rangle. \tag{57}
\end{aligned}
$$

Taking a local orthonormal basis of $\mathcal{D}$, where for $1 \le i \le \frac{p}{2}$ we have $\mathcal{E}_{i+p/2} = \tilde{T}_\xi^\sharp \mathcal{E}_i$, and using $(55)_1$, we obtain

$$
\begin{aligned}
\sum_{i=1}^p \phi(\tilde{T}_\xi^\sharp \mathcal{E}_i, \mathcal{E}_i) &= \sum_{i=1}^{p/2} \phi(\tilde{T}_\xi^\sharp \mathcal{E}_i, \mathcal{E}_i) + \sum_{i=1}^{p/2} \phi((\tilde{T}_\xi^\sharp)^2 \mathcal{E}_i, \tilde{T}_\xi^\sharp \mathcal{E}_i) \\
&= \sum_{i=1}^{p/2} \phi(\tilde{T}_\xi^\sharp \mathcal{E}_i, \mathcal{E}_i) - \sum_{i=1}^{p/2} \phi(\mathcal{E}_i, \tilde{T}_\xi^\sharp \mathcal{E}_i) = 0,
\end{aligned}
$$

as ϕ is symmetric in its arguments. It follows from (57) that

$$\mathrm{Tr}_g\,\chi = -\,\mathrm{Tr}_g\,\widetilde{\mathcal{T}}^\flat = -\langle\tilde{T},\tilde{T}\rangle = -p,$$

and taking trace of (56) we obtain that

$$\mathrm{div}((\mathrm{Tr}^\perp\mathfrak{T})^\top) = \frac{p(p+5)}{p+4}.$$

By the Divergence Theorem this cannot hold on closed M. On the other hand, if M is not closed, let $\mathfrak{T}_\xi\xi = 0$ and for all $X,Y,Z \in \mathcal{D}$ let $(\mathfrak{T}_\xi X)^\top = 0$, $(\mathfrak{T}_X\xi)^\top = 0$,

$$(\mathfrak{T}_X Y)^\top = \frac{1}{2}g(X,Y)f\xi + \tilde{T}(Y,X),$$

where $f \in C^\infty(M)$, and let $\langle\mathfrak{T}_\xi X,Y\rangle = \langle\tilde{T}_\xi^\sharp X,Y\rangle$, and $\langle\mathfrak{T}_X Y,Z\rangle = \omega(X,Y,Z)$, where ω is any 3-form. Then (56) holds if and only if

$$\xi(f) = \frac{2(p+5)}{p+4}.$$

$\square$

    Corollary 7 can be viewed as an integrability condition for (47a). Below we give examples of $\mathfrak{T}$, constructed for metrics g that satisfy (50) with particular form of χ, obtaining pairs $(g,\mathfrak{T})$ that are critical points of (6) with respect to variations of $\mathfrak{T}$ and $g^\perp$-variations of metric.

Proposition 8. *Let $n,p > 1$ and $H \neq 0$ everywhere on M. For any g such that $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are totally umbilical and (50) holds with $\chi = 0$, there exists a contorsion tensor $\mathfrak{T}$ such that $\mathfrak{T}_X Y \in \mathfrak{X}^\perp$ for all $X,Y \in \mathfrak{X}^\perp$ and $(g,\mathfrak{T})$ is critical for the action (6) with respect to $g^\perp$-variations of metric and arbitrary variations of $\mathfrak{T}$.*

Proof. Suppose that $\mathfrak{T}_X Y \in \mathfrak{X}^\perp$ for all $X,Y \in \mathfrak{X}^\perp$. Then $\phi^\top = 0$, $\chi = 0$, see definitions (46) (because $\langle\mathfrak{T}_j E_a, \mathcal{E}_i\rangle = -\langle\mathfrak{T}_j\mathcal{E}_i, E_a\rangle = 0$), $(\mathrm{Tr}^\perp\mathfrak{T})^\top = 0$, from equations for critical connections it follows that $\mathcal{D}$ is integrable and (47a) is an algebraic equation for symmetric $(0,2)$-tensor ϕ:

$$-\frac{8n-3}{n}\,H^\flat \otimes H^\flat + \Big(\frac{3p-3}{p}\,\mathrm{div}\,\tilde{H} + \frac{2n-1}{n}\,\langle H,H\rangle\Big)\,g^\perp - \frac{1}{2}\,\Upsilon_{T,T} - \langle\phi,H\rangle = 0. \tag{58}$$

For $H \neq 0$, we can always find ϕ (and then $\mathfrak{T}$) satisfying (58). Clearly, such ϕ is not unique. $\square$

Proposition 9. *Let $n,p > 1$ and $H \neq 0$ everywhere on M. For any g such that $\widetilde{\mathcal{D}}$ is totally umbilical and $\mathcal{D}$ is totally geodesic and (50) holds with $\chi = -\widetilde{\mathcal{T}}^\flat$, there exists a contorsion tensor $\mathfrak{T}$ such that $(\mathfrak{T}_X\,\xi)^\perp = \tilde{T}_\xi^\sharp X$ for all $X \in \mathfrak{X}^\perp$, $\xi \in \mathfrak{X}^\top$, and a pair $(g,\mathfrak{T})$ is critical for the action (6) with respect to $g^\perp$-variations of metric and arbitrary variations of $\mathfrak{T}$.*

Proof. For $(\mathfrak{T}_i E_a)^\perp = \tilde{T}_a^\sharp \mathcal{E}_i$ we have for $X, Y \in \mathfrak{X}^\perp$:

$$\chi(X, Y) = \sum\nolimits_{a,j} \langle \tilde{T}_a^\sharp \mathcal{E}_j, X \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_j, Y \rangle = -\widetilde{T}^\flat(X, Y).$$

Since $\langle \mathfrak{T}_i \mathcal{E}_i, E_a \rangle = -\langle \mathfrak{T}_i E_a, \mathcal{E}_i \rangle = -\langle \tilde{T}_a^\sharp \mathcal{E}_i, \mathcal{E}_i \rangle = 0$, we also get $(\mathrm{Tr}^\perp \mathfrak{T})^\top = 0 = \tilde{H}$ and similarly, $\phi^\top = 0$. So, (47a) has the following form:

$$-\frac{8n-3}{n} H^\flat \otimes H^\flat - \frac{1}{2}\Upsilon_{T,T} - 5\widetilde{T}^\flat + \frac{2n-1}{n}\langle H, H\rangle g^\perp - \langle \phi, H\rangle = 0, \quad (59)$$

Again, we get an algebraic equation for symmetric tensor ϕ, which admits many solutions. □

Note that in Propostions 8 and 9 instead of condition $H \neq 0$ everywhere on M, we can assume that at those points of M, where $H = 0$ the metric g satisfies (58) and (59) with $H = 0$ (then these equations do not contain ϕ).

Example 4. Let $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ be totally umbilical, $n, p > 1$, $\mathcal{D}$ integrable and (50) hold with $\chi = 0$. With these assumptions we can construct a simple example of $\mathfrak{T}$ that satisfies the Euler–Lagrange equations (47b–i) and (49) in some domain. Let U be a neighborhood of $p \in M$; we choose any local adapted orthonormal frame $(E_a, \mathcal{E}_i)$ on U. Then, due to $\phi(X, Y) = \phi(X^\perp, Y^\perp)$, we have

$$(\mathrm{div}\,\phi^\top)(\mathcal{E}_i, \mathcal{E}_j) = \sum\nolimits_a \langle \nabla_{E_a}(\phi^\top(\mathcal{E}_i, \mathcal{E}_j)), E_a \rangle + \sum\nolimits_k \langle \nabla_{\mathcal{E}_k}(\phi^\top(\mathcal{E}_i, \mathcal{E}_j)), \mathcal{E}_k \rangle$$
$$-\sum\nolimits_{a,m}\langle \phi^\top(\mathcal{E}_i, \mathcal{E}_m), E_a\rangle\langle \nabla_{E_a}\mathcal{E}_j, \mathcal{E}_m\rangle - \sum\nolimits_{a,m}\langle \phi^\top(\mathcal{E}_m, \mathcal{E}_j), E_a\rangle\langle \nabla_{E_a}\mathcal{E}_i, \mathcal{E}_m\rangle$$
$$-\sum\nolimits_{k,m}\langle \phi^\top(\mathcal{E}_i, \mathcal{E}_m), \mathcal{E}_k\rangle\langle \nabla_{\mathcal{E}_k}\mathcal{E}_j, \mathcal{E}_m\rangle - \sum\nolimits_{k,m}\langle \phi^\top(\mathcal{E}_m, \mathcal{E}_j), \mathcal{E}_k\rangle\langle \nabla_{\mathcal{E}_k}\mathcal{E}_i, \mathcal{E}_m\rangle.$$

We define components of $\mathfrak{T}$ with respect to the adapted frame on U. Let $(\mathfrak{T}_i\mathcal{E}_j - \mathfrak{T}_j\mathcal{E}_i)^\top = 0$ for $i \neq j$ and let $(\mathfrak{T}_i E_a)^\top$, $(\mathfrak{T}_a E_b)^\perp$ and $(\mathfrak{T}_a \mathcal{E}_i)^\perp$ be such that (47c,e,f,h) hold on U. For all $(i, j) \neq (p, p)$, consider (49) evaluated on $(\mathcal{E}_i, \mathcal{E}_j)$ as a system of linear, non-homogeneous, first-order PDEs for $\{\phi(\mathcal{E}_i, \mathcal{E}_j), (i, j) \neq (p, p)\}$, assume in this system that $\phi(\mathcal{E}_p, \mathcal{E}_p) = \frac{n-1}{n}H - \tilde{H} - \sum_{i=1}^{p-1}\phi(\mathcal{E}_i, \mathcal{E}_i)$, and let $\{\phi_{ij}, (i, j) \neq (p, p)\}$ be any local solution of this system of PDEs on (a subset of) U. Let $\mathfrak{T}_i\mathcal{E}_j + \mathfrak{T}_j\mathcal{E}_i = \phi_{ij}$ for $(i, j) \neq (p, p)$ and let $\mathfrak{T}_p\mathcal{E}_p = \frac{1}{2}(\frac{n-1}{n}H - \tilde{H} - \sum_{i=1}^{p-1}\phi_{ii})$, then (47d, i) hold. By the assumption that (50) holds and the fact that (49) is a linear, non-homogeneous equation for ϕ, (49) evaluated on $(\mathcal{E}_p, \mathcal{E}_p)$ will also be satisfied. Thus, equations (47b–i) and (49) hold on (a subset of) U for $\mathfrak{T}$ constructed above.

Note that when we consider adapted variations, we also have the equation dual (with respect to interchanging $\widetilde{\mathcal{D}}$ and $\mathcal{D}$) to (47a), so we can mix different assumptions from the above examples for different distributions, e.g., conditions $(\mathfrak{T}_i E_a)^\perp = \tilde{T}_a^\sharp \mathcal{E}_i$ and $\mathfrak{T}_X Y \in \mathfrak{X}^\top$ for $X, Y \in \mathfrak{X}^\top$.

3.4. Semi-symmetric Connections

The following connections are metric compatible, see [33]. Using variations of $\mathfrak{T}$ in this class, we obtain example with explicitly given tensor $\overline{\mathrm{Ric}}_{\mathcal{D}}$.

Definition 5. An affine connection $\bar{\nabla}$ on M is *semi-symmetric* if its torsion tensor S satisfies $S(X, Y) = \omega(Y)X - \omega(X)Y$, where ω is a one-form on M. For (M, g) we have

$$\bar{\nabla}_X Y = \nabla_X Y + \langle U, Y \rangle X - \langle X, Y \rangle U, \tag{60}$$

where $U = \omega^\sharp$ is the dual vector field.

We find Euler–Lagrange equations of (4) as a particular case of (30a–h), using variations of $\mathfrak{T}$ corresponding to semi-symmetric connections. Now we consider variations of a semi-symmetric connection only among connections also satisfying (60) for some U.

Proposition 10. *A semi-symmetric connection $\bar{\nabla}$ on $(M, g, \mathcal{D})$ satisfying (60) is critical for the action (4) with fixed g among all semi-symmetric connections if and only if*

$$2p(n-1)U^\top - (n-p)\tilde{H} = -(\mathfrak{a}/2)\, s^\top, \quad 2n(p-1)U^\perp - (p-n)H = -(\mathfrak{a}/2)\, s^\perp, \tag{61}$$

where $s^\top = (s(\cdot, \cdot))^\top$ and $s^\perp = (s(\cdot, \cdot))^\perp$. In particular, if $n = p = 1$ and $s = 0$ (no spin) then every semi-symmetric connection is critical among all such connections, because in this case we have $Q = 0$ in (18).

Proof. Let U_t, $t \in (-\epsilon, \epsilon)$, be a family of compactly supported vector fields on M, and let $U = U_0$ and $\dot{U} = \partial_t U_t|_{t=0}$. Then for a fixed metric g, from (82) we obtain

$$\partial_t Q(U_t)|_{t=0} = (p-n)\langle \dot{U}, \tilde{H} \rangle + 2p(n-1)\langle U^\top, \dot{U} \rangle + \langle \dot{U}, H \rangle(n-p)$$
$$+ 2n(p-1)\langle U^\perp, \dot{U} \rangle.$$

Separating parts with $(\dot{U})^\top$ and $(\dot{U})^\perp$, we get

$$\partial_t Q(U_t)|_{t=0} = \langle \dot{U}, \ (p-n)\tilde{H} + 2p(n-1)U^\top \rangle + \langle \dot{U}, \ (n-p)H + 2n(p-1)U^\perp \rangle,$$

from which (61) follow. $\qquad\square$

Remark 4. Using computations from Lemma 6, we can show that if a semi-symmetric connection $\bar{\nabla}$ on $(M, g, \mathcal{D})$ is critical for the action (6) with fixed g, then both $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are integrable and totally geodesic. Indeed, let $\bar{\nabla}$ be given by (60) and satisfy (47b–g) and conditions (47h), i.e., it is critical for action (6) with fixed g. We find from (85) that both $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are integrable. Moreover, if $n = p = 1$ then (84) and its dual with (47b–g) yield $H = 0 = \tilde{H}$ and $U = 0$ (i.e., the connection $\bar{\nabla}$ becomes the Levi-Civita connection). If $n > 2$ and $p > 2$ we also have $H = 0 = \tilde{H}$ and $U = 0$, in this case using also (47h). If $n = 1$ and $p > 1$ we obtain from (47d) that $U^\perp = 0$ and from

$(47\text{h})_1$ that $H = 0$, moreover as both distributions are totally umbilical by Corollary 1, it follows that they are totally geodesic.

Theorem 6. *A pair $(g, \mathfrak{T})$, where $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ and $\mathfrak{T}$ corresponds to a semi-symmetric connection on M defined by (60), is critical for (6) with respect to volume-preserving $g^{\pitchfork}$-variations of metric and variations of $\mathfrak{T}$ corresponding to semi-symmetric connections if and only if the following Euler–Lagrange equations are satisfied:*

$$r_{\mathcal{D}} - \langle \tilde{h}, \tilde{H} \rangle + \widetilde{\mathcal{A}}^{\flat} - \widetilde{\mathcal{T}}^{\flat} + \Psi + \widetilde{\mathcal{K}}^{\flat} - \mathrm{Def}_{\mathcal{D}} H + H^{\flat} \otimes H^{\flat} - \frac{1}{2} \Upsilon_{h,h} - \frac{1}{2} \Upsilon_{T,T}$$

$$- \frac{1}{2} \left(\mathrm{S_{mix}} + \mathrm{div}(\tilde{H} - H) \right) g^{\perp} - \frac{p-n}{4} (\mathrm{div}\, U^{\top}) g^{\perp} + \frac{n(p-1)}{2} U^{\perp \flat} \otimes U^{\perp \flat} = \lambda\, g^{\perp},$$

$$\tag{62a}$$

$$4 \langle \theta, \tilde{H} \rangle + 2(\mathrm{div}(\alpha - \tilde{\theta}))_{\mid V} + 2\langle \tilde{\theta} - \tilde{\alpha}, H \rangle + 2\, H^{\flat} \odot \tilde{H}^{\flat} - 2\, \tilde{\delta}_H$$

$$+ 4\, \Upsilon_{\tilde{\alpha}, \theta} + 2 \Upsilon_{\alpha, \tilde{\alpha}} + 2\, \Upsilon_{\tilde{\theta}, \theta} + \frac{1}{2}\,(n-p)\tilde{\delta}_{U^{\perp}} + \frac{1}{2}\,(n-p)\langle \tilde{\alpha} - \tilde{\theta}, U^{\perp} \rangle$$

$$- (p-n)\langle \theta, U^{\top} \rangle - p(n-1) U^{\top \flat} \otimes U^{\perp \flat} = 0,$$

$$\tag{62b}$$

and

$$2p(n-1)\, U^{\top} - (n-p)\tilde{H} = 0, \quad 2n(p-1)\, U^{\perp} - (p-n)H = 0. \tag{63}$$

Proof. By Proposition 2 and (83), we obtain

$$\partial_t \int_M (\bar{\mathrm{S}}_{\mathrm{mix}} - \mathrm{S}_{\mathrm{mix}})\, \mathrm{d\, vol}_g = \int_M \Big\langle \frac{1}{4}\,(p-n)(\mathrm{div}\, U^{\top})g^{\perp} - (p-n)\langle \theta, U^{\top} \rangle$$

$$- \frac{1}{2}\, n(p-1) U^{\perp \flat} \otimes U^{\perp \flat} - p(n-1) U^{\top \flat} \otimes U^{\perp \flat},\ B \Big\rangle \mathrm{d\, vol}_g\,.$$

Using (27a,b) give rise to (62a,b). Finally, notice that (63) is (61) for vacuum space-time. $\qquad\square$

Although generally $\overline{\mathrm{Ric}}_{\mathcal{D}}$ in (8) has a long expression and is not given here, for particular case of semi-symmetric connections, due to Theorem 6, we present the mixed Ricci tensor explicitly as

$$\begin{cases} \overline{\mathrm{Ric}}_{\mathcal{D}\mid \mathcal{D} \times \mathcal{D}} = \mathrm{Ric}_{\mathcal{D}\mid \mathcal{D} \times \mathcal{D}} + \frac{1}{2}\, n(p-1) U^{\perp \flat} \otimes U^{\perp \flat} - \frac{1}{4}\,(p-n)(\mathrm{div}\, U^{\top})\, g^{\perp} \\ \quad + \frac{Z}{2-n-p}\, g^{\perp}, \\ \overline{\mathrm{Ric}}_{\mathcal{D}\mid V} = \mathrm{Ric}_{\mathcal{D}\mid V} - \frac{1}{2}\,(n-p)\big(\tilde{\delta}_{U^{\perp}} + \langle \tilde{\alpha} - \tilde{\theta}, U^{\perp} \rangle\big) + (p-n)\langle \theta, U^{\top} \rangle \\ \quad + p(n-1) U^{\top \flat} \otimes U^{\perp \flat}, \\ \overline{\mathrm{Ric}}_{\mathcal{D}\mid \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} = \mathrm{Ric}_{\mathcal{D}\mid \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} + \frac{1}{2}\, p(n-1) U^{\top \flat} \otimes U^{\top \flat} - \frac{1}{4}\,(n-p)(\mathrm{div}\, U^{\perp})\, g^{\top} \\ \quad + \frac{Z}{2-n-p}\, g^{\top}, \end{cases} \tag{64}$$

also $\bar{\mathrm{S}}_{\mathcal{D}} = \mathrm{Tr}_g\, \overline{\mathrm{Ric}}_{\mathcal{D}} = \mathrm{S}_{\mathcal{D}} + \frac{2}{2-n-p}\, Z$, where $\mathrm{Ric}_{\mathcal{D}}$ and $\mathrm{S}_{\mathcal{D}}$ as in Definition 3, $n + p > 2$ and

$$Z = \frac{1}{2}\, n(p-1)\|U^{\perp}\|^2 + \frac{1}{2}\, p(n-1)\|U^{\top}\|^2 - \frac{1}{4}\, p(p-n)\, \mathrm{div}\, U^{\top} - \frac{1}{4}\, n(n-p)\, \mathrm{div}\, U^{\perp}.$$

This is because $\overline{\mathrm{Ric}}_{\mathcal{D}} - \frac{1}{2}\operatorname{Tr}(\overline{\mathrm{Ric}}_{\mathcal{D}})g = 0$ is equivalent to all three Euler–Lagrange equations for (6).

Example 5. For a *space-time* (M^{p+1}, g) endowed with $\widetilde{\mathcal{D}}$ spanned by a timelike unit vector field N, see Example 1, the tensor $\overline{\mathrm{Ric}}_{\mathcal{D}}$ has the following particular form (i.e., (64) with $n = 1$):

$$\begin{cases} \overline{\mathrm{Ric}}_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} = \mathrm{Ric}_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} + \frac{1}{2}(p-1)U^{\perp\flat}\otimes U^{\perp\flat} - \frac{1}{4}(p-1)(\operatorname{div}U^{\top})\,g^{\perp} \\ \quad + \frac{Z}{1-p}\,g^{\perp}, \\ \overline{\mathrm{Ric}}_{\mathcal{D}\,|\,V} = \mathrm{Ric}_{\mathcal{D}\,|\,V} - \frac{1}{2}(1-p)\big(\tilde{\delta}_{U^{\perp}} + \langle\tilde{\alpha}-\tilde{\theta}, U^{\perp}\rangle\big), \\ \overline{\mathrm{Ric}}_{\mathcal{D}\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} = \mathrm{Ric}_{\mathcal{D}\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} - \frac{1}{4}\epsilon_N(1-p)(\operatorname{div}U^{\perp}) + \epsilon_N\frac{Z}{1-p}, \end{cases}$$

and $\overline{\mathrm{S}}_{\mathcal{D}} = \mathrm{S}_{\mathcal{D}} + \frac{2\,\epsilon_N Z}{1-p}$, where $Z = \frac{1}{4}(p-1)\big(2\,\|U^{\perp}\|^2 - p\,\operatorname{div}U^{\top} + \operatorname{div}U^{\perp}\big)$, see (29). Note that $\theta = 0$ and $2\tilde{\delta}_{U^{\perp}}(N, \cdot) = (\nabla_N(U^{\perp}))^{\perp\flat}$.

Remark 5. By Proposition 10, also (61) holds, which allows us to simplify the Euler–Lagrange equations of Theorem 6 as discussed below. If $n = p = 1$ then (61) does not give any restrictions for U and all terms containing U vanish in (62a,b)—as expected from the last sentence in Proposition 10.

If $n = 1$ and $p > 1$ then by (61) we have $\tilde{H} = 0$ and $U^{\perp} = \frac{1}{2}H$, while $U^{\top}$ can be arbitrary. We also have $-\frac{1}{2}\Upsilon_{h,h} = -H^{\flat}\otimes H^{\flat}$, and (62a) becomes

$$-\operatorname{div}\tilde{h} - \widetilde{\mathcal{K}}^{\flat} + 2\widetilde{\mathcal{T}}^{\flat} - \frac{p-1}{4}H^{\flat}\otimes H^{\flat}$$
$$+ \frac{1}{2}\Big(\mathrm{S}_{\mathrm{mix}} + \operatorname{div}(\tilde{H}-H) + \frac{p-1}{2}\operatorname{div}U^{\top}\Big)g^{\perp} = \lambda\,g^{\perp},$$

where we replaced $r_{\mathcal{D}}$ by $\operatorname{div}\tilde{h}$ (with additional terms) according to (14), and for (62b) we have

$$2\,(\operatorname{div}(\alpha-\tilde{\theta}))_{|V} + \frac{7+p}{4}\langle\tilde{\theta}-\tilde{\alpha}, H\rangle - \frac{7+p}{4}\tilde{\delta}_H + 2\Upsilon_{\alpha,\tilde{\alpha}} = 0. \qquad (65)$$

Let $N \in \widetilde{\mathcal{D}}$ and $X \in \mathcal{D}$. Using results and notation from [26], we have the following:

$$2(\operatorname{div}\tilde{\theta})(X, N) = (\operatorname{div}\tilde{T}_N^{\sharp})(X) + \langle\tilde{T}_N^{\sharp}H, X\rangle,$$
$$2(\operatorname{div}\alpha)(X, N) = \langle\nabla_N H - \tilde{\tau}_1 H, X\rangle,$$
$$2\Upsilon_{\alpha,\tilde{\alpha}}(X, N) = \langle\tilde{A}_N H, X\rangle,$$
$$2\tilde{\delta}_H(X, N) = \langle\nabla_N H, X\rangle,$$
$$2\langle\tilde{\theta}-\tilde{\alpha}, H\rangle(X, N) = -\langle\tilde{T}_N^{\sharp}H + \tilde{A}_N H, X\rangle,$$

where $\tilde{\tau}_1 = \operatorname{Tr}\tilde{A}_N$. Hence, (65) holds if and only if for unit $N \in \widetilde{\mathcal{D}}$ and all $X \in \mathcal{D}$ we have

$$\frac{1-p}{8}\,\langle\nabla_N H, X\rangle - \langle\tilde\tau_1 H, X\rangle - (\operatorname{div}\tilde T_N^\sharp)(X) - \frac{15+p}{8}\,\langle\tilde T_N^\sharp H, X\rangle$$

$$+\frac{1-p}{8}\,\langle\tilde A_N H, X\rangle = 0.$$

If $n > 1$ and $p > 1$, then using (61) we reduce (62a) to the following:

$$-\operatorname{div}\tilde h - \widetilde{\mathcal K}^\flat + \frac12\,\Upsilon_{h,h} + \frac12\,\Upsilon_{T,T} + 2\,\tilde{\mathcal T}^\flat + \frac12\left(\mathrm S_{\mathrm{mix}} + \operatorname{div}(\tilde H - H)\right)g^\perp$$

$$-\frac{(p-n)^2}{8p(n-1)}\,(\operatorname{div}\tilde H)\,g^\perp - \frac{(p-n)^2 + 8n(p-1)}{8n(p-1)}\,H^\flat\otimes H^\flat = \lambda\,g^\perp,$$

and we reduce (62b) to the following:

$$4\,\Upsilon_{\tilde\alpha,\theta} + 2(\operatorname{div}(\alpha-\tilde\theta))_{\mid\mathrm V} + 2\,\Upsilon_{\alpha,\tilde\alpha} + 2\,\Upsilon_{\tilde\theta,\theta} - \frac{(p-n)^2 + 8n(p-1)}{4n(p-1)}\,\tilde\delta_H$$

$$-\frac{(p-n)^2 + 8n(p-1)}{4n(p-1)}\,\langle\tilde\alpha - \tilde\theta, H\rangle + \frac{(n-p)^2 + 8p(n-1)}{2p(n-1)}\,\langle\theta, \tilde H\rangle$$

$$+\frac{(p-n)^2}{4n(p-1)}\,H^\flat\odot\tilde H^\flat = 0.$$

Note that for vacuum space-time the distributions $\widetilde{\mathcal D}$ and $\mathcal D$ don't need to be umbilical to admit $(g, \mathfrak T)$ critical for (6) among all metrics and semi-symmetric connections.

4. Auxiliary Lemmas

Lemma 1. *For any variation g_t of metric and a t-dependent vector field X on M, we have*

$$\partial_t\,(\operatorname{div} X) = \operatorname{div}(\partial_t X) + \frac12\,X(\operatorname{Tr}_g B).$$

Proof. Differentiating the formula (20) and using (24), we get

$$\partial_t\big((\operatorname{div} X)\,\mathrm d\operatorname{vol}_g\big) = \big(\partial_t\,(\operatorname{div} X) + \frac12\,(\operatorname{div} X)\operatorname{Tr}_g B\big)\,\mathrm d\operatorname{vol}_g,$$

$$\partial_t\big(\mathcal L_X(\mathrm d\operatorname{vol}_g)\big) = \big(\operatorname{div}(\partial_t X) + \frac12\,X(\operatorname{Tr}_g B) + \frac12\,(\operatorname{div} X)\operatorname{Tr}_g B\big)\,\mathrm d\operatorname{vol}_g.$$

From this the claim follows. $\square$

Define symmetric $(1,2)$-tensors L, G, F, by the following formulas:

$$L(X,Y) = \frac14(\Theta^*_{X^\perp}Y^\perp + \Theta^{\wedge*}_{X^\perp}Y^\perp + \Theta^*_{Y^\perp}X^\perp + \Theta^{\wedge*}_{Y^\perp}X^\perp),$$

$$G(X,Y) = \frac14(\Theta^*_{X^\perp}Y^\top + \Theta^{\wedge*}_{X^\perp}Y^\top + \Theta^{\wedge*}_{Y^\perp}X^\top + \Theta^*_{Y^\perp}X^\top),$$

$$F(X,Y) = \frac14(\Theta^*_{X^\top}Y^\perp + \Theta^{\wedge*}_{X^\top}Y^\perp - \Theta_{X^\top}Y^\perp - \Theta^\wedge_{X^\top}Y^\perp$$

$$+\,\Theta^*_{Y^\top}X^\perp + \Theta^{\wedge*}_{Y^\top}X^\perp - \Theta_{Y^\top}X^\perp - \Theta^\wedge_{Y^\top}X^\perp), \qquad (66)$$

where $\Theta = \mathfrak{T} - \mathfrak{T}^* + \mathfrak{T}^\wedge - \mathfrak{T}^{*\wedge}$ and for any $(1,2)$-tensor P we have $\langle P_X^{\wedge *} Y, Z \rangle = \langle P_X^\wedge Z, Y \rangle = \langle P_Z X, Y \rangle = \langle P_Z^* Y, X \rangle$, for all $X, Y \in \mathfrak{X}_M$.

The following equalities (and similar formulas for $\Upsilon_{\alpha,\tilde{\alpha}}$, $\Upsilon_{\theta,\tilde{\alpha}}$, etc.) will be used (here S is a symmetric $(0,2)$-tensor, recall Remark 1 for other notational conventions):

$$\langle \langle \alpha, \tilde{H} \rangle, S \rangle = \sum_{a,i} \langle A_i(E_a), \tilde{H} \rangle S(E_a, \mathcal{E}_i),$$

$$\langle \Upsilon_{\alpha,\theta}, S \rangle = \sum_{a,i} S(A_i(E_a), T_i^\sharp(E_a)),$$

$$\Upsilon_{\alpha,\tilde{\theta}}(X,Y) = \frac{1}{2} \sum_{a,i} \langle X, A_i E_a \rangle \langle Y, \tilde{T}_a^\sharp \mathcal{E}_i \rangle, \quad X \in \mathfrak{X}^\top, \quad Y \in \mathfrak{X}^\perp.$$

The variations of components of Q in (18) (used in previous sections) are collected in the following three lemmas; the results for $g^\top$ variations are dual to $g^\perp$-parts in results for $g^\pitchfork$-variations.

Lemma 2. *For any $g^\pitchfork$-variation of metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ we have*

$$\partial_t \operatorname{Tr}^\top \mathfrak{T} = 0, \quad \partial_t \operatorname{Tr}^\perp \mathfrak{T} = -\sum_i \big(\frac{1}{2}(\mathfrak{T}_i + \mathfrak{T}_i^\wedge)(B^\sharp \mathcal{E}_i)^\perp + (\mathfrak{T}_i + \mathfrak{T}_i^\wedge)(B^\sharp \mathcal{E}_i)^\top \big),$$

$$\partial_t \operatorname{Tr}^\top \mathfrak{T}^* = \sum_a [\mathfrak{T}_a^*, B^\sharp] E_a,$$

$$\partial_t \operatorname{Tr}^\perp \mathfrak{T}^* = \sum_i \big([\mathfrak{T}_i^*, B^\sharp] \mathcal{E}_i - \frac{1}{2}(\mathfrak{T}_i^* + \mathfrak{T}_i^{*\wedge})(B^\sharp \mathcal{E}_i)^\perp - (\mathfrak{T}_i^* + \mathfrak{T}_i^{*\wedge})(B^\sharp \mathcal{E}_i)^\top \big).$$

Proof. For any variation g_t of metric and $X, Y \in \mathfrak{X}_M$ we have

$$(\partial_t \mathfrak{T}^\wedge)_X Y = (\partial_t \mathfrak{T})_Y X = 0, \quad (\partial_t \mathfrak{T}^*)_X = [\mathfrak{T}_X^*, B^\sharp],$$

where the first formula is obvious, the second one follows from $(19)_1$, equality $\partial_t \mathfrak{T} = 0$ and

$$\langle \mathfrak{T}_X^* B^\sharp(Y), Z \rangle = \langle \mathfrak{T}_X Z, B^\sharp(Y) \rangle = B(\mathfrak{T}_X Z, Y) = \partial_t \langle \mathfrak{T}_X Z, Y \rangle = \partial_t \langle \mathfrak{T}_X^* Y, Z \rangle$$
$$= B(\mathfrak{T}_X^* Y, Z) + \langle \partial_t \mathfrak{T}_X^* Y, Z \rangle = \langle B^\sharp \mathfrak{T}_X^* Y, Z \rangle + \langle \partial_t \mathfrak{T}_X^* Y, Z \rangle.$$

Using the above and (22) completes the proof. $\qquad\square$

Lemma 2 is used in the proof of the following

Lemma 3. *For $g^\pitchfork$-variation g_t of metric on $(M, \widetilde{\mathcal{D}}, g, \bar{\nabla} = \nabla + \mathfrak{T})$ we have*

$$\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{|V} = -\sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathfrak{T}_j^* E_a, \mathfrak{T}_a \mathcal{E}_i \rangle$$
$$+ \sum B(\mathcal{E}_i, E_b) \big(\langle \mathfrak{T}_j^* \mathcal{E}_i, \mathfrak{T}_b \mathcal{E}_j \rangle - \langle \mathfrak{T}_a^* \mathcal{E}_i, \mathfrak{T}_b E_a \rangle$$
$$- \langle \mathfrak{T}_b^* E_a, \mathfrak{T}_a \mathcal{E}_i \rangle - \langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a E_b \rangle \big), \tag{67}$$

$$\partial_t \langle \Theta, A \rangle = -2 \sum B(\mathcal{E}_i, \mathcal{E}_j) \big(\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathcal{E}_j, \mathfrak{T}_a E_b \rangle$$
$$- \frac{1}{2} \langle h(E_a, E_b), \mathcal{E}_j \rangle (\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle + \langle \Theta_b \mathcal{E}_i + \Theta_i E_b, E_a \rangle) \big)$$
$$+ \sum B(\mathcal{E}_i, E_b) \big(\langle \Theta_a \mathcal{E}_j + \Theta_j E_a, \mathcal{E}_i \rangle \langle h(E_a, E_b), \mathcal{E}_j \rangle$$
$$- \langle \Theta_a E_b + \Theta_b E_a, E_c \rangle \langle h(E_a, E_c), \mathcal{E}_i \rangle + 2 \langle h(E_a, E_b), \mathcal{E}_j \rangle \langle \mathcal{E}_j, \mathfrak{T}_a \mathcal{E}_i \rangle$$

$$-\frac{1}{2}\left\langle(\tilde{A}-\tilde{T}^{\sharp})_a\mathcal{E}_i,\mathcal{E}_j\right\rangle(\langle\Theta_a\mathcal{E}_j+\Theta_j E_a,E_b\rangle+\langle\Theta_b\mathcal{E}_j+\Theta_j E_b,E_a\rangle)$$

$$-2\left\langle h(E_b,E_a),\mathcal{E}_j\right\rangle\langle\mathcal{E}_i,\mathfrak{T}_j E_a\rangle+2\left\langle h(E_a,E_b),\mathcal{E}_j\right\rangle\langle E_a,\mathfrak{T}_j\mathcal{E}_i\rangle$$

$$-2\left\langle h(E_a,E_c),\mathcal{E}_i\right\rangle\langle E_b,\mathfrak{T}_a E_c\rangle)+\operatorname{div}^{\top}\langle B_{|V},G\rangle-\langle B_{|V},\operatorname{div}^{\top}G\rangle, \tag{68}$$

$$\partial_t\langle\Theta,T^{\sharp}\rangle=-2\sum B(\mathcal{E}_i,\mathcal{E}_j)\langle T(E_a,E_b),\mathcal{E}_i\rangle\langle\mathcal{E}_j,\mathfrak{T}_a E_b\rangle+\sum B(\mathcal{E}_i,E_b)\big(\langle\Theta_a\mathcal{E}_j$$

$$+\Theta_j E_a,\mathcal{E}_i\rangle\langle T(E_a,E_b),\mathcal{E}_j\rangle-2\left\langle T(E_a,E_c),\mathcal{E}_i\right\rangle\langle E_b,\mathfrak{T}_a E_c\rangle$$

$$-\langle\Theta_a E_b+\Theta_b E_a,E_c\rangle\langle T(E_a,E_c),\mathcal{E}_i\rangle+2\langle T(E_a,E_b),\mathcal{E}_j\rangle\langle\mathcal{E}_j,\mathfrak{T}_a\mathcal{E}_i\rangle$$

$$-2\langle T(E_b,E_a),\mathcal{E}_j\rangle\langle\mathcal{E}_i,\mathfrak{T}_j E_a\rangle+2\langle T(E_a,E_b),\mathcal{E}_j\rangle\langle E_a,\mathfrak{T}_j\mathcal{E}_i\rangle), \tag{69}$$

$$\partial_t\langle\Theta,\tilde{T}^{\sharp}\rangle=\sum B(\mathcal{E}_i,\mathcal{E}_j)\big(2\left\langle\tilde{T}(\mathcal{E}_k,\mathcal{E}_j),E_a\right\rangle\langle\mathcal{E}_k,\mathfrak{T}_a\mathcal{E}_i\rangle$$

$$-2\langle\tilde{T}(\mathcal{E}_i,\mathcal{E}_k),E_a\rangle\langle\mathcal{E}_j,\mathfrak{T}_a\mathcal{E}_k\rangle+2\langle\tilde{T}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle\langle E_a,\mathfrak{T}_k\mathcal{E}_i\rangle$$

$$-\frac{1}{2}\left\langle\Theta_a\mathcal{E}_j+\Theta_j E_a,\mathcal{E}_k\right\rangle\langle\tilde{T}(\mathcal{E}_i,\mathcal{E}_k),E_a\rangle+\frac{1}{2}\left\langle\Theta_a\mathcal{E}_k+\Theta_k E_a,\mathcal{E}_i\right\rangle\langle\tilde{T}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle$$

$$-\frac{1}{2}\left(\langle\Theta_a\mathcal{E}_i+\Theta_i E_a,\mathcal{E}_k\rangle-\langle\Theta_a\mathcal{E}_k+\Theta_k E_a,\mathcal{E}_i\rangle\right)\langle E_a,\tilde{T}(\mathcal{E}_j,\mathcal{E}_k)\rangle\big)$$

$$+\sum B(\mathcal{E}_i,E_b)\big(\frac{1}{2}\left(\langle\Theta_a\mathcal{E}_i+\Theta_i E_a,\mathcal{E}_j\rangle-\langle\Theta_a\mathcal{E}_j+\Theta_j E_a,\mathcal{E}_i\rangle\right)\langle E_a,(A+T^{\sharp})_j E_b\rangle$$

$$-2\left\langle\tilde{T}(\mathcal{E}_i,\mathcal{E}_j),E_a\right\rangle\langle E_b,\mathfrak{T}_a\mathcal{E}_j\rangle+2\left\langle\tilde{T}(\mathcal{E}_j,\mathcal{E}_i),E_a\right\rangle\langle E_a,\mathfrak{T}_j E_b\rangle$$

$$+2\left\langle\tilde{T}(\mathcal{E}_j,\mathcal{E}_i),E_a\right\rangle\langle\mathcal{E}_j,\mathfrak{T}_a E_b\rangle-2\left\langle\tilde{T}(\mathcal{E}_k,\mathcal{E}_j),E_b\right\rangle\langle\mathcal{E}_i,\mathfrak{T}_k\mathcal{E}_j\rangle$$

$$-\langle\Theta_a E_b+\Theta_b E_a,\mathcal{E}_j\rangle\langle\tilde{T}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle\big)+\operatorname{div}^{\perp}\langle B_{|V},F\rangle-\langle B_{|V},\operatorname{div}^{\perp}F\rangle, \tag{70}$$

$$\partial_t\langle\Theta,\tilde{A}\rangle=\sum B(\mathcal{E}_i,\mathcal{E}_j)\big(\frac{1}{2}\langle\Theta_k E_a+\Theta_a\mathcal{E}_k,\mathcal{E}_i\rangle\langle\tilde{h}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle$$

$$-\frac{1}{2}\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_k\rangle\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_k),E_a\rangle-2\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_k),E_a\rangle\langle\mathcal{E}_j,\mathfrak{T}_a\mathcal{E}_k\rangle$$

$$-(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_k\rangle+\langle\Theta_k E_a+\Theta_a\mathcal{E}_k,\mathcal{E}_i\rangle)\langle\tilde{h}(\mathcal{E}_j,\mathcal{E}_k),E_a\rangle$$

$$+(\langle\Theta_k E_a+\Theta_a\mathcal{E}_k,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_k\rangle)\langle(\tilde{A}_a-\tilde{T}^{\sharp}_a)\mathcal{E}_i,\mathcal{E}_k\rangle$$

$$+2\langle\tilde{h}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle\langle\mathcal{E}_k,\mathfrak{T}_a\mathcal{E}_i\rangle+2\langle\tilde{h}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle\langle E_a,\mathfrak{T}_k\mathcal{E}_i\rangle\big)$$

$$+\sum B(\mathcal{E}_i,E_b)\big((\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle+\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle)\langle(A_j+T^{\sharp}_j)E_b,E_a\rangle$$

$$-(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle(A_j-T^{\sharp}_j)E_b,E_a\rangle$$

$$-2\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle\langle E_b,\mathfrak{T}_a\mathcal{E}_j\rangle+2\langle\tilde{h}(\mathcal{E}_j,\mathcal{E}_i),E_a\rangle\langle E_a,\mathfrak{T}_j E_b\rangle$$

$$+2\langle\tilde{h}(\mathcal{E}_j,\mathcal{E}_i),E_a\rangle\langle\mathcal{E}_j,\mathfrak{T}_a E_b\rangle-2\langle\tilde{h}(\mathcal{E}_k,\mathcal{E}_j),E_b\rangle\langle\mathcal{E}_i,\mathfrak{T}_k\mathcal{E}_j\rangle$$

$$-\langle\Theta_b E_a+\Theta_a E_b,\mathcal{E}_j\rangle\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle\big)-2\operatorname{div}^{\top}\langle B,L\rangle+2\langle B,\operatorname{div}^{\top}L\rangle, \tag{71}$$

$$\partial_t\langle\operatorname{Tr}^{\top}\mathfrak{T},\operatorname{Tr}^{\perp}\mathfrak{T}^*\rangle=\frac{1}{2}\sum B(\mathcal{E}_i,\mathcal{E}_j)(\langle\operatorname{Tr}^{\top}\mathfrak{T},\mathfrak{T}^*_i\mathcal{E}_j-\mathfrak{T}^*_j\mathcal{E}_i\rangle)$$

$$-\sum B(\mathcal{E}_i,E_b)\langle\operatorname{Tr}^{\top}\mathfrak{T},\mathfrak{T}^*_b\mathcal{E}_i\rangle, \tag{72}$$

$$\partial_t\langle\operatorname{Tr}^{\top}\mathfrak{T}^*,\operatorname{Tr}^{\perp}\mathfrak{T}\rangle=-\frac{1}{2}\sum B(\mathcal{E}_i,\mathcal{E}_j)\langle\mathfrak{T}_j\mathcal{E}_i+\mathfrak{T}_i\mathcal{E}_j,\operatorname{Tr}^{\top}\mathfrak{T}^*\rangle$$

$$+\sum B(\mathcal{E}_i,E_b)\big(\langle\operatorname{Tr}^{\perp}\mathfrak{T},\mathfrak{T}^*_b\mathcal{E}_i\rangle-\langle\mathfrak{T}_b\mathcal{E}_i+\mathfrak{T}_i E_b,\operatorname{Tr}^{\top}\mathfrak{T}^*\rangle\big), \tag{73}$$

$$\partial_t\langle\,Tr^{\top}(\mathfrak{T}^*-\mathfrak{T}),\tilde{H}-H\rangle=\sum B(\mathcal{E}_i,\mathcal{E}_j)\big(-\frac{1}{2}\,\delta_{ij}\,div((Tr^{\top}(\mathfrak{T}^*-\mathfrak{T}))^{\top})$$

$$-\langle H, \mathcal{E}_j\rangle\langle Tr^\top(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle - \langle H, \mathcal{E}_j\rangle\langle Tr^\top(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle + \langle Tr^\top\mathfrak{T}^*, \mathcal{E}_j\rangle\langle\mathcal{E}_i, H\rangle\big)$$

$$+\sum B(\mathcal{E}_i, E_b)\big(\langle\tilde{H}, E_b\rangle\langle Tr^\top(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle - \langle H, \mathcal{E}_i\rangle\langle Tr^\top(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle$$

$$+\langle\mathfrak{T}_b^*\mathcal{E}_i, \tilde{H} - H\rangle + \langle Tr^\top\mathfrak{T}^*, E_b\rangle\langle\mathcal{E}_i, H\rangle + 2\langle T_i^\sharp E_b, Tr^\top(\mathfrak{T}^* - \mathfrak{T})\rangle$$

$$+\langle\mathcal{E}_i, (Tr^\top(\mathfrak{T}^* - \mathfrak{T}))^\perp\rangle\langle\tilde{H}, E_b\rangle - \langle H, \mathcal{E}_i\rangle\langle Tr^\top(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle$$

$$-\langle Tr^\top\mathfrak{T}^*, \mathcal{E}_i\rangle\langle E_b, \tilde{H}\rangle - \langle\nabla_b((Tr^\top(\mathfrak{T}^* - \mathfrak{T}))^\perp), \mathcal{E}_i\rangle - \langle\tilde{A}_b\mathcal{E}_i - \tilde{T}_b^\sharp\mathcal{E}_i, Tr^\top(\mathfrak{T}^* - \mathfrak{T})\rangle\big)$$

$$+div\big((B^\sharp((Tr^\top(\mathfrak{T}^* - \mathfrak{T}))^\perp))^\top - \frac{1}{2}(Tr_\mathcal{D}B)(Tr^\top(\mathfrak{T}^* - \mathfrak{T}))^\top\big), \tag{74}$$

$$\partial_t\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H\rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j)\big(\langle\mathfrak{T}_j^*\mathcal{E}_i, \tilde{H} - H\rangle$$

$$+\langle\,\mathrm{Tr}^\perp\mathfrak{T}^*, \mathcal{E}_i\rangle\langle H, \mathcal{E}_j\rangle - \langle H, \mathcal{E}_j\rangle\,\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle$$

$$-\frac{1}{2}\,\delta_{ij}\,\mathrm{div}((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\top) - \langle H, \mathcal{E}_j\rangle\,\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle\big)$$

$$+\sum B(\mathcal{E}_i, E_b)\big(\langle\tilde{H}, E_b\rangle\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle - \langle H, \mathcal{E}_i\rangle\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle$$

$$+\langle\,\mathrm{Tr}^\perp\mathfrak{T}^*, E_b\rangle\langle H, \mathcal{E}_i\rangle + \langle\mathfrak{T}_i^* E_b, \tilde{H} - H\rangle - \langle\,\mathrm{Tr}^\perp\mathfrak{T}^*, \mathcal{E}_i\rangle\langle\tilde{H}, E_b\rangle$$

$$+2\langle T_i^\sharp E_b, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle + \langle\mathcal{E}_i, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle\langle\tilde{H}, E_b\rangle - \langle\nabla_b((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\perp), \mathcal{E}_i\rangle$$

$$-\langle\tilde{A}_b\mathcal{E}_i - \tilde{T}_b^\sharp\mathcal{E}_i, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle - \langle H, \mathcal{E}_i\rangle\langle\,\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle\big)$$

$$+\mathrm{div}\big((B^\sharp((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\perp))^\top - \frac{1}{2}(\mathrm{Tr}_\mathcal{D}\,B)(\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\top\big). \tag{75}$$

Proof. To obtain $\partial_t\Theta$, we compute for $X, Y, Z \in \mathfrak{X}_M$:

$$\partial_t\langle\mathfrak{T}_X^{*\wedge}Y, Z\rangle = B(\mathfrak{T}_X^{*\wedge}Y, Z) + \langle(\partial_t\mathfrak{T}^{*\wedge})_X Y, Z\rangle.$$

On the other hand,

$$\partial_t\langle\mathfrak{T}_X^{*\wedge}Y, Z\rangle = \partial_t\langle\mathfrak{T}_Y^* X, Z\rangle = B(\mathfrak{T}_Y^* X, Z) + \langle\partial_t(\mathfrak{T}_Y^* X), Z\rangle = \langle\mathfrak{T}_Y^* B^\sharp X, Z\rangle,$$

so

$$(\partial_t\mathfrak{T}^{*\wedge})_X Y = \mathfrak{T}_Y^* B^\sharp X - B^\sharp\mathfrak{T}_Y^* X.$$

From this we obtain

$$(\partial_t\Theta)_X Y = -(\partial_t\mathfrak{T}^*)_X Y - \partial_t(\mathfrak{T}^{*\wedge})_X Y$$

$$= -\mathfrak{T}_X^* B^\sharp Y + B^\sharp\mathfrak{T}_X^* Y - \mathfrak{T}_Y^* B^\sharp X + B^\sharp\mathfrak{T}_Y^* X. \tag{76}$$

We shall use Proposition 3 and the fact that for $g^\perp$-variations $B(X, Y) = 0$ for $X, Y \in \mathfrak{X}^\top$.

Proof of (67). We have $\langle\mathfrak{T}^*, \mathfrak{T}^\wedge\rangle_{|V} = \sum\langle\mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_i E_a\rangle + \sum\langle\mathfrak{T}_i^* E_a, \mathfrak{T}_a\mathcal{E}_i\rangle$, so

$$\partial_t\langle\mathfrak{T}^*, \mathfrak{T}^\wedge\rangle_{|V} = \sum\big[B(\mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_i E_a) + B(\mathfrak{T}_i^* E_a, \mathfrak{T}_a\mathcal{E}_i) + \langle\mathfrak{T}_a^*\partial_t\mathcal{E}_i, \mathfrak{T}_i E_a\rangle$$

$$+\langle\mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_{\partial_t\mathcal{E}_i}E_a\rangle + \langle\mathfrak{T}_{\partial_t\mathcal{E}_i}^* E_a, \mathfrak{T}_a\mathcal{E}_i\rangle + \langle\mathfrak{T}_i^* E_a, \mathfrak{T}_a\partial_t\mathcal{E}_i\rangle$$

$$+\langle(\partial_t\mathfrak{T}^*)_a\mathcal{E}_i, \mathfrak{T}_i E_a\rangle + \langle(\partial_t\mathfrak{T}^*)_i E_a, \mathfrak{T}_a\mathcal{E}_i\rangle\big].$$

We compute 8 terms above separately:

$$\sum B(\mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_i E_a) = \sum\big[B(\mathcal{E}_j, E_b)\langle\mathfrak{T}_a^*\mathcal{E}_i, E_b\rangle\langle\mathfrak{T}_i E_a, \mathcal{E}_j\rangle$$

$$+ B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_a^*\mathcal{E}_i, \mathcal{E}_j\rangle\langle \mathfrak{T}_i E_a, E_b\rangle + B(\mathcal{E}_j, \mathcal{E}_k)\langle \mathfrak{T}_a^*\mathcal{E}_i, \mathcal{E}_k\rangle\langle \mathfrak{T}_i E_a, \mathcal{E}_j\rangle \big],$$

$$\sum B(\mathfrak{T}_i^* E_a, \mathfrak{T}_a\mathcal{E}_i) = \sum \big[B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, E_b\rangle\langle \mathfrak{T}_a\mathcal{E}_i, \mathcal{E}_j\rangle$$
$$+ B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_j\rangle\langle \mathfrak{T}_a\mathcal{E}_i, E_b\rangle + B(\mathcal{E}_k, \mathcal{E}_j)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_j\rangle\langle \mathfrak{T}_a\mathcal{E}_i, \mathcal{E}_k\rangle \big],$$

$$\sum \langle \mathfrak{T}_a^*\partial_t\mathcal{E}_i, \mathfrak{T}_i E_a\rangle = -\sum \big[B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_a^* E_b, \mathfrak{T}_i E_a\rangle + \frac{1}{2} B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_a^*\mathcal{E}_j, \mathfrak{T}_i E_a\rangle \big],$$

$$\sum \langle \mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_{\partial_t\mathcal{E}_i} E_a\rangle = -\sum \big[B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_b E_a\rangle + \frac{1}{2} B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_a^*\mathcal{E}_i, \mathfrak{T}_j E_a\rangle \big],$$

$$\sum \langle \mathfrak{T}_{\partial_t\mathcal{E}_i}^* E_a, \mathfrak{T}_a\mathcal{E}_i\rangle = -\sum \big[B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_b^* E_a, \mathfrak{T}_a\mathcal{E}_i\rangle + \frac{1}{2} B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_j^* E_a, \mathfrak{T}_a\mathcal{E}_i\rangle \big],$$

$$\sum \langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a\partial_t\mathcal{E}_i\rangle = -\sum \big[B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a E_b\rangle + \frac{1}{2} B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_i^* E_a, \mathfrak{T}_a\mathcal{E}_j\rangle \big],$$

$$\sum \langle (\partial_t\mathfrak{T}^*)_a\mathcal{E}_i, \mathfrak{T}_i E_a\rangle = \sum \big[B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_a^* E_b, \mathfrak{T}_i E_a\rangle + B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_a^*\mathcal{E}_j, \mathfrak{T}_i E_a\rangle$$
$$- B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_a^*\mathcal{E}_i, E_b\rangle\langle \mathcal{E}_j, \mathfrak{T}_i E_a\rangle - B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_a^*\mathcal{E}_k, \mathcal{E}_j\rangle\langle \mathcal{E}_i, \mathfrak{T}_k E_a\rangle$$
$$- B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_a^*\mathcal{E}_i, \mathcal{E}_j\rangle\langle E_b, \mathfrak{T}_i E_a\rangle \big],$$

$$\sum \langle (\partial_t\mathfrak{T}^*)_i E_a, \mathfrak{T}_a\mathcal{E}_i\rangle = \sum \big[B(\mathcal{E}_j, E_a)\langle \mathfrak{T}_i^*\mathcal{E}_j, \mathfrak{T}_a\mathcal{E}_i\rangle$$
$$- B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, E_b\rangle\langle \mathcal{E}_j, \mathfrak{T}_a\mathcal{E}_i\rangle - B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_j\rangle\langle E_b, \mathfrak{T}_a\mathcal{E}_i\rangle$$
$$- B(\mathcal{E}_j, \mathcal{E}_k)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_j\rangle\langle \mathcal{E}_k, \mathfrak{T}_a\mathcal{E}_i\rangle \big].$$

Summing the 8 terms computed above and simplifying, we obtain (67).
Proof of (68). We have

$$\langle \Theta, A\rangle = \sum \langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle\langle h(E_a, E_b), \mathcal{E}_i\rangle.$$

So

$$\partial_t\langle \Theta, A\rangle = \sum \big[B(\Theta_a\mathcal{E}_i + \Theta_i E_a, E_b)\langle h(E_a, E_b), \mathcal{E}_i\rangle + \langle \Theta_a\mathcal{E}_i$$
$$+ \Theta_i E_a, E_b\rangle B(h(E_a, E_b), \mathcal{E}_i) + \langle \Theta_a(\partial_t\mathcal{E}_i) + \Theta_{\partial_t\mathcal{E}_i} E_a, E_b\rangle\langle h(E_a, E_b), \mathcal{E}_i\rangle$$
$$+ \langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle\langle \partial_t h(E_a, E_b), \mathcal{E}_i\rangle + \langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle\langle h(E_a, E_b), \partial_t\mathcal{E}_i\rangle$$
$$+ \langle (\partial_t\Theta)_a\mathcal{E}_i + (\partial_t\Theta)_i E_a, E_b\rangle\langle h(E_a, E_b), \mathcal{E}_i\rangle \big].$$

We start from the fourth term of the 6 terms above. Then, from [26],

$$\sum \langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle\langle \partial_t h(E_a, E_b), \mathcal{E}_i\rangle = \sum \frac{1}{2}\big[\langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle$$
$$+ \langle \Theta_b\mathcal{E}_i + \Theta_i E_b, E_a\rangle \big]\big(\nabla_a B(E_b, \mathcal{E}_i) - B(h(E_a, E_b), \mathcal{E}_i) + B(\nabla_i E_a, E_b)\big).$$

We have

$$\frac{1}{2}\sum \big(\langle \Theta_a\mathcal{E}_i + \Theta_i E_a, E_b\rangle + \langle \Theta_b\mathcal{E}_i + \Theta_i E_b, E_a\rangle\big)\nabla_a B(E_b, \mathcal{E}_i)$$
$$= \operatorname{div}^\top\langle B_{|V}, G\rangle - \langle B_{|V}, \operatorname{div}^\top G\rangle,$$

because

$$\langle B_{|V}, \operatorname{div} G\rangle = \frac{1}{2}\sum \big(\langle \nabla_a\Theta_i^{\wedge *} E_b, E_a\rangle + \langle \nabla_a\Theta_i^* E_b, E_a\rangle\big) B(E_b, \mathcal{E}_i).$$

We also have

$$-\sum \frac{1}{2}\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle + \langle \Theta_b \mathcal{E}_i + \Theta_i E_b, E_a \rangle\big) B(h(E_a, E_b), \mathcal{E}_i)$$
$$= -\frac{1}{2}\sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathcal{E}_j, h(E_a, E_b)\rangle\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle + \langle \Theta_b \mathcal{E}_i + \Theta_i E_b, E_a \rangle\big),$$
$$\sum \frac{1}{2}\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle + \langle \Theta_b \mathcal{E}_i + \Theta_i E_b, E_a \rangle\big) B(\nabla_i E_a, E_b)$$
$$= -\frac{1}{2}\sum B(\mathcal{E}_j, E_b)\langle (\tilde{A} - \tilde{T}^\sharp)_a \mathcal{E}_j, \mathcal{E}_i\rangle\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle + \langle \Theta_b \mathcal{E}_i + \Theta_i E_b, E_a \rangle\big).$$

Now we consider other terms of $\partial_t \langle \Theta, A \rangle$. For the fifth term we have

$$\sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle h(E_a, E_b), \partial_t \mathcal{E}_i \rangle$$
$$= -\frac{1}{2}\sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle h(E_a, E_b), \mathcal{E}_j \rangle.$$

For the first, second and third terms we have

$$\sum B(\Theta_a \mathcal{E}_i + \Theta_i E_a, E_b)\langle h(E_a, E_b), \mathcal{E}_i \rangle$$
$$= \sum B(\mathcal{E}_j, E_b)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle \langle h(E_a, E_b), \mathcal{E}_i \rangle,$$
$$\sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle B(h(E_a, E_b), \mathcal{E}_i)$$
$$= \sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle h(E_a, E_b), \mathcal{E}_j \rangle,$$
$$\langle \Theta_a(\partial_t \mathcal{E}_i) + \Theta_{\partial_t \mathcal{E}_i} E_a, E_b \rangle = -\frac{1}{2}\sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \Theta_a \mathcal{E}_j + \Theta_j E_a, E_b \rangle$$
$$- \sum B(\mathcal{E}_i, E_c)\langle \Theta_a E_c + \Theta_c E_a, E_b \rangle.$$

Using (76), we have

$$\sum \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, E_b \rangle$$
$$= \sum \big[-2B(\mathcal{E}_i, E_c)\langle \mathfrak{T}_a^* E_c, E_b \rangle - 2B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_a^* \mathcal{E}_j, E_b \rangle + 2B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_j \rangle$$
$$- 2B(\mathcal{E}_j, E_a)\langle \mathfrak{T}_i^* \mathcal{E}_j, E_b \rangle + 2B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_j \rangle \big].$$

Hence, for the sixth term of $\partial_t \langle \Theta, A \rangle$, we have

$$\sum \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, E_b \rangle \langle h(E_a, E_b), \mathcal{E}_i \rangle$$
$$= \sum [-2B(\mathcal{E}_i, E_c)\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a^* E_c, E_b \rangle$$
$$- 2B(\mathcal{E}_i, \mathcal{E}_j)\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a^* \mathcal{E}_j, E_b \rangle + 2B(\mathcal{E}_j, E_b)\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_j \rangle$$
$$- 2B(\mathcal{E}_j, E_a)\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_i^* \mathcal{E}_j, E_b \rangle + 2B(\mathcal{E}_j, E_b)\langle h(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_i^* E_a, \mathcal{E}_j \rangle].$$

So finally we get (68).
Proof of (69). We have

$$\langle \Theta, T^\sharp \rangle = \sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle,$$

thus

$$\partial_t \langle \Theta, T^\sharp \rangle = \sum \big[B(\Theta_a \mathcal{E}_i + \Theta_i E_a, E_b) \langle T(E_a, E_b), \mathcal{E}_i \rangle$$
$$+ \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle T(E_a, E_b), \partial_t \mathcal{E}_i \rangle + \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle$$
$$+ \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle B(T(E_a, E_b), \mathcal{E}_i) + \langle \Theta_a (\partial_t \mathcal{E}_i) + \Theta_{\partial_t \mathcal{E}_i} E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle \big],$$

because $\partial_t T = 0$. We compute 5 terms above separately:

$$\sum B(\Theta_a \mathcal{E}_i + \Theta_i E_a, E_b) = \sum B(\mathcal{E}_j, E_b) \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle,$$

$$\sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle B(T(E_a, E_b), \mathcal{E}_i)$$
$$= \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \Theta_a \mathcal{E}_j + \Theta_j E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle,$$

$$\sum \langle \Theta_a (\partial_t \mathcal{E}_i) + \Theta_{\partial_t \mathcal{E}_i} E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle = -\frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \Theta_a \mathcal{E}_j$$
$$+ \Theta_j E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle - \sum B(\mathcal{E}_i, E_b) \langle \Theta_a E_b + \Theta_b E_a, E_c \rangle \langle T(E_a, E_c), \mathcal{E}_i \rangle,$$

$$\sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle T(E_a, E_b), \partial_t \mathcal{E}_i \rangle$$
$$= -\frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_j \rangle,$$

$$\sum \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle$$
$$= \sum [-2B(\mathcal{E}_i, E_c) \langle \mathfrak{T}_a^* E_c, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle$$
$$- 2B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathfrak{T}_a^* \mathcal{E}_j, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle + 2B(\mathcal{E}_j, E_b) \langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_j \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle$$
$$- 2B(\mathcal{E}_j, E_a) \langle \mathfrak{T}_i^* \mathcal{E}_j, E_b \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle + 2B(\mathcal{E}_j, E_b) \langle \mathfrak{T}_i^* E_a, \mathcal{E}_j \rangle \langle T(E_a, E_b), \mathcal{E}_i \rangle].$$

Finally, we get (69).

Proof of (70). We have $\langle \Theta, \tilde{T}^\sharp \rangle = \sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle \langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle$. Now we compute

$$\partial_t \langle \Theta, \tilde{T}^\sharp \rangle$$
$$= \sum [B(\Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j) \langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle + \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle B(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a)$$
$$+ \langle \Theta_a (\partial_t \mathcal{E}_i) + \Theta_{\partial_t \mathcal{E}_i} E_a, \mathcal{E}_j \rangle \langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle + \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \partial_t \mathcal{E}_j \rangle \langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle$$
$$+ \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle \langle \partial_t \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle + \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, \mathcal{E}_j \rangle \langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle].$$
$$(77)$$

Let $U : \mathcal{D} \times \mathcal{D} \to \widetilde{\mathcal{D}}$ be a $(1, 2)$-tensor, given by $\langle U_i \mathcal{E}_j, E_a \rangle = \frac{1}{2} (\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle - \langle \Theta_a \mathcal{E}_j + \Theta_j E_a, \mathcal{E}_i \rangle)$. We compute the fifth term in $\partial_t \langle \Theta, \tilde{T}^\sharp \rangle$:

$$2 \sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j \rangle \langle \partial_t \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle = 2 \sum \langle U_i \mathcal{E}_j, E_a \rangle \langle \partial_t \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a \rangle$$
$$= \sum \langle U_i \mathcal{E}_j, E_a \rangle \big(2 \langle \tilde{T}(-\frac{1}{2}(B^\sharp \mathcal{E}_i)^\perp, \mathcal{E}_j), E_a \rangle + 2 \langle \tilde{T}(\mathcal{E}_i, -\frac{1}{2}(B^\sharp \mathcal{E}_j)^\perp), E_a \rangle$$
$$+ \langle \nabla_{(B^\sharp \mathcal{E}_j)^\top} \mathcal{E}_i - \nabla_{(B^\sharp \mathcal{E}_i)^\top} \mathcal{E}_j, E_a \rangle + \langle \nabla_j ((B^\sharp \mathcal{E}_i)^\top) - \nabla_i ((B^\sharp \mathcal{E}_j)^\top), E_a \rangle \big),$$
$$\sum \langle U_i \mathcal{E}_j, E_a \rangle \langle \tilde{T}(-(B^\sharp \mathcal{E}_i)^\perp, \mathcal{E}_j), E_a \rangle = -\sum B(\mathcal{E}_i, \mathcal{E}_k) \langle U_i \mathcal{E}_j, \tilde{T}(\mathcal{E}_k, \mathcal{E}_j) \rangle$$

$$= -\frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_k)\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j\rangle - \langle \Theta_a \mathcal{E}_j + \Theta_j E_a, \mathcal{E}_i\rangle\big)\langle E_a, \tilde{T}(\mathcal{E}_k, \mathcal{E}_j)\rangle,$$

$$-\sum \langle U_i \mathcal{E}_j, E_a\rangle\langle \nabla_{(B^\sharp \mathcal{E}_i)^\top} \mathcal{E}_j, E_a\rangle = \sum B(\mathcal{E}_i, E_b)\langle U_i \mathcal{E}_j, (A + T^\sharp)_j E_b\rangle$$

$$= \frac{1}{2} \sum B(\mathcal{E}_i, E_b)\big(\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j\rangle - \langle \Theta_a \mathcal{E}_j + \Theta_j E_a, \mathcal{E}_i\rangle\big)\langle E_a, (A + T^\sharp)_j E_b\rangle,$$

$$-\sum \langle U_i \mathcal{E}_j, E_a\rangle\langle \nabla_i ((B^\sharp \mathcal{E}_j)^\top), E_a\rangle$$

$$= \sum \big[\langle \nabla_i (B(\mathcal{E}_j, E_a)U_j^* E_a), \mathcal{E}_i\rangle - B(\mathcal{E}_j, E_a)\langle \nabla_i U_j^* E_a, \mathcal{E}_i\rangle\big],$$

where $\langle U_j \mathcal{E}_i, E_a\rangle = \langle U_j^* E_a, \mathcal{E}_i\rangle$. Note that

$$\langle U_j^* E_a, \mathcal{E}_i\rangle = \frac{1}{2}\,\langle \Theta_a^* \mathcal{E}_j + \Theta_a^{\wedge *} \mathcal{E}_j - \Theta_a \mathcal{E}_j - \Theta_a^{\wedge} \mathcal{E}_j, \mathcal{E}_i\rangle,$$

thus, using (1,2)-tensor F defined in (66), we can write

$$-\sum \langle U_i \mathcal{E}_j, E_a\rangle\langle \nabla_i((B^\sharp \mathcal{E}_j)^\top), E_a\rangle = \mathrm{div}^\perp(\langle B_{|V}, F\rangle) - \langle B_{|V}, \mathrm{div}^\perp F\rangle.$$

For the first four terms of $\partial_t \langle \Theta, \tilde{T}^\sharp\rangle$, see (77), we obtain:

$$B(\Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j) = B(\mathcal{E}_j, \mathcal{E}_k)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_k\rangle + B(\mathcal{E}_j, E_b)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b\rangle,$$

$$\sum \langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_j\rangle B(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a) = 0,$$

$$\langle \Theta_a(\partial_t \mathcal{E}_i) + \Theta_{\partial_t \mathcal{E}_i} E_a, \mathcal{E}_j\rangle = -\frac{1}{2}\,B(\mathcal{E}_i, \mathcal{E}_k)\langle \Theta_a \mathcal{E}_k + \Theta_k E_a, \mathcal{E}_j\rangle$$

$$-B(\mathcal{E}_i, E_b)\langle \Theta_a E_b + \Theta_b E_a, \mathcal{E}_j\rangle,$$

$$\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \partial_t \mathcal{E}_j\rangle = -\frac{1}{2}\,B(\mathcal{E}_j, \mathcal{E}_k)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, \mathcal{E}_k\rangle$$

$$-B(\mathcal{E}_j, E_b)\langle \Theta_a \mathcal{E}_i + \Theta_i E_a, E_b\rangle.$$

Using (76), we consider

$$\langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, \mathcal{E}_j\rangle = \langle -\mathfrak{T}_a^* B^\sharp \mathcal{E}_i + B^\sharp \mathfrak{T}_a^* \mathcal{E}_i - \mathfrak{T}_i^* B^\sharp E_a + B^\sharp \mathfrak{T}_i^* E_a, \mathcal{E}_j\rangle$$

$$+\langle -\mathfrak{T}_i^* B^\sharp E_a + B^\sharp \mathfrak{T}_i^* E_a - \mathfrak{T}_a^* B^\sharp \mathcal{E}_i + B^\sharp \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_j\rangle,$$

which can be simplified to the following:

$$\langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, \mathcal{E}_j\rangle = -2 \sum_k B(\mathcal{E}_i, \mathcal{E}_k)\langle \mathfrak{T}_a^* \mathcal{E}_k, \mathcal{E}_j\rangle$$

$$-2 \sum_b B(\mathcal{E}_i, E_b)\langle \mathfrak{T}_a^* E_b, \mathcal{E}_j\rangle + 2 \sum_k B(\mathcal{E}_k, \mathcal{E}_j)\langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_k\rangle$$

$$+2 \sum_b B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_a^* \mathcal{E}_i, E_b\rangle - 2 \sum_k B(\mathcal{E}_k, E_a)\langle \mathfrak{T}_i^* \mathcal{E}_k, \mathcal{E}_j\rangle$$

$$+2 \sum_b B(\mathcal{E}_j, E_b)\langle \mathfrak{T}_i^* E_a, E_b\rangle + 2 \sum_k B(\mathcal{E}_k, \mathcal{E}_j)\langle \mathfrak{T}_i^* E_a, \mathcal{E}_k\rangle.$$

Hence, the sixth term in $\partial_t \langle \Theta, \tilde{T}^\sharp\rangle$ is:

$$\sum \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, \mathcal{E}_j\rangle\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle$$

$$= \sum [-2 B(\mathcal{E}_i, \mathcal{E}_k)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_k, \mathcal{E}_j\rangle$$

$$-2B(\mathcal{E}_i, E_b)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* E_b, \mathcal{E}_j\rangle + 2B(\mathcal{E}_k, \mathcal{E}_j)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_k\rangle$$
$$+2B(\mathcal{E}_j, E_b)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_i, E_b\rangle - 2B(\mathcal{E}_k, E_a)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* \mathcal{E}_k, \mathcal{E}_j\rangle$$
$$+2B(\mathcal{E}_j, E_b)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* E_a, E_b\rangle + 2B(\mathcal{E}_k, \mathcal{E}_j)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* E_a, \mathcal{E}_k\rangle].$$

Finally, we get (70).

Proof of (71). We have

$$\langle \Theta, \tilde{A}\rangle = \sum \langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle.$$

Hence

$$\partial_t \langle \Theta, \tilde{A}\rangle$$
$$= \sum [B(\Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle + \langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle B(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a)$$
$$+\langle \Theta_{\partial_t \mathcal{E}_i} E_a + \Theta_a(\partial_t \mathcal{E}_i), \mathcal{E}_j\rangle\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle + \langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \partial_t \mathcal{E}_j\rangle\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle$$
$$+\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle\langle \partial_t \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle + \langle (\partial_t \Theta)_i E_a + (\partial_t \Theta)_a \mathcal{E}_i, \mathcal{E}_j\rangle\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle].$$

We shall denote by (h) the fifth of the above 6 terms, and write it as sum of seven terms (h1) to (h7):

$$\sum \langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle\langle \partial_t \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle$$
$$= \sum \Big[-\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)\nabla_a B(\mathcal{E}_i, \mathcal{E}_j)$$
$$-\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)\langle \tilde{h}(B^\sharp \mathcal{E}_i, \mathcal{E}_j) + \tilde{h}(\mathcal{E}_i, B^\sharp \mathcal{E}_j), E_a\rangle$$
$$-\frac{1}{2}\big(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle\big)\langle \nabla_i((B^\sharp \mathcal{E}_j)^\top) + \nabla_j((B^\sharp \mathcal{E}_i)^\top), E_a\rangle$$
$$-\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)\langle \nabla_{(B^\sharp \mathcal{E}_j)^\top} \mathcal{E}_i + \nabla_{(B^\sharp \mathcal{E}_i)^\top} \mathcal{E}_j, E_a\rangle$$
$$+\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)(\nabla_i B(\mathcal{E}_j, E_a) + \nabla_j B(\mathcal{E}_i, E_a))$$
$$-\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)(B(\nabla_i E_a, \mathcal{E}_j) + B(\nabla_j E_a, \mathcal{E}_i))$$
$$+\frac{1}{2}(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)(B(\nabla_a \mathcal{E}_i, \mathcal{E}_j) + B(\nabla_a \mathcal{E}_j, \mathcal{E}_i))\Big].$$

We have for the term (h1) above:

$$-\frac{1}{2}\sum(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)\nabla_a B(\mathcal{E}_i, \mathcal{E}_j)$$
$$= \sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \nabla_a(\Theta_i^* \mathcal{E}_j + \Theta_i^{\wedge *}\mathcal{E}_j), E_a\rangle$$
$$- \sum \langle \nabla_a\big(B(\mathcal{E}_i, \mathcal{E}_j)(\Theta_i^* \mathcal{E}_j + \Theta_i^{\wedge *}\mathcal{E}_j)\big), E_a\rangle,$$

which can be written as

$$-\frac{1}{2}\sum(\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle + \langle \Theta_j E_a + \Theta_a \mathcal{E}_j, \mathcal{E}_i\rangle)\nabla_a B(\mathcal{E}_i, \mathcal{E}_j)$$
$$= -2\operatorname{div}^\top \langle B, L\rangle + 2\langle B, \operatorname{div}^\top L\rangle.$$

For (h2):

$$-\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle\tilde{h}(B^\sharp\mathcal{E}_i,\mathcal{E}_j)+\tilde{h}(\mathcal{E}_i,B^\sharp\mathcal{E}_j),E_a\rangle$$
$$=-\sum B(\mathcal{E}_i,\mathcal{E}_k)(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle\tilde{h}(\mathcal{E}_k,\mathcal{E}_j),E_a\rangle.$$

Note that for (h3) we can assume $\nabla_X E_a\in\mathcal{D}$ for all $X\in TM$ at a point, where we compute the formula, and hence

$$-\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle\nabla_i((B^\sharp\mathcal{E}_j)^\top)+\nabla_j((B^\sharp\mathcal{E}_i)^\top),E_a\rangle$$
$$=-\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\nabla_i B(E_a,\mathcal{E}_j).$$

For (h5), analogously,

$$\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)(\nabla_i B(\mathcal{E}_j,E_a)+\nabla_j B(\mathcal{E}_i,E_a))$$
$$=\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\nabla_i B(\mathcal{E}_j,E_a),$$

so (h3)+(h5)=0. For (h4) we have

$$-\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle\nabla_{(B^\sharp\mathcal{E}_j)^\top}\mathcal{E}_i+\nabla_{(B^\sharp\mathcal{E}_i)^\top}\mathcal{E}_j,E_a\rangle$$
$$=\sum B(\mathcal{E}_j,E_b)(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle(A_i+T_i^\sharp)E_b,E_a\rangle.$$

For (h6) term we have

$$-\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)(B(\nabla_i E_a,\mathcal{E}_j)+B(\nabla_j E_a,\mathcal{E}_i))$$
$$=\sum B(\mathcal{E}_k,\mathcal{E}_j)(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle(\tilde{A}_a-\tilde{T}_a^\sharp)\mathcal{E}_k,\mathcal{E}_i\rangle,$$

and (h7) term can be written as

$$\frac{1}{2}\sum(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)(B(\nabla_a\mathcal{E}_i,\mathcal{E}_j)+B(\nabla_a\mathcal{E}_j,\mathcal{E}_i))$$
$$=-\sum B(E_b,\mathcal{E}_i)(\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle+\langle\Theta_j E_a+\Theta_a\mathcal{E}_j,\mathcal{E}_i\rangle)\langle(A_j-T_j^\sharp)E_b,E_a\rangle.$$

Now we compute other terms of $\partial_t\langle\Theta,\tilde{A}\rangle$. Recall that those 6 terms are

$$\partial_t\langle\Theta,\tilde{A}\rangle=\sum\big[B(\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j)\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle$$
$$+\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle B(\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a)+\langle\Theta_{\partial_t\mathcal{E}_i}E_a+\Theta_a(\partial_t\mathcal{E}_i),\mathcal{E}_j\rangle\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle$$
$$+\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\partial_t\mathcal{E}_j\rangle\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle+\langle\Theta_i E_a+\Theta_a\mathcal{E}_i,\mathcal{E}_j\rangle\langle\partial_t\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle$$
$$+\langle(\partial_t\Theta)_i E_a+(\partial_t\Theta)_a\mathcal{E}_i,\mathcal{E}_j\rangle\langle\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\rangle\big].$$

For the first and second terms of the above $\partial_t\langle\Theta,\tilde{A}\rangle$ we have

$$B(\Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j)$$
$$= \sum B(\mathcal{E}_j, \mathcal{E}_k)\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_k\rangle + \sum B(\mathcal{E}_j, E_b)\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, E_b\rangle,$$
$$\sum \langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_j\rangle B(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a) = 0,$$

because $B = 0$ on $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$. For the third and fourth terms we have:

$$\langle \Theta_{\partial_t \mathcal{E}_i} E_a + \Theta_a(\partial_t \mathcal{E}_i), \mathcal{E}_j\rangle$$
$$= \sum [-\frac{1}{2} B(\mathcal{E}_i, \mathcal{E}_k)\langle \Theta_k E_a + \Theta_a \mathcal{E}_k, \mathcal{E}_j\rangle - B(\mathcal{E}_i, E_b)\langle \Theta_b E_a + \Theta_a E_b, \mathcal{E}_j\rangle],$$
$$\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \partial_t \mathcal{E}_j\rangle$$
$$= \sum [-\frac{1}{2} B(\mathcal{E}_j, \mathcal{E}_k)\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, \mathcal{E}_k\rangle - B(\mathcal{E}_j, E_b)\langle \Theta_i E_a + \Theta_a \mathcal{E}_i, E_b\rangle].$$

For the sixth term, note that

$$\sum \langle (\partial_t \Theta)_a \mathcal{E}_i + (\partial_t \Theta)_i E_a, \mathcal{E}_j\rangle\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle$$
$$= \sum [-2B(\mathcal{E}_i, \mathcal{E}_k)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_k, \mathcal{E}_j\rangle$$
$$- 2B(\mathcal{E}_i, E_b)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* E_b, \mathcal{E}_j\rangle + 2B(\mathcal{E}_k, \mathcal{E}_j)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_i, \mathcal{E}_k\rangle$$
$$+ 2B(\mathcal{E}_j, E_b)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_a^* \mathcal{E}_i, E_b\rangle - 2B(\mathcal{E}_k, E_a)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* \mathcal{E}_k, \mathcal{E}_j\rangle$$
$$+ 2B(\mathcal{E}_j, E_b)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* E_a, E_b\rangle + 2B(\mathcal{E}_k, \mathcal{E}_j)\langle \tilde{h}(\mathcal{E}_i, \mathcal{E}_j), E_a\rangle\langle \mathfrak{T}_i^* E_a, \mathcal{E}_k\rangle].$$

Finally, we get (71).
Proof of (72) **and** (73) is straightforward.
Proof of (74) **and** (75). The variation formulas for these terms appear in the following part of Q in (18):

$$-\langle \mathrm{Tr}^\top \mathfrak{T} - \mathrm{Tr}^\perp \mathfrak{T} + \mathrm{Tr}^\perp \mathfrak{T}^* - \mathrm{Tr}^\top \mathfrak{T}^*, \tilde{H} - H\rangle$$
$$= \langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H\rangle + \langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H\rangle.$$

We have

$$\partial_t \langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H\rangle = B(\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H)$$
$$+ \sum \langle (\partial_t \mathfrak{T}^*)_k \mathcal{E}_k, \tilde{H} - H\rangle + \langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \partial_t \tilde{H}\rangle - \langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \partial_t H\rangle,$$
$$B(\mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H) = \sum B(\mathcal{E}_i, E_b)\big(\langle \tilde{H}, E_b\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle$$
$$-\langle H, \mathcal{E}_i\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle\big) - \sum B(\mathcal{E}_i, \mathcal{E}_j)\langle H, \mathcal{E}_j\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle.$$

Then we have

$$\sum \langle (\partial_t \mathfrak{T}^*)_a E_a, \tilde{H} - H\rangle = \sum B(\mathcal{E}_i, E_b)\big(\langle \mathfrak{T}_b^* \mathcal{E}_i, \tilde{H} - H\rangle + \langle \mathrm{Tr}^\top \mathfrak{T}^*, E_b\rangle\langle \mathcal{E}_i, H\rangle$$
$$-\langle \mathrm{Tr}^\top \mathfrak{T}^*, \mathcal{E}_i\rangle\langle E_b, \tilde{H}\rangle\big) + \sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathrm{Tr}^\top \mathfrak{T}^*, \mathcal{E}_j\rangle\langle \mathcal{E}_i, H\rangle,$$
$$\sum \langle (\partial_t \mathfrak{T}^*)_i \mathcal{E}_i, \tilde{H} - H\rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j)\big(\langle \mathfrak{T}_j^* \mathcal{E}_i, \tilde{H} - H\rangle + \langle \mathrm{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i\rangle\langle H, \mathcal{E}_j\rangle\big)$$
$$+ \sum B(\mathcal{E}_i, E_b)\big(\langle \mathfrak{T}_i^* E_b, \tilde{H} - H\rangle + \langle \mathrm{Tr}^\perp \mathfrak{T}^*, E_b\rangle\langle H, \mathcal{E}_i\rangle - \langle \mathrm{Tr}^\perp \mathfrak{T}^*, \mathcal{E}_i\rangle\langle \tilde{H}, E_b\rangle\big).$$

Next, we shall use equations (20) and (21) from [26]:

$$\langle \partial_t \tilde{H}, X \rangle = \langle\, 2\langle \theta, X^\top \rangle,\, B \rangle - \frac{1}{2}\, X^\top (\mathrm{Tr}_{\mathcal{D}}\, B),$$

$$\langle \partial_t H, X \rangle = \mathrm{div}(B^\sharp(X^\perp))^\top + \langle B^\sharp(X^\perp), \tilde{H} \rangle - \langle B^\sharp(X^\perp), H \rangle - \langle \tilde{\delta}_{X^\perp}, B \rangle$$
$$- \langle\langle \tilde{\alpha} - \tilde{\theta}, X^\perp \rangle, B \rangle - B(H, X^\top).$$

We have

$$\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \partial_t \tilde{H} \rangle = 2 \sum B(\mathcal{E}_i, E_b)\langle T_i^\sharp E_b, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle$$

$$- \frac{1}{2}\sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathcal{E}_i, \mathcal{E}_j \rangle\, \mathrm{div}((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\top) - \mathrm{div}(\frac{1}{2}\, (\mathrm{Tr}_{\mathcal{D}}\, B)(\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\top).$$

Finally,

$$\langle \partial_t H, \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}) \rangle = \mathrm{div}((B^\sharp((\mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}))^\perp))^\top)$$

$$+ \sum [B(\mathcal{E}_i, E_b)\langle \mathcal{E}_i, \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T})\rangle\langle \tilde{H}, E_b\rangle - B(\mathcal{E}_i, \mathcal{E}_j)\langle H, \mathcal{E}_j\rangle\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle$$

$$- B(\mathcal{E}_i, E_b)\langle H, \mathcal{E}_i\rangle\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle - B(\mathcal{E}_i, E_b)\langle \nabla_b((\mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}))^\perp), \mathcal{E}_i\rangle$$

$$- B(\mathcal{E}_i, E_b)\langle(\tilde{A}_b - \tilde{T}_b^\sharp)\mathcal{E}_i, \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T})\rangle - B(\mathcal{E}_i, E_b)\langle H, \mathcal{E}_i\rangle\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle],$$

$$\langle \partial_t H, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}) \rangle = \mathrm{div}((B^\sharp((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\perp))^\top)$$

$$+ \sum [B(\mathcal{E}_i, E_b)\langle \mathcal{E}_i, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle\langle \tilde{H}, E_b\rangle - B(\mathcal{E}_i, \mathcal{E}_j)\langle H, \mathcal{E}_j\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \mathcal{E}_i\rangle$$

$$- B(\mathcal{E}_i, E_b)\langle H, \mathcal{E}_i\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle - B(\mathcal{E}_i, E_b)\langle \nabla_b((\mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}))^\perp), \mathcal{E}_i\rangle$$

$$- B(\mathcal{E}_i, E_b)\langle(\tilde{A}_b - \tilde{T}_b^\sharp)\mathcal{E}_i, \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T})\rangle - B(\mathcal{E}_i, E_b)\langle H, \mathcal{E}_i\rangle\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), E_b\rangle].$$

Summing $\partial_t\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle$ and $\partial_t\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle$, we obtain (74) and (75). $\qquad\square$

We have the following results for critical metric connections and $g^\perp$-variations (see Definition 2), that can be considered as a special case of Lemma 3 (see (46) for definitions of tensors χ and ϕ, that appear below).

Lemma 4. *Let $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ be both totally umbilical distributions on (M, g). Let g_t be a $g^\perp$-variation of metric g and $\nabla + \mathfrak{T}$ be a metric connection: $\mathfrak{T}^* = -\mathfrak{T}$. If $\mathfrak{T}$ is a critical point for (6) with fixed g, then, up to divergences of compactly supported vector fields, the following formulas hold:*

$$\partial_t\langle \mathrm{Tr}^\top(\mathfrak{T}^* - \mathfrak{T}),\ \tilde{H} - H \rangle = \langle B,\ 3\, H^\flat \odot (\mathrm{Tr}^\top \mathfrak{T})^{\perp\flat}$$
$$+ \frac{p-1}{p}\, (\mathrm{div}\,\tilde{H}) g^\perp \rangle, \tag{78a}$$

$$\partial_t\langle \mathrm{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \langle B, 3\frac{n-1}{n} H^\flat \otimes H^\flat - \frac{1}{2}\langle \phi, \tilde{H} - H \rangle$$
$$+ \mathrm{div}((\mathrm{Tr}^\perp \mathfrak{T})^\top) g^\perp \rangle, \tag{78b}$$

$$\partial_t\langle \mathrm{Tr}^\top \mathfrak{T}, \mathrm{Tr}^\perp \mathfrak{T}^* \rangle = 0, \tag{78c}$$

$$\partial_t\langle \mathrm{Tr}^\top \mathfrak{T}^*, \mathrm{Tr}^\perp \mathfrak{T} \rangle = \langle B,\ \frac{1}{2}\, \langle \phi, \mathrm{Tr}^\top \mathfrak{T} \rangle \rangle, \tag{78d}$$

$$\partial_t \langle \Theta, A \rangle = \langle B, \, \frac{2}{n} \, H^\flat \odot (\mathrm{Tr}^\top \mathfrak{T})^{\perp \flat} \rangle, \tag{78e}$$

$$\partial_t \langle \Theta, \tilde{A} \rangle = \langle B, \, \frac{1}{p} \, \langle \phi, \tilde{H} \rangle + 2 \, \mathrm{div} \, L^\top + 8\chi + 8\widetilde{\mathcal{T}}^\flat \rangle, \tag{78f}$$

$$\partial_t \langle \Theta, T^\sharp \rangle = \langle B, \, \Upsilon_{T,T} \rangle, \tag{78g}$$

$$\partial_t \langle \Theta, \tilde{T}^\sharp \rangle = \langle B, \, 12 \, \widetilde{\mathcal{T}}^\flat + 2\chi \rangle, \tag{78h}$$

$$\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = \langle B, \, \frac{1}{2} \, \Upsilon_{T,T} - 2\widetilde{\mathcal{T}}^\flat - \chi \rangle. \tag{78i}$$

Proof. First we adapt the results of Lemma 3 to the case of $g^\perp$-variation and totally umbilical distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. Then we shall use the Euler–Lagrange equations (31a–j), which for a metric connection have the following form:

$$(\mathfrak{T}_V \, U - \mathfrak{T}_U \, V)^\top = 2 \, \tilde{T}(U, V), \tag{79a}$$

$$\langle (\mathfrak{T}_U - T_U^\sharp) X, \, Y \rangle = 0, \tag{79b}$$

$$(\mathrm{Tr}^\perp \mathfrak{T})^\perp = \frac{n-1}{n} H, \tag{79c}$$

$$(\mathfrak{T}_Y \, X - \mathfrak{T}_X \, Y)^\perp = 2 \, T(X, Y), \tag{79d}$$

$$\langle (\mathfrak{T}_X - T_X^\sharp) \, U, \, V \rangle = 0, \tag{79e}$$

$$(\mathrm{Tr}^\top \mathfrak{T})^\top = \frac{p-1}{p} \, \tilde{H}, \tag{79f}$$

for all $X, Y \in \widetilde{\mathcal{D}}$ and $U, V \in \mathcal{D}$, and

$$(\mathrm{Tr}^\perp \mathfrak{T})^\top = -\tilde{H} \quad \text{for} \quad n > 1, \qquad (\mathrm{Tr}^\top \mathfrak{T})^\perp = -H \quad \text{for} \quad p > 1.$$

The last two equations require special assumptions on dimensions of the distributions—we shall not use them in this proof. For metric connections we also have

$$\Theta = \Theta^\wedge = 2 \, (\mathfrak{T} + \mathfrak{T}^\wedge).$$

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions, using (79f), we obtain

$$\partial_t \langle \mathrm{Tr}^\top (\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \big(3 \langle H, \mathcal{E}_j \rangle \langle \mathrm{Tr}^\top \mathfrak{T}, \mathcal{E}_i \rangle$$

$$+ \frac{p-1}{p} \delta_{ij} \, \mathrm{div} \, \tilde{H} \Big) + \mathrm{div} \left(\frac{p-1}{p} (\mathrm{Tr}_\mathcal{D} B) \tilde{H} - 2 (B^\sharp (\mathrm{Tr}^\top \mathfrak{T})^\perp)^\top \right).$$

Writing divergence of compactly supported vector field as div Z, we finally get

$$\partial_t \langle \mathrm{Tr}^\top (\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle$$

$$= \sum B(\mathcal{E}_i, \mathcal{E}_j) \Big(3 \, \langle H, \mathcal{E}_j \rangle \langle \mathrm{Tr}^\top \mathfrak{T}, \mathcal{E}_i \rangle + \frac{p-1}{p} \, \delta_{ij} \, \mathrm{div} \, \tilde{H} \Big) + \mathrm{div} \, Z.$$

Without explicitly using the orthonormal frame, we can write the above as (78a).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions, using (79c), we have:

$$\partial_t \langle \operatorname{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \Big(3\, \frac{n-1}{n}\, \langle H, \mathcal{E}_j \rangle \langle H, \mathcal{E}_i \rangle$$

$$- \frac{1}{2}\, \langle \mathfrak{T}_j \mathcal{E}_i, \tilde{H} - H \rangle - \frac{1}{2}\, \langle \mathfrak{T}_i \mathcal{E}_j, \tilde{H} - H \rangle + \delta_{ij}\, \operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) \Big)$$

$$+ \operatorname{div}\big((\operatorname{Tr}_{\mathcal{D}} B)(\operatorname{Tr}^\perp \mathfrak{T})^\top - 2(B^\sharp(\operatorname{Tr}^\perp \mathfrak{T})^\perp)^\top \big).$$

Writing divergence of compactly supported vector field as div Z, we finally get

$$\partial_t \langle \operatorname{Tr}^\perp(\mathfrak{T}^* - \mathfrak{T}), \tilde{H} - H \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \Big(3\frac{n-1}{n}\, \langle H, \mathcal{E}_j \rangle \langle H, \mathcal{E}_i \rangle$$

$$- \frac{1}{2}\, \langle \mathfrak{T}_j \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_j, \tilde{H} - H \rangle + \delta_{ij}\, \operatorname{div}((\operatorname{Tr}^\perp \mathfrak{T})^\top) \Big) + \operatorname{div} Z.$$

Without explicitly using the orthonormal frame, we can write the above as (78b).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions:

$$\partial_t \langle \operatorname{Tr}^\top \mathfrak{T}, \operatorname{Tr}^\perp \mathfrak{T}^* \rangle = \frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \operatorname{Tr}^\top \mathfrak{T},\ \mathfrak{T}_i^* \mathcal{E}_j - \mathfrak{T}_j^* \mathcal{E}_i \rangle = 0,$$

as $B(\mathcal{E}_i, \mathcal{E}_j)$ is symmetric and $\mathfrak{T}_i^* \mathcal{E}_j - \mathfrak{T}_j^* \mathcal{E}_i$ is antisymmetric in i, j.

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions:

$$\partial_t \langle \operatorname{Tr}^\top \mathfrak{T}^*, \operatorname{Tr}^\perp \mathfrak{T} \rangle = \frac{1}{2} \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \phi(\mathcal{E}_i, \mathcal{E}_j), \operatorname{Tr}^\top \mathfrak{T} \rangle.$$

Without explicitly using the orthonormal frame, we can write the above as (78d).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions, using (79b), we have:

$$\partial_t \langle \Theta, A \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j) \big(2\, \langle \mathcal{E}_j, H/n \rangle \langle \operatorname{Tr}^\top \mathfrak{T}, \mathcal{E}_i \rangle - 4 \langle \mathcal{E}_j, H/n \rangle \langle T_i^\sharp E_a, E_a \rangle \big)$$

$$= 2 \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle \mathcal{E}_j, H/n \rangle \langle \operatorname{Tr}^\top \mathfrak{T}, \mathcal{E}_i \rangle.$$

Without explicitly using the orthonormal frame, we can write the above as (78e).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions:

$$\partial_t \langle \Theta, \tilde{A} \rangle = -2 \operatorname{div}^\top \langle B, L \rangle + 2 \langle B, \operatorname{div}^\top L \rangle$$

$$+ 4 \sum B(\mathcal{E}_i, \mathcal{E}_j) \big(\langle \mathfrak{T}_k \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_k, E_a \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_j, \mathcal{E}_k \rangle - 2 \langle \mathfrak{T}_j E_a, \mathcal{E}_i \rangle \langle \tilde{H}/p, E_a \rangle \big).$$

Using symmetry $B(X, Y) = B(Y, X)$ for $X, Y \in \mathfrak{X}^\perp$, we obtain:

$$\partial_t \langle \Theta, \tilde{A} \rangle = -2 \operatorname{div}^\top \langle B, L \rangle + 2 \langle B, \operatorname{div}^\top L \rangle$$
$$+ \sum B(\mathcal{E}_i, \mathcal{E}_j) \big[\frac{1}{p} \langle \mathfrak{T}_j \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_j, \tilde{H} \rangle + 4 \langle \mathfrak{T}_k \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_k, \tilde{T}(\mathcal{E}_j, \mathcal{E}_k) \rangle \big].$$

Note that

$$\langle \mathfrak{T}_k \mathcal{E}_i + \mathfrak{T}_i \mathcal{E}_k, \tilde{T}(\mathcal{E}_j, \mathcal{E}_k) \rangle = \langle \mathfrak{T}_k E_a, \mathcal{E}_i \rangle \langle \tilde{T}_a^\sharp \mathcal{E}_k, \mathcal{E}_j \rangle - \langle \mathfrak{T}_i E_a, \tilde{T}_a^\sharp \mathcal{E}_j \rangle.$$

By the above, we can write $\partial_t \langle \Theta, \tilde{A} \rangle$ as

$$\partial_t \langle \Theta, \tilde{A} \rangle = -2 \operatorname{div}^\top \langle B, L \rangle + \langle B, \frac{1}{p} \langle \phi, \tilde{H} \rangle + 2 \operatorname{div}^\top L - 4\,\psi$$
$$+ 4 \sum_{a,j} (\mathfrak{T}_j E_a)^{\perp \flat} \odot (\tilde{T}_a^\sharp \mathcal{E}_j)^{\perp \flat} \rangle, \tag{80}$$

where

$$\psi(X, Y) = \frac{1}{2} \sum_a \big(\langle \mathfrak{T}_{X^\perp} E_a, \tilde{T}_a^\sharp(Y^\perp) \rangle + \langle \mathfrak{T}_{Y^\perp} E_a, \tilde{T}_a^\sharp(X^\perp) \rangle \big).$$

We claim that ψ can be written in terms of tensor χ introduced in (46). Indeed, for arbitrary symmetric (0,2)-tensor $B : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ we have

$$\langle B, \psi \rangle = \langle B, \ -2\tilde{T}^\flat - \sum_{a,j} (\mathfrak{T}_j E_a)^{\perp \flat} \odot (\tilde{T}_a^\sharp \mathcal{E}_j)^{\perp \flat} \rangle.$$

Using (46), we obtain

$$\psi = -2\,\tilde{T}^\flat - \chi. \tag{81}$$

Using the following computation:

$$\langle B, \operatorname{div}^\top L \rangle = \langle B, \operatorname{div}^\top L^\top + \operatorname{div}^\top L^\perp \rangle$$
$$= \langle B, \operatorname{div} L^\top \rangle + \langle B, \langle L^\top, \tilde{H} \rangle \rangle - \langle B, \langle L^\perp, H \rangle \rangle,$$
$$\operatorname{div}^\top \langle B, L \rangle = \operatorname{div} \langle B, L \rangle - \operatorname{div}^\perp \langle B, L \rangle$$
$$= \operatorname{div} \langle B, L \rangle + \langle B, \langle L^\top, \tilde{H} \rangle \rangle - \operatorname{div}^\perp \langle B, L^\perp \rangle,$$

we obtain

$$- \operatorname{div}^\top \langle B, L \rangle + \langle B, \operatorname{div}^\top L \rangle = - \operatorname{div} \langle B, L^\top \rangle + \langle B, \operatorname{div} L^\top \rangle,$$

which, together with (80)–(81), up to divergence of a compactly supported vector field, yields (78f).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions we have

$$\partial_t \langle \Theta, T^\sharp \rangle = 2 \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle T(E_a, E_b), \mathcal{E}_i \rangle \langle \mathfrak{T}_a \mathcal{E}_j, E_b \rangle.$$

Using (79d), we obtain:

$$\partial_t \langle \Theta, T^\sharp \rangle = 2 \sum B(\mathcal{E}_i, \mathcal{E}_j) \langle T(E_a, E_b), \mathcal{E}_i \rangle \langle T(E_a, E_b), \mathcal{E}_j \rangle.$$

Without explicitly using the orthonormal frame, we can write the above as (78g).

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions we have

$$\partial_t \langle \Theta, \tilde{T}^\sharp \rangle = -\sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \tilde{T}(\mathcal{E}_i, \mathcal{E}_k), E_a\rangle (4\langle \mathfrak{T}_a \mathcal{E}_j + \mathfrak{T}_j E_a, \mathcal{E}_k\rangle - 2\langle \mathfrak{T}_k E_a, \mathcal{E}_j\rangle).$$

Using (79e), we obtain:

$$\partial_t \langle \Theta, \tilde{T}^\sharp \rangle = \sum B(\mathcal{E}_i, \mathcal{E}_j)(4\langle (\tilde{T}_a^\sharp)^2 \mathcal{E}_j, \mathcal{E}_i\rangle - 4\langle \mathfrak{T}_j E_a, \tilde{T}_a^\sharp \mathcal{E}_i\rangle - 2\langle \tilde{T}_a^\sharp \mathcal{E}_j, \mathcal{E}_i\rangle\langle \mathfrak{T}_j E_a, \mathcal{E}_k\rangle).$$

Next, we have

$$\partial_t \langle \Theta, \tilde{T}^\sharp \rangle = \langle B, \ 4\tilde{T}^\flat - 4\psi - 2\sum\nolimits_{a,j}(\mathfrak{T}_j E_a)^{\perp\flat} \odot (\tilde{T}_a^\sharp \mathcal{E}_j)^{\perp\flat}\rangle.$$

For metric connections, $g^\perp$-variations of metric and totally umbilical distributions we have:

$$\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = \sum B(\mathcal{E}_i, \mathcal{E}_j)\langle \mathfrak{T}_j E_a, \mathfrak{T}_a \mathcal{E}_i\rangle.$$

Using (79b,d,e), we obtain the following:

$$\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = \sum B(\mathcal{E}_i, \mathcal{E}_j)\left(\langle T(E_a, E_b), \mathcal{E}_j\rangle\langle T(E_a, E_b), \mathcal{E}_i\rangle + \langle \mathfrak{T}_j E_a, \tilde{T}_a^\sharp \mathcal{E}_i\rangle \right).$$

We can write, $\partial_t \langle \mathfrak{T}^*, \mathfrak{T}^\wedge \rangle_{\,|\,V} = \langle B, \frac{1}{2}\Upsilon_{T,T} + \psi\rangle$, which, together with (81), yields (78i). $\qquad\qquad\square$

Lemma 5. *Let g_t be a $g^\perp$-variation of $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$, let $\mathfrak{T}$ be the contorsion tensor of a metric connection that is critical for* (6) *with fixed g, and let $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ be totally umbilical distributions. Then, up to divergences of compactly supported vector fields, for Q given by* (18) *we have*

$$-\partial_t Q = \left\langle\!\left\langle \phi, \frac{p+2}{2p}\tilde{H} - \frac{1}{2}H + \frac{1}{2}\,\mathrm{Tr}^\top \mathfrak{T}\right\rangle - 2\,\mathrm{div}\,\phi^\top + 7\chi + \frac{3n+2}{n}H^\flat \odot (\mathrm{Tr}^\top \mathfrak{T})^{\perp\flat} \right.$$
$$\left. -\,\mathrm{div}((\mathrm{Tr}^\perp \mathfrak{T})^\top)\,g^\perp + \frac{p-1}{p}(\mathrm{div}\,\tilde{H})g^\perp - 3\frac{n-1}{n}H^\flat \otimes H^\flat + 2\tilde{T}^\flat + \frac{3}{2}\Upsilon_{T,T},\, B \right\rangle.$$

Proof. Recall that

$$L(X,Y) = \frac{1}{4}(\Theta^*_{X^\perp}Y^\perp + \Theta^{\wedge *}_{X^\perp}Y^\perp + \Theta^*_{Y^\perp}X^\perp + \Theta^{\wedge *}_{Y^\perp}X^\perp),$$

and let $L^\perp(X,Y) = (L(X,Y))^\perp$ and $L^\top(X,Y) = (L(X,Y))^\top$ for $X,Y \in \mathfrak{X}_M$. We have $L = L^\top + L^\perp$. Note that $\langle L^\perp(X,Y), Z\rangle = \langle L^\perp(X^\perp, Y^\perp), Z^\perp\rangle$ and for metric connections

$$\langle \mathfrak{T}^{\wedge *}_X Y, Z\rangle = \langle \mathfrak{T}^\wedge_X Z, Y\rangle = \langle \mathfrak{T}_Z X, Y\rangle = -\langle \mathfrak{T}_Z Y, X\rangle$$
$$= -\langle \mathfrak{T}^\wedge_Y Z, X\rangle = -\langle \mathfrak{T}^{\wedge *}_Y X, Z\rangle = -\langle \mathfrak{T}^{\wedge *\wedge}_X Y, Z\rangle,$$

for all $X, Y, Z \in \mathfrak{X}_M$, so

$$4\langle L(X,Y), Z\rangle = \langle Z, \Theta^*_X Y + \Theta^{\wedge *}_X Y + \Theta^*_Y X + \Theta^{\wedge *}_Y X\rangle = -4\langle Z, \mathfrak{T}_X Y + \mathfrak{T}^\wedge_X Y\rangle.$$

Hence, $L^\perp = -(\mathfrak{T} + \mathfrak{T}^\wedge)^\perp$ and $L^\top = -(\mathfrak{T} + \mathfrak{T}^\wedge)^\top$ and for metric connections we obtain $L = -\phi$, see (46), which together with Lemma 4 yields the claim.
$\qquad\qquad\square$

Lemma 6. *Let $\bar{\nabla}$ be a semi-symmetric connection on $(M, g, \mathcal{D})$. Then:*

   a) Formula (18) *reduces to*

$$Q = (n - p)\langle U, H - \tilde{H}\rangle + np\langle U, U\rangle - n\langle U^\perp, U^\perp\rangle - p\langle U^\top, U^\top\rangle. \quad (82)$$

b) For any $g^\pitchfork$-variation of metric g and Q given by (82)*, up to divergences of compactly supported vector fields we have*

$$\partial_t Q(g_t)|_{t=0} = \langle\, B,\ -(n-p)\tilde{\delta}_{U^\perp} - (n-p)\langle\tilde{\alpha} - \tilde{\theta}, U^\perp\rangle + 2(p-n)\langle\theta, U^\top\rangle$$
$$-\frac{1}{2}(p-n)(\operatorname{div} U^\top)g^\perp + n(p-1)U^{\perp\flat} \otimes U^{\perp\flat}$$
$$+2p(n-1)U^{\top\flat} \odot U^{\perp\flat}\,\rangle. \quad (83)$$

Proof. a) From (60) we obtain

$$\operatorname{Tr}^\top \mathfrak{T} = \sum_a \langle U, E_a\rangle E_a - \sum_a \langle E_a, E_a\rangle U = U^\top - n\,U. \quad (84)$$

Similarly, $\operatorname{Tr}^\perp \mathfrak{T} = U^\perp - p\,U$. We also have

$$\mathfrak{T}_a \mathcal{E}_i = \langle U, \mathcal{E}_i\rangle E_a, \quad \mathfrak{T}_i E_a = \langle U, E_a\rangle \mathcal{E}_i, \quad (85)$$

so we obtain $\langle \mathfrak{T}, \mathfrak{T}^\wedge\rangle_{|V} = 0$. Next, we have

$$\langle\, \operatorname{Tr}^\top \mathfrak{T} - \operatorname{Tr}^\perp \mathfrak{T}, H - \tilde{H}\rangle = (p - n - 1)\langle U^\perp, H\rangle + (n - p - 1)\langle U^\top, \tilde{H}\rangle.$$

We have $(\mathfrak{T} + \mathfrak{T}^\wedge)_i E_a = \langle U, E_a\rangle \mathcal{E}_i + \langle U, \mathcal{E}_i\rangle E_a$. Also

$$\langle\, \operatorname{Tr}^\top \mathfrak{T}, \operatorname{Tr}^\perp \mathfrak{T}\rangle = np\langle U, U\rangle - n\langle U^\perp, U^\perp\rangle - p\langle U^\top, U^\top\rangle.$$

Thus, $\langle \mathfrak{T} + \mathfrak{T}^\wedge, \tilde{A} - \tilde{T}^\sharp + A - T^\sharp\rangle = \langle H + \tilde{H}, U\rangle$. b) By [26, Lemma 3], we have:

$$\langle U^\perp,\ \partial_t U^\perp\rangle = \langle U^\perp,\ -(B^\sharp(U^\perp))^\top\rangle = 0,$$
$$\langle U^\top,\ \partial_t U^\top\rangle = \langle U^\top, B^\sharp(U^\perp)\rangle = \langle B, U^{\top\flat} \odot U^{\perp\flat}\rangle.$$

Similarly, by [26, Eq. (20) and Eq. (21)], we have

$$\langle\partial_t \tilde{H}, U\rangle = \operatorname{div}((\operatorname{Tr}_\mathcal{D} B)U^\top) + \langle\, B, 2\langle\theta, U^\top\rangle - \frac{1}{2}(\operatorname{div} U^\top)g^\perp\rangle,$$
$$\langle\partial_t H, U\rangle = \operatorname{div}((B^\sharp(U^\perp))^\top) + \langle B, U^\perp \odot (\tilde{H} - H) - U^\top \odot H - \tilde{\delta}_{U^\perp}$$
$$-\langle\tilde{\alpha} - \tilde{\theta}, U^\perp\rangle\,\rangle.$$

Omitting divergences of compactly supported vector fields and using $B|_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} = 0$, we obtain

$$\partial_t Q(g_t)|_{t=0} = (n-p)B(U, H - \tilde{H}) + (n-p)\langle U, \partial_t H\rangle - (n-p)\langle\partial_t\tilde{H}, U\rangle$$
$$+np\,B(U,U) - nB(U^\perp, U^\perp) - 2n\langle\partial_t U^\perp, U\rangle - pB(U^\top, U^\top) - 2p\langle\partial_t U^\top, U^\top\rangle,$$

that reduces to (83). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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