Contractive Iterated Function Systems Enriched with Nonexpansive Maps

Filip Strobін

Abstract. Motivated by a recent paper of Leśniak and Snigireva [*Iterated function systems enriched with symmetry*, preprint], we investigate the properties of the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ of an IFS $\mathcal{F}$ enriched by some other IFS $\mathcal{G}$. We show that in natural cases, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is in fact the attractor of certain IFSs related naturally with the IFSs $\mathcal{F}$ and $\mathcal{G}$. We also give an example when $A_{\mathcal{F} \cup \mathcal{G}}^*$ is not compact, yet still being the attractor of considered related IFSs. Finally, we use presented machinery to prove that the so called *lower transition attractors* due to Vince are semiattractors of enriched IFSs.

Mathematics Subject Classification. Primary 28A80; Secondary 47H09.

Keywords. Iterated function systems, attractors, semiattractors, groups of isometries.

1. Introduction

In the paper we study the following problem: Assume that $\mathcal{F}$ is an iterated function system (IFS for short) which has the attractor $A_{\mathcal{F}}$ or the semiattractor $A_{\mathcal{F}}^*$, and let $\mathcal{G}$ be any IFS on the same underlying space. What can we say about the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ of the IFS $\mathcal{F} \cup \mathcal{G}$?

The motivation of our studies came from the recent paper of Leśniak and Snigireva [16], in which the authors consider the case when $\mathcal{F}$ is a finite family of Banach contractions and $\mathcal{G}$ is singleton consisting of periodic symmetry (that is, the isometry whose certain iteration is the identity) with a fixed point. Roughly speaking, the main results of [16] show that in considered cases, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is in fact the attractor of certain IFSs which

are somehow related with $\mathcal{F}$ and $\mathcal{G}$. Moreover, the authors of [16] prove that the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ can be recovered by the famous chaos game algorithm.

In the present paper, we extend the framework. Instead of a single periodic isometry, in our main results we assume that $\mathcal{G}$ is just an IFS consisting of nonexpansive maps. We also weaken the contractive assumptions on maps from the IFS $\mathcal{F}$ and allow $\mathcal{F}$ to be infinite. Let us remark that dealing with the theory of infinite IFSs is necessary in considered problem—on the one hand, in the reasonings we come across to monoids and groups generated by the IFS $\mathcal{G}$ which in most cases are infinite, and on the other hand, we give a relatively simple example in which the semiattractor $A_{\mathcal{G} \cup \mathcal{F}}^*$ is closed and bounded, yet not compact, so cannot be an attractor of a finite IFS.

The paper is organized as follows:

In the next section we present basic definitions and facts, among them the notion of iterated function system, its attractor and semiattractor in the sense of Lasota and Myjak. We also present basic results on the existence of attractors and semiattractors.

Then, in Sect. 3, we present the first main result of the paper. Namely, using just set-algebraic reasonings, we present different descriptions of the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ in a very general framework—we just assume that the semiattractor $A_{\mathcal{F}}^*$ of the IFS $\mathcal{F}$ exists. More precisely, we show that the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the semiattractor (or some its modification) of certain IFSs, related somehow with $\mathcal{F}$ and $\mathcal{G}$.

In Sect. 4 we prove that in natural case when $\mathcal{F}$ admits the attractor $A_{\mathcal{F}}$ and $\mathcal{G}$ consists of nonexpansive maps, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is in fact the attractor of IFSs considered in Sect. 3. Additional properties of $A_{\mathcal{F} \cup \mathcal{G}}^*$ are shown under stronger assumption that the monoid generated by $\mathcal{G}$ is a group. We also give some sufficient conditions under which the semiattractor is compact.

In Sect. 5 we present a relatively natural example of IFSs $\mathcal{F}$ and $\mathcal{G}$ which satisfy the assumptions considered in Sect. 4, but whose semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is not compact. In fact, the IFS $\mathcal{F}$ consists of just one constant map, and $\mathcal{G}$ consists of the so-called rotative isometry (which, in a sense, can be considered as “almost periodic isometry”). This example shows limitations in searching for compact semiattractors of enriching IFSs.

Then, in Sect. 6 we present some natural examples of enriched IFSs on the plane and their attractors. We focus on the ones which cannot be handled by the framework from the paper [16], that is, we will enrich IFSs $\mathcal{F}$ by maps which are not periodic isometries.

Finally, in Sect. 7 we use presented machinery to prove that *lower transition attractors* considered recently by Vince [21] are attractors of enriched IFSs.

At the end of the introductory part, let us remark that the authors of [16] were mainly interested in the case when the obtained semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is symmetric w.r.t. the maps from $\mathcal{G}$, i.e., when $g(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*$ for

$g \in \mathcal{G}$. As they proved, this is the case provided that $\mathcal{G}$ consists of the periodic symmetry. As we show, the reason is that the family of all compositions of g forms a group. If this is not the case, then we may lose this property, nevertheless, many other properties investigated in [16] are still valid.

2. Preliminaries

2.1. Iterated Function Systems and Hutchinson Operators

Given a metric space X , by 2^X , $\mathcal{K}(X)$ and $\mathcal{C}_b(X)$ we will denote the hyperspaces of all subsets of X , all nonempty and compact subsets of X , and nonempty closed and bounded subsets of X , respectively. The spaces $\mathcal{K}(X)$, $\mathcal{C}_b(X)$ will be considered as metric spaces endowed with the Hausdorff–Pompeiu metric h . It is well known, that $\mathcal{K}(X)$ and $\mathcal{C}_b(X)$ are complete provided that X is complete. For $A \subset X$, by $\text{cl } A$ we will denote the closure of the set A .

Definition 2.1. By an *iterated function system* (IFS for short) we will mean any family $\mathcal{F}$ of continuous selfmaps of a metric space X . If $\mathcal{F}$ is an IFS, then by $\overline{F}_{\mathcal{F}}$ we will denote the *Hutchinson operator* adjusted to $\mathcal{F}$, that is, the map $\overline{F}_{\mathcal{F}} : 2^X \rightarrow 2^X$ defined by:

$$\forall_{A \subset X} \overline{F}_{\mathcal{F}}(A) = \text{cl} \left(\bigcup_{f \in \mathcal{F}} f(A) \right)$$

We will say that an IFS $\mathcal{F}$ is

- *bounded*, if for every bounded set D , the set $\overline{F}_{\mathcal{F}}(D)$ is also bounded;
- *compact*, if for every compact set D , the set $\overline{F}_{\mathcal{F}}(D)$ is also compact.

In other words, an IFS $\mathcal{F}$ is bounded iff the restriction $(\overline{F}_{\mathcal{F}})|_{\mathcal{C}_b(X)} : \mathcal{C}_b(X) \rightarrow \mathcal{C}_b(X)$, and is compact iff $(\overline{F}_{\mathcal{F}})|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$.

Remark 2.2. (1) Each finite IFS $\mathcal{F} = \{f_i : i \in I\}$ is compact and for every $K \in \mathcal{K}(X)$, we have that $\overline{F}_{\mathcal{F}}(K) = \bigcup_{i \in I} f_i(K)$ (that is, we may omit the closure).

- (2) Each finite IFS consisting of Lipschitz maps is bounded.
- (3) Let $\mathcal{F}$ be an IFS so that $\sup\{\text{Lip}(f) : f \in \mathcal{F}\} < \infty$. Then the following conditions are equivalent:
 - (i) $\mathcal{F}$ is bounded;
 - (ii) there exists $z_0 \in X$ such that the orbit $\{f(z_0) : f \in \mathcal{F}\}$ is bounded.
- (4) Let $\mathcal{F}$ be an IFS so that $\sup\{\text{Lip}(f) : f \in \mathcal{F}\} < \infty$. If $\mathcal{F}$ is compact, then it is bounded.

Items (1) and (2) are obvious (in fact, (2) follows from (4)) and (4) follows from (3). We just make a small comment on (3). The implication (i) $\rightarrow$ (ii) is trivial. To see the opposite one, take any nonempty bounded set $D \subset X$

and let $L := \sup\{d(x, z_0) : x \in D\}$, $M := \text{diam}(\{f(z_0) : f \in \mathcal{F}\})$ and $N := \sup\{\text{Lip}(f) : f \in \mathcal{F}\}$. For every $x, y \in D$ and $f, g \in \mathcal{F}$, we have

$$\begin{aligned} d(f(x), g(y)) &\leq d(f(x), f(z_0)) + d(f(z_0), g(z_0)) + d(g(z_0), g(y)) \\ &\leq Nd(x, z_0) + M + Nd(y, z_0) \leq M + 2LN \end{aligned}$$

Hence $\text{diam}(\overline{F}_{\mathcal{F}}(D)) \leq M + 2LN < \infty$.

Remark 2.3. Most authors assume that IFSs are finite families of maps and that the Hutchinson operators are defined on the hyperspaces $\mathcal{K}(X)$ or (rarely) on $\mathcal{C}_b(X)$. As mentioned in the Introduction, our setting naturally moves the discussion to infinite IFSs. Such IFSs were considered in the literature, for example in [8, 9, 12, 17] or [20]. In fact, item (3) of the above Remark for non-expansive maps was noted in [12] (see also [1]).

We will also make use of the alternative version of the Hutchinson operator.

Definition 2.4. Let $\mathcal{F}$ be an IFS on a metric space X . The map $F_{\mathcal{F}} : 2^X \rightarrow 2^X$ defined by:

$$\forall A \subset X \quad F_{\mathcal{F}}(A) = \bigcup_{f \in \mathcal{F}} f(A)$$

will be called the *weak Hutchinson operator of $\mathcal{F}$* .

Remark 2.5. According to Remark 2.2 (1), if $\mathcal{F}$ is finite, then $\mathcal{F}$ is compact and the Hutchinson operator coincides with the weak Hutchinson operator on the hyperspace $\mathcal{K}(X)$. This is also the case for certain infinite IFSs. For example, if the parameter space I is a compact topological space and $\mathcal{F} = \{f_i : i \in I\}$ consisting of contractive maps is such that the map $X \times I \ni (x, i) \rightarrow f_i(x) \in X$ is continuous, then for any nonempty and compact K , the map $i \rightarrow f_i(K)$ is continuous and hence $\bigcup_{i \in I} f_i(K)$ itself is also compact. Such a version of infinite IFSs were considered for example in [17].

Clearly, in the case when the IFS $\mathcal{F}$ is finite, .

The next lemma shows some relationships between the Hutchinson operator and the weak Hutchinson operator. It is a folklore, we give the short proof for the sake of completeness.

Lemma 2.6. *In the above framework, for every $n \in \mathbb{N}$ and $A \subset X$,*

$$\text{cl}(F_{\mathcal{F}}^{(n)}(A)) = \overline{F}_{\mathcal{F}}^{(n)}(A) = \overline{F}_{\mathcal{F}}^{(n)}(\text{cl}(A)).$$

Proof. We just have to prove the equality

$$\text{cl}(F_{\mathcal{F}}^{(n)}(A)) = \overline{F}_{\mathcal{F}}^{(n)}(\text{cl}(A)).$$

By continuity of maps from $\mathcal{F}$ and standard properties of the closure operator, we have

$$\text{cl}(F_{\mathcal{F}}(A)) = \text{cl}\left(\bigcup_{f \in \mathcal{F}} f(A)\right) = \text{cl}\left(\bigcup_{f \in \mathcal{F}} f(\text{cl } A)\right) = \overline{F}_{\mathcal{F}}(\text{cl } A).$$

Now assume that the assertion holds for some $n \in \mathbb{N}$. Then by the assumption and the above calculations,

$$\begin{aligned} \text{cl}(F_{\mathcal{F}}^{(n+1)}(A)) &= \text{cl}(F_{\mathcal{F}}(F_{\mathcal{F}}^{(n)}(A))) = \overline{F}_{\mathcal{F}}(\text{cl}(F_{\mathcal{F}}^{(n)}(A))) = \overline{F}_{\mathcal{F}}(\overline{F}_{\mathcal{F}}^{(n)}(\text{cl } A)) \\ &= \overline{F}_{\mathcal{F}}^{(n+1)}(\text{cl } A) \end{aligned}$$

and the result follows. $\square$

We end this section with a simple lemma, which gives a description of the n -th iteration of the weak Hutchinson operator. We skip its proof as it follows directly from definition of the Hutchinson operator.

Lemma 2.7. *Let $\mathcal{F} = \{f_i : i \in I\}$, $\mathcal{G} = \{g_j : j \in J\}$ be IFSs on the same metric space X . Then for every $A \subset X$ and a natural number n , the n -th iterate of $F_{\mathcal{F} \cup \mathcal{G}}$ of a set A is given by:*

$$\begin{aligned} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A) &= \bigcup_{p \in \mathbb{N}} \bigcup_{k_1 + \dots + k_p + s_1 + \dots + s_p = n} \\ &\bigcup_{i_1^1, \dots, i_{k_1}^1, \dots, i_1^p, \dots, i_{k_p}^p \in I} \bigcup_{j_1^1, \dots, j_{s_1}^1, \dots, j_1^p, \dots, j_{s_p}^p \in J} \\ &f_{i_1^1} \circ \dots \circ f_{i_{k_1}^1} \circ g_{j_1^1} \circ \dots \circ g_{j_{s_1}^1} \circ \dots \circ f_{i_1^p} \circ \dots \circ f_{i_{k_p}^p} \circ g_{j_1^p} \circ \dots \circ g_{j_{s_p}^p}(A) \end{aligned} \quad (1)$$

where we agree that k_1 can equal 0 (and then $f_{i_1^1} \circ \dots \circ f_{i_{k_1}^1}$ disappears), or s_p can equal 0 (and then $g_{j_1^p} \circ \dots \circ g_{j_{s_p}^p}$ disappears).

2.2. Hutchinson Attractors of Iterated Function Systems

The classical Hutchinson result [2, 11] states that:

Theorem 2.8. *Assume that $\mathcal{F}$ is an IFS on a complete metric space X consisting of finitely many Banach contractions (that is, the Lipschitz constant $\text{Lip}(f) < 1$ for $f \in \mathcal{F}$). Then there exists the unique set $A_{\mathcal{F}} \in \mathcal{K}(X)$ which is $\mathcal{F}$ -invariant, i.e.,*

$$A_{\mathcal{F}} = \overline{F}_{\mathcal{F}}(A_{\mathcal{F}}) = \bigcup_{f \in \mathcal{F}} f(A_{\mathcal{F}}).$$

Moreover, for every $D \in \mathcal{C}_b(X)$, the sequence of iterates of the Hutchinson operator $\overline{F}_{\mathcal{F}}^{(n)}(D)$ converges to $A_{\mathcal{F}}$ w.r.t. the Hausdorff–Pompeiu metric.

As we will recall in the following, the above result can be strengthened in various ways—we can weaken the contractive assumptions and allow $\mathcal{F}$ to be infinite. On the other hand, we may lose compactness of $A_{\mathcal{F}}$. In order to make suitable formulation, we should give some further notations.

Definition 2.9. A selfmap $f : X \rightarrow X$ of a metric space X is called a *Matkowski contraction*, if for some nondecreasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ so that $\varphi^{(n)}(t) \rightarrow 0$ for $t > 0$, it holds

$$\forall_{x,y \in X} d(f(x), f(y)) \leq \varphi(d(x, y)).$$

In that case, the map φ will be called a *witness* for f .

Note that a witness function φ for f necessarily satisfies $\varphi(t) < t$ for $t > 0$, so each Matkowski contraction is nonexpansive map. On the other hand, if the Lipschitz constant $\text{Lip}(f) < 1$, then f is a Matkowski contraction—simply take the function $\varphi(t) := \text{Lip}(f)t$ as a witness.

As proved by Matkowski [19], each Matkowski contraction on a complete metric space satisfies the thesis of the Banach fixed point theorem. In fact, this result is one of the strongest generalizations of the Banach theorem. For example, Rakotch contractions or Browder contractions are Matkowski contractions. For a discussion on various types of contractiveness from the perspective of the fixed point theory, see for example [13].

Definition 2.10. We say that an IFS $\mathcal{F}$ is a *Matkowski contractive*, if $\mathcal{F}$ is bounded and all maps from $\mathcal{F}$ are Matkowski contractions with the same witness.

Now we give the promised extension of Theorem 2.8. It is rather a folklore. However, in the literature we could only find its more specialized versions (for example, restricted to countable IFSs), hence we give the sketch of the proof.

Theorem 2.11. Assume that $\mathcal{F}$ is a Matkowski contractive IFS on a complete metric space X . The following conditions hold.

- (1) There exists the unique set $A_{\mathcal{F}} \in \mathcal{C}_b(X)$ such that

$$A_{\mathcal{F}} = \overline{F}_{\mathcal{F}}(A_{\mathcal{F}}) = \text{cl} \left(\bigcup_{f \in \mathcal{F}} f(A_{\mathcal{F}}) \right).$$

Moreover, for every $D \in \mathcal{C}_b(X)$, the sequence of iterates of the Hutchinson operator $\overline{F}_{\mathcal{F}}^{(n)}(D)$ converges to $A_{\mathcal{F}}$ w.r.t. the Hausdorff–Pompeiu metric.

- (2) If additionally the IFS $\mathcal{F}$ is compact, then $A_{\mathcal{F}}$ is also compact.

Proof. In a standard way (see a.e., [10, Proposition 1]) we can show that for every $f \in \mathcal{F}$ and $D, E \in \mathcal{C}_b(X)$, it holds

$$h(f(D), f(E)) \leq \varphi(h(D, E))$$

Hence

$$\begin{aligned} h(\overline{F}_{\mathcal{F}}(D), \overline{F}_{\mathcal{F}}(E)) &= h\left(\text{cl} \left(\bigcup_{f \in \mathcal{F}} f(D) \right), \text{cl} \left(\bigcup_{f \in \mathcal{F}} f(E) \right)\right) \\ &\leq \sup \{h(f(D), f(E)) : f \in \mathcal{F}\} \leq \varphi(h(D, E)). \end{aligned}$$

Hence $\overline{F}_{\mathcal{F}}$ is a Matkowski contraction on the hyperspace $\mathcal{C}_b(X)$ w.r.t. the Hausdorff–Pompeiu metric and the existence and uniqueness of $A_{\mathcal{F}}$ follows from the mentioned Matkowski fixed point theorem.

Now if we additionally assume that $\mathcal{F}$ is compact, then we can use the Matkowski theorem for the restriction $(\overline{F}_{\mathcal{F}})|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ and justify that the (necessarily unique) set $A_{\mathcal{F}}$ is compact. $\square$

Definition 2.12. The set $A_{\mathcal{F}}$ which satisfies the assertion of item (1) of the above theorem will be called *the Hutchinson attractor of $\mathcal{F}$* .

Remark 2.13. As proved in [12], if in Theorem 2.11 we additionally assume that the witness φ satisfies the following condition

$$\limsup_{t \rightarrow \infty} (t - \varphi(t)) = \infty$$

(which is the case for example if $\sup\{\text{Lip}(f) : f \in \mathcal{F}\} < 1$), then there exists a nonempty and bounded set $\tilde{A}$ such that $F_{\mathcal{F}}(\tilde{A}) = \tilde{A}$. Clearly, the attractor of $\mathcal{F}$ is then $A_{\mathcal{F}} = \text{cl}(\tilde{A})$. On the other hand, in [9] there is given a sufficient condition for the attractor $A_{\mathcal{F}}$ to satisfy $F_{\mathcal{F}}(A_{\mathcal{F}}) = A_{\mathcal{F}}$.

Remark 2.14. Observe that if $A_{\mathcal{F}}$ is the Hutchinson attractor of an IFS $\mathcal{F}$, then for every closed and bounded set $D \subset X$, there exists closed and bounded set E such that $D \subset E$ and $F_{\mathcal{F}}(E) \subset E$. Indeed, it is enough to consider the set

$$E := \text{cl} \left(D \cup \bigcup_{n \in \mathbb{N}} \overline{F}_{\mathcal{F}}^{(n)}(D) \right) = A_{\mathcal{F}} \cup D \cup \bigcup_{n \in \mathbb{N}} \overline{F}_{\mathcal{F}}^{(n)}(D).$$

Note that the set E can be considered as the attractor of $\mathcal{F}$ with condensation D (or inhomogeneous attractor); see, i.e., [2]. Indeed, it is easy to see that

$$\overline{F}_{\mathcal{F}}(E) \cup D = E.$$

Remark 2.15. Let us commend that the class of Hutchinson attractors is very large. In fact, every closed and bounded subset of a metric space $A \subset X$ is the Hutchinson attractor of the IFS $\mathcal{F} = \{f_y : y \in A\}$, where $f_y(x) := y$ for $x \in X$. Moreover, if A is additionally separable, then in a similar way we can show that A is the Hutchinson attractor of some countable IFS. On the other hand, in the main results of the paper, we will prove that considered sets are Hutchinson attractors of certain IFSs given a priori.

The following remark explains some nuances in the definition of Matkowski contractiveness and Theorem 2.11.

Remark 2.16. Assume that $\mathcal{F}$ is an IFS on a complete metric space X .

1. If $\mathcal{F}$ is finite and consists of Matkowski contractions, then $\mathcal{F}$ is Matkowski contractive. Indeed, it is bounded and its common witness is the function $\max\{\varphi_f : f \in \mathcal{F}\}$, where φ_f is a witness for f ; cf. [10].

2. If $\mathcal{F}$ is infinite and Matkowski contractive, then the attractor $A_{\mathcal{F}}$ can be noncompact, cf. Remark 2.15.
3. If $\mathcal{F}$ is infinite, bounded and consists of Matkowski contractions, then it may be not Matkowski contractive. For example, for $a \in [0, 1)$, take $f_a(x) = ax$, $x \in \mathbb{R}$, and consider $\mathcal{F} := \{f_a, a \in [0, 1)\}$. Then $\mathcal{F}$ is bounded and each element of $\mathcal{F}$ is a Matkowski contraction ($\text{Lip}(f_a) = a < 1$), but $\mathcal{F}$ is not Matkowski contractive. In fact, $\mathcal{F}$ -invariant sets are sets of the form $[a, b]$, where $a \leq 0 \leq b$, so $\mathcal{F}$ does not have any Hutchinson attractor.
4. If $\mathcal{F}$ consists of Matkowski contractions with the same witness, then it may not be Matkowski contractive as it may be not bounded. For example, for $k \in \mathbb{N}$, take $f_k(x) = \frac{1}{2}x + k$, $x \in \mathbb{R}$, and consider $\mathcal{F} := \{f_k, k \in \mathbb{N}\}$. Then each element of $\mathcal{F}$ is a Matkowski contraction (as $\text{Lip}(f_k) = \frac{1}{2}$), but $\mathcal{F}$ is not bounded. Observe that here each $\mathcal{F}$ -invariant set is necessarily unbounded, so $\mathcal{F}$ does not have any Hutchinson attractor.

2.3. Semiattractors of Iterated Function Systems

We first give a definition of Kuratowski limits (see, e.g., [4, 14] for detailed discussion).

Definition 2.17. Let (S_n) be a sequence of subsets of a metric space X . The *upper Kuratowski limit* of (S_n) is defined by

$$\text{Ls}(S_n) := \{y \in X : y = \lim_{n \rightarrow \infty} x_n \text{ for some sequence } (x_n) \\ \text{so that } x_n \in S_n \text{ for infinitely many } n \in \mathbb{N}\}$$

and the *lower Kuratowski limit* of (S_n) is defined by

$$\text{Li}(S_n) := \{y \in X : y = \lim_{n \rightarrow \infty} x_n \text{ for some sequence } (x_n) \\ \text{so that } x_n \in S_n \text{ for all } n \in \mathbb{N}\}$$

If $\text{Ls}(S_n) = \text{Li}(S_n)$, then its common value, denoted by $\text{Lt}(S_n)$, is called *the Kuratowski limit* of (S_n) .

Now we recall the notion of semiattractor of an IFS. We refer the reader to [14] or [16] for details.

Definition 2.18. Let $\mathcal{F}$ be an IFS on a metric space X . If the set

$$\bigcap_{x \in X} \text{Li } F_{\mathcal{F}}^{(n)}(\{x\})$$

is nonempty, then we call it as the *semiattractor* of $\mathcal{F}$, and will denote by $A_{\mathcal{F}}^*$.

The following result lists basic properties of semiattractors, that will be of importance later on:

Theorem 2.19. Let $\mathcal{F}$ be an IFS on a metric space X with the semiattractor $A_{\mathcal{F}}^*$.

1. The semiattractor $A_{\mathcal{F}}^*$ is $\overline{F}_{\mathcal{F}}$ -invariant, that is, $\overline{F}_{\mathcal{F}}(A_{\mathcal{F}}^*) = A_{\mathcal{F}}^*$.

2. If $\mathcal{G}$ is any IFS on X , then the IFS $\mathcal{F} \cup \mathcal{G}$ has semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ and

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \text{cl} \left(\bigcup_{n=1}^{\infty} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right) = \text{cl} \left(\bigcup_{n=1}^{\infty} \overline{F}_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right).$$

In particular, $A_{\mathcal{F}}^* \subset A_{\mathcal{F} \cup \mathcal{G}}^*$.

3. If $A_{\mathcal{F}}^*$ is compact, then it is the attractor of the IFS $\mathcal{F}|_{A_{\mathcal{F}}^*}$ comprised of restrictions of maps from $\mathcal{F}$ to the attractor $A_{\mathcal{F}}^*$, i.e., $\mathcal{F}|_{A_{\mathcal{F}}^*} = \{f|_{A_{\mathcal{F}}^*} : f \in \mathcal{F}\}$.
4. If $\mathcal{F}$ has the Hutchinson attractor $A_{\mathcal{F}}$, then $A_{\mathcal{F}}^* = A_{\mathcal{F}}$.

Proof. Item 1. follows from [14, Theorem 3.1]. The first equality in item 2. follows from [14, Theorem 3.2] and the second one from Lemma 2.6. Item 4. is true because the Hausdorff–Pompeiu convergence of closed sets, $h(F_{\mathcal{F}}^{(n)}(\{x\}), A_{\mathcal{F}}) \rightarrow 0$, implies the Kuratowski convergence:

$$\text{Lt}(F_{\mathcal{F}}^{(n)}(\{x\})) = A_{\mathcal{F}}.$$

Finally, item 3. follows as Kuratowski convergence in compact space implies Hausdorff–Pompeiu convergence. $\square$

Let us end this section with a result on a relationship between the semi-attractor of an infinite, countable IFS $\mathcal{F}$ and its finite “sub-IFSs” $\mathcal{F}' \subset \mathcal{F}$ (see [8] or [20] for further discussion).

Proposition 2.20. Assume that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is a countable IFS and for every $n \in \mathbb{N}$, set $\mathcal{F}_n := \{f_1, \dots, f_n\}$. Assume that $\mathcal{F}$ and all $\mathcal{F}_n$ s have semiattractors $A_{\mathcal{F}}^*$, $A_{\mathcal{F}_n}^*$, respectively. Then $A_{\mathcal{F}}^* = \text{cl}(\bigcup_{n \in \mathbb{N}} A_{\mathcal{F}_n}^*)$.

Proof. Put $A^* := \text{cl}(\bigcup_{n \in \mathbb{N}} A_{\mathcal{F}_n}^*)$. The inclusion $A^* \subset A_{\mathcal{F}}^*$ is clear. We will show the opposite one. By [20, Proposition 3.6], the set A^* is $\mathcal{F}$ -invariant. Hence for every $n \in \mathbb{N}$,

$$F_{\mathcal{F}}^{(n)}(A_{\mathcal{F}_1}) \subset F_{\mathcal{F}}^{(n)}(A^*) \subset \overline{F}_{\mathcal{F}}^{(n)}(A^*) = A^*$$

and, by Theorem 2.19, we have the inclusion $A_{\mathcal{F}_1}^* \subset A^*$. $\square$

Remark 2.21. The above result shows that the semiattractors (or attractors) of countable infinite IFSs can be approximated by the semiattractors (or attractors) of their finite sub-IFSs. Let us note, however, that there exist infinite IFSs with attractors such that their finite sub-IFSs do not have attractors (see [15, Example 5]).

2.4. Chaos Game and Disjunctive Sequences

The chaos game algorithm is a classical and simple algorithm for getting images of Hutchinson attractors of contractive IFSs. In classical setting, when $\mathcal{F} = \{f_i : i \in I\}$ is a finite IFS, we pick any point $x_0 \in X$, then we choose randomly a map f_{i_1} from the IFS $\mathcal{F} = \{f_i : i \in I\}$ and define $x_1 := f_{i_1}(x_0)$. Then we choose randomly a map f_{i_2} and define $x_2 := f_{i_2}(x_1)$, and so on. Then, (with a

probability one) for appropriate big $N < M$, the set $\{x_n : N \leq n \leq M\}$ should approximate the attractor $A_{\mathcal{F}}$. The simple explanation (see [18]) is that with the probability one, we choose a sequence $\mathbf{i} = (i_1, i_2, \dots)$, such that each finite sequence of elements from I appears in this sequence infinitely many times. Hence if $x_0 \in A_{\mathcal{F}}$ (we can make this assumptions for simplicity), then in each set of the form

$$f_{j_1} \circ \dots \circ f_{j_n}(A_{\mathcal{F}})$$

we will find a point from our sequence (x_n) (even infinitely many such points).

This idea leads to deterministic version of the chaos game algorithm—we choose a sequence $\mathbf{i} = (i_1, i_2, \dots)$ with the above property and generate the sequence (x_n) according to the formula $x_n := f_{i_n}(x_{n-1})$. Such an approach can be adjusted to any countable IFS, and was studied in [15] (see also [5] for more general cases).

Let us make some precise definitions:

Definition 2.22. Let $\mathcal{F} = \{f_i : i \in I\}$ be any IFS on a metric space X .

- The sequence $\mathbf{i} = (i_1, i_2, \dots)$ of elements of I with the property that each finite sequence appears in $\mathbf{i}$ infinitely many times will be called *disjunctive*.
- If (x_n) is defined according to $x_n := f_{i_n}(x_{n-1})$, where $x_0 \in X$ is any point, then $\mathbf{i}$ will be called as *the driver of (x_n)* .
- We will say that $(x_n) \subset X$ *recovers* $A \subset X$, if

$$\lim_{k \rightarrow \infty} h(\{x_n : n \geq k\}, A) = 0, \quad (2)$$

where h is the Hausdorff–Pompeiu metric.

Remark 2.23. Since any sequence $\mathbf{i}$ has countably many values, we see that disjunctive sequences exist only if I is countable. Thus in the given later Theorem 4.1 item (D) we deal just with countable IFSs.

The mentioned explanation of the chaos game algorithm can be reformulated in the following form: the Hutchinson attractor $A_{\mathcal{F}}$ is recovered by any sequence (x_n) with disjunctive driver.

Remark 2.24. Let us remark that in [16], a bit different definition of recovering is considered. Namely, it is stated that the sequence (x_n) recovers the semi-attractor $A_{\mathcal{F}}^*$, if the set $\omega((x_n))$ of limit points of the sequence (x_n) coincides with $A_{\mathcal{F}}^*$. Clearly, (2) for $A_{\mathcal{F}}^*$ implies that $\omega((x_n)) = A_{\mathcal{F}}^*$. On the other hand, in most cases the converse also holds (for example, if one of maps from an IFS is Matkowski contraction and the driver is disjunctive). For a discussion of the concept of recovering we refer to [6].

3. Description of Semiattractors of Enriched IFSs

Let us introduce the following notation: for an IFS $\mathcal{G} = \{g_j : j \in J\}$, let

$$\mathbb{M}(\mathcal{G}) := \{g_{j_1} \circ \cdots \circ g_{j_k} : k \in \mathbb{N}, j_1, \dots, j_k \in J\} \cup \{\text{Id}_X\}$$

where Id_X is the identity function on X . In other words, $\mathbb{M}(\mathcal{G})$ is the monoid generated by $\mathcal{G}$.

The following result shows that the semiattractor of $\mathcal{F} \cup \mathcal{G}$ can be described in several ways.

Theorem 3.1. *Let $\mathcal{F}, \mathcal{G}$ be IFSs on the same metric space X such that $\mathcal{F}$ has the semiattractor $A_{\mathcal{F}}^*$, and let $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the semiattractor of $\mathcal{F} \cup \mathcal{G}$. Then the following conditions hold:*

- (1) *Let $\mathcal{N}$ be any IFS such that $\mathcal{F} \cup \mathcal{G} \subset \mathcal{N} \subset \mathbb{M}(\mathcal{F} \cup \mathcal{G})$. Then $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the semiattractor of $\mathcal{N}$ and in particular:*

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \right).$$

- (2) *Let $\mathcal{M}_1 := \{g \circ f \circ h : f \in \mathcal{F}, g, h \in \mathbb{M}(\mathcal{G})\}$ and $\mathcal{M}_2 := \{g \circ f : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}$. Then $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the semiattractor of both $\mathcal{M}_1$ and $\mathcal{M}_2$, and in particular:*

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_1}^{(n)}(A_{\mathcal{F}}^*) \right) = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_2}^{(n)}(A_{\mathcal{F}}^*) \right).$$

- (3) *Let $\mathcal{H} := \{f \circ g : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}$. Then*
 (3i) *denoting by $A_{\mathcal{H}}^*$ the semiattractor of $\mathcal{H}$, it holds*

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \overline{F_{\mathbb{M}(\mathcal{G})}}(A_{\mathcal{H}}^*) = \text{cl} \left(\bigcup_{g \in \mathbb{M}(\mathcal{G})} g(A_{\mathcal{H}}^*) \right),$$

- (3ii) *the following inclusions hold*

$$A_{\mathcal{F}}^* \subset A_{\mathcal{H}}^* \subset A_{\mathcal{F} \cup \mathcal{G}}^*.$$

- (4) *Assume additionally that $\mathbb{M}(\mathcal{G})$ forms a group. Then the following conditions hold.*

- (4i) *For any $g \in \mathbb{M}(\mathcal{G})$, we have*

$$A_{\mathcal{F} \cup \mathcal{G}}^* = g(A_{\mathcal{F} \cup \mathcal{G}}^*).$$

- (4ii) *Let $\mathcal{R} := \{g \circ f \circ g^{-1} : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}$. Then $A_{\mathcal{F} \cup \mathcal{G}}^*$ is $\mathcal{R}$ -invariant, i.e.,*

$$\overline{F_{\mathcal{R}}}(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*. \quad (3)$$

Remark 3.2. Before we give a proof, let us mention that the IFSs $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{N} := \mathcal{F} \cup \mathbb{M}(\mathcal{G})$ were considered in the mentioned paper [16]. The theses of the above Theorem 3.1 are partial extensions of main results from that paper. In the next section we will formulate results under more restrictive assumptions, which will give full generalizations of those from [16].

Proof. The existence of the semiattractors of $\mathcal{N}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{H}$ is guaranteed by item 2. of Theorem 2.19 as the IFS $\mathcal{F}$ is contained in each of those IFSs.

We first prove item (1). In view of Theorem 2.19, we only have to prove that

$$\text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \right) = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right) \quad (4)$$

The inclusion “ $\supset$ ” follows easily from the fact that $\mathcal{F} \cup \mathcal{G} \subset \mathcal{N}$. Now to see the opposite inclusion, take any ingredient in the union $\bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*)$ of the form (1), that is, a set of the form

$$h_{i_1} \circ \cdots \circ h_{i_n}(A_{\mathcal{F}}^*)$$

(where we set the enumeration $\mathcal{N} = \{h_i : i \in J\}$). Then consider two cases:

Case 1. $h_{i_1} \circ \cdots \circ h_{i_n} \neq \text{Id}_X$. Then $h_{i_1} \circ \cdots \circ h_{i_n}(A_{\mathcal{F}}^*)$ can be rewritten as an ingredient of $F_{\mathcal{F} \cup \mathcal{G}}^{(p)}(A_{\mathcal{F}}^*)$ of the form (1) for sufficiently large p (indeed, we just have to rewrite each h_{i_k} as a composition of maps from $\mathcal{F}$ and $\mathcal{G}$). Thus

$$h_{i_1} \circ \cdots \circ h_{i_n}(A_{\mathcal{F}}^*) \subset \text{cl} \left(\bigcup_{k \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(k)}(A_{\mathcal{F}}^*) \right).$$

Case 2. $h_{i_1} \circ \cdots \circ h_{i_n} = \text{Id}_X$. Then

$$h_{i_1} \circ \cdots \circ h_{i_n}(A_{\mathcal{F}}^*) = A_{\mathcal{F}}^* \subset \overline{F}_{\mathcal{F} \cup \mathcal{G}}(A_{\mathcal{F}}^*) \subset \text{cl} \left(\bigcup_{k \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(k)}(A_{\mathcal{F}}^*) \right).$$

All in all, we proved that

$$\bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right),$$

which gives us the second inclusion in (4).

Now we prove item (2). In view of Theorem 2.19, we only have to prove that

$$\begin{aligned} \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right) &\subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_2}^{(n)}(A_{\mathcal{F}}^*) \right) \\ &\subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_1}^{(n)}(A_{\mathcal{F}}^*) \right) \subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right) \end{aligned} \quad (5)$$

Using item (1) for $\mathcal{N} := \mathbb{M}(\mathcal{G}) \cup \mathcal{F}$, and noting that

$$F_{\mathcal{M}_1}(A) \subset F_{\mathbb{M}(\mathcal{G})} \circ F_{\mathcal{F}} \circ F_{\mathbb{M}(\mathcal{G})}(A) \subset F_{\mathcal{N}}^{(3)}(A)$$

for every $A \subset X$, we see that for every $k \in \mathbb{N}$,

$$F_{\mathcal{M}_1}^{(k)}(A_{\mathcal{F}}^*) \subset F_{\mathcal{N}}^{(3k)}(A_{\mathcal{F}}^*) \subset \bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right)$$

where in the last equality we used (4). This gives the last inclusion “ $\subset$ ” in (5). The second inclusion follows from $\mathcal{M}_2 \subset \mathcal{M}_1$. Now we show the first one. Take any ingredient from the union $\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*)$ of the form (1). Consider two cases:

Case 1. The considered ingredient ends with some f from $\mathcal{F}$. Then we can rewrite it in the following way:

$$(\text{Id}_X \circ f_{i_1^1}) \circ \cdots \circ (\text{Id}_X \circ f_{i_{k_1}^1}) \circ (g_1 \circ f_{i_1^2}) \circ \cdots \circ (\text{Id}_X \circ f_{i_{k_p}^p})(A_{\mathcal{F}}^*)$$

that is, we replace each f_i either by $(\text{Id}_X \circ f_i)$, or, if it has elements from $\mathcal{G}$ on the left, by $(g \circ f_i)$, where g is an appropriate composition from maps from $\mathcal{G}$. In particular, this ingredient is a subset of some $F_{\mathcal{M}_2}^{(m)}(A_{\mathcal{F}}^*)$.

Case 2. The ingredient ends with some map g from $\mathcal{G}$. Then using similar replacement as in the Case 1 and using the fact that $A_{\mathcal{F}}^* = \overline{F}_{\mathcal{F}}(A_{\mathcal{F}}^*)$, we can proceed in the following way:

$$\begin{aligned} f_{i_1^1} \circ \cdots \circ g_p(A_{\mathcal{F}}^*) &= f_{i_1^1} \circ \cdots \circ g_p \left(\text{cl} \left(\bigcup_{f \in \mathcal{F}} f(A_{\mathcal{F}}^*) \right) \right) \\ &\subset \text{cl} \left(\bigcup_{f \in \mathcal{F}} f_{i_1^1} \circ \cdots \circ g_p \circ f(A_{\mathcal{F}}^*) \right) \\ &= \text{cl} \left(\bigcup_{f \in \mathcal{F}} (\text{Id}_X \circ f_{i_1^1}) \circ \cdots \circ (g_p \circ f)(A_{\mathcal{F}}^*) \right) \\ &\subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_2}^{(n)}(A_{\mathcal{F}}^*) \right) \end{aligned}$$

All in all, we proved that

$$\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_2}^{(n)}(A_{\mathcal{F}}^*) \right)$$

which gives us the first inclusion in (5).

Now we take care of item (3). By Theorem 2.19 and the fact that $\mathcal{F} \subset \mathcal{H}$, we see that the set

$$\text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{H}}^{(n)}(A_{\mathcal{F}}^*) \right)$$

is the semiattractor of $\mathcal{H}$. Hence we have to prove that

$$\overline{F}_{\mathbb{M}(\mathcal{G})} \left(\text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{H}}^{(n)}(A_{\mathcal{F}}^*) \right) \right) = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right). \quad (6)$$

Again, using item (1) for $\mathcal{N} := \mathcal{F} \cup \mathbb{M}(\mathcal{G})$, we see that for any $k \in \mathbb{N}$,

$$\begin{aligned} F_{\mathbb{M}(\mathcal{G})}(F_{\mathcal{H}}^{(k)}(A_{\mathcal{F}}^*)) &\subset F_{\mathbb{M}(\mathcal{G})} \circ (F_{\mathcal{F}} \circ F_{\mathbb{M}(\mathcal{G})})^{(k)}(A_{\mathcal{F}}^*) \subset \bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \\ &\subset \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{N}}^{(n)}(A_{\mathcal{F}}^*) \right) = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \right) \end{aligned}$$

so using Lemma 2.6, we get the inclusion “ $\subset$ ” in (6). We will show the opposite one. Take any ingredient from the union $\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*)$ of the form (1). Consider two cases:

Case 1. There appears some map f from $\mathcal{F}$ in the composition in (1). Then we can rewrite it in the following way:

$$\text{Id}_X \circ (f_{i_1^1} \circ \text{Id}_X) \circ \cdots \circ (f_{i_{k_1}^1} \circ g_1) \circ (f_{i_1^{p_1}} \circ \text{Id}_X) \circ \cdots \circ (f_{i_{k_p}^{p_p}} \circ g_l)(A_{\mathcal{F}}^*)$$

that is, we add Id_X at the beginning (if it is needed) and replace each f_i either by $(f_i \circ \text{Id}_X)$, or, if it has elements from $\mathcal{G}$ on the right, by $(f_i \circ g)$, where g is an appropriate composition from maps from $\mathcal{G}$. In particular, this ingredient is a subset of some $F_{\mathbb{M}(\mathcal{G})}(F_{\mathcal{H}}^{(m)}(A_{\mathcal{F}}^*)) \subset \overline{F}_{\mathbb{M}(\mathcal{G})}(\text{cl}(\bigcup_{n \in \mathbb{N}} F_{\mathcal{H}}^{(n)}(A_{\mathcal{F}}^*)))$.

Case 2. The ingredient is of the form $g_{i_1} \circ \cdots \circ g_{i_n}(A_{\mathcal{F}}^*)$. Then we also have

$$g_{i_1} \circ \cdots \circ g_{i_n}(A_{\mathcal{F}}^*) = g_{i_1} \circ \cdots \circ g_{i_n}(\overline{F}_{\mathcal{F}}(A_{\mathcal{F}}^*)) \subset \overline{F}_{\mathbb{M}(\mathcal{G})} \left(\text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{H}}^{(n)}(A_{\mathcal{F}}^*) \right) \right).$$

All in all, we proved that

$$\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}^*) \subset \overline{F}_{\mathbb{M}(\mathcal{G})} \left(\text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{H}}^{(n)}(A_{\mathcal{F}}^*) \right) \right)$$

which gives us the second inclusion in (6) and the proof of item (3i) is finished. Now the inclusion $A_{\mathcal{H}}^* \subset A_{\mathcal{F} \cup \mathcal{G}}^*$ follows from item (3i) and the inclusion $A_{\mathcal{F}}^* \subset A_{\mathcal{H}}^*$ follows from the fact that $\mathcal{F} \subset \mathcal{H}$ (see the second part of Theorem 2.19, item 2.).

Now we move to item (4). We first show (4i). Since $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the semiattractor of $\mathcal{F} \cup \mathbb{M}(\mathcal{G})$ (see item (1)), we have $g(A_{\mathcal{F} \cup \mathcal{G}}^*) \subset A_{\mathcal{F} \cup \mathcal{G}}^*$ for every $g \in \mathbb{M}(\mathcal{G})$. Hence, taking inverse function g^{-1} , we also have $A_{\mathcal{F} \cup \mathcal{G}}^* = g(g^{-1}(A_{\mathcal{F} \cup \mathcal{G}}^*)) \subset g(A_{\mathcal{F} \cup \mathcal{G}}^*)$.

Now we prove item (4ii). Since $\mathcal{R} \subset \mathcal{M}_1$ from item (2), we see that

$$\overline{F}_{\mathcal{R}}(A_{\mathcal{F} \cup \mathcal{G}}^*) \subset \overline{F}_{\mathcal{M}_1}(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*.$$

It remains to show the opposite inclusion. Take any ingredient of the union $\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_1}^{(n)}(A_{\mathcal{F}}^*)$, that is, the set

$$(g_1 \circ f_1 \circ h_1) \circ (g_2 \circ f_2 \circ h_2) \circ \cdots \circ (g_k \circ f_k \circ h_k)(A_{\mathcal{F}}^*).$$

We can rewrite it, and using item (1), proceed as follows:

$$\begin{aligned} & (g_1 \circ f_1 \circ g_1^{-1}) \circ (g_1 \circ h_1 \circ g_2 \circ f_2 \circ h_2) \circ \cdots \circ (g_k \circ f_k \circ h_k)(A_{\mathcal{F}}^*) \\ & \subset F_{\mathcal{R}} \left(A_{\mathcal{F}}^* \cup \bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathbb{M}(\mathcal{G})}^{(n)}(A_{\mathcal{F}}^*) \right) = F_{\mathcal{R}} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathbb{M}(\mathcal{G})}^{(n)}(A_{\mathcal{F}}^*) \right) \subset \overline{F}_{\mathcal{R}}(A_{\mathcal{F} \cup \mathcal{G}}^*). \end{aligned}$$

Hence using item (2), we have

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{M}_1}^{(n)}(A_{\mathcal{F}}^*) \right) \subset \overline{F}_{\mathcal{R}}(A_{\mathcal{F} \cup \mathcal{G}}^*)$$

and this gives us the remaining inclusion in (3). $\square$

Example 3.3. Let $f(x) := 0$ and $g(x) := x + 1$ for $x \in \mathbb{R}$, and set $\mathcal{F} := \{f\}$ and $\mathcal{G} := \{g\}$. It is easy to see that the following hold:

- (a) The Hutchinson attractor of $\mathcal{F}$ is $A_{\mathcal{F}} = \{0\}$;
- (b) The semiattractor of $\mathcal{F} \cup \mathcal{G}$ is $A_{\mathcal{F} \cup \mathcal{G}}^* = \mathbb{N} \cup \{0\}$;
- (c) The IFS $\mathcal{H} := \{f \circ h : h \in \mathbb{M}(\mathcal{G})\} = \mathcal{F}$ and its Hutchinson attractor equals $A_{\mathcal{H}} = \{0\}$.

Now if $\mathcal{G}'$ is a proper subset of $\mathbb{M}(\mathcal{G})$, then the semiattractors of the IFSs $\mathcal{M}'_1 := \{h' \circ f \circ h : h' \in \mathcal{G}', h \in \mathbb{M}(\mathcal{G})\}$ and $\mathcal{M}'_2 := \{h' \circ f : h' \in \mathcal{G}'\}$ equals $\{h'(0) : h' \in \mathcal{G}'\} \neq \mathbb{N} \cup \{0\} = A_{\mathcal{F} \cup \mathcal{G}}^*$. In particular, we cannot replace $\mathbb{M}(\mathcal{G})$ by its proper subset in definition of the IFSs $\mathcal{M}_1$ and $\mathcal{M}_2$ in Theorem 3.1 item (2).

Similarly, for the same IFS $\mathcal{G}'$, we see that $\overline{F}_{\mathcal{G}'}(A_{\mathcal{H}}) = \{h'(0) : h' \in \mathcal{G}'\}$. In particular, we cannot replace the IFS $\mathbb{M}(\mathcal{G})$ in Theorem 3.1 item (3) by some its proper subset.

We end this section with the question concerning item (4) of Theorem 3.1.

Problem 3.4. *In the framework of item (4) of Theorem 3.1, is the set $A_{\mathcal{F} \cup \mathcal{G}}^*$ the semiattractor of the IFS $\mathcal{R}$?*

Note that in the next section we give sufficient conditions for which this is the case.

4. Enriching Matkowski Contractive IFSs with Nonexpansive Mappings

In this section we prove that under additional contractive assumptions on IFSs $\mathcal{F}$ and $\mathcal{G}$, we can extend the assertions of Theorem 3.1 by observing that considered IFSs $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{H}$ and $\mathcal{R}$ are actually Matkowski contractive.

Theorem 4.1. Assume that $\mathcal{F}, \mathcal{G}$ are IFSs on a complete metric space such that

- $\mathcal{F}$ is Matkowski contractive;
- $\mathcal{G}$ consists of nonexpansive maps;
- the monoid $\mathbb{M}(\mathcal{G})$ is a bounded IFS.

Let $A_{\mathcal{F} \cup \mathcal{G}}^*$ be the semiattractor of $\mathcal{F} \cup \mathcal{G}$ and $A_{\mathcal{F}}$ be the Hutchinson attractor of $\mathcal{F}$. Then the following conditions hold:

- (A) The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is closed and bounded.
 (B) Define

$$\mathcal{M}_1 := \{g \circ f \circ h : f \in \mathcal{F}, g, h \in \mathbb{M}(\mathcal{G})\} \text{ and } \mathcal{M}_2 := \{g \circ f : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}.$$

Then:

- (Bi) $\mathcal{M}_1, \mathcal{M}_2$ are Matkowski contractive (with the same witness as for $\mathcal{F}$).
 (Bii) The Hutchinson attractors of $\mathcal{M}_1$ and $\mathcal{M}_2$ coincides with $A_{\mathcal{F} \cup \mathcal{G}}^*$; that is, $A_{\mathcal{M}_1} = A_{\mathcal{M}_2} = A_{\mathcal{F} \cup \mathcal{G}}^*$;
 (C) Define

$$\mathcal{H} := \{f \circ g : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}.$$

Then:

- (Ci) $\mathcal{H}$ is Matkowski contractive (with the same witness as for $\mathcal{F}$).
 (Cii) $A_{\mathcal{F}} \subset A_{\mathcal{H}} \subset A_{\mathcal{F} \cup \mathcal{G}}^*$, where $A_{\mathcal{F}}, A_{\mathcal{H}}$ are the Hutchinson attractors of $\mathcal{F}$ and $\mathcal{H}$, respectively;
 (Ciii)

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \overline{F_{\mathbb{M}(\mathcal{G})}}(A_{\mathcal{H}}).$$

- (D) Assume additionally that $\mathcal{F} \cup \mathcal{G}$ is countable. Then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is recovered by any sequence (x_n) with disjunctive driver, that is,

$$\lim_{k \rightarrow \infty} h(\{x_n : n \geq k\}, A_{\mathcal{F} \cup \mathcal{G}}^*) = 0.$$

- (E) Assume additionally that $\mathcal{F}$ and $\mathbb{M}(\mathcal{G})$ are compact. Then:
 (Ei) The IFSs $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{H}$ are compact;
 (Eii) The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is compact.

Proof. Item (A) follows from (B) (and (C))

Now we prove (B). We first show that $\mathcal{M}_1$ is a bounded IFS. Take any closed and bounded set $D \subset X$. We have

$$\overline{F_{\mathcal{M}_1}}(D) = \text{cl}(F_{\mathcal{M}_1}(D)) \subset \text{cl}\left(F_{\mathcal{F} \cup \mathbb{M}(\mathcal{G})}^{(3)}(D)\right)$$

and the latter set is bounded as $\mathcal{F} \cup \mathbb{M}(\mathcal{G})$ is bounded. Hence $\mathcal{M}_1$ is bounded. Now take $f \in \mathcal{F}$ and $g, h \in \mathbb{M}(\mathcal{G})$, and let φ be a common witness for maps from $\mathcal{F}$. Then for every $x, y \in X$, we have

$$\begin{aligned} d(g \circ f \circ h(x), g \circ f \circ h(y)) &\leq d(f \circ h(x), f \circ h(y)) \\ &\leq \varphi(d(g(x), g(y))) \leq \varphi(d(x, y)) \end{aligned}$$

so $g \circ f \circ h$ is Matkowski contraction with witness φ . Similarly we deal with the IFS $\mathcal{M}_2$. This ends the proof of (Bi). Item (Bii) follows from Theorem 3.1 item (2), as in this case the semiattractors of $\mathcal{M}_1$ and $\mathcal{M}_2$ are their Hutchinson attractors.

Now we prove (C). Item (Ci) can be seen in the same way as (Bi). Now by Theorem 3.1 item (3ii), we get (Cii). Finally, (Ciii) follows from Theorem 3.1 item (3i).

To see (D), first fix an enumeration $\mathcal{F} \cup \mathcal{G} = \{h_i : i \in I\}$ and let φ be a witness function of maps from $\mathcal{F}$. Then we have (see a.e., [15])

$$d(x_n, A_{\mathcal{F} \cup \mathcal{G}}^*) \leq \varphi^{(t_n)}(\text{diam}(\{x_0\} \cup A_{\mathcal{F} \cup \mathcal{G}}^*))$$

where t_n is the number of maps from $\mathcal{F}$ used in the driver of (x_n) up to n -th position.

Finally, choose any $\varepsilon > 0$.

Since $(t_n) \rightarrow \infty$ when $n \rightarrow 0$ (as the driver is disjunctive), we have that $d(x_n, A_{\mathcal{F} \cup \mathcal{G}}^*) \rightarrow 0$ and hence we can find $k_1 \in \mathbb{N}$ such that for every $n \geq k_1$, we have

$$d(x_n, A_{\mathcal{F} \cup \mathcal{G}}^*) < \varepsilon$$

so for every $k \geq k_1$, it holds

$$\sup\{d(x_n, A_{\mathcal{F} \cup \mathcal{G}}^*) : n \geq k\} \leq \varepsilon. \quad (7)$$

Fix any $k \geq k_1$.

Now choose a closed and bounded set D such that $x_0 \in D$, $A_{\mathcal{F} \cup \mathcal{G}}^* \subset D$ and $\overline{F}_{\mathcal{M}_2}(D) \subset D$ (this can be done since $\mathcal{M}_2$ is Matkowski contractive, see Remark 2.14). Then the whole sequence $(x_n) \subset D$. Take any $z \in A_{\mathcal{F} \cup \mathcal{G}}^*$ and find $k_0 \in \mathbb{N}$ so that

$$\varphi^{(k_0)}(\text{diam}(D)) < \frac{\varepsilon}{2}.$$

By Theorem 3.1 item (2) and Theorem 2.19 item 2., and the fact that the sequence $(\overline{F}_{\mathcal{M}_2}^{(n)}(A_{\mathcal{F}}))$ is nondecreasing, we can find $j \geq k_0$ and $z_1 \in \overline{F}_{\mathcal{M}_2}^{(j)}(A_{\mathcal{F}})$ such that $d(z, z_1) < \frac{\varepsilon}{4}$. Then by Lemma 2.6, we can find $z' \in F_{\mathcal{M}_2}^{(j)}(A_{\mathcal{F}})$ such that $d(z_1, z') < \frac{\varepsilon}{4}$. Then

$$d(z, z') \leq d(z, z_1) + d(z_1, z') < \frac{\varepsilon}{2}.$$

Now since $z' \in F_{\mathcal{M}_2}^{(j)}(A_{\mathcal{F}})$, we can find maps $f_1, \dots, f_j \in \mathcal{F}$ and $g_1, \dots, g_j \in \mathbb{M}(\mathcal{G})$ such that

$$z' \in (g_1 \circ f_1) \circ \dots \circ (g_j \circ f_j)(A_{\mathcal{F}}).$$

Now, as each g_i is a composition of maps from $\mathcal{G}$, we can go back to enumeration of $\mathcal{F} \cup \mathcal{G} = \{h_i : i \in I\}$, and find $i_1, \dots, i_p$ such that

$$(g_1 \circ f_1) \circ \dots \circ (g_j \circ f_j) = h_{i_1} \circ \dots \circ h_{i_p}.$$

Since the driver of (x_n) is disjunctive, there is an n_0 , which can be assumed to be greater than earlier taken k , such that

$$x_{n_0} = h_{i_1} \circ \cdots \circ h_{i_p}(x_{n_0-p}).$$

Hence

$$x_{n_0} \in h_{i_1} \circ \cdots \circ h_{i_p}(D) = (g_1 \circ f_1) \circ \cdots \circ (g_j \circ f_j)(D).$$

As the attractor $A_{\mathcal{F}} \subset A_{\mathcal{F} \cup \mathcal{G}}^* \subset D$, we see that also the earlier chosen point z'

$$z' \in (g_1 \circ f_1) \circ \cdots \circ (g_j \circ f_j)(D).$$

All in all,

$$\begin{aligned} d(z, \{x_n : n \geq k\}) &\leq d(z, x_{n_0}) \leq d(z, z') + d(z', x_{n_0}) \leq \frac{\varepsilon}{2} \\ &\quad + \text{diam}(g_1 \circ f_1) \circ \cdots \circ (g_j \circ f_j)(D)) \\ &\leq \frac{\varepsilon}{2} + \varphi^{(j)}(\text{diam}(D)) \leq \varepsilon. \end{aligned}$$

Hence

$$\sup\{d(z, \{x_n : n \geq k\}) : z \in A_{\mathcal{F} \cup \mathcal{G}}^*\} \leq \varepsilon. \quad (8)$$

By (7) and (8), we have that $h(\{x_n : n \geq k\}, A_{\mathcal{F} \cup \mathcal{G}}^*) \leq \varepsilon$ and the result follows.

Finally, observe that (Ei) is clear, and (Eii) follows from (Ei) and the fact that the Hutchinson attractor of a compact IFS is compact (Theorem 2.11 item (2)). $\square$

As a simple corollary of items (A) and (E) of the above theorem, we get:

Theorem 4.2. *Assume that $\mathcal{F}, \mathcal{G}$ are IFSs on a complete metric space such that $\mathcal{F}$ is Matkowski contractive, $\mathcal{G}$ consists of nonexpansive maps and the monoid $\mathbb{M}(\mathcal{G})$ is bounded. Assume that one of the following conditions hold:*

- (i) $\mathcal{F}$ and $\mathbb{M}(\mathcal{G})$ are finite;
- (ii) the space X is proper, that is, each closed and bounded subset of X is compact.

Then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^$ is compact.*

Remark 4.3. In view of Theorem 2.19, item 3., the compactness of semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ guarantees that $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the Hutchinson attractor of the IFS $(\mathcal{F} \cup \mathcal{G})|_{A_{\mathcal{F} \cup \mathcal{G}}^*}$ comprised of the restrictions of maps from $\mathcal{F} \cup \mathcal{G}$ to the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$. The example of noncompact semiattractor for finite IFS $\mathcal{F} \cup \mathcal{G}$ that is given in Sect. 5 shows that it may not be the case if we do not have compactness.

By Remark 2.2, item (3), we see that in Theorem 4.1, the assumption that $\mathbb{M}(\mathcal{G})$ is bounded means exactly that there exists $z_0 \in X$ such that the orbit $\{g(z_0) : g \in \mathbb{M}(\mathcal{G})\}$ is bounded. In the next result we show that boundedness of $\mathbb{M}(\mathcal{G})$ is essential.

Proposition 4.4. *Assume that $\mathcal{F}$ is an IFS on a metric space with the semiattractor $A_{\mathcal{F}}^*$, and let $\mathcal{G}$ be an IFS consisting of nonexpansive maps so that $\mathbb{M}(\mathcal{G})$ is unbounded. Then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is unbounded.*

Proof. By Theorem 3.1 item (3i), the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^* = \overline{F}_{\mathbb{M}(\mathcal{G})}(A_{\mathcal{H}}^*)$, so it must be unbounded as it contains an unbounded orbit $\{g(z_0) : g \in \mathbb{M}(\mathcal{G})\}$, where z_0 is any point of $A_{\mathcal{H}}^*$. $\square$

Now the question arises, whether, under the assumptions of Theorem 4.1, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is automatically the Hutchinson attractor of $\mathcal{F} \cup \mathcal{G}$. The answer is negative.

Example 4.5. Let $f(x) := 0$ and $g(x) := -x$ for $x \in \mathbb{R}$. Then $\mathcal{F} := \{f\}$ is Matkowski contractive, $\mathcal{G} := \{g\}$ consists of nonexpansive map, $\mathbb{M}(\mathcal{G}) = \{g, \text{Id}_X\}$ is bounded and $A_{\mathcal{F} \cup \mathcal{G}}^* = \{0\}$. However, $\{0\}$ is not the Hutchinson attractor of $\mathcal{F} \cup \mathcal{G}$, as every interval $[-a, a]$ is $\mathcal{F} \cup \mathcal{G}$ -invariant.

In the next theorem we study the case when the monoid of $\mathbb{M}(\mathcal{G})$ is actually a group. Note that this assumption together with nonexpansiveness of elements of $\mathcal{G}$, force that $\mathbb{M}(\mathcal{G})$ is group of isometries, so we will formulate it in such a way.

Theorem 4.6. *Assume that $\mathcal{F}, \mathcal{G}$ are IFSs on a complete metric space such that:*

- $\mathcal{F}$ is Matkowski contractive;
- the monoid $\mathbb{M}(\mathcal{G})$ is bounded and is a group of isometries.

Let $A_{\mathcal{F} \cup \mathcal{G}}^$ be the semiattractor of $\mathcal{F} \cup \mathcal{G}$. The the following conditions hold:*

(1) *Define*

$$\mathcal{R} := \{g \circ f \circ g^{-1} : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}.$$

Then

- (1i) *the IFS $\mathcal{R}$ is Matkowski contractive (with the same witness as for $\mathcal{F}$);*
- (1ii) *the Hutchinson attractor of $\mathcal{R}$ coincides with $A_{\mathcal{F} \cup \mathcal{G}}^*$; that is, $A_{\mathcal{R}} = A_{\mathcal{F} \cup \mathcal{G}}^*$;*
- (1iii) *for every $g \in \mathbb{M}(\mathcal{G})$, we have*

$$g(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*.$$

(2) *Define*

$$\mathcal{H} := \{f \circ g : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}.$$

Then the following conditions are equivalent:

- (i) $A_{\mathcal{H}} = A_{\mathcal{F} \cup \mathcal{G}}^*$;
- (ii) $\overline{F}_{\mathcal{H}}(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*$;
- (iii) $A_{\mathcal{F}} = A_{\mathcal{F} \cup \mathcal{G}}^*$;
- (iv) $g(A_{\mathcal{F}}) = A_{\mathcal{F}}$ for every $g \in \mathcal{G}$;

(v) $g(A_{\mathcal{F}}) = A_{\mathcal{F}}$ for every $g \in \mathbb{M}(\mathcal{G})$.

Proof. Since $\mathcal{R} \subset \mathcal{M}_1 := \{g \circ f \circ h : f \in \mathcal{F}, g, h \in \mathbb{M}(\mathcal{G})\}$, item (1i) follows from Theorem 4.1 item (B). From item (A) of Theorem 4.1 we also know that $A_{\mathcal{F} \cup \mathcal{G}}^*$ is bounded. Hence to prove that it is the Hutchinson attractor of $\mathcal{R}$, it is enough to show that it is $\mathcal{R}$ -invariant. However, this is precisely stated in Theorem 3.1 (4ii). Item (1iii) is stated in Theorem 3.1, item (4ii).

Now we prove (2). The equivalence of (i) and (ii) follows from the fact that $A_{\mathcal{F} \cup \mathcal{G}}^*$ is closed and bounded. The equivalence of (iv) and (v) follows from definition of $\mathbb{M}(\mathcal{G})$. The implication (iii) $\Rightarrow$ (i) follows from Theorem 4.1 item (C). The implication (iii) $\Rightarrow$ (v) follows from item (1). Now to see (iv) $\Rightarrow$ (iii), observe that

$$A_{\mathcal{F}} = \text{cl} \left(F_{\mathcal{F}}(A_{\mathcal{F}}) \cup \bigcup_{g \in \mathcal{G}} g(A_{\mathcal{F}}) \right) = \overline{F}_{\mathcal{F} \cup \mathcal{G}}(A_{\mathcal{F}}).$$

Hence by Lemma 2.6 and standard properties of closures, we have

$$\begin{aligned} A_{\mathcal{F} \cup \mathcal{G}}^* &= \text{cl} \left(\bigcup_{n \in \mathbb{N}} F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}) \right) = \text{cl} \left(\bigcup_{n \in \mathbb{N}} \text{cl} \left(F_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}) \right) \right) \\ &= \text{cl} \left(\bigcup_{n \in \mathbb{N}} \overline{F}_{\mathcal{F} \cup \mathcal{G}}^{(n)}(A_{\mathcal{F}}) \right) = A_{\mathcal{F}}. \end{aligned}$$

(in fact, we could also use the fact that $A_{\mathcal{F}}$ is closed and bounded $\mathcal{F} \cup \mathcal{G}$ -invariant set). Finally, we prove the remaining implication (i) $\Rightarrow$ (iii). Let $\tilde{A} = \bigcup_{g \in \mathbb{M}(\mathcal{G})} g(A_{\mathcal{H}})$. Then by Lemma 2.6 and Theorem 3.1 (3), we have

$$\begin{aligned} \overline{F}_{\mathcal{F}}(A_{\mathcal{F} \cup \mathcal{G}}^*) &= \overline{F}_{\mathcal{F}}(\text{cl } \tilde{A}) = \overline{F}_{\mathcal{F}}(\tilde{A}) = \text{cl} \left(\bigcup_{f \in \mathcal{F}} f(\tilde{A}) \right) \\ &= \text{cl} \left(\bigcup_{f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})} f \circ g(A_{\mathcal{H}}) \right) = \overline{F}_{\mathcal{H}}(A_{\mathcal{H}}) = A_{\mathcal{H}} = A_{\mathcal{F} \cup \mathcal{G}}^*. \end{aligned}$$

Hence $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the attractor of $\mathcal{F}$ as it is closed and bounded. $\square$

Remark 4.7. In the framework of the above theorem, if one of the following conditions hold:

- (a) $\mathcal{G}$ consists of periodic isometries;
- (b) $\mathcal{G}$ consists of nonexpansive maps and is closed under taking inverse functions,

then $\mathbb{M}(\mathcal{G})$ is a group.

Theorem 4.6 and the fact that the attractor of a compact IFS is compact, gives another sufficient condition for compactness of the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$:

Corollary 4.8. *Assume that $\mathcal{F}, \mathcal{G}$ are IFSs on a complete metric space such that $\mathcal{F}$ is Matkowski contractive and the monoid $\mathbb{M}(\mathcal{G})$ is bounded and is a group of isometries. If $\mathcal{F}$ is compact and any of equivalent conditions (i)–(v) from item (2) of Theorem 4.6 holds, then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is compact.*

In the next section we will give a relatively simple example in which the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is not compact. We now show that main results from [16] can be obtained as corollaries of our theorems. Below we let $g^{(0)}$ to be the identity map.

Corollary 4.9 [16, Theorem 10, Proposition 11, Theorem 14]. *Let $\mathcal{F}$ be a finite IFS consisting of Banach contractions (that is, $\text{Lip}(f) < 1$ for $f \in \mathcal{F}$) and $\mathcal{G} := \{g\}$, where g is a p -periodic isometry with a fixed point. Then the following conditions hold:*

- (i) *The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is compact and $g^{(k)}(A_{\mathcal{F} \cup \mathcal{G}}^*) = A_{\mathcal{F} \cup \mathcal{G}}^*$ for every $k = 0, \dots, p-1$.*
- (ii) *The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ coincides with the Hutchinson attractors of the IFSs $\{g^{(k)} \circ f : f \in \mathcal{F}, k = 0, \dots, p-1\}$, $\{g^k \circ f \circ g^{-k} : f \in \mathcal{F}, k = 0, \dots, p-1\}$, $\{g^k \circ f \circ g^j : f \in \mathcal{F}, k, j = 0, \dots, p-1\}$.*
- (iii) *The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^* = \bigcup_{k=0}^{p-1} g^{(k)}(A_{\mathcal{H}})$, where $A_{\mathcal{H}}$ is the attractor of the IFS $\{f \circ g^{(k)} : f \in \mathcal{F}, k = 0, \dots, p-1\}$.*
- (iv) *The semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is recovered by any sequence (x_n) with disjunctive driver.*

Finally, we present an example which shows that the assumption on non-expansiveness of maps from $\mathcal{G}$ is important in the above discussion:

Example 4.10. Let $f(x) := 1$ and $g(x) := 2x$ for $x \in \mathbb{R}$, and set $\mathcal{F} := \{f\}$ and $\mathcal{G} := \{g\}$. Then the Hutchinson attractor $A_{\mathcal{F}} = \{1\}$ and the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^* = \{2^n : n \in \mathbb{N} \cup \{0\}\}$, so it is not bounded.

5. Example of Noncompact Bounded Semi-attractor $A_{\mathcal{F} \cup \mathcal{G}}^*$

We say that a selfmap $f : X \rightarrow X$ of a metric space X is *rotative* (see [7]), if there exist $n \in \mathbb{N}$ and $\alpha < n$ such that for every $x \in X$,

$$d(f^{(n)}(x), x) \leq \alpha d(f(x), x).$$

In such case we call f as an (α, n) -rotative.

Let $X := \ell_\infty(\mathbb{C})$, that is, X is the Banach space of all bounded complex sequences endowed with the supremum norm $\|\cdot\|$. Set

$$\forall_{(x_k) \in X} f((x_k)) := (1, 1, \dots)$$

and let $\mathcal{F} := \{f\}$. For every $\delta > 0$, we will construct a $(\delta, 2)$ -rotative bijective linear isometry $T : X \rightarrow X$ such that, setting $\mathcal{G} := \{T\}$, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is not compact. This shows that in the thesis of Theorem 4.1 item (A)

we cannot guarantee that the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is compact (note here that the monoid $\mathbb{M}(\mathcal{G})$ is bounded as its maps have common fixed point 0).

Choose any positive $\delta < 1$ and a natural number M such that for every $\beta \in [0, \frac{2\pi}{M}]$, we have

$$|-1 - e^{i(\pi-\beta)}| < \delta. \quad (9)$$

Clearly, this also gives

$$|e^{i(2\pi-\beta)} - 1| < \delta. \quad (10)$$

Now define $T : X \rightarrow X$ in the following way:

$$\forall_{(x_k) \in X} T((x_k)) = \left(x_k e^{i\left(\pi - \frac{\pi}{k+M}\right)} \right).$$

In other words, each coordinate x_k of a given sequence is rotated around 0 counterclockwise by the angle $\pi - \frac{\pi}{k+M}$. We first prove that T is $(\delta, 2)$ -rotative bijective linear isometry.

Linearity and bijectivity of T is obvious. Now if $(x_k) \in X$, we have

$$\|Tx\| = \sup \left\{ \left| x_k e^{i\left(\pi - \frac{\pi}{k+M}\right)} \right| : k \in \mathbb{N} \right\} = \sup \{ |x_k| : k \in \mathbb{N} \} = \|x\|.$$

Hence T is isometry also. Now we show that T is $(\delta, 2)$ -rotative. At first observe that for every $k \in \mathbb{N}$, we have $\frac{\pi}{M+k} \leq \frac{2\pi}{M}$, so by (9) and the triangle inequality:

$$\left| e^{i\left(\pi - \frac{\pi}{k+M}\right)} - 1 \right| \geq |-1 - 1| - \left| -1 - e^{i\left(\pi - \frac{\pi}{M+1}\right)} \right| > 2 - \delta$$

On the other hand, as $\frac{2\pi}{M+k} \leq \frac{2\pi}{M}$, we have by (10),

$$\left| e^{i\left(2\pi - \frac{2\pi}{M+k}\right)} - 1 \right| < \delta.$$

Now if $x = (x_k) \in X$, then we have

$$\begin{aligned} \|T^{(2)}(x) - x\| &= \left\| \left(x_k e^{i\left(2\pi - \frac{2\pi}{M+k}\right)} \right) - (x_k) \right\| = \sup \left\{ \left| x_k e^{i\left(2\pi - \frac{2\pi}{M+k}\right)} - x_k \right| : k \in \mathbb{N} \right\} \\ &= \sup \left\{ |x_k| \left| e^{i\left(2\pi - \frac{2\pi}{M+k}\right)} - 1 \right| : k \in \mathbb{N} \right\} \\ &\leq \sup \{ |x_k| \delta : k \in \mathbb{N} \} = \delta \sup \{ |x_k| : k \in \mathbb{N} \} \\ &\leq \delta \sup \{ |x_k| (2 - \delta) : k \in \mathbb{N} \} \leq \delta \sup \left\{ |x_k| \left| e^{i\left(\pi - \frac{\pi}{M+k}\right)} - 1 \right| : k \in \mathbb{N} \right\} \\ &= \delta \|T(x) - x\|. \end{aligned}$$

Hence T is $(\delta, 2)$ -rotative.

Now choose $z_0 := (1, 1, 1, \dots)$. Then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ (where $\mathcal{F} = \{f\}$ and $\mathcal{G} = \{T\}$) equals

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \text{cl}\{T^n(z_0) : n \in \mathbb{N} \cup \{0\}\}.$$

To prove that it is not compact, it is enough to show that there exists a sequence (p_k) such that for $k \neq n$,

$$\|T^{(p_k)}(z_0) - T^{(p_n)}(z_0)\| \geq 1$$

Let (n_k) be the sequence of natural numbers such that for every $k \in \mathbb{N}$,

$$M + n_{k+1} \geq 2(M + n_k) \quad (11)$$

and

$$M + n_k \text{ is even.}$$

Finally, for $k \in \mathbb{N}$, let $p_k := M + n_k$. Choose $1 \leq j < k$. Then the n_k -th coordinate of $T^{p_k}(z_0)$ equals

$$e^{ip_k\left(\pi - \frac{\pi}{M+n_k}\right)} = e^{i(p_k\pi - \pi)} = e^{-i\pi} = -1$$

and the n_k -th coordinate of $T^{p_j}(z_0)$ equals:

$$e^{ip_j\left(\pi - \frac{\pi}{M+n_k}\right)} = e^{i\left(p_j\pi - \frac{p_j\pi}{M+n_k}\right)} = e^{-i\frac{p_j}{p_k}\pi}.$$

Since, by (11), $-\frac{p_j}{p_k}\pi \in [-\frac{\pi}{2}, 0]$, we see that

$$\|T^{(p_k)}(z_0) - T^{(p_j)}(z_0)\| \geq \left| e^{ip_k\left(\pi - \frac{\pi}{M+n_k}\right)} - e^{ip_j\left(\pi - \frac{\pi}{M+n_k}\right)} \right| = |-1 - e^{-i\frac{p_j}{p_k}\pi}| \geq 1.$$

The example presented above works on the nonseparable space $\ell_\infty(\mathbb{C})$. If the underlying space X is finitely dimensional, then $A_{\mathcal{F} \cup \mathcal{G}}^*$ is compact as X is proper. However, the following question is open:

Problem 5.1. *Is there a separable complete metric space X and IFSs $\mathcal{F}, \mathcal{G}$ on X which satisfy the assumptions of Theorem 4.1 and such that the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is not compact?*

6. Illustrative Examples on the Plane

In this section we give some simple examples on the plane which illustrates our main results. We will define $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{G}$ consists of one rotation on irrational angle, and the attractor $A_{\mathcal{H}}$ (of the IFS $\mathcal{H} := \{f \circ g : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}$) lies on the line $\mathbb{R} \times \{0\}$. Hence, by Theorem 4.1, item (C), the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ equals $\{z \cdot e^{i\beta} : z \in A_{\mathcal{H}}, \beta \in [0, 2\pi]\}$ (as each orbit of rotation on irrational angle is dense in the circle), that is, the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ is the attractor $A_{\mathcal{H}}$ wrapped around the origin (Fig. 1).

The pictures we present were obtained by the chaos game algorithm. Note that the probability of the choice of the rotation map was much bigger than the maps from $\mathcal{F}$ (circa 1 : 700).

FIGURE 1. The semiattractor of $\mathcal{F} \cup \mathcal{G}$ from Example 6.1

Example 6.1 [3, Section 3, p. 310]. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z) := 1$ and $g(z) := z \cdot e^{i\alpha}$ for $z \in \mathbb{C}$, where α is such that $\frac{\alpha}{\pi}$ is irrational, and set $\mathcal{F} := \{f\}$ and $\mathcal{G} := \{g\}$. Clearly, the Hutchinson attractor $A_{\mathcal{H}} = \{1\}$, and the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^* = \{z \in \mathbb{C} : |z| = 1\} =: S$ is the unit circle.

Remark 6.2. Although the unit circle S is the attractor of some finite contractive IFS, if $\mathcal{F}$ is a finite IFS consisting of Banach contracting affine maps and $\mathcal{G}$ consists of a single periodic isometry, then the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$ cannot equal to S . We will show it. Clearly, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is an affine map so that $f(S) \subset S$, then $f(S)$ is either the whole S or it is a single point. Hence, if f is a Banach contraction (which in this case means exactly that it is a Matkowski contraction), then the second possibility holds. Now let:

- $\mathcal{F}$ be a finite IFS consisting of affine Banach contractions maps;
- $\mathcal{G}$ consists of a single periodic isometry;
- maps from $\mathcal{F}$ and $\mathcal{G}$ transforms S into S .

Then $\mathcal{H} = \{f \circ g : f \in \mathcal{F}, g \in \mathbb{M}(\mathcal{G})\}$ consists of finitely many affine Banach contractions which transforms S into S . From the above observation we see that $\overline{F}_{\mathcal{H}}(S)$ must be finite, and hence also the attractor $A_{\mathcal{H}}$ is finite (as $A_{\mathcal{H}} \subset \overline{F}_{\mathcal{H}}(S)$). Thus also the set $\overline{F}_{\mathbb{M}(\mathcal{G})}(A_{\mathcal{H}})$ is finite and cannot be equal to S (Fig. 2).

Example 6.3. Define $f_1, f_2 : \mathbb{C} \rightarrow \mathbb{C}$ by $f_1(x + iy) := \frac{1}{3}x - \frac{2}{3}$ and $f_2(x + iy) := \frac{1}{3}x + \frac{2}{3}$, for $x, y \in \mathbb{R}$. Moreover, let $g(z) := z \cdot e^{i\alpha}$ for $z \in \mathbb{S}$, where α is such that $\frac{\alpha}{\pi}$ is irrational. Define $\mathcal{F} := \{f_1, f_2\}$ and $\mathcal{G} := \{g\}$. Clearly, the attractor of $\mathcal{F}$ is the Cantor set stretched on the interval $[-1, 1]$. We will show, however,

that the attractor of $\mathcal{H} = \{f \circ h : f \in \mathcal{F}, h \in \mathbb{M}(\mathcal{G})\}$ equals

$$A_{\mathcal{H}} = \left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]. \quad (12)$$

It is sufficient to show that this set is $\mathcal{H}$ -invariant. Observe for any $x \in \mathbb{R}$ and any $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} f_1 \circ g^{(k)}(x) &= f_1(x \cdot \cos(k\alpha) - 0 \cdot \sin(k\alpha) + i(0 \cdot \cos(k\alpha) + x \cdot \sin(k\alpha))) \\ &= \frac{1}{3}x \cdot \cos(k\alpha) - \frac{2}{3} \end{aligned}$$

and

$$\begin{aligned} f_2 \circ (g^{(k)})(x) &= f_2(x \cdot \cos(k\alpha) - 0 \cdot \sin(k\alpha) + i(0 \cdot \cos(k\alpha) + x \cdot \sin(k\alpha))) \\ &= \frac{1}{3}x \cdot \cos(k\alpha) + \frac{2}{3}. \end{aligned}$$

Hence, setting $a_k := \cos(k\alpha)$, we get

$$\begin{aligned} f_1 \circ (g^{(k)}) \left(\left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right] \right) \\ = \left[-\frac{1}{3}a_k - \frac{2}{3}, -\frac{1}{9}a_k - \frac{2}{3}\right] \cup \left[\frac{1}{9}a_k - \frac{2}{3}, \frac{1}{3}a_k - \frac{2}{3}\right] \end{aligned}$$

where we use the convention that if $a > b$, then $[a, b] := [b, a]$. Hence

$$\begin{aligned} \text{cl} \left(\bigcup_{k \in \mathbb{N} \cup \{0\}} f_1 \circ (g^{(k)}) \left(\left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right] \right) \right) \\ = \text{cl} \left(\bigcup_{|a| \leq 1} \left[-\frac{1}{3}a - \frac{2}{3}, -\frac{1}{9}a - \frac{2}{3}\right] \right) \\ = \left[-1, -\frac{1}{3}\right]. \end{aligned}$$

In the same way we prove

$$\text{cl} \left(\bigcup_{k \in \mathbb{N} \cup \{0\}} f_2 \circ (g^{(k)}) \left(\left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right] \right) \right) = \left[\frac{1}{3}, 1\right],$$

which gives us (12). Finally, we have

$$A_{\mathcal{F} \cup \mathcal{G}}^* = \left\{ z \cdot e^{i\beta} : z \in \left[\frac{1}{3}, 1\right], \beta \in [0, 2\pi] \right\}.$$

Remark 6.4. In the framework of Example 6.3, for every $n \in \mathbb{N}$, let $\mathcal{F}^{(n)} := \{f_{i_1} \circ \dots \circ f_{i_n} : i_1, \dots, i_n = 1, 2\}$ and $\mathcal{H}^{(n)} := \{f \circ h : f \in \mathcal{F}^{(n)}, h \in \mathbb{M}(\mathcal{G})\}$. In a similar way as in this example, we can see that in such a case, $A_{\mathcal{H}_n}$ is the $(n+1)$ -th step in the construction of the Cantor set on $[-1, 1]$ (for example,

FIGURE 2. The semiattractor of $\mathcal{F} \cup \mathcal{G}$ from Example 6.3 FIGURE 3. The semiattractor of $\mathcal{F}^{(2)} \cup \mathcal{G}$ from Remark 6.4

$A_{\mathcal{H}_2} = [-1, -\frac{7}{9}] \cup [-\frac{5}{9}, -\frac{1}{3}] \cup [\frac{1}{3}, \frac{5}{9}] \cup [\frac{7}{9}, 1]$, and so on). Hence we can get as the semiattractor $A_{\mathcal{F} \cup \mathcal{G}}^*$, the approximation of the Cantor set wrapped around the origin (Fig. 3).

7. Lower Transition Attractors as Semi-attractors of Enriched IFSs

We start with giving some further denotations from [21]. Throughout this section, let

$$\mathcal{F} = \{g, f_1, \dots, f_N\}$$

be an IFS on a Euclidean space $\mathbb{R}^d$ consisting of similarity transformations, and such that

$$\max\{\text{Lip}(f_1), \dots, \text{Lip}(f_N)\} < \text{Lip}(g).$$

Fix $a_*, a_1, \dots, a_N \in X$ so that $f_i(a_i) = 0$ for $i = 1, \dots, N$, and also $g(a_*) = 0$, and set $t_0 := \frac{1}{\text{Lip}(g)}$. Now for any $i = 1, \dots, N$, any $t \geq 0$ and $x \in X$, define

$$f_{(i,t)}(x) := tf_i(x) + a_i$$

and

$$g_t(x) := tg(x) + a_*.$$

Then the family $\mathcal{F}_t := \{g_t, f_{(1,t)}, \dots, f_{(N,t)}\}$, $t \geq 0$ is called in [21] as a *bounded one-parameter similarity family* (see [21, Definition 8.1, Definition 5.1 and Proposition 5.4]).

A compact set $A \subset X$ is called $(\mathcal{F}_{t_0}, a_*)$ -invariant, if $\overline{F}_{\mathcal{F}_{t_0}}(A) = A$ and $a_* \in A$.

Definition 7.1 [21, Definition 8.2]. In the above framework, the set

$$A_{\mathcal{F},*} := \bigcap \{A \subset X : A \text{ is } (\mathcal{F}_{t_0}, a_*)\text{-invariant}\}$$

will be called the *lower transition attractor* of $(\mathcal{F}_t)$.

It is clear that from the perspective of investigations of lower transition attractors, we can assume that $t_0 = 1$ (just consider the scaled maps $t_0 f_1, \dots, t_0 f_N, t_0 g$). In such a case, it is easy to see that the maps g and g_1 are isometries. The following result shows that the lower transition attractor is the semi-attractor of $\mathcal{F}_1 \cup \{h\}$ for certain appropriate constant map h and, in consequence, the Hutchinson attractor of certain IFSs.

Proposition 7.2. Assume that $\mathcal{F}_t = \{g_t, f_{(1,t)}, \dots, f_{(N,t)}\}$, $t \geq 0$, is a bounded one-parameter similarity family as above with $t_0 = 1$, and let $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the constant map defined by

$$h(x) := a_*$$

Then the lower transition attractor $A_{\mathcal{F},*}$ exists and:

- (1) is the semi-attractor of the IFS $\mathcal{F}_1 \cup \{h\}$;

(2) is the Hutchinson attractor of each of the IFSs

$$\mathcal{M}_1 := \{g_1^{(m)} \circ f_{(i,1)} \circ g_1^{(n)} : i = 1, \dots, N, m, n \in \mathbb{N} \cup \{0\}\} \cup \{h\}$$

and

$$\mathcal{M}_2 := \{g_1^{(m)} \circ f_{(i,1)} : i = 1, \dots, N, m \in \mathbb{N} \cup \{0\}\} \cup \{h\},$$

which are compact and such that

$$\sup\{\text{Lip}(f) : f \in \mathcal{M}_i\} \leq \max\{\text{Lip}(f_1), \dots, \text{Lip}(f_N)\}$$

(in particular, they are Matkowski contractive).

Proof. At first observe that the IFS $\mathbb{M}(\{g_1\}) = \{g_1^{(m)} : m \in \mathbb{N} \cup \{0\}\}$ is bounded (as $g_1^{(m)}(a_*) = a_*$; recall Remark 2.2), hence compact (as $\mathbb{R}^d$ is a proper space). Hence, using Theorem 4.1 items (A) and (E) for $\{f_{(1,1)}, \dots, f_{(N,1)}, h\}$ and $\{g_1\}$, we see that the semiattractor $A_{\mathcal{F}_1 \cup \{h\}}^*$ is compact. Moreover, using Theorem 2.19 for IFSs $\{h\}$ and $\mathcal{F}_1$, we see that the semiattractor of $\mathcal{F}_1 \cup \{h\}$ equals

$$A_{\mathcal{F}_1 \cup \{h\}}^* = \text{cl} \left(\bigcup_{n=1}^{\infty} F_{\mathcal{F}_1 \cup \{h\}}^{(n)}(\{a_*\}) \right).$$

Now since a_* is a fixed point of g_1 , we see that $a_* \in F_{\mathcal{F}_1}^{(n)}(\{a_*\})$ for every $n \in \mathbb{N}$, so for every $n \in \mathbb{N}$,

$$F_{\mathcal{F}_1}^{(n)}(\{a_*\}) = F_{\mathcal{F}_1 \cup \{h\}}^{(n)}(\{a_*\})$$

and hence

$$A_{\mathcal{F}_1 \cup \{h\}}^* = \text{cl} \left(\bigcup_{n=1}^{\infty} F_{\mathcal{F}_1}^{(n)}(\{a_*\}) \right).$$

Thus using [21, Theorem 8.2 (f)], we see that $\text{cl}(\bigcup_{n=1}^{\infty} F_{\mathcal{F}_1}^{(n)}(\{a_*\}))$ is the lower transition attractor of $(\mathcal{F}_t)$. This ends the proof of item (1).

Now, using Theorem 4.1 item (B) for IFSs $\{f_{(1,1)}, \dots, f_{(N,1)}, h\}$ and $\{g_1\}$, we see that the set $A_{\mathcal{F},*}$, as the semiattractor of $\mathcal{F}_1 \cup \{h\}$, is the attractor of the Matkowski contractive IFSs

$$\begin{aligned} \mathcal{M}'_1 &:= \{g_1^{(m)} \circ f_{(i,1)} \circ g_1^{(n)} : i = 1, \dots, N, m, n \in \mathbb{N} \cup \{0\}\} \\ &\cup \{g_1^{(m)} \circ h \circ g_1^{(n)} : i = 1, \dots, N, m, n \in \mathbb{N} \cup \{0\}\}. \end{aligned}$$

and

$$\mathcal{M}'_2 := \{g_1^{(m)} \circ f_{(i,1)} : i = 1, \dots, N, m \in \mathbb{N} \cup \{0\}\} \cup \{g_1^{(m)} \circ h : m \in \mathbb{N} \cup \{0\}\}.$$

whose witness is the same as for $\mathcal{F}_1$, which means that

$$\sup\{\text{Lip}(f) : f \in \mathcal{M}'_i\} \leq \max\{\text{Lip}(f_1), \dots, \text{Lip}(f_N)\}$$

However, since $h(x) = a_*$ and a_* is a fixed point of g_1 , we see that $\mathcal{M}'_1 = \mathcal{M}_1$ and $\mathcal{M}'_2 = \mathcal{M}_2$, so we get the assertion (2). $\square$

Remark 7.3. It can be viewed that the above presented proof can be extended into wider setting. In forthcoming paper we are going to investigate the existence of lower transition attractors (as well as upper transition attractors) from [21] for more general families $\mathcal{F}_t$.

Acknowledgements

I would like to thank Krzysztof Leśniak and Nina Snigireva for presenting the topic and many fruitful discussions and suggestions, and to the anonymous referee for several important remarks and pointing out some missprints.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] Angelos, J., Grossman, G., Kaufmann, E., Lenker, T., Rakesh, L.: Limit cycles for successive projections onto hyperplanes in $\mathbb{R}^n$. *Linear Algebra Appl.* **285**, 201–228 (1998)
- [2] Barnsley, M.F.: *Fractals Everywhere*. Academic Press Professional, Boston (1993)
- [3] Barnsley, M.F., Vince, A.: Developments in fractal geometry. *Bull. Math. Sci.* **3**, 299–348 (2013)
- [4] Beer, G.: *Topologies on Closed and Closed and Convex Sets*. Kluwer, Dordrecht (1993)
- [5] Fitzsimmons, M., Kunze, H.: Small and minimal attractors of an IFS. *Commun. Nonlinear Sci. Numer. Simul.* **85**, 105277 (2020)
- [6] Fitzsimmons, M., Kunze, H.: Necessary/sufficient conditions for the chaos game to render quasi attractors. *Commun. Nonlinear Sci. Numer. Simul.* **90**, 13 (2020)
- [7] Goebel, K., Koter, M.: Fixed points of rotative Lipschitzian mappings. *Rend. Sem. Mat. Fis. Milano* **51**, 145–156 (1981)
- [8] Fernau, H.: Infinite iterated function systems. *Math. Nachr.* **170**, 79–91 (1994)
- [9] Hata, M.: On the structure of self-similar sets. *Jpn. J. Appl. Math.* **2**, 381–414 (1985)

- [10] Hata, M.: On some properties of set dynamical systems. *Proc. Jpn. Acad. Ser. A* **61**(4), 99–102 (1985)
- [11] Hutchinson, J.: Fractals and self-similarity. *Indiana Univ. Math. J.* **30**(5), 713–747 (1981)
- [12] Gwóźdź-Lukawska, G., Jachymski, J.: The Hutchinson–Barnsley theory for infinite iterated function systems. *Bull. Aust. Math. Soc.* **72**(3), 441–454 (2005)
- [13] Jachymski, J., Jóźwik, I.: Nonlinear contractive conditions: a comparison and related problems. *Pol. Acad. Sci.* **77**, 123–146 (2007)
- [14] Lasota, A., Myjak, J.: Attractors of multifunctions. *Bull. Pol. Acad. Sci. Math.* **48**(3), 319–334 (2000)
- [15] Leśniak, K.: Random iteration for infinite nonexpansive iterated function systems. *Chaos* **25**, 083117 (2015)
- [16] Leśniak, K., Snigireva, N.: Iterated function systems enriched with symmetry. preprint
- [17] Lewellen, G.B.: Self-similarity. *Rocky Mt. J. Math.* **23**(3), 1023–1040 (1993)
- [18] Martyn, T.: The chaos game revisited: yet another, but a trivial proof of the algorithm’s correctness. *Appl. Math. Lett.* **25**(2), 206–208 (2012)
- [19] Matkowski, J.: Integrable solutions of functional equations. *Diss. Math.* **127**, 68 (1975)
- [20] Seclean, N.A.: The existence of the attractor of countable iterated function systems. *Mediterr. J. Math.* **9**(1), 61–79 (2012)
- [21] Vince, A.: Thresholds for one-parameter families of affine iterated function systems. *Nonlinearity* **33**(12), 6541–6563 (2020)

Filip Strobin
Institute of Mathematics
Lodz University of Technology
Wólczańska 215
93-005 Łódź
Poland
e-mail: filip.strobin@p.lodz.pl

Received: September 21, 2020.

Accepted: June 7, 2021.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.