



“I have Learned Non-Euclidean Geometry Just from this Book” - Some facets of the correspondence between Friedrich Engel and David Hilbert

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Abstract. Friedrich Engel and David Hilbert learned to know each other at Leipzig in 1885 and exchanged letters in particular during the next 15 years which contain interesting information on the academic life of mathematicians at the end of the 19th century. In the present article we will mainly discuss a statement by Hilbert himself on Moritz Pasch’s influence on his views of geometry, and on personnel politics concerning Hermann Minkowski and Eduard Study but also Engel himself.

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To the Memory of Heinrich Wefelscheid

In the 1980ies I learned to know Heinrich Wefelscheid when he visited Münster university for one of the great Saturday colloquiums on theoretical mathematics that were organized there each summer. Even if our mathematical interests in the strict sense were not too close to each other, we stayed in contact, for

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example, in 1997 he formally introduced me to Mathematische Gesellschaft in Hamburg.

We once met again at the celebration of Günter Pickert's (1917–2015) ninetieth birthday at Gießen university in 2007. Our common interests in the history of mathematics led to a discussion on David Hilbert's (1862–1943) "Grundlagen der Geometrie" [6]. I mentioned that I had found the German original of the quote in the title of the present article in a letter of January 14, 1894 from Hilbert to Friedrich Engel (1861–1941) [5, Sign.: NE180610] and had informed Dov Tamari (1911–2006) about it who utilized it several times in his book on Moritz Pasch (1843–1930) [13, pp. iv, xix–xx, 113–114]. The quote refers to Hilbert's learning of non-Euclidean geometry from Pasch's "Vorlesungen über neuere Geometrie" [9] and to the importance of this book for the axiomatization of geometry. It is given in context in the beginning of Sect. 3.

At the end of our conversation Heinrich Wefelscheid invited me to give a talk on the correspondence between Engel and Hilbert at the 35th "Arbeitstagung über Geometrie und Algebra" in the beginning of the next year. (This was the one with the "Festkolloquium" at the occasion of Helmut Karzel's (*1928) eightieth birthday.) The talk took place in the conference room of Dresdner Bank at Pariser Platz in Berlin on March 3, 2008. After it, Heinrich Wefelscheid expressed his interest not only in its part on the foundations of geometry, but also in the part concerned with how to get a professorship in mathematics. He suggested that I should submit a written version of my talk to Results in Mathematics.

Several circumstances during the last years have hindered me from writing up this manuscript until this very sad occasion. But it seems appropriate to me to commemorate Heinrich Wefelscheid in this way, who was not only an active mathematician himself, but also an effective supporter of present mathematicians in need and a keeper of the memory of past mathematicians.

1. Engel, Hilbert, and their Correspondence

Friedrich Engel and David Hilbert were almost exactly of the same age: Engel was born on December 26, 1861, Hilbert less than one month later, on January 23, 1862. Also their academic studies had a similar structure: Both started them at the university next to their birthplace, Engel at Leipzig, Hilbert at Königsberg, and spent one semester abroad before finishing their dissertation, Engel at Berlin, Hilbert at Heidelberg. Felix Klein (1849–1925) was the doctoral father of neither of them, formally seen, but had decisive influence on the academic careers of both of them.

Engel and Hilbert learned to know each other at Leipzig university in autumn 1885. Hilbert had received his doctorate at Königsberg university in the beginning of that year under the supervision of Ferdinand (since 1918: von)

Lindemann (1852–1939) and spent the winter term 1885/86 with the latter’s doctoral father Klein at Leipzig. (For this part of Hilbert’s life cf. also [10].) Engel had taken his doctorate at Leipzig already in 1883 with Adolph Mayer (1839–1908), spent the academic year 1884/85 with Sophus Lie (1842–1899) at Christiania, and already taken his “Habilitation” in June 1885 after his return to Leipzig. (For a complete account of Engel’s biography cf., e. g., [12].)

Despite the formally different stages of their careers the two almost equally aged young gentlemen came in close contact soon. On a personal level this is indicated by their use of the informal “Du” as form of address and “Dein Freund” (= “Your friend”) as closing of letters which were reserved to very closed acquaintances in academic circles in those years. But also their scientific exchange at that time was intensive even if this fact seems to have escaped the attention of the research on Hilbert up to now.

In particular, it was Engel who introduced Hilbert to the theory of transformation groups. In his first letter to Engel dated April 8, 1886 Hilbert wrote with respect to a paper (probably [3]) that the former had sent to him [5, Sign.: NE180601]:

“freut man sich doch stets dann am meisten über die Zusendung einer mathematischen Abhandlung, wenn man ihrem Gegenstande und ihrer Betrachtungsweise nicht so ganz ferne steht. Dass letzteres nicht der Fall ist, habe ich, wie du weisst, Deinen liebenswürdigen Bemühungen während meines Aufenthaltes in Leipzig zu verdanken.”
(= “one is always most pleased about the sending of a mathematical treatise when one is not so far away from its subject and its point of view. That the latter is not the case I have to thank your gracious efforts during my stay in Leipzig as you know.”)

(This was not the only case of Engel influencing other mathematicians. After Engel’s leave for Greifswald Felix Hausdorff (1868–1942) payed tribute to him in a letter of May 13, 1904 as the broadest and less fanatic of the docents at Leipzig [5, Sign.: NE160232].)

The correspondence between Engel and Hilbert started in spring 1886 when Hilbert went to Paris for a study stay during the summer term which Klein had arranged for him and Eduard Study (1862–1930). Study would later on become like a brother to Engel, for their relation cf. [15].

Altogether 27 writings from Hilbert to Engel have remained in the possession of the “Mathematische Seminar” of Justus-Liebig-Universität Gießen, they are now preserved in the archive of the university [5, Sign.: NE1806XX]. Furthermore, 30 writings from Engel to Hilbert have remained, they are kept in the “Abteilung für Handschriften und Seltene Drucke” of Niedersächsische Staats- und Universitätsbibliothek Göttingen [7, Sign.: Cod. Ms. D. Hilbert 94].

Because of the wealth of this historical resource the following exposition will have to restrict to only some aspects mentioned in the correspondence.

On the one hand this will be Hilbert's interest in "non-Euclidean", in order to be more precise: not necessarily Euclidean geometry, which finally led to his treatise "Grundlagen der Geometrie" [6] on the foundations of geometry and the influence on him in this respect. On the other hand some personnel matters will be discussed which are related to Engel himself, Hermann Minkowski (1864–1909), and Study.

2. Foundations of Geometry

Hilbert's "Grundlagen der Geometrie" [6], which were published in 1899 for the first time, had an eminent influence on the development of mathematics in the 20th century. Here the axiomatic standpoint in the modern sense is developed for the first time in a consequent and complete manner.

Already Otto Blumenthal (1876–1944) has mentioned in his Hilbert biography that Pasch had exerted some influence on Hilbert's exposition [1, p. 403]. More detailed in this respect is Michael Markus Toepell's (*1951) dissertation [14] on the origin of the "Grundlagen" [6]. Based on Hilbert's scientific estate, he has analyzed the impact of Pasch's "Vorlesungen über neuere Geometrie" (= "Lectures on recent geometry") [9] from which the axioms of order come which were explicitly mentioned by Blumenthal already. Furthermore, Pasch's book of 1882 contains the first steps to axiomatize geometry in a modern sense, even if Pasch's view of the "axioms" is that of the fundamental central theorems of an already existing theory whereas Hilbert later on went more or less the other way round and defined the theory by the choice of its axioms. This aspect is worked out in particular by Walter S. Contro in [2].

All these historical studies are almost exclusively based on scientific texts, for example publications, lecture notes or private notebooks. But the Engel-Hilbert correspondence provides one concise statement by Hilbert on his dependence on Pasch: Hilbert wrote this letter at Königsberg on January 14, 1894. Its relevant paragraphs begin with a congratulation to Engel at the occasion of the publication of the last of the three volumes of his joint enterprise with Lie on the "Theorie der Transformationsgruppen" (= "Theory of transformation groups") [8]. They read as follows [5, Sign.: NE180610]:

"Dass Du mit der Herausgabe des Lie'schen Werkes fertig bist ist Dir wohl wie ein vom Herzen gefallener Stein? Übrigens konntest Du nicht im 3ten Bande Deinen Einfluss auf Lie geltend machen, dass er die persönlichen Gehässigkeiten unterdrückte? Was die Sache selbst betrifft, so habe ich mir vorgenommen, Dir im nächsten Semester meine Meinung einmal ausführlicher zu schreiben. Ich habe nämlich eine Vorlesung über Nicht-Euklidische Geometrie angekündigt und möchte bei dieser Gelegenheit auch die Lie'sche Auffassung gründlich studiren. Ich glaube wohl, dass Lie in manchem harten

Urtheil recht hat, doch habe ich z. B. das Buch von Pasch Neuere Geometrie von Lie nicht besprochen gefunden. Ich habe Nicht-Euklidische Geometrie lediglich aus diesem Buch gelernt, und bin überzeugt, dass in diesem Buche Alles correct ist, und das es wirklich die einfachsten Axiome der Geometrie vollständig aufzählt, was mir Lie noch lange nicht erreicht zu haben scheint.

Freilich muss man bei Pasch in den späteren Abschnitten noch Manches selbst ergänzen. Indem ich hoffe Dir bald einmal Genaueres hierüber schreiben zu können, bin ich mit besten Grüßen

Dein Freund Hilbert.”

(= “Is it like a stone that has fallen from your heart that you have finished publishing Lie’s work? By the way, could you not use your influence on Lie in Volume 3 to suppress the personal venom? As for the matter itself, I have decided to write you my opinion in more detail in the next semester. I have announced a lecture on non-Euclidean geometry, and I would like to take this opportunity to study Lie’s conception thoroughly. I think that Lie is right in some harsh judgments, but I have not found, for example, Pasch’s book *Neuere Geometrie*, discussed by Lie. I have learned non-Euclidean geometry just from this book, and I am convinced that everything in this book is correct and that it really lists the simplest axioms of geometry completely, which Lie does not seem to have achieved by far.

Of course, with Pasch you have to complete some things yourself in the later sections. Hoping to be able to write you more details about this soon, I am with best regards

Your friend Hilbert.”)

The detailed statement announced by Hilbert could unfortunately not been found in Engel’s estate, be it that Hilbert did not find the time to write it up, be it that it was not preserved.

Note that the lecture course on non-Euclidean geometry for the summer term 1894 that Hilbert refers to in the letter is not the course “Elemente der Euklidischen Geometrie” (= “Elements of Euclidean geometry”) that Blumenthal mentions [1, p. 402]. As already pointed out by Toepell, e. g. [14, pp. 1–2], this confirms that Hilbert’s interest in the foundations of geometry dates from very early times.

This will have made it all the easier for Engel to raise Hilbert’s interest in the theory of transformation groups as already mentioned in Sect. 2. Furthermore, non-Euclidean geometry stayed a topic in the correspondence of the two friends. Engel, for example, sent Hilbert a copy of his 1898 book [4], on

Nikolai Ivanovich Lobachevsky (1793–1856). Hilbert expressed his thanks in a letter of February 15, 1899 [5, Sign.: NE180614]:

“Es ist mir das Geschenk von um so grösserem Werte, als ich mich schon seit längerer Zeit mit den Grundlagen der Geometrie beschäftige und insbesondere die Arbeiten von Lobatschewskij studiere – bisher allerdings nur die nicht-russischen.” (= “It is a gift of all the greater value to me as I have been occupied with the foundations of geometry for a long time and have been studying the works of Lobachevsky in particular – so far, however, only the non-Russian ones.”)

As to Euclidean geometry Engel wrote to Hilbert two days later [7, Sign.: Cod. Ms. D. Hilbert 94, 10/1]:

“Dass Du Dich jetzt wieder lebhaft mit den Grundlagen der Geometrie beschäftigst, habe ich von Liebmann erfahren, dank dessen Vermittelung ich auch die autographierte Ausarbeitung Deiner Vorlesung bekomme. Was ich bis jetzt davon gesehen habe – leider habe ich nur ganz flüchtige Blicke hineinwerfen können – hat mich sehr begierig gemacht, das Ganze genauer zu studiren, doch wird das wohl noch nicht so bald geschehen können.” (= “I learned from Liebmann that you are now lively again dealing with the fundamentals of geometry, thanks to whom I also get the autographed elaboration of your lecture. What I’ve seen of it so far – unfortunately I have only been able to glimpse at it – has made me very eager for studying the whole thing more closely, but that will probably not be able to happen anytime soon.”)

The lecture course took place in the winter term 1898/99 and is well known as a stepstone towards Hilbert’s “Grundlagen der Geometrie” [6], cf. [1, p. 402], [14, Chap. 4, 5]. The elaboration of the course had been undertaken by Hans von Schaper (1875–1913), Hilbert’s first assistant.

Just the next day, February 18, 1899, Hilbert replied to Engel’s announcement that he would be glad to have his comments on the text, but warned him that it was only a preliminary version for the audience of his lecture course [5, Sign.: NE180615]. That very day, however, Lie died at Christiania and Engel testified in his answer on February 23, 1899 [7, Sign.: Cod. Ms. D. Hilbert 94, 11]:

“Augenblicklich stehe ich noch ganz unter dem Eindrucke der Nachricht von Lies Tode.” (= “At the moment I’m still completely under the impression of the news of Lie’s death.”)

The personal relation between Engel and Lie had deteriorated during the last years of the latter’s life because of quarrels on priority which partially originated from Lie’s mental problems resulting from his pernicious anemia. But Lie’s death struck Engel heavily and may have kept him from sending his

remarks on the lecture elaboration to Hilbert. At any rate, the next letter from Engel to Hilbert preserved in the Hilbert estate is a postcard dated May 28, 1900 [7, Sign.: Cod. Ms. D. Hilbert 94, 11]. At that time the “autographed elaboration of your lecture” had since long been transferred to Hilbert’s “Festschrift” for the Gauss-Weber memorial, his “Grundlagen der Geometrie” [6].

3. The Associate Professorship for Minkowski at Königsberg

In 1892 Hilbert had become associate (“Extraordinarius”) and in 1893 full professor (“Ordinarius”) at Königsberg university. Immediately, he began negotiations with the responsible authorities how to fill his own succession as associate professor there. His desired candidate was his close collaborator and friend Minkowski who had left Königsberg for Bonn in 1887 where he held an associate professorship just since 1893.

In the end, Hilbert was successful in this respect, but his biographer Blumenthal made a cryptic remark on this enterprise in Hilbert’s “Lebensgeschichte” [1, p. 396], which indicates some trouble:

“Es gelang ihm [= Hilbert], trotz einiger verwickelter Personalfragen, Minkowski Ostern 1894 als Extraordinarius nach Königsberg zu ziehen.” (= “He [= Hilbert] succeeded in moving Minkowski to Königsberg as an associate professor at Easter 1894, despite some complex personnel issues.”)

The solution of this mystery can be found in the correspondence between Engel and Hilbert, too: The missing third, so say, is Eduard Study, who, on the one hand, had accompanied Hilbert on his journey to Paris in summer 1886, cf. Sect. 2, and, on the other hand, had become a very close friend to Engel in the meantime. Even if Study was only few months younger than Engel and Hilbert, his academic career had much slower advanced than that of his two friends: In 1894, Hilbert had a full professorship at Königsberg, Engel was not that well off but at least had a permanent position and the formal title of “etatmäßiger Honorarprofessor” at Leipzig.

Study, however, who had taken his doctorate at Munich, had been ordered by Klein to the suicide mission of giving a firm foundation for the calculus of enumerative geometry. (It would take about half a century from that time before Bartel Leendert van der Waerden (1903–1996) could give the first general results in this direction.) Even if Study could present a partial result for his “Habilitation” in June 1885 at Leipzig, he was not able to advance in this direction, turned away in anger from Klein, and also left Saxony for Prussia in 1888. He went to Marburg where he at least received a grant, which, however, was no permanent solution. So Study decided in 1893 to move to the United States for the winter term 1893/94 as lecturer at Johns Hopkins University

– which also was a temporal position only, as he had to admit to Engel in a letter. (For details cf. [15, Sect. 1].)

On January 12, 1894 Engel urgently wrote to Hilbert [7, Sign.: Cod. Ms. D. Hilbert 94, 5]:

“Lieber Freund!

Ich habe eben von Study einen Brief erhalten, der mich ordentlich erschreckt hat. Er hat sich offenbar in der letzten Zeit körperlich sehr schlecht befunden. Dadurch und durch ungünstige Nachrichten über das Befinden der Seinigen [...] ist seine Stimmung furchtbar gedrückt [...].

Unter diesen Umständen möchte ich gerne authentische Auskunft darüber haben, ob für ihn wirklich in Königsberg Aussichten sind. In Halle hörte ich, dass Study, Minkowski und Burkhardt der Reihe nach vorgeschlagen wären und Wiener erzählte, Althoff habe ihm gesagt, er hoffe Study nach Königsberg zu bringen.”

(= “Dear friend!

I have just received a letter from Study that really frightened me. Apparently he has been in very poor physical condition lately. Because of this and because of unfavorable news about the condition of his family [...], his mood is terribly depressed, [...].

Under these circumstances I would like to have authentic information about whether there really are prospects for him in Königsberg. In Halle I heard that Study, Minkowski and Burkhardt had been proposed one after the other, and Wiener said that Althoff had told him he hoped to bring Study to Königsberg.”)

Hilbert reacted immediately on January 14, 1894 and informed Engel that things were a little bit different, – “some complex personnel issues”, as Blumenthal writes – but that his negotiations with the notorious Prussian Ministerial Director Friedrich Althoff (1839–1908) had resulted in a perspective for Study, even though not at Königsberg [5, Sign.: NE180610]:

“Lieber Engel.

Die Lage Studys finde auch ich höchst beklagenswerth. Die hiesige Professur ist bereits durch Minkowski, der an erster Stelle vorgeschlagen war, besetzt; doch bieten sich jetzt um so sichrere Aussichten für Study in Bonn. Study war nämlich an zweiter Stelle hier vorgeschlagen und Althoff wollte nun Minkowski auf unserer beiderseitigen grossen Wunsch hierher berufen, und Study, falls Lipschitz auf den Tausch eingehen sollte, nach Bonn setzen. Die Einwilligung Lipschitz ist nun jedenfalls erfolgt, und möglicherweise befindet sich

Study bereits im Besitz des ministeriellen Schreibens. [...] Alles was ich Dir mittheilte [beruht] auf einer bestimmten Verabredung von Althoff, der mich zu diesem Zweck während der Weihnachtsferien nach Berlin beordert hatte.”

(= “Dear Engel.

I also find Study’s situation extremely deplorable. The professorship here has already been filled by Minkowski, who was suggested in the first place, but there are now all the more secure prospects for Study in Bonn. Study was proposed here in second place and Althoff now wanted to call Minkowski here, at our mutual request, and, if Lipschitz should accept the exchange, he wanted to send Study to Bonn. In any case, Lipschitz’s consent has now been given, and Study may already be in possession of the ministerial letter. [...] [E]verything I communicated to you is based on a certain appointment with Althoff, who had ordered me to Berlin for this purpose during the Christmas break.”)

As a matter of fact, after some weeks Study received a call to an associate professorship at Bonn university, accepted it, and, after a change to Greifswald university as full professor from 1897 to 1904, returned to Bonn as full professor until his retirement in 1927.

4. Hilbert’s Plan of an Associate Professorship for Engel at Göttingen

The last section has given proof of Hilbert’s influence on Althoff and of his successful negotiations with the authorities even at a very early stage of his career. But he soon had to realize that there was someone who could master him:

In 1895 Hilbert accepted a call to Göttingen university as successor to Heinrich Weber (1842–1913) by thus fulfilling a long-lasting wish of his doctoral grandfather Klein. At that time Arthur Schoenflies (1853–1928) held an associate professorship there but left it in 1899 for a full professorship at Königsberg university. According to the tradition at Göttingen Hilbert and Klein alone as the two full professors of mathematics would have made a suggestion for Schoenflies’ succession, and both wanted a geometer, but in detail their interests were diametrically opposed: Klein wished a specialist for “Darstellende Geometrie” (= “descriptive geometry”) whereas Hilbert was interested in a representative of modern geometry – namely Engel.

Klein may have been afraid that Hilbert could have convinced Althoff of his interests, in particular since Klein himself had been an advocate of the Lie school long time ago. So he called other troops for help, namely the entire Göttingen “Philosophische Fakultät” (= “philosophical faculty”) that on the

one hand could be interested in taking part in the process and that on the other hand could more easily be steered by Klein than his colleague Hilbert who formally had equal rights.

In a letter of February 15, 1899 Hilbert had to inform Engel, who had not had any knowledge of Hilbert's ideas before, on the failure of his plans [5, Sign.: NE180614]:

“Dass ich nicht eher an Dich geschrieben habe, hat seinen Grund darin, dass ich hoffte, Dir eine Mittheilung machen zu können, die mir und hoffentlich auch Dir erfreulich gewesen sein würde. Es besteht hier der Usus, der freilich von nun an durch eine Fakultätssitzung in voriger Woche aufgehoben worden ist, dass Extraordinariate ohne Zuthun der Fakultät wiederbesetzt werden. Eine Einigung über die Besetzung war aber zwischen Klein und mir in sofern nicht zu erzielen, als Klein unter allen Umständen einen Mathematiker wünschte, der am Polytechnikum darstellende Geometrie getrieben hat und ich mich gegen diese beschränkende und einseitige Meinung aussprach. Ich habe von vorne herein allen Kollegen, die ein Interesse hatten erklärt, dass, wenn ich zu bestimmen hätte, ich den Versuch machen würde, Dich zu gewinnen; in der That hätte ich, wenn es gelungen wäre, Dich zur Uebersiedlung hierher als Extraordinarius zu bewegen, dies als einen grossen Gewinn für unsere Universität angesehen.”

(= “The reason I did not write to you earlier is because I hoped to be able to give you a message that would have been pleasant to me and, hopefully, to you too. There is a custom here, which has now been repealed by a faculty meeting last week however, that associate professorships can be filled again without the involvement of the faculty. However, an agreement on the occupation could not be reached between Klein and me in so far as Klein wanted a mathematician under all circumstances who did descriptive geometry at a Polytechnic and I spoke out against this limiting and one-sided opinion. Right from the start I explained to all colleagues who were interested that if I had to decide, I would try to win you over; in fact, if one had succeeded in persuading you to move here as an associate professor, I would have regarded this as a great gain for our university.”)

The transfer of the right to propose the candidates for associate professorships to the “Philosophische Fakultät”, however, turned out to be a Pyrrhic victory for Klein: His favorite candidate, Georg Scheffers (1866–1945), proved being not suitable for the position during his performance visit, and Gino Fano (1871–1952), Klein's second choice, accepted a professorship at Messina

instead. In the end, Friedrich Schilling (1868–1950) got the position, but left so little impression during his five years stay at Göttingen that he is not mentioned in the list of professors [11, pp. 120–128].

Even if these events delayed Lie theory from being pursued more intensively in Göttingen, the following letters between Engel and Hilbert disclose the prospective payment for an associate professor of mathematics at Göttingen: In his reply to Hilbert's letter quoted above Engel posed the question whether the position would have been financially attractive [7, Sign.: Cod. Ms. D.Hilbert 94, 10/1]. And he was really impressed [7, Sign.: Cod. Ms. D.Hilbert 94, 11] when Hilbert informed him that after some time he would have received 4000 Marks basic salary plus 540 Marks additionally for housing plus at least 1500 Marks for his teaching activities ("Collegengelder", Hilbert made a Freudian(?) error here and wrote "Collegengelder", "colleagues' money") [5, Sign.: NE180615].

Ending this article, a question asked by Heinrich Wefelscheid is answered: According to the official conversion factor of Deutsche Bundesbank these about 6000 Marks would be equivalent to only 42000 Euros of 2021, but this factor is calculated for the mere cost of living. Engel could almost have bought a house from one annual pay, or, putting things differently, it was about three times as high as that of a teacher at a primary school.

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