A Unified Approach to the Decomposition Theorems in Baer $*$ -Rings

Zbigniew Burdak, Marek Kosiek, Patryk Pagacz^{ }, and
Marek Słociński

Abstract. The aim of the paper is to generalize decomposition theorems showed in Bagheri-Bardi et al. (Linear Algebra Appl 583:102–118, 2019; Linear Algebra Appl 539:117–133, 2018) by a unified approach. We show a general decomposition theorem with respect to a hereditary property. Then the vast majority of decompositions known in the algebra of Hilbert space operators is generalized to elements of Baer $*$ -rings by this theorem. The theorem yields also results which are new in the algebra of bounded Hilbert space operators. Additionally, the model of summands in Wold–Słociński decomposition is given in Baer $*$ -rings.

Mathematics Subject Classification. 16W10.

Keywords. Baer $*$ -rings, wold decomposition, canonical decomposition.

1. Preliminaries

In the recent papers [1, 2] the authors noticed the important role of an algebraic structure in several results on decompositions in the algebra of bounded linear operators on Hilbert space. As a consequence they manage to generalize those results to Baer $*$ -rings. We show a general decomposition theorem which yields the vast majority of decompositions with respect to hereditary properties. In particular, for the algebra of bounded Hilbert space operators they imply known decompositions as well as some new. Since our proofs are purely algebraic, the results are formulated in Baer $*$ -rings.

Research was supported by the Ministry of Science and Higher Education of the Republic of Poland.

Let R be a $*$ -ring with a unity 1 and let $\tilde{R} \subset R$ denotes the set of all projections (self-adjoint idempotents). Further $S^r := \{x \in R : sx = 0 \text{ for all } s \in S\}$ and similarly defined S^l are the right and the left annihilator of $S \subset R$. Recall that R is a Rickart $*$ -ring if the right annihilator of each element is the right principal ideal generated by a projection. The ring R is called a Baer $*$ -ring if this property extends on subsets. Since R is a $*$ -ring, the left annihilator of each element is also the left principal ideal generated by a projection. For a Rickart $*$ -ring the set $\tilde{R}$ is a lattice, for a Baer $*$ -ring the lattice $\tilde{R}$ is complete. Since the projection generating annihilator of x in Rickart $*$ -ring is unique, we may denote $\{x\}^l = R(1 - [x])$ where $[x] \in \tilde{R}$ is called the left projection of x . The projection $[x]$ is the minimal one satisfying $[x]x = x$ and $\{x\}^l = \{[x]\}^l$. It follows that $\{x\}^r = (1 - [x^*])R$ and $[x^*]$ is the right projection of x (the minimal one satisfying $x[x^*] = x$ and $\{x\}^r = \{[x^*]\}^r$).

For any $p \in \tilde{R}$ the set pRp is a ring with the unity p and it is called a corner of R . If R is a Rickart $*$ -ring or a Baer $*$ -ring then their corners are of the same type.

Definition 1.1. Let $x \in R$ and $p \in \tilde{R}$.

- The compression of x to p is $pxp \in pRp$.
- The element x is p invariant if $(1 - p) \in \{xp\}^l$. Then $xp (= pxp)$ is the compression of x to p .
- The projection p decomposes x between two summands

$$x = xp + x(1 - p) = pxp + (1 - p)x(1 - p)$$

if and only if x is p and $1 - p$ invariant.

All the above is defined for a subset $S \subset R$ by means that the respective condition holds for any $x \in S$.

Note that p decomposes S if and only if $px = xp$ for any $x \in S$. In other words, $p \in S'$ (the commutant of S).

More generally we can consider a unity decomposition (factorization) $\{p_i\}_1^n \subset \tilde{R}$, i.e. a set of pairwise orthogonal projections such that $\sum_{i=1}^n p_i = 1$. In Baer $*$ -rings a unity decomposition may be infinite where $\sum_{i=1}^\infty p_i := \sup\{p_1 + \dots + p_i : i \in \mathbb{Z}_+\}$. A sequence of pairwise orthogonal projections $\{p_i\}_{i=1}^\infty$ (possibly $n = \infty$ in Baer $*$ -rings) decomposes x if $\{p_i\}_{i=1}^\infty \subset \{x\}'$ and $x = \sum_{i=1}^\infty xp_i$ ($= x \sup\{p_1 + \dots + p_i : i \in \mathbb{Z}_+\}$ for $n = \infty$.) If a unity decomposition $\{p_i\}_1^n \subset Z(R)$ (centre of R), then any $x \in R$ may be decomposed as $x = x \sum_{i=1}^n p_i = \sum_{i=1}^n xp_i$. Hence $R = \sum_{i=1}^n p_i R p_i$ is a decomposition of R . It is clear that only projections in the centre provide decompositions of R .

2. Decomposition

Since now on we assume R to be a Baer $*$ -ring. Recall from [9, Theorem 20] or [3, Proposition 4.5]:

Theorem 2.1. *The commutant of a $*$ -subset of a Baer $*$ -ring is a Baer $*$ -subring with unambiguous sups and infs.*

In preliminaries we defined $\sum_{i=0}^{\infty} xp_i := x \sum_{i=0}^{\infty} p_i$ for a set of pairwise orthogonal projections $\{p_i\}$. However, if xp_i are projections, then they are pairwise orthogonal and the left hand sum makes sense on its own. Hence, we need to check that the definition causes no ambiguity. It follows by Corollary 2.2(1) provided we check that if xp_i are projections, then x is a projection commuting with all p_i -s. Indeed, $(xp_i)^2 = xp_i = p_i x^*$ yields $xp_i = p_i xp_i$ and $x^2 p_i = xp_i$. Hence, $\{p_i\}_{i \geq 0} \subset \{x^2 - x\}^r = \{[(x^2 - x)^*]\}^r$ and so $1 = \sup\{p_i\} \leq 1 - [(x^2 - x)^*]$ implies $x^2 = x$. Similarly one can check that $x^* = x$.

The next result follows from Theorem 2.1 and [2, equality (1)]. Precisely, [2, equality (1)] is showed for isometries, but it extends to projections.

Corollary 2.2. *Infinite sums admit the following properties:*

- (1) *If a projection p commutes with pairwise orthogonal projections $\{p_i\}_{i \geq 1}$, then p commutes with $\sum_{i=1}^{\infty} p_i$ and $p \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} pp_i$.*
- (2) *If $p_i q_j = 0$ for any $i, j \in \mathbb{Z}_+$, then $(\sum_{i=1}^{\infty} p_i)(\sum_{j=1}^{\infty} q_j) = 0$ where $\{p_i\}_{i \geq 1}, \{q_j\}_{j \geq 1}$ are sets of pairwise orthogonal projections.*
- (3) *If $\{p_{(i,j)}\}_{(i,j) \in \mathbb{Z}_+^2}$ is a set of pairwise orthogonal projections, then*

$$\sum_{(i,j) \in \mathbb{Z}_+^2} p_{(i,j)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{(i,j)}.$$

If x, y commute, then not necessarily $[x]$ commutes with y . Indeed, a non unitary isometry y commutes with itself but $[y]y = y \neq y[y]$.

Lemma 2.3. *If $x \in \{y, y^*\}'$, then $[x] \in \{y, y^*, [y]\}'$.*

Proof. Since y commutes with x we get

$$0 = xy - yx = [x]xy - y[x]x = [x]yx - y[x]x = ([x]y - y[x])x.$$

In other words, $[x]y - y[x] \in \{x\}^l = R(1 - [x])$ and so

$$[x]y - y[x] = ([x]y - y[x])(1 - [x]) = [x]y - [x]y[x].$$

Reducing $[x]y$ we get $y[x] = [x]y[x]$. Since x commutes also with y^* , by similar arguments $y^*[x] = [x]y^*[x]$. Hence $[x]y = (y^*[x])^* = ([x]y^*[x])^* = [x]y[x] = y[x]$. Since $[x]$ is selfadjoint it commutes also with y^* .

We have showed that if $x \in \{y, y^*\}'$, then $[x] \in \{y, y^*\}'$. Replacing x by y and y by $[x]$ we get that if $y \in \{[x]\}'$, then $[y] \in \{[x]\}'$. Since we have already showed that $y \in \{[x]\}'$, the proof is complete. $\square$

By [1, Lemma 3.10] and Lemma 2.3 we get

Corollary 2.4. *Let p be a selfadjoint element commuting with an arbitrary x . Then p commutes with $[x]$, x commutes with $[p]$ and $[x]$ commutes with $[p]$. Moreover, if p is a projection, then $[px] = p[x]$.*

Let us generalize a definition of a hereditary property, which is known in the algebra of Hilbert space operators, to Baer $\ast$ -rings. The property may concern a single element like unitarity or several elements like commutativity.

Definition 2.5. Let $\mathcal{P}$ denotes a property of elements in a Baer $\ast$ -ring.

- The property $\mathcal{P}$ is called hereditary if for any $\{x_k : k \in K\} \subset R$ having the property $\mathcal{P}$ and any $p \in \tilde{R} \cap \{x_k : k \in K\}'$ the set of compressions $\{px_k : k \in K\}$ have the property $\mathcal{P}$ relative to the corner ring pRp .
- A set of elements $\{x_k : k \in K\}$ completely does not have the property $\mathcal{P}$ (is completely non- $\mathcal{P}$) if and only if the only $p \in \tilde{R} \cap \{x_k : k \in K\}'$ such that the set of compressions $\{px_k : k \in K\}$ have the property $\mathcal{P}$ relative to the corner ring pRp is $p = 0$.

In Definition 2.5 the property $\mathcal{P}$ for the set of compressions is considered in the corner pRp , not in R . Let us explain the difference on examples. Let $u \in R$ be unitary, so $uu^* = u^*u = 1$. Let p be a projection commuting with u . Then $(pu)^*pu = pu(pu)^* = p$. Since p is a unity in the corner pRp the element pu is unitary relative to pRp . However, if $p \neq 1$ then pu is not unitary relative to R . Some properties means the same relative to R and pRp . Then the ambiguity appears in the second part of Definition 2.5. As an example, if $x \in pRp$ is normal relative to pRp (i.e. $xx^* = x^*x$), then it is normal relative to R . On the other hand, let $x \in pRp$ be completely non-normal relative to pRp . Thus $1 - p$ commutes with x and $(1 - p)x = 0$, so $(1 - p)x$ is normal relative to $(1 - p)R(1 - p)$. Hence x is not completely non-normal relative to R for $p \neq 1$.

Similarly like examples above, many properties may be defined by algebraic expressions. Let us introduce some description which may be used to the most of hereditary properties.

Definition 2.6. Let it be given a Baer $\ast$ -ring R . Let R^n (where n is not necessarily finite) be an R -module with natural operations $\mathbf{x} + \mathbf{y} = \{x_k + y_k\}_k$, $a\mathbf{x} = \{ax_k\}_k$, $\mathbf{x}a = \{x_ka\}_k$ for any $\mathbf{x} = \{x_k\}_k, \mathbf{y} = \{y_k\}_k \in R^n$ and $a \in R$.

Let $\mathbf{F} = \{F_i\}_{i \in I}$ be a family of functions $F_i : R^n \mapsto R$ such that

$$F_i(q\mathbf{x}) = qF_i(\mathbf{x}) = F_i(\mathbf{x})q \quad (2.1)$$

for any $\mathbf{x} \in R^n$ and any $q \in \tilde{R} \cap \mathbf{x}'$, where $\mathbf{x}'$ is the commutant of $\mathbf{x}$ viewed as the commutant of a subset of R .

We say that $\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to R if $\mathbf{x} \in \bigcap_{i \in I} \ker F_i$.

The property $\mathcal{P}_{\mathbf{F}}$ should be viewed as a property of elements in R rather than a property of an element in R^n . From (2.1) it is a hereditary property. Moreover, by the condition (2.1) functions F_i preserve commutativity with q . The functions defined using algebraic expressions of elements in $\mathbf{x}$, their adjoints or possibly operations of taking the left or the right projection satisfy (2.1).

Now we state the main theorem of the paper. Let us explain the reason why in the result below, $p\mathbf{x}$ is claimed to have the property $\mathcal{P}_{\mathbf{F}}$ relative to the both R and pRp , but $(1-p)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(1-p)R(1-p)$ only. By Definition 2.6 clearly $\mathbf{x} \in pRp$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to pRp if and only if it has the property $\mathcal{P}_{\mathbf{F}}$ relative to R . On the other hand, note that by (2.1) we get $F_i(0) = 0$. Hence any $\mathbf{x} \in pRp$ ($p \neq 1$) is not completely non- $\mathcal{P}_{\mathbf{F}}$ relative to R since $(1-p)\mathbf{x} = 0$.

Theorem 2.7. *Let R be a Baer $*$ -ring and $\mathcal{P}_{\mathbf{F}}$ be a property defined by a family $\mathbf{F}$ as in Definition 2.6.*

For any $\mathbf{x} \in R^n$ there is a unique projection $p \in \mathbf{x}'$ such that:

- $p\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to R and pRp ,
- $(1-p)\mathbf{x}$ completely does not have the property $\mathcal{P}_{\mathbf{F}}$ relative to $(1-p)R(1-p)$.

Proof. Let $p = \sup\{q \in \tilde{R} \cap \mathbf{x}', q \leq 1 - [F_i(\mathbf{x})] \text{ for } i \in I\}$. Since projections are selfadjoint, $\tilde{R} \cap \mathbf{x}' = \tilde{R} \cap (\mathbf{x} \cup \mathbf{x}^*)'$. Hence, by Theorem 2.1 $p \in (\mathbf{x} \cup \mathbf{x}^*)'$. On the other hand $p \leq 1 - [F_i(\mathbf{x})]$ yields $p \in \{[F_i(\mathbf{x})]\}^l = \{F_i(\mathbf{x})\}^l$ and so $F_i(p\mathbf{x}) = pF_i(\mathbf{x}) = 0$ for all $i \in I$. In conclusion $p\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to R and so relative to pRp .

Let $r \in \mathbf{x}'$ be a projection such that the compression $r\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$. Let us show that $r \leq p$. The element $p+r$ is not necessarily an idempotent, but it is selfadjoint, it commutes with elements in $\mathbf{x}$, and by the assumptions,

$$(p+r)F_i(\mathbf{x}) = pF_i(\mathbf{x}) + rF_i(\mathbf{x}) = F_i(p\mathbf{x}) + F_i(r\mathbf{x}) = 0$$

for any $i \in I$. Hence $F_i(\mathbf{x}) \in \{p+r\}^r = \{[p+r]\}^r$ yields $[p+r] \in \{F_i(\mathbf{x})\}^l = \{[F_i(\mathbf{x})]\}^l$ for any $i \in I$. Consequently $[p+r] \leq 1 - [F_i(\mathbf{x})]$. On the other hand, by Corollary 2.4, $[p+r] \in \mathbf{x}'$. Eventually, $[p+r]$ is in the set of which supremum is equal to p , so $[p+r] \leq p$. Hence $1-p \in \{[p+r]\}^r = \{p+r\}^r$ and consequently $0 = (p+r)(1-p) = r(1-p)$ yields $r \leq p$.

Let r be a projection in $(1-p)R(1-p)$ commuting with $(1-p)\mathbf{x}$ such that $r(1-p)\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$. Since projections in a corner are precisely projections in the whole ring majorised by the projection generating the corner we have $r \leq 1-p$. Hence, since $(1-p) \in \mathbf{x}'$, we get $r\mathbf{x} = r(1-p)\mathbf{x} = (1-p)\mathbf{x}r = \mathbf{x}(1-p)r = \mathbf{x}r$. Consequently, $r \in \mathbf{x}'$. Moreover $0 = F_i(r(1-p)\mathbf{x}) = F_i(r\mathbf{x})$ for any $i \in I$. Hence $r\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$, so $r \leq p$ which by $r \in (1-p)R(1-p)$ yields $r = 0$. Thus $(1-p)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$.

For uniqueness of p assume that r is a projection that decomposes $\mathbf{x}$ between objects having the property $\mathcal{P}_{\mathbf{F}}$ and completely not having it. Since $r\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$, by the previous part of the proof $r \leq p$. Hence p and r commute, so $p(1-r) = p-r \leq 1-r$ is a projection in $\mathbf{x}'$, it belongs to the corner $(1-r)R(1-r)$ and

$$F_i((p-r)\mathbf{x}) = (p-r)F_i(\mathbf{x}) = pF_i(\mathbf{x}) - rF_i(\mathbf{x}) = F_i(p\mathbf{x}) - F_i(r\mathbf{x}) = 0$$

for any $i \in I$. In other words, the compression of $(1 - r)\mathbf{x}$ given by $(p - r)$ have the property $\mathcal{P}_{\mathbf{F}}$. However, by assumption on r the only compression of $(1 - r)\mathbf{x}$ having the property $\mathcal{P}_{\mathbf{F}}$ is the trivial one. In conclusion $p - r = 0$, so p is unique. $\square$

The uniqueness of the projection p in Theorem 2.7 implies that p and $1 - p$ are the maximal projections with the respective properties. Precisely:

Corollary 2.8. *Let R be a Baer $*$ -ring, $\mathcal{P}_{\mathbf{F}}$ be a property defined by a family $\mathbf{F}$ as in Definition 2.6, $\mathbf{x} \in R^n$ and $p \in \mathbf{x}'$ be the unique projection obtained in Theorem 2.7. Then*

- p is the maximal projection commuting with $\mathbf{x}$ such that the compression $p\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to R and pRp ,
- $1 - p$ is the maximal projection commuting with $\mathbf{x}$ such that the compression $(1 - p)\mathbf{x}$ completely does not have the property $\mathcal{P}_{\mathbf{F}}$ relative to $(1 - p)R(1 - p)$.

Proof. The maximality of p follows directly from its definition in the proof of Theorem 2.7.

For the maximality of $1 - p$ consider an arbitrary $q \in \mathbf{x}'$ such that $(1 - q)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(1 - q)R(1 - q)$. Let r be the unique projection obtained by Theorem 2.7 for the compression $q\mathbf{x}$ in the corner ring qRq , so $r q \mathbf{x} = r \mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to qRq and $r q R q r = r R r$ and $(q - r)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(q - r)R(q - r)$. We may use $r\mathbf{x}$ as a compression since $r \in \mathbf{x}'$. Indeed, since $r \leq q$ and r commutes with $q\mathbf{x}$ we get $r\mathbf{x} = r q \mathbf{x} = q \mathbf{x} r = \mathbf{x} q r = \mathbf{x} r$.

Clearly, $r\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ also relative to R . By the maximality of p we get $r \leq p$ and so $p - r$ is a projection commuting with $\mathbf{x}$. Since $p - r \leq p$ and the property $\mathcal{P}_{\mathbf{F}}$ is hereditary the compression $(p - r)\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to $(p - r)R(p - r)$. On the other hand, $p - r \leq 1 - r$. If we show that $(1 - r)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(1 - r)R(1 - r)$, we get $p - r = 0$. Consequently $p = r \leq q$ yields $1 - q \leq 1 - p$ which proves the maximality of $1 - p$.

It has left to show that $(1 - r)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(1 - r)R(1 - r)$. Assume s is a projection commuting with $\mathbf{x}$, majorized by $1 - r$ and such that $s\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ relative to sRs . Note that qsq is not necessarily a projection, but it is selfadjoint. Moreover, since $q, s \in \mathbf{x}'$ we get $qsq \in \mathbf{x}'$ and by Corollary 2.4 also $[qsq] \in \mathbf{x}'$. Moreover, by (2.1) we get

$$qsqF_i(\mathbf{x}) = qsF_i(q\mathbf{x}) = qsF_i(\mathbf{x})q = qF_i(s\mathbf{x})q = q0q = 0$$

for any $i \in I$. Hence $\{F_i(\mathbf{x})\}_{i \in I} \subset \{qsq\}^r = \{[qsq]\}^r$ and consequently

$$F_i([qsq]\mathbf{x}) = [qsq]F_i(\mathbf{x}) = 0$$

for any $i \in I$. In other words $[qsq]\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ in $[qsq]R[qsq]$. On the other hand, $qsqr = qsrq = q0q = 0$ yields $r \in \{qsq\}^r = \{[qsq]\}^r$ and

consequently $[qsq] < q - r$. Since $(q - r)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(q - r)R(q - r)$ we get $[qsq] = 0$ which yields $0 = qsq = qs^2q = (qs)(qs)^*$. Since the involution in Baer $*$ -rings is proper we get $qs = 0$. Consequently, $s \leq 1 - q$ and since $s\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$ in sRs and $(1 - q)\mathbf{x}$ is completely non- $\mathcal{P}_{\mathbf{F}}$ relative to $(1 - q)R(1 - q)$ we get $s = 0$. $\square$

One can consider $\mathbf{F}_J = \{F_i\}_{i \in J}$ where $J \subset I$ and the corresponding property $\mathcal{P}_{\mathbf{F}_J}$. Then, by Theorem 2.7 there is the respective projection p_J . In the following Proposition 2.9 we show that projections p_J corresponding to various sets J commute to each other. By this commutativity we are able to get decompositions among more than two summands and so we gain more detailed descriptions. In the next section we show several applications of this fact. More precisely, we extend several classical results known in the algebra of bounded linear operators on a Hilbert space $\mathcal{B}(H)$ to Baer $*$ -rings. Moreover, we get some new results also in $\mathcal{B}(H)$.

Proposition 2.9. *Suppose we have functions $F_i : R^n \mapsto R$ for $i \in I_1 \cup I_2$, and the corresponding families $\mathbf{F} = \{F_i\}_{i \in I_1 \cup I_2}$, $\mathbf{F}_1 = \{F_i\}_{i \in I_1}$, $\mathbf{F}_2 = \{F_i\}_{i \in I_2}$.*

Let $\mathbf{x} \in R^n$ and let p_{12} , p_1 , p_2 be projections decomposing $\mathbf{x}$ with a correspondence to the properties $\mathcal{P}_{\mathbf{F}}$, $\mathcal{P}_{\mathbf{F}_1}$, $\mathcal{P}_{\mathbf{F}_2}$, respectively, as in Theorem 2.7.

The projection p_1 commutes with p_2 and $p_1p_2 = p_{12}$.

Proof. It is clear that $p_{12} \leq p_1$ by which they commute and so $p_1(1 - p_{12})$ is a well defined projection. Consider an arbitrary projection $q \in \mathbf{x}'$ such that $q \leq p_1(1 - p_{12})$. Note that if $q \leq 1 - [F_j(\mathbf{x})]$ for all $j \in I_1 \cup I_2$, then, since p_{12} is defined as the supremum of such projections (see proof of Theorem 2.7), we get $q \leq p_{12}$. However, we assumed $q \leq p_1(1 - p_{12})$, so in particular $q \leq 1 - p_{12}$. Hence, either $q = 0$ or there is $j_0 \in I_1 \cup I_2$ such that $q \not\leq 1 - [F_{j_0}(\mathbf{x})]$ (equivalently $q[F_{j_0}(\mathbf{x})] \neq 0$). On the other hand, since $q \leq p_1$ the compression $q\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}_1}$ so we get $0 = F_i(q\mathbf{x}) = qF_i(\mathbf{x})$ for any $i \in I_1$. In other words $q \in \{F_i(\mathbf{x})\}^l = \{[F_i(\mathbf{x})]\}^l$ which is equivalent to $q \leq 1 - [F_i(\mathbf{x})]$ for any $i \in I_1$. Hence, if $q \neq 0$ there is $j_0 \in (I_1 \cup I_2) \setminus I_1 \subset I_2$ such that $q[F_{j_0}(\mathbf{x})] \neq 0$. Hence $F_{j_0}(q\mathbf{x}) \neq 0$. Indeed, if $0 = F_{j_0}(q\mathbf{x}) = qF_{j_0}(\mathbf{x})$, then by the left annihilators equality $q[F_{j_0}(\mathbf{x})] = 0$ which is not true. Since q was arbitrary, $p_1(1 - p_{12})\mathbf{x}$ completely does not have the property $\mathcal{P}_{\mathbf{F}_2}$. Consequently $p_1(1 - p_{12}) \leq 1 - p_2$, so $0 = p_2p_1(1 - p_{12}) = p_2p_1 - p_2p_1p_{12} = p_2p_1 - p_{12}$ where the last equality follows by $p_{12} \leq p_1, p_{12} \leq p_2$. Hence $p_2p_1 = p_{12}$ and since the product of projections is a projection if and only if they commute we get $p_1p_2 = p_2p_1$. $\square$

Note that if $I_1 \cap I_2 \neq \emptyset$, then we may consider $\hat{\mathbf{F}} = \{F_i\}_{i \in I_1 \cap I_2}$ and $\hat{\mathbf{F}}_1 = \{F_i\}_{i \in I_1 \setminus I_2}$ and the corresponding projections $\hat{p}, \hat{p}_1$. Clearly $p_1 \leq \hat{p}, p_2 \leq \hat{p}, p_1 \leq \hat{p}_1$ and $p_1 = \hat{p}_1\hat{p}$. Consequently $p_1p_2 = \hat{p}_1\hat{p}$. In other words it is enough to consider disjoint sets I_1, I_2 or more precisely disjoint families $\mathbf{F}_1, \mathbf{F}_2$.

3. Applications

In this section we derive several decompositions from Theorem 2.7. The condition $pF_i(\mathbf{x}) = F_i(p\mathbf{x})$ makes $\mathcal{P}_{\mathbf{F}}$ a hereditary property. By hereditariness, $(1-p)\mathbf{x}$ completely does not have the property $\mathcal{P}_{\mathbf{F}}$ relative to $(1-p)R(1-p)$ if and only if p is the maximal (so unique) projection such that $p\mathbf{x}$ has the property $\mathcal{P}_{\mathbf{F}}$. Hence the existence of the maximal projection in some statements in this section is equivalent to the existence of the corresponding decomposition. Let us give a little leeway that for non hereditary property the maximality does not imply the uniqueness of the corresponding decomposition – there may exist different decompositions between a part having the property and the one completely not having it. The reason is that the maximality of the projection may be considered only as a maximal element of some chain without uniqueness. For example, the property of being a bilateral shift is non hereditary. There may exist different maximal bilateral shift parts of the same unitary operator on a Hilbert space. We skip details since it requires Spectral Theorem and is far from the subject of the article.

In this section we recall or adopt from the algebra of bounded linear operators on Hilbert spaces several properties of Baer $*$ -ring elements. Let us start with the basic ones. Recall that an element $x \in R$ is called normal, a partial isometry, an isometry, a unitary element if $xx^* = x^*x$, $xx^*x = x$, $x^*x = 1$, $x^*x = xx^* = 1$, respectively.

Theorem 3.1. *For any x in a Baer $*$ -ring R there are the maximal projections p_n, p_p, p_i, p_u commuting with x such that:*

- $p_n x$ is normal,
- $p_p x$ is a partial isometry,
- $p_i x$ is an isometry,
- $p_u x$ is unitary

all relative to the proper corner ring.

Proof. The result follows from Theorem 2.7 where:

- $F(x) = xx^* - x^*x$ and $\mathbf{x} = x$ for p_n ,
- $F(x) = x - xx^*x$ and $\mathbf{x} = x$ for p_p ,
- $F(x, y) = y - x^*x$ and $\mathbf{x} = (x, 1)$ for p_i ,
- $F_1(x, y) = y - x^*x$, $F_2(x, y) = y - xx^*$ and $\mathbf{x} = (x, 1)$ for p_u .

□

Note that, Theorem 3.1 states the respective properties only relative to the corners while Theorem 2.7 yields the result relative to the both, the whole ring and the corresponding corner. Since Theorem 3.1 follows from Theorem 2.7 one may find it strange. However, the interpretation of the property $\mathcal{P}_{\mathbf{F}}$ depends on the ring and in the case of the whole ring it may not be the one considered in Theorem 3.1. In the cases of p_n and p_p the interpretations are the same relative to the whole ring and the corner. Indeed, $p_n x$ is normal

and $p_p x$ is a partial isometry also relative to the whole ring R . In the case of p_i (and similarly p_u) the interpretation differs. Note that $F(p_i(x, 1)) = F(p_i x, p_i) = 0$ yields $(p_i x)^* p_i x = p_i$. Hence, the corresponding property $\mathcal{P}_F$ may be interpreted as $p_i x$ is an isometry only if p_i is the unity which is the case of the corner ring $p_i R p_i$. Relative to the whole ring $\mathcal{P}_F$ means that $p_i x$ is a partial isometry but not an isometry (unless $p_i = 1$).

The next result is formulated for a general element of a Baer $*$ -ring, but it can be viewed in the context of Halmos-Wallen-Foiaş result on power partial isometries [7, Theorem].

Corollary 3.2. *For any x in a Baer $*$ -ring there is a unity decomposition*

$$p_u + p_{pi} + p_{pci} + p_r = 1$$

such that $p_u, p_{pi}, p_{pci}, p_r \in \{x\}'$ and

- $p_u x$ is unitary,
- $p_{pi} x$ is a pure isometry,
- $p_{pci} x$ is a pure co-isometry,
- $p_r x, p_r x^*$ are completely non-isometric.

Proof. Indeed, let p_u, p_i be as in Theorem 3.1 for x while p_{ci} be an isometric projection calculated for x^* in Theorem 3.1. By Proposition 2.9 $p_i p_{ci} = p_{ci} p_i = p_u$. Hence $p_{pi} = p_i(1 - p_u)$ and $p_{pci} = p_{ci}(1 - p_u)$ are well defined and orthogonal to each other. It remains to define $p_r = 1 - p_u - p_{pi} - p_{pci}$. Since p_r is orthogonal to p_i and p_{ci} it compress x and x^* to completely non-isometric elements. $\square$

For power partial isometries the result is finer, the last part is described as truncated shifts [1, 7].

Theorem 2.7 may be successfully applied to pairs (more generally sets) of elements. It works well, nevertheless the considered property describes a relation between/among elements (f.e. commutativity) or characterizes elements (f.e. normality). The following result on the double commutativity may be modified to the commutativity. Recall that elements in a pair (x, y) doubly commute if $x \in \{y, y^*\}'$.

Theorem 3.3. *For any pair (x, y) of arbitrary elements in R there is a unique projection $p \in \{x, y\}'$ such that elements in (px, py) doubly commute and elements in $((1 - p)x, (1 - p)y)$ completely do not doubly commute.*

Proof. It follows from Theorem 2.7 for $F_1(x, y) = xy - yx, F_2(x, y) = xy^* - y^*x$. $\square$

Much wider class are compatible pairs. The concept of compatibility was introduced for isometries on Hilbert spaces by Horák and Müller in [8]. It naturally extends to general pairs of elements in Baer $*$ -rings.

Definition 3.4. A pair (x, y) is compatible if $\{[x^m] : m \geq 1\} \subset \{[y^n] : n \geq 1\}'$.

The following corollary is obvious for isometries in $\mathcal{B}(H)$, while in Baer $*$ -rings it follows from Lemma 2.3.

Corollary 3.5. *Any doubly commuting pair is compatible.*

An example of compatible, completely non doubly commuting pair is (x, x) where x is a non unitary isometry. Other examples can be found in papers on operators on Hilbert spaces [4, 5, 8].

The next result shows a decomposition of an arbitrary pair between a compatible pair and a completely non compatible pair.

Theorem 3.6. *For any $x, y \in R$ there is $p \in \{x, y\}'$ such that (px, py) is a compatible pair and $((1-p)x, (1-p)y)$ is completely non compatible.*

Proof. It follows from Theorem 2.7 for $F_{m,n}(x, y) = [x^m][y^n] - [y^n][x^m]$ for $m, n \in \mathbb{Z}_+$ where $F_{m,n}(px, py) = pF_{m,n}(x, y)$ by Corollary 2.4. $\square$

As a conclusion of Corollary 3.5 and Theorems 3.3 and 3.6 we get a decomposition of an arbitrary pair of elements among three compressions.

Theorem 3.7. *For any pair (x, y) of arbitrary elements in R there are unique projections $p, q \in \{x, y\}'$ where $p \leq q$ such that*

- (px, py) doubly commute,
- $(q(1-p)x, q(1-p)y)$ are compatible, completely non doubly commuting,
- $((1-q)x, (1-q)y)$ is completely non compatible.

A very rich class of examples of Baer $*$ -rings are bounded operators on Hilbert spaces $\mathcal{B}(H)$. Theorem 3.3 in $\mathcal{B}(H)$ is known. However, to the authors knowledge, compatibility was defined only for isometries so far. Hence Theorem 3.6 is new also in $\mathcal{B}(H)$.

The compatibility does not imply the commutativity. We give an example. Recall that two projections are equivalent if there is a partial isometry having them as the left and the right projection.

Example 3.8. Let $p_{i,j}$ be a set of pairwise orthogonal and equivalent projections for $i, j = 1, 2, 3$ and let $x_{i,j}, y_{i,j}$ be partial isometries such that $[x_{i,j}^*] = [y_{i,j}^*] = p_{i,j}$, $[x_{i,j}] = p_{i+1,j}$, $[y_{i,j}] = p_{i,j+1}$ for $i, j = 1, 2$. Define

$$x = \sum_{i,j=1,2} x_{i,j}, \quad y = \sum_{i,j=1,2} y_{i,j}.$$

Let us check that x, y are compatible. From the equality $[x^*] = \sum_{i,j=1,2} [x_{i,j}^*]$ and by similar equalities for $[x], [y^*], [y]$ we get

$$[x^*] = [y^*] = \sum_{i,j=1,2} p_{i,j}, \quad [x] = \sum_{\substack{i=2,3 \\ j=1,2}} p_{i,j}, \quad [y] = \sum_{\substack{i=1,2 \\ j=2,3}} p_{i,j}.$$

By orthogonality of projections $p_{i,j}$ one may check that $x^2 = \sum_{j=1,2} x_{2,j}x_{1,j}$, $y^2 = \sum_{i=1,2} y_{i,2}y_{i,1}$ and $x^n = y^n = 0$ for $n \geq 3$. Hence $[x^2] = p_{3,1} + p_{3,2}$, $[y^2] = p_{1,3} + p_{2,3}$, $[x^n] = [y^n] = 0$ for $n \geq 3$ which yields compatibility.

Consider $\bar{y} = y_{1,1} + y_{1,2} + uy_{2,1} + y_{2,2}$ where $u \neq p_{2,2}$ is a partial isometry such that $[u] = [u^*] = p_{2,2}$ (in other words compression of u to $p_{2,2}Rp_{2,2}$ is unitary, not the unity). Note that $[y^n] = [\bar{y}^n]$ for any $n \geq 0$ so $x, \bar{y}$ are compatible as well as x, y are. Let us show that at most one of pairs x, y and $x, \bar{y}$ may commute. Indeed $yp_{1,1} = y_{2,1}x_{1,1}p_{1,1} \neq uy_{2,1}x_{1,1}p_{1,1} = \bar{y}xp_{1,1}$ while $xyp_{1,1} = x_{1,2}y_{1,1}p_{1,1} = x\bar{y}p_{1,1}$. Hence at most one of equalities $xy = yx, x\bar{y} = \bar{y}x$ may hold.

One may ask about a quaternary decomposition with respect to commutativity and compatibility as in Theorem 3.9 below. The answer is affirmative, but not obvious even in $\mathcal{B}(H)$. It follows from Proposition 2.9.

Theorem 3.9. *For any pair (x, y) of arbitrary elements in R there is a unique unity decomposition among $p_{11}, p_{1,0}, p_{01}, p_{00} \in \{x, y\}'$ such that*

- $(p_{11}x, p_{11}y)$ commute and are compatible,
- $(p_{10}x, p_{10}y)$ commute and are completely non compatible,
- $(p_{10}x, p_{10}y)$ completely do not commute and are compatible,
- $(p_{10}x, p_{10}y)$ completely do not commute and are completely non compatible.

Proof. Let $I_1 = \{1\}$ and $F_1(x, y) = xy - yx$ and $I_2 = \mathbb{Z}_+^2$ and $F_{m,n}(x, y) = [x^m][y^n] - [y^n][x^m]$ as in the proof of Theorem 3.6. Obviously the corresponding properties $\mathcal{P}_{F_1}$ and $\mathcal{P}_{F_2}$ defined as in Theorem 2.7 are the commutativity and the compatibility, respectively. On the other hand, by Proposition 2.9 projections p_{cm}, p_{cp} corresponding to $\mathcal{P}_{F_1}, \mathcal{P}_{F_2}$ commute. Hence $p_{11} = p_{cm}p_{cp}$, $p_{10} = p_{cm}(1 - p_{cp})$, $p_{01} = (1 - p_{cm})p_{cp}$, $p_{00} = (1 - p_{cm})(1 - p_{cp})$ provide the decomposition as required in the statement. $\square$

Let $\mathcal{P}$ be a property characterizing individual elements (f.e. normality). Recall that a set S completely does not have the property $\mathcal{P}$ (f.e. is completely not normal) if for any $0 \neq p \in S'$ there is at least one $x \in S$ such that pxp does not have the property (at least one pxp is not normal). We extend results of Theorem 3.1 on subsets. We show the decomposition with respect to normality. Other results may be proved similarly.

Corollary 3.10. *Let $S \subset R$ where R is a Baer $*$ -ring. There is a maximal projection $p \in S'$ such that pS ($= pSp$) is normal (a set of normal elements).*

Proof. It is enough to take $F_s : R^S \ni \mathbf{x} \mapsto x_s^*x_s - x_sx_s^*$ for any $s \in S$ and apply Theorem 2.7. $\square$

Let us finish this section by a generalization of Wold, Helson-Lowdenslager, Suciú result [10, Theorem 3]. For those reason we extend the concept of a quasi-unitary semigroup of isometries to Baer $*$ -rings.

Definition 3.11. Let G be an abelian group and let $S \subset G$ be a semigroup such that $S \cap S^{-1} = \{1\}$ and $SS^{-1} = G$. Denote by $\{x_s\}_S$ a semigroup of isometries in a Baer $*$ -ring R (i.e. $x_1 = 1, x_sx_r = x_{sr}$).

We call a semigroup $\{x_s\}_S$ quasi-unitary if $\sup\{[x_g^*x_s] : g^{-1}s \notin S^{-1}\} = 1$. A semigroup is purely quasi-unitary if it is quasi-unitary and completely non unitary.

A completely non quasi-unitary group is called strange.

Theorem 3.12. *Let $\{x_s\}_S$ be a semigroup of isometries in a Baer $*$ -ring R . There are projections $p_u, p_{pqu}, p_s \in \{x_s\}'_S$ such that $p_u + p_{pqu} + p_s = 1$ and*

- $p_u x_s$ is unitary for every $s \in S$,
- $\{p_{pqu} x_s\}_S$ is purely quasi-unitary,
- $\{p_s x_s\}_S$ is strange.

Proof. Define $F(\mathbf{x}) = 1 - \sup\{[x_g^*x_s] : g^{-1}s \notin S^{-1}\}$ where $\mathbf{x} \in R^S, \mathbf{x} = \{x_s\}_S$. Note that by Corollaries 2.2 and 2.4, $pF(\mathbf{x}) = F(p\mathbf{x})$ for any projection $p \in \mathbf{x}'$. Hence, by Theorem 2.7 we get a projection p_{qu} which is the maximal one compressing the semigroup to a quasi-unitary semigroup. Similarly like in Corollary 3.10 we consider a family of functions $F_s : R^S \ni \mathbf{x} \mapsto 1 - x_s x_s^*$ and get a projection p_u . It is clear that $p_u \leq p_{qu}$ and so $p_{pqu} = p_{qu}(1 - p_u)$ is a well defined projection compressing the semigroup to a purely quasi-unitary semigroup. Clearly $p_s = 1 - p_{qu}$ compress the semigroup to a strange semigroup. $\square$

4. Multiple Canonical Decomposition

Consider a property of a single element. Assume there is a pair (x, y) such that each of its elements admits a decomposition between the summand having the property and the one completely not having it. We may usually find also a decomposition of the pair (x, y) between the pair having the property and the one completely not having it as in Corollary 3.10 for example. However, the fact that the pair completely does not have the property does not say much about individual elements in the pair. Indeed, consider as an example the property of being normal. A normal element and a completely non-normal element as well as two completely non-normal elements form completely non-normal pairs. Hence a pair completely not having a given property requires finer description. Wold, Helson-Lowdenslager, Suciú result recalled in the previous section is one of the first attempts to give a characterization of this type. The best result would be a quaternary decomposition, as defined:

Definition 4.1. A canonical decomposition of a pair (x, y) with respect to a property $\mathcal{P}$ characterizing single elements x, y is a quaternary decomposition

$$p_{11} + p_{10} + p_{01} + p_{00} = 1$$

where $p_{11}, p_{10}, p_{01}, p_{00} \in \{x, y\}'$ are such that

- each of $p_{11}x, p_{11}y$ has the property $\mathcal{P}$,
- $p_{10}x$ has the property $\mathcal{P}$, $p_{10}y$ completely does not have the property $\mathcal{P}$,
- $p_{01}x$ completely does not have the property $\mathcal{P}$, $p_{01}y$ has the property $\mathcal{P}$,

- each of $p_{00}x, p_{00}y$ completely does not have the property $\mathcal{P}$.

Unfortunately, a general pair may not admit a canonical decomposition. Let us explain why Proposition 2.9 does not work for canonical decompositions. Consider once again the property of being normal. By Proposition 2.9 projections p_x, p_y corresponding (in the sense of Theorem 2.7) to $F_x(x, y) = x^*x - xx^*, F_y(x, y) = y^*y - yy^*$ do commute. Hence $p_x p_y$ is a projection. It can be checked that it is the maximal projection where both compressions are normal. However, $p_x(1 - p_y)$ compress x to a normal element but $p_x(1 - p_y)y$ is not necessarily a completely non-normal element. Indeed, there may exist a projection $0 \neq q \leq p_x(1 - p_y)$ commuting with y where qy is normal but q does not commute with x . To be precise, in the decomposition of x we consider the projection corresponding to $F_x(x) = x^*x - xx^*$ instead of $F_x(x, y) = x^*x - xx^*$. The formula is the same. The difference is that the respective supremum is taken in the set of projections commuting only with x in the first case and with both (x, y) in the second case. Hence the projection corresponding to $F_x(x)$ may majorize the one corresponding to $F_x(x, y)$. Let us formulate a result similar to [6, Corollary (2.3)].

Proposition 4.2. *Let R be a Baer $*$ -ring and let $\mathcal{P}_{\mathbf{F}}$ be a hereditary property defined by a family $\mathbf{F} := \{F_i : R \mapsto R\}_{i \in I}$ satisfying (2.1) as in Definition 2.6.*

Then:

- *there are maximal projections $p_x \in \{x\}'$ and $p_y \in \{y\}'$ such that $p_x x, p_y y$ have the property $\mathcal{P}_{\mathbf{F}}$,*
- *there are maximal projections $q_x, q_y \in \{x, y\}'$ such that $q_x x, q_y y$ have the property $\mathcal{P}_{\mathbf{F}}$,*
- *$q_x \leq p_x, q_y \leq p_y$.*

Moreover, the following conditions are equivalent:

- *(x, y) admits a canonical decomposition with respect to the property $\mathcal{P}_{\mathbf{F}}$,*
- *$p_x, p_y \in \{x, y\}'$,*
- *$p_x = q_x, p_y = q_y$.*

Proof. In fact the first part has been explained before the proposition. Precisely, the existence of p_x, p_y follows from Theorem 2.7 for $\{F_i\}_{i \in I}$. Define

$$F_{i1}(x, y) := F_i(x), \quad F_{i2}(x, y) := F_i(y)$$

for $i \in I$. Then Proposition 2.9 for $I_1 = I \times \{1\}, I_2 = I \times \{2\}$ and $\mathbf{x} = (x, y)$ yields the existence of q_x, q_y which commute. To see that $q_x \leq p_x$ recall from the proof of Theorem 2.7 that

$$p_x = \sup\{q \in \tilde{R} \cap \{x\}' : q < 1 - [F_i(x)]\}.$$

Note that $q_x \leq 1 - [F_{i1}(x, y)] = 1 - [F_i(x)]$ and obviously $q_x \in \tilde{R} \cap \{x\}'$. Hence, q_x belongs to the set above, so $q_x \leq p_x$. Similarly $q_y \leq p_y$.

For the second part, denote by $p_{11} + p_{10} + p_{01} + p_{00} = 1$ the canonical decomposition of the pair (x, y) . If it exists, then $p_x = p_{11} + p_{10}, p_y = p_{11} + p_{01}$

so they commute with both x, y . If $p_x \in \{x, y\}'$ then, similarly as we showed $q_x \leq p_x$ we may show the reverse inequality. If $p_x = q_x, p_y = q_y$ then p_x, p_y commute. One can check that $p_{11} = p_x p_y, p_{10} = p_x(1 - p_y), p_{01} = (1 - p_x)p_y, p_{00} = (1 - p_x)(1 - p_y)$ is the canonical decomposition with respect to the property $\mathcal{P}_F$. $\square$

Recall that any pair of doubly commuting operators in $\mathcal{B}(H)$ admits a canonical decomposition with respect to any hereditary property [6, Corollary 2.4]. Unfortunately, the proof uses Double Commutant Theorem and may not be directly extended to Baer $\ast$ -rings. In the case of Baer $\ast$ -rings such a result, if it is correct, requires a different proof. By Theorem 2.1, if p_x is obtained as a supremum of projections commuting with y , then it commutes with y as well (the notation as in Proposition 4.2). Such condition is used in [2] to show that a pair of doubly commuting isometries admits a canonical decomposition with respect to unitarity. However, it is not the only way. One can imagine a set of projections $p_i \in \{x\}'$ such that $p_i y = y p_{i+1}$, but $p_i \notin \{y\}'$. Then $\sup\{p_i\} \in \{y\}'$.

Recall that the decomposition of an isometry with respect to unitarity is called Wold decomposition and the corresponding canonical decomposition of a pair is called Wold–Słociński decomposition. Such results in Baer $\ast$ -rings are showed in [2]. Recall that an isometry x is called a unilateral shift if the projections $[x^n(1 - [x])]$ are pairwise orthogonal for $n \geq 0$ and $\sum_{n=0}^{\infty} [x^n(1 - [x])] = 1$.

Theorem 4.3. ([2, Theorem 2.4]) *Let x be an isometry in a Baer $\ast$ -ring R . Then there is a unique projection $p_u \in \{x\}'$ such that,*

- *the compression $p_u x$ is unitary and,*
- *the compression $(1 - p_u)x$ is a unilateral shift.*

The existence of Wold decomposition follows from Theorem 3.1. The important advantage of the result in [2] is that a completely non unitary isometry is described as a unilateral shift. The generalization of Wold–Słociński decomposition to Baer $\ast$ -rings [2, Theorem 3.2] is only a decomposition. The models of all summands in $\mathcal{B}(H)$ are known and can be generalized to Baer $\ast$ -rings.

Theorem 4.4. *Let (x, y) be a pair of doubly commuting isometries in a Baer $\ast$ -ring. There are unique $p_{uu}, p_{us}, p_{su}, p_{ss} \in \{x, y\}'$ such that*

$$p_{uu} + p_{us} + p_{su} + p_{ss} = 1$$

and

- *$(p_{uu}x, p_{uu}y)$ is a pair of unitary elements,*
- *$p_{us}x$ is unitary, $p_{us}y$ is a unilateral shift and*

$$p_{us}x = \sum_{i=0}^{\infty} p_{us}x[y^i(1 - [y])],$$

- $p_{su}x$ is a unilateral shift, $p_{su}y$ is unitary and

$$p_{su}y = \sum_{i=0}^{\infty} p_{su}y[x^i(1 - [x])],$$

- $(p_{ss}x, p_{ss}y)$ is a pair of unilateral shifts and

$$p_{ss} = \sum_{m,n \geq 0} [p_{ss}x^m y^n (1 - [x])(1 - [y])].$$

Proof. The existence of the decomposition is showed in [2, Theorem 3.2]. It was not emphasized that it is unique. However, since unitarity is a hereditary property it is unique. A precise proof is a consequence of the uniqueness in Theorem 2.7. Indeed, by Corollary 3.10, there is the maximal (so unique) projection p_{uu} compressing (x, y) to unitary elements (normal isometries are unitary elements). On the other hand the sum $p_{uu} + p_{us}$ is the maximal projection compressing x to a unitary element, so $p_{uu} + p_{us}$ is also unique by Theorem 3.1. Hence p_{us} is unique as well as is, by similar arguments, p_{su} . Consequently also p_{ss} is unique.

Since x, y doubly commute, x commutes with $[y^n]$ for any n as well as y commutes with $[x^m]$ for any $m \geq 0$ by Lemma 2.3.

Let us describe the compression $(p_{us}x, p_{us}y)$ (and similarly $(p_{su}x, p_{su}y)$). By the above, since x commutes also with p_{us} it doubly commutes with $p_{us}y^m(1 - [y])$ and, by Lemma 2.3, it commutes with $[p_{us}y^m(1 - [y])]$. Note that $(p_{us}y)^i(1 - [p_{us}y]) = p_{us}y^i(1 - [y])$. Hence, since $p_{us}y$ is a unilateral shift (relative to $p_{us}Rp_{us}$), $\sum_{i=0}^{\infty} [p_{us}y^i(1 - [y])] = p_{us}$. Since, by Corollary 2.4, $[p_{us}y^i(1 - [y])] = p_{us}[y^i(1 - [y])]$ we get

$$p_{us}x = \sum_{i=0}^{\infty} p_{us}x[p_{us}y^i(1 - [y])] = \sum_{i=0}^{\infty} p_{us}x[y^i(1 - [y])]$$

which is a decomposition of $p_{us}x$.

Let us show the last part. Since $p_{ss}x, p_{ss}y$ are unilateral shifts (relative to $p_{ss}Rp_{ss}$), by Corollary 2.2

$$\begin{aligned} p_{ss} &= p_{ss}^2 = \left(\sum_{m=0}^{\infty} [p_{ss}x^m(1 - [x])] \right) \left(\sum_{n=0}^{\infty} [p_{ss}y^n(1 - [y])] \right) \\ &= \sum_{m,n \geq 0} [p_{ss}x^m(1 - [x])][p_{ss}y^n(1 - [y])]. \end{aligned}$$

It has left to show that

$$[p_{ss}x^m(1 - [x])][p_{ss}y^n(1 - [y])] = [p_{ss}x^m y^n(1 - [x])(1 - [y])]$$

which, by Corollary 2.4, is equivalent to

$$p_{ss}[x^m(1 - [x])][y^n(1 - [y])] = p_{ss}[x^m y^n(1 - [x])(1 - [y])].$$

Note that $x^m(1 - [x]), y^n(1 - [y])$ are partial isometries. Moreover, as $(1 - [x])(1 - [y])$ is a projection, also $x^m y^n(1 - [x])(1 - [y])$ is a partial isometry.

Recall that for a partial isometry $z \in R$ we have the equality $[z] = zz^*$. Hence one can check that

$$\begin{aligned}[x^m(1 - [x])] &= [x^m] - [x^{m+1}], \\ [y^n(1 - [y])] &= [y^n] - [y^{n+1}], \\ [x^m y^n (1 - [x])(1 - [y])] &= ([x^m] - [x^{m+1}])([y^n] - [y^{n+1}])\end{aligned}$$

which finishes the proof. $\square$

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] Bagheri-Bardi, G.A., Elyaspour, A., Esslamzadeh, G.: The role of algebraic structure in the invariant subspace theory. *Linear Algebra Appl.* **583**, 102–118 (2019)
- [2] Bagheri-Bardi, G.A., Elyaspour, A., Esslamzadeh, G.H.: Wold-type decompositions in Baer $\ast$ -rings. *Linear Algebra Appl.* **539**, 117–133 (2018)
- [3] Berberian, S.K.: *Baer $\ast$ -Rings*. Grundlehren der mathematischen Wissenschaften, vol. 195. Springer, Berlin, Heidelberg (1972)
- [4] Burdak, Z., Kosiek, M., Pagacz, P., Słociński, M.: On the commuting isometries. *Linear Algebra Appl.* **516**, 167–185 (2017)
- [5] Burdak, Z., Kosiek, M., Słociński, M.: Compatible pairs of commuting isometries. *Linear Algebra Appl.* **479**, 216–259 (2015)
- [6] Catepillan, X., Ptak, M., Szymanski, W.: Multiple canonical decompositions of families of operators and a model of quasinormal families. *Proc. Am. Math. Soc.* **121**(4), 1165 (2006)
- [7] Halmos, P.R., Wallen, L.J.: Powers of partial isometries. *J. Math. Mech.* **19**, 657–663 (1970)
- [8] Horák, K., Müller, V.: Functional model for commuting isometries. *Czechoslov. Math. J.* **39**(2), 370–379 (1989)
- [9] Kaplansky, I.: *Rings of Operators*. Benjamin, New York (1968)
- [10] Suciú, I.: On the semi-groups of isometries. *Stud. Math.* **30**(1), 101–110 (1968)

Zbigniew Burdak
Department of Applied Mathematics
University of Agriculture
ul. Balicka 253c
30-198 Kraków
Poland
e-mail: rmburdak@cyf-kr.edu.pl

Marek Kosiek, Patryk Pagacz and Marek Słociński
Wydział Matematyki i Informatyki
Uniwersytet Jagielloński
ul. Prof. St. Łojasiewicza 6
30-348 Kraków
Poland
e-mail: Marek.Kosiek@im.uj.edu.pl;
Patryk.Pagacz@im.uj.edu.pl;
Marek.Slocinski@im.uj.edu.pl

Received: September 21, 2020.

Accepted: April 15, 2021.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.