Quantitative Estimates for Nonlinear Sampling Kantorovich Operators

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Abstract. In this paper, we establish quantitative estimates for nonlinear sampling Kantorovich operators in terms of the modulus of smoothness in the setting of Orlicz spaces. This general frame allows us to directly deduce some quantitative estimates of approximation in L^p -spaces, $1 \leq p < \infty$, and in other well-known instances of Orlicz spaces, such as the Zygmund and the exponential spaces. Further, the qualitative order of approximation has been obtained assuming f in suitable Lipschitz classes. The above estimates achieved in the general setting of Orlicz spaces, have been also improved in the L^p -case, using a direct approach suitable to this context. At the end, we consider the particular cases of the nonlinear sampling Kantorovich operators constructed by using some special kernels.

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1. Introduction

The theory of nonlinear integral operators in connection with approximation problems has been started by Musielak (see e.g. [31–34]) and subsequently, it has been extensively developed by Bardaro et al. in [8], and also studied in various papers by other authors (see e.g. [3–6, 9, 10, 16, 17, 22, 30, 38]).

The linear version of the sampling Kantorovich type operators has been first introduced in [7] in one-dimensional setting and there, some approximation results in the setting of Orlicz spaces $L^\varphi(\mathbb{R})$ have been achieved.

It is well-known that, Orlicz spaces are very general spaces including, among its various special cases, the L^p -spaces (see e.g. [8, 35, 36]). Subsequently, these operators were extended in [39, 40] to the nonlinear case. The order of approximation for nonlinear sampling Kantorovich operators have been studied in [22] considering functions in suitable Lipschitz classes both in the space of uniformly continuous and bounded functions and in Orlicz spaces. Results concerning the multidimensional version of these operators have been obtained in [24].

In the last forty years, the study of approximation results by sampling-type operators (in linear and nonlinear cases) has been a wide research area both from a theoretical and an application point of view, such as signal and image processing. In particular, sampling type operators (in their multivariate version) can be used in order to reconstruct and approximate images, see e.g. [18, 19, 29, 37].

Concerning the problem of the order of approximation for the (linear) sampling Kantorovich operators, a quantitative estimate in the setting of Orlicz spaces in terms of modulus of smoothness has been very recently established in [26]. On the other hand, quantitative estimates with respect to the Jordan variation for sampling-type operators have been obtained in [2] exploiting a suitable modulus of smoothness for the space of absolutely continuous functions $AC(\mathbb{R})$. However, a quantitative approach for nonlinear sampling Kantorovich operators has not been addressed as yet.

In the present paper, we prove some quantitative estimates for the nonlinear sampling Kantorovich operators using the modulus of smoothness of $L^p(\mathbb{R})$. Further, the qualitative order of approximation is established for functions belonging to suitable Lipschitz classes. In the particular case of L^p -approximation, we directly established a quantitative estimate for the order of approximation, with the main purpose to obtain a sharper estimate than that one achieved in the general case. If the latter estimate is applied for the linear version of the sampling Kantorovich operators, we become able to improve the result that could be derived from Theorem 3.1 of [26]. Finally, we give some concrete examples of nonlinear sampling Kantorovich operators constructed by using Fejér and B-spline kernels, establishing some particular results in these instances.

2. Preliminaries

In this section, we recall the necessary background material related to Orlicz spaces used throughout the paper.

We denote by $C(\mathbb{R})$ the space of all uniformly continuous and bounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$ endowed with the norm $\|\cdot\|_\infty$. Also, $C_c(\mathbb{R})$ is the subspace of $C(\mathbb{R})$ consisting of functions with compact support and $M(\mathbb{R})$ is the linear space of Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ (or $\mathbb{C}$).

A function $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to be a φ -function if it satisfies the following conditions:

- (Φ1) φ is a continuous function with $\varphi(0) = 0$;
- (Φ2) φ is a non-decreasing function and $\varphi(u) > 0$ for every $u > 0$;
- (Φ3) $\lim_{u \rightarrow +\infty} \varphi(u) = +\infty$.

Let us introduce the functional I^φ associated to the φ -function φ and defined by

$$I^\varphi[f] := \int_{\mathbb{R}} \varphi(|f(x)|) dx,$$

for every $f \in M(\mathbb{R})$. It is well-known that I^φ is a modular functional (see, e.g., [8, 36]) and the Orlicz space generated by φ is defined by

$$L^\varphi(\mathbb{R}) := \{f \in M(\mathbb{R}) : I^\varphi[\lambda f] < +\infty, \text{ for some } \lambda > 0\}.$$

$L^\varphi(\mathbb{R})$ contains the subspace $E^\varphi(\mathbb{R})$ of all finite elements of $L^\varphi(\mathbb{R})$, i.e.,

$$E^\varphi(\mathbb{R}) = \{f \in L^\varphi(\mathbb{R}) : I^\varphi[\lambda f] < +\infty, \text{ for every } \lambda > 0\}.$$

In general $E^\varphi(\mathbb{R})$ is a proper subspace of $L^\varphi(\mathbb{R})$ and they coincide if and only if the so-called Δ_2 -condition on φ is satisfied, i.e. there exists a constant $M > 0$ such that

$$\frac{\varphi(2u)}{\varphi(u)} \leq M, \text{ for every } u > 0. \quad (\Delta_2)$$

Examples of φ -functions satisfying Δ_2 -condition are $\varphi(u) = u^p$ for $1 \leq p < \infty$ (in this case $L^\varphi(\mathbb{R}) = L^p(\mathbb{R})$) or $\varphi_{\alpha, \beta}(u) = u^\alpha \ln^\beta(e + u)$ for $\alpha \geq 1, \beta > 0$ (in this case, the Orlicz spaces $L^{\alpha, \beta}(\mathbb{R})$ is the interpolation space $L^\alpha \log^\beta L(\mathbb{R})$).

A concept of convergence in Orlicz spaces, called *modular convergence*, was introduced in [35].

We say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is modularly convergent to $f \in L^\varphi(\mathbb{R})$, if there exists $\lambda > 0$ such that

$$I^\varphi[\lambda(f_w - f)] = \int_{\mathbb{R}} \varphi(\lambda|f_w(x) - f(x)|) dx \rightarrow 0, \text{ as } w \rightarrow +\infty. \quad (2.1)$$

Now, we can recall the definition of the modulus of smoothness in Orlicz spaces $L^\varphi(\mathbb{R})$, with respect to the modular I^φ . For any fixed $f \in L^\varphi(\mathbb{R})$ and for a suitable $\lambda > 0$, we denote

$$\omega(f, \delta)_\varphi := \sup_{|t| \leq \delta} I^\varphi[\lambda(f(\cdot + t) - f(t))], \quad (2.2)$$

with $\delta > 0$.

In order to recall the class of operators we work with, we need some additional concepts.

Let $\Pi := (t_k)_{k \in \mathbb{Z}}$ be a sequence of real numbers such that $-\infty < t_k < t_{k+1} < +\infty$ for every $k \in \mathbb{Z}$, $\lim_{k \rightarrow \pm\infty} t_k = \pm\infty$ and there are two positive constants Δ, δ such that $\delta \leq \Delta_k := t_{k+1} - t_k \leq \Delta$, for every $k \in \mathbb{Z}$.

In what follows, a function $\chi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ will be called a *kernel* if it satisfies the following conditions:

- ($\chi 1$) $k \rightarrow \chi(wx - t_k, u) \in \ell^1(\mathbb{Z})$, for every $(x, u) \in \mathbb{R}^2$ and $w > 0$;
- ($\chi 2$) $\chi(x, 0) = 0$, for every $x \in \mathbb{R}$;
- ($\chi 3$) χ is a (L, ψ) -Lipschitz kernel, i.e., there exist a measurable function $L : \mathbb{R} \rightarrow \mathbb{R}_0^+$ and a φ -function ψ , such that

$$|\chi(x, u) - \chi(x, v)| \leq L(x) \psi(|u - v|),$$

for every $x, u, v \in \mathbb{R}$;

- ($\chi 4$) there exists $\theta_0 > 0$ such that

$$\mathcal{T}_w(x) := \sup_{u \neq 0} \left| \frac{1}{u} \sum_{k \in \mathbb{Z}} \chi(wx - t_k, u) - 1 \right| = \mathcal{O}(w^{-\theta_0})$$

as $w \rightarrow +\infty$, uniformly with respect to $x \in \mathbb{R}$.

Moreover, we assume that the function L of condition ($\chi 3$) satisfies the following properties:

- (L1) $L \in L^1(\mathbb{R})$ and is bounded in a neighborhood of the origin;
- (L2) there exists $\beta_0 > 0$ such that

$$m_{\beta_0, \Pi}(L) := \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) |wx - t_k|^{\beta_0} < +\infty,$$

i.e., the absolute moment of order β_0 is finite.

Then, nonlinear sampling Kantorovich operators for a given kernel χ are defined by

$$(S_w f)(x) := \sum_{k \in \mathbb{Z}} \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u) du \right), \quad x \in \mathbb{R},$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $x \in \mathbb{R}$.

The following lemma will be needed in the proof of our main theorems.

Lemma 2.1. (see [7]) *Let L be a function satisfying conditions (L1) and (L2). Then, we have*

$$m_{0, \Pi}(L) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} L(u - t_k) < +\infty.$$

3. Main Results

In this section, firstly we give the following quantitative estimate for nonlinear sampling Kantorovich operators by using the modulus of smoothness in Orlicz spaces. For this aim, we need a growth condition on the composition of the function φ , which generates the Orlicz space and the function ψ of condition ($\chi 3$).

Namely, given a φ -function φ , we require that there exists a φ -function η such that, for every $\lambda \in (0, 1)$, there exists a constant $C_\lambda \in (0, 1)$ satisfying

$$\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u), \quad (\text{H})$$

for every $u \in \mathbb{R}_0^+$, where ψ is the φ -function of condition $(\chi 3)$ (see [8, 21, 39]).

Theorem 3.1. *Let φ be a convex φ -function. Suppose that φ satisfies condition (H) with η convex, $f \in L^{\varphi+\eta}(\mathbb{R})$ and also for any fixed $0 < \alpha < 1$, we have*

$$w \int_{|y| > 1/w^\alpha} L(wy) dy \leq M_1 w^{-\alpha_0}, \text{ as } w \rightarrow +\infty, \quad (3.1)$$

for suitable positive constants M_1, α_0 depending on α and L . Then, there exist $\mu > 0$ and $\lambda_0 > 0$ such that

$$\begin{aligned} I^\varphi[\mu(S_w f - f)] &\leq \frac{\|L\|_1}{3\delta m_{0,\Pi}(L)} \omega(f, 1/w^\alpha)_\eta + \frac{M_1 I^\eta[\lambda_0 f]}{3\delta m_{0,\Pi}(L)} w^{-\alpha_0} \\ &\quad + \frac{\Delta}{3\delta} \omega(f, \Delta/w)_\eta + \frac{I^\varphi[\lambda_0 f]}{3} w^{-\theta_0}, \end{aligned}$$

for every sufficiently large $w > 0$, where $m_{0,\Pi}(L) < +\infty$ in view of Lemma 2.1 and $\theta_0 > 0$ is the constant of condition $(\chi 4)$. Particularly, if $\mu > 0$ is sufficiently small, this inequality implies the modular convergence of nonlinear sampling Kantorovich operators $S_w f$ to f .

Proof. Let $\lambda_0 > 0$ such that $I^\varphi[\lambda_0 f] < +\infty$. Also, we fix $\lambda > 0$ such that

$$\lambda < \min \left\{ 1, \frac{\lambda_0}{2} \right\}.$$

In correspondence to λ , by condition (H) there exists $C_\lambda \in (0, 1)$ such that $\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u)$, $u \in \mathbb{R}_0^+$. By condition $(\chi 4)$, there exists $M_2 > 0$ such that

$$\mathcal{T}_w(x) \leq M_2 w^{-\theta_0},$$

uniformly with respect to $x \in \mathbb{R}$, for sufficiently large $w > 0$. Let us choose $\mu > 0$ such that

$$\mu \leq \min \left\{ \frac{C_\lambda}{3m_{0,\Pi}(L)}, \frac{\lambda_0}{3M_2} \right\}.$$

Taking into account that φ is convex and non-decreasing, for $\mu > 0$, we can write

$$\begin{aligned}
 & I^\varphi [\mu (S_w f - f)] \\
 & \leq \frac{1}{3} \left\{ \int_{\mathbb{R}} \varphi \left(3\mu \left| (S_w f)(x) - \sum_{k \in \mathbb{Z}} \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) \right| \right) dx \right. \\
 & \quad + \int_{\mathbb{R}} \varphi \left(3\mu \left| \sum_{k \in \mathbb{Z}} \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) \right. \right. \\
 & \quad \left. \left. - \sum_{k \in \mathbb{Z}} \chi (wx - t_k, f(x)) \right| \right) dx + \int_{\mathbb{R}} \varphi \left(3\mu \left| \sum_{k \in \mathbb{Z}} \chi (wx - t_k, f(x)) - f(x) \right| \right) dx \right\} \\
 & =: I_1 + I_2 + I_3.
 \end{aligned}$$

Now, we estimate I_1 . Applying condition $(\chi 3)$, we have

$$\begin{aligned}
 3I_1 &= \int_{\mathbb{R}} \varphi \left(3\mu \left| (S_w f)(x) - \sum_{k \in \mathbb{Z}} \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) \right| \right) dx \\
 &\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} \left| \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u) du \right) \right. \right. \\
 &\quad \left. \left. - \chi \left(wx - t_k, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) \right| \right) dx \\
 &\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} L(wx - t_k) \psi \left(\left| \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} [f(u) - f(u + x - t_k/w)] du \right| \right) \right) dx \\
 &\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} L(wx - t_k) \psi \left(\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)| du \right) \right) dx.
 \end{aligned}$$

Applying Jensen inequality twice (see e.g., [20]), the change of variable $y = x - t_k/w$, condition (H) and Fubini-Tonelli theorem, we obtain

$3I_1$

$$\begin{aligned}
&\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \varphi \left(3\mu_{m_{0,\Pi}(L)} \psi \left(\frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)| du \right) \right) dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wx - t_k) \varphi \left(3\mu_{m_{0,\Pi}(L)} \psi \left(\frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)| du \right) \right) dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wy) \varphi \left(3\mu_{m_{0,\Pi}(L)} \psi \left(\frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + y)| du \right) \right) dy \\
&\leq \frac{1}{m_{0,\Pi}(L)} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wy) \varphi \left(C_\lambda \psi \left(\frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + y)| du \right) \right) dy \\
&\leq \frac{1}{m_{0,\Pi}(L)} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wy) \eta \left(\lambda \frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + y)| du \right) dy \\
&\leq \frac{1}{m_{0,\Pi}(L)} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wy) \frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} \eta(\lambda |f(u) - f(u + y)|) du dy \\
&\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} \int_{\mathbb{R}} wL(wy) \sum_{k \in \mathbb{Z}} \int_{t_k/w}^{t_{k+1}/w} \eta(\lambda |f(u) - f(u + y)|) du dy \\
&\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} \int_{\mathbb{R}} wL(wy) \left[\int_{\mathbb{R}} \eta(\lambda |f(u) - f(u + y)|) du \right] dy \\
&\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} \int_{\mathbb{R}} wL(wy) I^\eta[\lambda(f(\cdot) - f(\cdot + y))] dy =: J.
\end{aligned}$$

Now, let $0 < \alpha < 1$ be fixed. We now split the above integral J as

$$\begin{aligned}
J &:= \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \left\{ \int_{|y| \leq 1/w^\alpha} + \int_{|y| > 1/w^\alpha} \right\} L(wy) I^\eta[\lambda(f(\cdot) - f(\cdot + y))] dy \\
&=: J_1 + J_2.
\end{aligned}$$

For J_1 , one has

$$\begin{aligned}
J_1 &= \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y| \leq 1/w^\alpha} L(wy) I^\eta[\lambda(f(\cdot) - f(\cdot + y))] dy \\
&\leq \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y| \leq 1/w^\alpha} L(wy) \omega(f, |y|)_\eta dy \\
&\leq \omega(f, 1/w^\alpha)_\eta \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y| \leq 1/w^\alpha} L(wy) dy \\
&\leq \omega(f, 1/w^\alpha)_\eta \frac{\delta^{-1}}{m_{0,\Pi}(L)} \|L\|_1.
\end{aligned}$$

On the other hand, taking into account that I^η is convex (since η is convex), for J_2 we can write

$$\begin{aligned} J_2 &= \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y|>1/w^\alpha} L(wy) I^\eta [\lambda(f(\cdot) - f(\cdot + y))] dy \\ &\leq \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y|>1/w^\alpha} L(wy) \frac{1}{2} \{I^\eta [2\lambda f(\cdot)] + I^\eta [2\lambda f(\cdot + y)]\} dy. \end{aligned}$$

Moreover, it can be easily seen that

$$I^\eta [2\lambda f(\cdot)] = I^\eta [2\lambda f(\cdot + y)],$$

for every y . Therefore, by assumption (3.1), we have

$$\begin{aligned} J_2 &\leq \frac{w\delta^{-1}}{m_{0,\Pi}(L)} \int_{|y|>1/w^\alpha} L(wy) I^\eta [2\lambda f] dy \\ &\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} I^\eta [\lambda_0 f] M_1 w^{-\alpha_0}, \text{ as } w \rightarrow +\infty. \end{aligned}$$

Now we estimate I_2 .

$$\begin{aligned} 3I_2 &= \int_{\mathbb{R}} \varphi \left(3\mu \left| \sum_{k \in \mathbb{Z}} \chi \left(wx - t_k, \frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) - \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) \right| \right) dx \\ &\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} \left| \chi \left(wx - t_k, \frac{w}{\Delta_k^{t_k/w}} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du \right) - \chi(wx - t_k, f(x)) \right| \right) dx. \end{aligned}$$

Using condition $(\chi 3)$, the change of variable $y = u - t_k/w$, Jensen inequality and condition (H), we have

$$\begin{aligned}
3I_2 &\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} L(wx - t_k) \psi \left(\left| \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u + x - t_k/w) du - f(x) \right| \right) \right) dx \\
&\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} L(wx - t_k) \psi \left(\left| \frac{w}{\Delta_k} \int_0^{\Delta_k/w} [f(x+y) - f(x)] dy \right| \right) \right) dx \\
&\leq \int_{\mathbb{R}} \varphi \left(3\mu \sum_{k \in \mathbb{Z}} L(wx - t_k) \psi \left(\frac{w}{\Delta_k} \int_0^{\Delta_k/w} |f(x+y) - f(x)| dy \right) \right) dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \varphi \left(3\mu m_{0,\Pi}(L) \psi \left(\frac{w}{\Delta_k} \int_0^{\Delta_k/w} |f(x+y) - f(x)| dy \right) \right) dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \varphi \left(C_\lambda \psi \left(\frac{w}{\Delta_k} \int_0^{\Delta_k/w} |f(x+y) - f(x)| dy \right) \right) dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \eta \left(\lambda \frac{w}{\Delta_k} \int_0^{\Delta_k/w} |f(x+y) - f(x)| dy \right) dx.
\end{aligned}$$

Using Jensen inequality and Fubini-Tonelli theorem, we get

$$\begin{aligned}
3I_2 &\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \left[\frac{w}{\Delta_k} \int_0^{\Delta_k/w} \eta(\lambda |f(x+y) - f(x)|) dy \right] dx \\
&\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} \int_{\mathbb{R}} w \sum_{k \in \mathbb{Z}} L(wx - t_k) \left[\int_0^{\Delta/w} \eta(\lambda |f(x+y) - f(x)|) dy \right] dx \\
&\leq \frac{\delta^{-1}}{m_{0,\Pi}(L)} \int_{\mathbb{R}} w m_{0,\Pi}(L) \left[\int_0^{\Delta/w} \eta(\lambda |f(x+y) - f(x)|) dy \right] dx \\
&= \delta^{-1} w \int_0^{\Delta/w} \left[\int_{\mathbb{R}} \eta(\lambda |f(x+y) - f(x)|) dx \right] dy \\
&= \delta^{-1} w \int_0^{\Delta/w} I^\eta[\lambda(f(\cdot + y) - f(\cdot))] dy \leq \delta^{-1} \omega(f, \Delta/w)_\eta w \int_0^{\Delta/w} dy \\
&= \delta^{-1} \Delta \omega(f, \Delta/w)_\eta.
\end{aligned}$$

For I_3 , denoted by $A_0 \subseteq \mathbb{R}$ the set of all points of $\mathbb{R}$ for which $f \neq 0$ almost everywhere, we obtain

$$\begin{aligned} 3I_3 &= \int_{A_0} \varphi \left(3\mu \left| \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) - f(x) \right| \right) dx \\ &= \int_{A_0} \varphi \left(3\mu |f(x)| \left| \frac{1}{f(x)} \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) - 1 \right| \right) dx \\ &\leq \int_{A_0} \varphi(3\mu |f(x)| \mathcal{T}_w(x)) dx. \end{aligned}$$

By the convexity of φ and condition $(\chi 4)$, we have

$$\begin{aligned} 3I_3 &\leq \int_{A_0} \varphi(3\mu M_2 w^{-\theta_0} |f(x)|) dx \leq w^{-\theta_0} \int_{A_0} \varphi(3M_2 \mu |f(x)|) dx \\ &\leq w^{-\theta_0} \int_{\mathbb{R}} \varphi(3M_2 \mu |f(x)|) dx \leq w^{-\theta_0} I^\varphi[3M_2 \mu f] \leq w^{-\theta_0} I^\varphi[\lambda_0 f] < +\infty, \end{aligned}$$

for positive constants M_2, θ_0 . This completes the proof. $\square$

Note that, condition (3.1) is satisfied when, for instance, the kernel χ satisfies condition $(\chi 3)$ with L having compact support, e.g. $\text{supp } L \subset [-B, B]$, $B > 0$. Indeed,

$$w \int_{|y| > 1/w^\alpha} L(wy) dy = w \int_{|u| > w^{1-\alpha}} L(u) du = 0,$$

for sufficiently large $w > B^{1/(1-\alpha)}$. Moreover, in this case, condition (L2) is satisfied for every $\beta_0 > 0$. Then, we get the following.

Corollary 3.1. *Let χ be a kernel satisfying condition $(\chi 3)$ with L having compact support. Moreover, let φ be a convex φ -function satisfying condition (H) with η convex and $f \in L^{\varphi+\eta}(\mathbb{R})$. Then, for every $0 < \alpha < 1$, there exist*

constant $\mu > 0$ and $\lambda_0 > 0$ such that

$$I^\varphi[\mu(S_w f - f)] \leq \frac{\|L\|_1}{3\delta m_{0,\Pi}(L)} \omega(f, 1/w^\alpha)_\eta + \frac{\Delta}{3\delta} \omega(f, \Delta/w)_\eta + \frac{I^\varphi[\lambda_0 f]}{3} w^{-\theta_0},$$

for every sufficiently large $w > 0$, where $m_{0,\Pi}(L) < +\infty$ in view of Lemma 2.1 and $\theta_0 > 0$ is the constant of condition $(\chi 4)$.

Note that, if L has not compact support, we may require the following sufficient condition:

$$M_\nu(L) := \int_{\mathbb{R}} L(u) |u|^\nu du < +\infty, \quad \nu > 0, \quad (3.2)$$

which imply assumption (3.1). In this case, for every $0 < \alpha < 1$, we get

$$\begin{aligned} w \int_{|y| > 1/w^\alpha} L(wy) dy &= \int_{|u| > w^{1-\alpha}} L(u) du = \int_{|u| > w^{1-\alpha}} \frac{|u|^\nu}{|u|^\nu} L(u) du \\ &\leq \frac{1}{w^{(1-\alpha)\nu}} \int_{|u| > w^{1-\alpha}} |u|^\nu L(u) du \leq \frac{M_\nu(L)}{w^{(1-\alpha)\nu}} = \mathcal{O}\left(w^{(\alpha-1)\nu}\right), \end{aligned}$$

for sufficiently large $w > 0$. Hence, (3.1) is satisfied with $\alpha_0 = (1 - \alpha)\nu$ and $M_1 = M_\nu(L)$.

We now recall the definition of Lipschitz classes in Orlicz spaces. We define by $Lip_\varphi(\nu)$, $0 < \nu \leq 1$, the set of all functions $f \in M(\mathbb{R})$ such that, there exists $\lambda > 0$ with:

$$I^\varphi[\lambda(f(\cdot) - f(\cdot + t))] = \int_{\mathbb{R}} \varphi(\lambda|f(x) - f(x + t)|) dx = \mathcal{O}(|t|^\nu),$$

as $t \rightarrow 0$. In this context, from Theorem 3.1 we immediately get the following corollary.

Corollary 3.2. *Under the assumptions of Theorem 3.1 with $0 < \alpha < 1$, and for any $f \in Lip_\eta(\nu)$, $0 < \nu \leq 1$, there exist $K > 0$, $\mu > 0$ such that*

$$I^\varphi[\mu(S_w f - f)] \leq K w^{-\ell},$$

for sufficiently large $w > 0$, where $\ell := \min\{\alpha\nu, \alpha_0, \theta_0\}$.

Now, we consider below some particular cases of Orlicz spaces.

Let $\varphi(u) = u^p$, $u \in \mathbb{R}_0^+$, $p \geq 1$. Then, the Orlicz space $L^\varphi(\mathbb{R})$ coincides with the space $L^p(\mathbb{R})$. If $\psi(u) = u^{q/p}$, $1 \leq q \leq p$, condition (H) it turns out to be satisfied with $\eta(u) = u^q$ and $C_\lambda = \lambda^{q/p}$. In this case, we have $L^{\varphi+\eta}(\mathbb{R}) = L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ (which is, in fact, a proper subspace of $L^p(\mathbb{R})$), and obviously Theorem 3.1 and its corollary hold.

From the theory developed in [39], we know that if the function ψ of condition (χ_3) is $\psi(u) = u$, $u \in \mathbb{R}$, and $\varphi(u) = u^p$, $1 \leq p < \infty$, the operators S_w maps the whole space $L^p(\mathbb{R})$ into itself and therefore, we can obtain, as particular case, a quantitative estimate in $L^p(\mathbb{R})$. But since the φ -modulus of smoothness does not satisfy the well-known property $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$, satisfied e.g., by the ω_p -modulus of smoothness below defined, we can proceed using a direct approach and estimating the term $S_w f - f$ with respect to the p -norm. For the above purpose, we need to recall the definition of the L^p -modulus of smoothness of order one given as

$$\omega_p(f, \delta) := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p = \sup_{|h| \leq \delta} \left(\int_{\mathbb{R}} |f(t+h) - f(t)|^p dt \right)^{1/p},$$

with $\delta > 0$, $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$.

We can prove the following estimate.

Theorem 3.2. Suppose that $M_p(L) < +\infty$, $1 \leq p < \infty$. Then, for every $f \in L^p(\mathbb{R})$, the following quantitative estimate holds

$$\|S_w f - f\|_p \leq \delta^{-1/p} [2m_{0,\Pi}(L)]^{(p-1)/p} [\|L\|_1 + M_p(L)]^{1/p} \omega_p(f, 1/w) \\ + \delta^{-1/p} m_{0,\Pi}(L) \Delta^{1/p} \omega_p(f, \Delta/w) + M_2 \|f\|_p w^{-\theta_0},$$

for sufficiently large $w > 0$, where $m_{0,\Pi}(L) < +\infty$ in view of Lemma 2.1 and $M_2, \theta_0 > 0$ are the constants of condition $(\chi 4)$.

Proof. Recalling that $I^\varphi[f] = \|f\|_p^p$, when $\varphi(u) = u^p$, proceeding as in the first part of the proof of Theorem 3.1, using the Minkowsky inequality, and that the function $|\cdot|^{1/p}$ is concave and hence sub-additive, we immediately obtain:

$$\|S_w f - f\|_p \leq \left(\int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} L(wx - t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)| du \right]^p dx \right)^{1/p} \\ + \left(\int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} L(wx - t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u + x - t_k/w) - f(x)| du \right]^p dx \right)^{1/p} \\ + \left(\int_{\mathbb{R}} \left| \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) - f(x) \right|^p dx \right)^{1/p} =: I_1 + I_2 + I_3.$$

We estimate I_1 . Applying Jensen inequality twice, Fubini-Tonelli theorem, and by the change of variable $y = x - t_k/w$, we get

$$I_1^p = \int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} L(wx - t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)| du \right]^p dx \\ \leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} m_{0,\Pi}(L) |f(u) - f(u + x - t_k/w)| du \right]^p dx \\ \leq m_{0,\Pi}(L)^{p-1} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wx - t_k) \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + x - t_k/w)|^p du \right]^p dx \\ \leq m_{0,\Pi}(L)^{p-1} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} L(wy) \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + y)|^p du \right]^p dy \\ \leq \delta^{-1} m_{0,\Pi}(L)^{p-1} \int_{\mathbb{R}} w L(wy) \left[\sum_{k \in \mathbb{Z}} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u + y)|^p du \right]^p dy \\ = \delta^{-1} m_{0,\Pi}(L)^{p-1} \int_{\mathbb{R}} w L(wy) \left[\int_{\mathbb{R}} |f(u) - f(u + y)|^p du \right]^p dy \\ \leq \delta^{-1} m_{0,\Pi}(L)^{p-1} \int_{\mathbb{R}} w L(wy) \omega_p(f, |y|)^p dy$$

$$\begin{aligned}
&\leq \delta^{-1} m_{0,\Pi}(L)^{p-1} \int_{\mathbb{R}} w L(wy) (1+w|y|)^p \omega_p\left(f, \frac{1}{w}\right)^p dy \\
&\leq \delta^{-1} m_{0,\Pi}(L)^{p-1} 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p \int_{\mathbb{R}} w L(wy) (1+(w|y|)^p) dy \\
&= \delta^{-1} m_{0,\Pi}(L)^{p-1} 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p \int_{\mathbb{R}} L(z) (1+|z|^p) dz \\
&= \delta^{-1} m_{0,\Pi}(L)^{p-1} 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p (\|L\|_1 + M_p(L)),
\end{aligned}$$

for every $w > 0$, where $\|L\|_1$ and $M_p(L)$ are both finite. Note that, in the above estimates we used the following well-known property of the modulus of smoothness

$$\omega_p(f, \lambda\delta) \leq (1+\lambda) \omega_p(f, \delta), \quad \lambda, \delta > 0.$$

Now we estimate I_2 . Using Jensen inequality twice, the change of variable $y = u - t_k/w$ and Fubini-Tonelli theorem, we have

$$\begin{aligned}
I_2^p &= \int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} L(wx - t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u + x - t_k/w) - f(x)| du \right]^p dx \\
&\leq \int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} L(wx - t_k) \frac{w}{\Delta_k} \int_0^{\Delta_k/w} |f(x+y) - f(x)| dy \right]^p dx \\
&\leq \frac{1}{m_{0,\Pi}(L)} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \left[\frac{w}{\Delta_k} \int_0^{\Delta_k/w} m_{0,\Pi}(L) |f(x+y) - f(x)| dy \right]^p dx \\
&\leq \delta^{-1} m_{0,\Pi}(L)^{p-1} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) \left[w \int_0^{\Delta/w} |f(x+y) - f(x)|^p dy \right] dx \\
&\leq \delta^{-1} m_{0,\Pi}(L)^p \int_{\mathbb{R}} w \int_0^{\Delta/w} |f(x+y) - f(x)|^p dy dx \\
&\leq \delta^{-1} m_{0,\Pi}(L)^p \int_0^{\Delta/w} w \left[\int_{\mathbb{R}} |f(x+y) - f(x)|^p dx \right] dy \\
&\leq \delta^{-1} m_{0,\Pi}(L)^p \int_0^{\Delta/w} w [\omega_p(f, \Delta/w)]^p dy = \delta^{-1} m_{0,\Pi}(L)^p \Delta [\omega_p(f, \Delta/w)]^p.
\end{aligned}$$

For I_3 , denoted by $A_0 \subseteq \mathbb{R}$ the set of all points of $\mathbb{R}$ for which $f \neq 0$ almost everywhere, we obtain

$$\begin{aligned} I_3^p &= \int_{A_0} \left| \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) - f(x) \right|^p dx \\ &= \int_{A_0} |f(x)|^p \left| \frac{1}{f(x)} \sum_{k \in \mathbb{Z}} \chi(wx - t_k, f(x)) - 1 \right|^p dx \leq \int_{A_0} |f(x)|^p [\mathcal{T}_w(x)]^p dx. \end{aligned}$$

From condition $(\chi 4)$, we have

$$\begin{aligned} I_3^p &\leq \int_{A_0} |f(x)|^p M_2^p w^{-p\theta_0} dx \leq M_2^p w^{-p\theta_0} \int_{\mathbb{R}} |f(x)|^p dx \\ &= M_2^p w^{-p\theta_0} \|f\|_p^p \end{aligned}$$

for positive constants M_2, θ_0 . This proves the theorem. $\square$

Now, denoting by $Lip(\alpha, p)$, $0 < \alpha \leq 1$, $p \geq 1$, the corresponding Lipschitz classes in $L^p(\mathbb{R})$, we can immediately state the following.

Corollary 3.3. *Suppose that $M_p(L) < +\infty$, for $1 \leq p < \infty$. Then, for every $f \in Lip(\alpha, p)$, with $0 < \alpha \leq 1$, we have*

$$\begin{aligned} \|S_w f - f\|_p &\leq \delta^{-1/p} [2m_{0,\Pi}(L)]^{(p-1)/p} [\|L\|_1 + M_p(L)]^{1/p} C_1 \frac{1}{w^\alpha} \\ &\quad + \delta^{-1/p} m_{0,\Pi}(L) C_1 \Delta^{1/p} \left(\frac{\Delta}{w} \right)^\alpha + M_2 \|f\|_p w^{-\theta_0}, \end{aligned}$$

for every sufficiently large $w > 0$, where $m_{0,\Pi}(L)$ is finite in view of Lemma 2.1 and, $C_1 > 0$, $M_2, \theta_0 > 0$ are the constants arising from the fact that $f \in Lip(\alpha, p)$ and from condition $(\chi 4)$, respectively.

Remark 3.1. Note that if the kernel χ is of the form $\chi(x, u) = L(x)u$, with L satisfying conditions $(L1)$ and $(L2)$, the above operators reduces to the linear case considered in [7]. In this situation, condition $(\chi 4)$ becomes

$$\mathcal{T}_w(x) = \left| \sum_{k \in \mathbb{Z}} L(wx - t_k) - 1 \right| = \mathcal{O}(w^{-\theta_0}), \text{ as } w \rightarrow +\infty, \quad (3.3)$$

uniformly with respect to $x \in \mathbb{R}$, for some $\theta_0 > 0$. Sometimes, a stronger condition [instead of (3.3)] is required, i.e., that

$$\sum_{k \in \mathbb{Z}} L(u - t_k) = 1, \quad (3.4)$$

for every $u \in \mathbb{R}$. In this case, condition $(\chi 4)$ holds for every $\theta_0 > 0$. When $t_k = k$ (uniform sampling scheme) and L is continuous, it is well known that

(3.4) is equivalent to

$$\widehat{L}(2\pi k) := \begin{cases} 0, & k \in \mathbb{Z} \setminus \{0\}, \\ 1, & k = 0, \end{cases}$$

where $\widehat{L}(v) := \int_{\mathbb{R}} L(u) e^{-ivu} du$, $v \in \mathbb{R}$, denotes the Fourier transform of L (see [7, 11, 23]).

The rate of approximation for (linear) sampling Kantorovich operators in various settings was studied in [23]. Also, a quantitative estimate for these operators was obtained in [26] by using the modulus of smoothness in Orlicz spaces.

The general setting of Orlicz spaces allows us to directly deduce the results concerning some quantitative estimates of approximation in L^p -spaces (as in Corollary 3.3), together with some other useful spaces, as for examples Zygmund spaces and the exponential spaces, defined in Sect. 2.

In the case of approximation by linear sampling Kantorovich operators, considered in Remark 3.1, and below denoted by S_w^* , we can immediately deduce, as a particular case, the following results.

Corollary 3.4. *Suppose that $M_p(L) < +\infty$, for $1 \leq p < \infty$. Then, for every $f \in L^p(\mathbb{R})$, there holds*

$$\begin{aligned} \|S_w^* f - f\|_p &\leq \delta^{-1/p} [2m_{0,\Pi}(L)]^{(p-1)/p} [\|L\|_1 + M_p(L)]^{1/p} \omega_p(f, 1/w) \\ &\quad + \delta^{-1/p} m_{0,\Pi}(L) \Delta^{1/p} \omega_p(f, \Delta/w), \end{aligned}$$

for sufficiently large $w > 0$, where $m_{0,\Pi}(L) < +\infty$. Moreover, if $f \in \text{Lip}(\alpha, p)$, with $0 < \alpha \leq 1$, we have

$$\begin{aligned} \|S_w^* f - f\|_p &\leq \delta^{-1/p} [2m_{0,\Pi}(L)]^{(p-1)/p} [\|L\|_1 + M_p(L)]^{1/p} C_1 \frac{1}{w^\alpha} \\ &\quad + \delta^{-1/p} m_{0,\Pi}(L) C_1 \Delta^{1/p} \left(\frac{\Delta}{w}\right)^\alpha, \end{aligned}$$

for sufficiently large $w > 0$, where $C_1 > 0$ is the constant arising from the fact that f belongs to $\text{Lip}(\alpha, p)$.

Note that, the estimates established in Corollary 3.4 are sharper than those achieved in the general case of Theorem 3.2.

4. Examples of Kernels

In this section, we give some concrete examples of the above nonlinear sampling Kantorovich operators describing a natural procedure to construct kernels. We will consider kernel functions of the form

$$\chi(wx - t_k, u) = L(wx - t_k) g_w(u),$$

where $(g_w)_{w>0}$ is a family of functions $g_w : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $g_w(u) \rightarrow u$ uniformly on $\mathbb{R}$ as $w \rightarrow +\infty$ and such that there exists a φ -function ψ with

$$|g_w(u) - g_w(v)| \leq \psi(|u - v|), \quad (4.1)$$

for every $u, v \in \mathbb{R}$ and $w > 0$. Hence, assumptions (χi) , $i = 1, \dots, 4$ and (Lj) , $j = 1, 2$ can be summarized as follows.

($\mathcal{L}1$) $k \rightarrow L(wx - t_k) \in \ell^1(\mathbb{Z})$, for every $x \in \mathbb{R}$ and $w > 0$, L is locally bounded in a neighborhood of the origin and there exists $\beta_0 > 0$ such that

$$m_{\beta_0, \Pi}(L) := \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} L(wx - t_k) |wx - t_k|^{\beta_0} < +\infty; \quad (4.2)$$

($\mathcal{L}2$) $g_w(0) = 0$, for every $w > 0$;

($\mathcal{L}3$) there exists $\theta_0 > 0$ such that

$$\mathcal{T}_w(x) := \sup_{u \neq 0} \left| \frac{g_w(u)}{u} \sum_{k \in \mathbb{Z}} L(wx - t_k) - 1 \right| = \mathcal{O}(w^{-\theta_0}),$$

as $w \rightarrow +\infty$, uniformly with respect to $x \in \mathbb{R}$.

Firstly, we show an example of sequence $(g_w)_{w>0}$ satisfying all the assumptions of the above theory.

Example 4.1. Let us define $g_w(u) = u^{1-1/w}$ for every $u \in (a, 1)$, with $0 < a < 1/e$ and $g_w(u) = u$ otherwise, for $w > 0$. It is easily seen that $g_w(u) \rightarrow u$ uniformly on $\mathbb{R}$, as $w \rightarrow +\infty$. Note that if the function L satisfies condition (3.4), assumption ($\mathcal{L}3$) holds for $\theta_0 = 1$. In fact, the function $g_w(u) - u$ on $(a, 1)$ achieves the maximum at $u_0 := \left(\frac{w-1}{w}\right)^w$ for sufficiently large $w > 0$ ($g_w(u) - u = 0$ otherwise), then we have for every $u \in \mathbb{R}$

$$|g_w(u) - u| \leq |g_w(u_0) - u_0| = \left(\frac{w-1}{w}\right)^w \left(\frac{1}{w-1}\right) \leq \frac{C}{w-1},$$

for sufficiently large $w > 0$ and for a suitable positive constant C . Then

$$\sup_{u \neq 0} \left| \frac{g_w(u)}{u} - 1 \right| = \sup_{u \in (a, 1)} \frac{1}{|u|} |g_w(u) - u| \leq a^{-1} \frac{C}{w-1} = \mathcal{O}(w^{-1}),$$

as $w \rightarrow +\infty$. Moreover, $g_w(u)$ satisfies (4.1) for sufficiently large $w > 0$ and ψ concave.

In addition, if we consider the particular case $g_w(u) = u$, $u \in \mathbb{R}$, $w > 0$, the function ψ corresponding to $\chi(x, u) = L(x)u$ is $\psi(u) = u$, $u \in \mathbb{R}_0^+$. In this case, our operators reduce to linear ones studied in details in [1, 7, 23] and as stated in Remark 3.1, condition ($\mathcal{L}3$) becomes

$$\mathcal{T}_w(x) = \left| \sum_{k \in \mathbb{Z}} L(wx - k) - 1 \right| = \mathcal{O}(w^{-\theta_0}), \text{ as } w \rightarrow +\infty,$$

uniformly with respect to $x \in \mathbb{R}$, which is fulfilled for every $\theta_0 > 0$, if $\sum_{k \in \mathbb{Z}} L(u - k) = 1$, for every $u \in \mathbb{R}$.

In order to construct a first example of function L , we consider the well-known Fejér kernel, of the form

$$F(x) := \frac{1}{2} \operatorname{sinc}^2\left(\frac{x}{2}\right), \quad x \in \mathbb{R},$$

where the sinc-function is the following

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0. \end{cases}$$

It can be easily seen that the function F is non-negative and bounded, belongs to $L^1(\mathbb{R})$, with $\|F\|_1 = m_{0,\Pi}(F) = 1$ and satisfies the moment condition in (4.2) for every $0 < \beta_0 < 1$ (see, e.g., [12, 22, 23]). In addition, it is easy to see that $M_\nu(F) < +\infty$, for any $0 < \nu < 1$, hence, as stated before, condition (3.1) is satisfied with $\alpha_0 = (1 - \alpha)\nu$, $0 < \alpha < 1$, and $M_1 = M_\nu(F)$. Furthermore, the Fourier transform of F is given by

$$\widehat{F}(v) := \begin{cases} 1 - |v/\pi|, & |v| \leq \pi, \\ 0, & |v| > \pi, \end{cases}$$

(see [11]). Then, by Remark 3.1, we have $\sum_{k \in \mathbb{Z}} F(u - k) = 1$ for every $u \in \mathbb{R}$. Consequently, condition (L3) reduces to $\sup_{u \neq 0} |g_w(u)/u - 1| = \mathcal{O}(w^{-\theta_0})$, as $w \rightarrow +\infty$, for some $\theta_0 > 0$, which is satisfied since $g_w(u) - u$ converges uniformly to zero, as $w \rightarrow +\infty$.

Then, considering e.g. a uniform sampling scheme, i.e., $t_k = k$, $k \in \mathbb{Z}$, the corresponding nonlinear sampling Kantorovich operators based on Fejér kernel are

$$(S_w^F f)(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \operatorname{sinc}^2\left(\frac{wx - k}{2}\right) g_w\left(w \int_{k/w}^{(k+1)/w} f(u) du\right), \quad x \in \mathbb{R},$$

for every $w > 0$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $x \in \mathbb{R}$. For $S_w^F f$, from Theorem 3.1, we obtain the following.

Corollary 4.1. *Let φ be a convex φ -function. Suppose that φ satisfies condition (H) with η convex and $f \in L^{\varphi+\eta}(\mathbb{R})$. Then, for every $0 < \alpha < 1$, there exist constant $\mu > 0$, $\lambda_0 > 0$, such that*

$$\begin{aligned} I^\varphi[\mu(S_w^F f - f)] \\ \leq \frac{1}{3} \left\{ \omega(f, 1/w^\alpha)_\eta + \tilde{K} I^\eta[\lambda_0 f] w^{-\alpha_0} + \omega(f, 1/w)_\eta + I^\varphi[\lambda_0 f] w^{-\theta_0} \right\}, \end{aligned}$$

for sufficiently large $w > 0$, where F is the Fejér kernel, $\tilde{K} > 0$ is a suitable constant, and $\theta_0 > 0$ is the parameter of assumption (L3).

Analogously, we may obtain a similar version of Corollary 3.2 for the operators $S_w^F f$.

The Fejér kernel has unbounded support. Thus, to reconstruct a given signal of f by means of $S_w^F f$, we need to compute an infinite number of mean values $w \int_{k/w}^{(k+1)/w} f(u) du$ in order to evaluate the above operators at any fixed point $x \in \mathbb{R}$. Therefore, for a practical application of the above sampling series with L having unbounded support, the sampling series must be truncated and this leads to truncation errors which worsen the quality of the reconstruction.

However, considering kernels with L having compact support, the truncation error can be avoided. In this case, the infinite sampling series computed at any fixed $x \in \mathbb{R}$ reduces to a finite one. Important examples of kernels with compact support can be generated by using the well-known B-spline of order $n \in \mathbb{N}$, given by

$$M_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{n}{2} + x - j\right)_+^{n-1},$$

where the function $(x)_+ := \max\{x, 0\}$ is the positive part of $x \in \mathbb{R}$ (see [7, 11, 25, 27, 28, 39]).

The Fourier transform of M_n is given by

$$\widehat{M_n}(v) := \text{sinc}^n\left(\frac{v}{2\pi}\right), \quad v \in \mathbb{R}.$$

Then, we have $\sum_{k \in \mathbb{Z}} M_n(u - k) = 1$, for every $u \in \mathbb{R}$, by Remark 3.1. Therefore, condition (L3) reduces to $\sup_{u \neq 0} |g_w(u)/u - 1| = O(w^{-\theta_0})$, as $w \rightarrow +\infty$, for some $\theta_0 > 0$, which is again satisfied. Obviously, each M_n is bounded on $\mathbb{R}$, with compact support on $[-n/2, n/2]$, and hence $M_n \in L^1(\mathbb{R})$, for all $n \in \mathbb{N}$, with $\|M_n\|_1 = m_{0,\Pi}(M_n) = 1$. Further, condition (L1) is satisfied for every $\beta_0 > 0$ (see [7]).

In this case, the nonlinear sampling Kantorovich operators based upon the B-spline kernel of order n , with $t_k = k$, $k \in \mathbb{Z}$, are given by

$$(S_w^{M_n} f)(x) = \sum_{k \in \mathbb{Z}} M_n(wx - k) g_w \left(w \int_{k/w}^{(k+1)/w} f(u) du \right), \quad x \in \mathbb{R},$$

for every $w > 0$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $x \in \mathbb{R}$. From Corollary 3.1, we obtain the following.

Corollary 4.2. *Let φ be a convex φ -function satisfying condition (H) with η convex and $f \in L^{\varphi+\eta}(\mathbb{R})$. Then, for every $0 < \alpha < 1$, there exist constants*

$\mu > 0$, and $\lambda_0 > 0$, such that

$$I^\varphi [\mu (S_w^{M_n} f - f)] \leq \frac{1}{3} \omega(f, 1/w^\alpha)_\eta + \frac{1}{3} \omega(f, 1/w)_\eta + \frac{I^\varphi[\lambda_0 f]}{3} w^{-\theta_0},$$

for sufficiently large $w > 0$, where $\theta_0 > 0$ is the constant of condition (L3).

As before, also for the operators $S_w^{M_n} f$ we may obtain a similar version of Corollary 3.2. For other useful examples of kernels, see e.g., [13–15].

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