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On the Factor Opposing the Lebesgue Norm in Generalized Grand Lebesgue Spaces

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Abstract. We prove that if $1 < p < \infty$ and $\delta :\,]0, p-1] \to\,]0, \infty[$ is continuous, nondecreasing, and satisfies the Δ_2 condition near the origin, then

$$\bar{\delta}(\varepsilon) := \left[\sup_{0 < \zeta < \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon} \approx \delta(\varepsilon), \qquad \varepsilon \in\,]0, p-1]. \qquad (*)$$

This result permits to clarify the assumptions on the increasing function against the Lebesgue norm in the definition of generalized grand Lebesgue spaces and to sharpen and simplify the statements of some known results concerning these spaces.

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1. Introduction

The first appearence of generalized grand Lebesgue spaces—a class of Banach function spaces, see e.g. the monograph by Bennett and Sharpley [4]—is in the final remark in [5], where the factor against the $L^{p-\varepsilon}$ norm has been further generalized from a power into an increasing function. The growing interest in literature for this class of spaces has been motivated either by their utility in the theory of PDEs (see e.g. [1,8,11]), either in Function Spaces theory (see [6,9,10,16–18,21]). We refer to [7] for a study of these spaces, to [13] for a survey, to [12] for a recent characterization of its norm in term of the decreasing

rearrangement. The increasing interest on these spaces led to "maximize" the generalization of the original norm

$$\|f\|_{L^{p)}(0,1)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}},$$

due to Iwaniec and Sbordone in 1992 (see [14]), into

$$\|f\|_{L^{p),\delta}(0,1)} = \operatorname*{ess\,sup}_{0<\varepsilon<p-1} \left(\delta(\varepsilon) \int_0^1 |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}},$$

where δ is a nonnegative measurable bounded function on $]0, p-1]$ (see [7, Theorem 2.1] for details; see [2,3,15,19] for a generalization also with respect to ε). Even without a deep knowledge of the literature on these spaces, already the expression of the norm $\|\cdot\|_{L^{p),\delta}(0,1)}$ suggests clearly that the most natural assumption on δ is the property to be nondecreasing of the function

$$\widehat{\delta}(\varepsilon) := \delta(\varepsilon)^{\frac{1}{p-\varepsilon}}, \qquad \varepsilon \in]0, p-1].$$

In [12, **3.8**] it has been observed that if $\widehat{\delta}$ is nondecreasing, then δ is nondecreasing, and that the viceversa does not hold. Hence it is not a surprise that the main results in [7,12] have the "strong" assumption on the monotonicity of $\widehat{\delta}$ and not on the monotonicity of δ.

The Δ_2 condition is a notion familiar for researchers working in Orlicz spaces (see e.g. [20]). When the so-called Δ_2 *condition near the origin* (we write $\delta \in \Delta_2$ if $\delta(2\varepsilon) \leq c\delta(\varepsilon)$ for ε small, for some $c > 1$) plays a role, again from [12, **3.8**] we know that under the "weak" assumption that δ is nondecreasing, $\delta \in \Delta_2 \Leftrightarrow \widehat{\delta} \in \Delta_2$.

The considerations above show that

$$\widehat{\delta} \; nondecreasing \qquad \text{and} \qquad \widehat{\delta} \in \Delta_2 \tag{1.1}$$

is a stronger assumption with respect to

$$\delta \; nondecreasing \qquad \text{and} \qquad \delta \in \Delta_2 \tag{1.2}$$

and the viceversa does not hold, a simple example being $\delta(\varepsilon) \equiv 1/2$.

The novelty of this paper is that, for the theory built on generalized grand Lebesgue spaces, the weaker assumption (1.2) is sufficient for all statements containing (1.1) as hypothesis: in fact, essentially, we prove (see Theorem 1 for the precise statement) that if (1.2) holds, then $\widehat{\delta}$ can be replaced, up to equivalence, by its nondecreasing envelope. We may therefore sharpen a number of results (see the last Sect. 4). In next Sect. 2 we state and prove Theorem 1, and in Sect. 3 we show that the result fails without the assumption Δ_2.

2. The Main Result

Theorem 1. *Let $1 < p < \infty$, and let $\delta :]0, p-1] \to]0, \infty[$ be continuous, nondecreasing, and satisfying the Δ_2 condition near the origin, i.e.*

$$\exists c > 1\,,\ \exists \varepsilon_0 \in]0, p-1] \quad \text{such that} \quad \delta(2\varepsilon) \le c\delta(\varepsilon) \qquad \forall\, 0 < \varepsilon < \varepsilon_0\,. \quad (2.1)$$

The function

$$\bar{\delta}(\varepsilon) := \left[\sup_{0 < \zeta < \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon}, \qquad \varepsilon \in]0, p-1] \qquad (2.2)$$

is such that for some $M > 1$

$$\delta(\varepsilon) \le \bar{\delta}(\varepsilon) \le M\delta(\varepsilon) \qquad \forall\, 0 < \varepsilon \le p-1. \qquad (2.3)$$

Proof. The left wing inequality in (2.3) is an immediate consequence of the continuity of δ: in fact,

$$\bar{\delta}(\varepsilon) = \left[\sup_{0 < \zeta < \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon} \ge \left[\delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \right]^{p-\varepsilon} = \delta(\varepsilon),$$

therefore the proof consists of showing the right wing inequality.

We begin observing that $\bar{\delta}$ is not affected, up to equivalence, multiplying δ by a positive constant k: assuming, without loss of generality, $k > 1$, we have

$$\bar{\delta}(\varepsilon) \le \overline{k\delta}(\varepsilon) = \left\{ \sup_{0 < \zeta < \varepsilon} [k\delta(\zeta)]^{\frac{1}{p-\zeta}} \right\}^{p-\varepsilon}$$

$$= \left[\sup_{0 < \zeta < \varepsilon} k^{\frac{1}{p-\zeta}} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon} \le \left[\sup_{0 < \zeta < \varepsilon} k\delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon}$$

$$\le k^p \left[\sup_{0 < \zeta < \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon} = k^p \bar{\delta}(\varepsilon) \qquad \forall\, 0 < \varepsilon \le p-1\,.$$

As a consequence, dividing δ by $2\delta(p-1)$, we may assume without loss of generality that $\delta(\varepsilon) \in]0, 1]$ for every $\varepsilon \in]0, p-1]$ and, since δ is nondecreasing, that $\delta_0 := \delta(0+) \le 1/2$. If $\delta_0 > 0$, then for some $\varepsilon_1 \in]0, p-1]$ we have

$$\delta_0 \le \delta(\zeta) < 2\delta_0 \qquad \forall\, 0 < \zeta < \varepsilon_1\,,$$

hence

$$\bar{\delta}(\varepsilon) = \left[\sup_{0 < \zeta < \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon} \le \left[\sup_{0 < \zeta < \varepsilon} (2\delta_0)^{\frac{1}{p-\zeta}} \right]^{p-\varepsilon}$$

$$= \sup_{0 < \zeta < \varepsilon} (2\delta_0)^{\frac{p-\varepsilon}{p-\zeta}} \le (2\delta_0)^{\frac{p-\varepsilon_1}{p}}$$

$$= (2\delta_0)^{\frac{-\varepsilon_1}{p}} \cdot (2\delta_0) = 2(2\delta_0)^{\frac{-\varepsilon_1}{p}} \delta_0 \le 2(2\delta_0)^{\frac{-\varepsilon_1}{p}} \delta(\varepsilon) \qquad \forall\, 0 < \varepsilon < \varepsilon_1\,;$$

on the other hand,

$$\bar{\delta}(\varepsilon) = \frac{\bar{\delta}(\varepsilon)}{\delta(\varepsilon)}\delta(\varepsilon) \leq \frac{1}{\delta(\varepsilon_1)}\delta(\varepsilon) \qquad \forall \varepsilon_1 < \varepsilon \leq p - 1. \tag{2.4}$$

The two relations above show that in the case $\delta_0 > 0$, then (2.3) holds with $M = \max\{2(2\delta_0)^{\frac{-\varepsilon_1}{p}}, 1/\delta(\varepsilon_1)\}$. In the following we may therefore assume that $\delta(0+) = 0$.

For every $\varepsilon \in]0, \min\{1, p-1\}[$ let $n = n(\varepsilon) \in \mathbb{N}$ be such that

$$2^{-n} \leq \varepsilon < 2^{-n+1}. \tag{2.5}$$

By continuity of δ, $\delta(0+) = 0$ and, on the other hand, we recall that $\delta(\zeta) > 0$ for $\zeta \in]0, p-1]$; therefore for every $\varepsilon \in]0, \min\{1, p-1\}[$ there exists $\zeta_\varepsilon \in]0, \varepsilon]$ such that

$$\sup_{0 < \zeta \leq \varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} = \delta(\zeta_\varepsilon)^{\frac{1}{p-\zeta_\varepsilon}}; \tag{2.6}$$

let $m = m(\varepsilon) \in \mathbb{N}$, $m \geq n$, be such that $2^{-m} \leq \zeta_\varepsilon < 2^{-m+1}$. Hence for every $\varepsilon \in]0, \min\{1, p-1\}[$ we have the existence of integers n, m such that $m \geq n$, both depending on ε, for which

$$\frac{\bar{\delta}(\varepsilon)}{\delta(\varepsilon)} \overset{(2.2)}{=} \frac{\left[\sup_{0<\zeta\leq\varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}}\right]^{p-\varepsilon}}{\delta(\varepsilon)} \overset{(2.6)}{=} \frac{\left[\delta(\zeta_\varepsilon)^{\frac{1}{p-\zeta_\varepsilon}}\right]^{p-\varepsilon}}{\delta(\varepsilon)} \leq \frac{\delta(2^{-m+1})^{\frac{p-2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})}$$

$$:= \mathbf{A},$$

where the last inequality is due to:

$$\zeta_\varepsilon < 2^{-m+1} \Rightarrow \delta(\zeta_\varepsilon) \leq \delta(2^{-m+1})$$
$$\varepsilon < 2^{-n+1} \Rightarrow p - \varepsilon > p - 2^{-n+1}$$
$$\zeta_\varepsilon \geq 2^{-m} \Rightarrow p - \zeta_\varepsilon \leq p - 2^{-m}$$
$$\varepsilon \geq 2^{-n} \Rightarrow \delta(\varepsilon) \geq \delta(2^{-n}).$$

We go on with the estimate as follows:

$$\mathbf{A} = \frac{\delta(2^{-m+1})^{\frac{p-2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})}$$

$$= \frac{\delta(2^{-m+1})}{\delta(2^{-n})}\delta(2^{-m+1})^{\frac{p-2^{-n+1}}{p-2^{-m}}-1}$$

$$= \frac{\delta(2^{-m+1})}{\delta(2^{-n})}\delta(2^{-m+1})^{\frac{2^{-m}-2^{-n+1}}{p-2^{-m}}}$$

$$= \frac{\delta(2^{-m+1})}{\delta(2^{-n})}\delta(2^{-m+1})^{\frac{2^{-m}}{p-2^{-m}}}\delta(2^{-m+1})^{-\frac{2^{-n+1}}{p-2^{-m}}} \leq \frac{\delta(2^{-m+1})^{1-\frac{2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})} := \mathbf{B},$$

where the last inequality is due to the fact that $0 < \delta(2^{-m+1}) < 1$ and $\frac{2^{-m}}{p-2^{-m}} > 0$.

Now fix any $n_0 \in \mathbb{N}$ such that

$$n_0 > 1 - \log_2 \left(\min\{\varepsilon_0, p - 1\} \right); \tag{2.7}$$

note that it depends only on p and δ and that from (2.7) we have

$$-n_0 + 1 < \log_2 \left(\min\{\varepsilon_0, p - 1\} \right),$$

from which

$$\varepsilon_2 := 2^{-n_0+1} < \min\{\varepsilon_0, p - 1\}, \tag{2.8}$$

where ε_0 is from assumption (2.1).

Consider for the moment ε in the interval $]0, \varepsilon_2[$, so that from (2.5)

$$2^{-n} \leq \varepsilon < 2^{-n_0+1};$$

this means that $-n < -n_0 + 1$, i.e. $n \geq n_0$, hence, from (2.8), we have

$$2^{-n} \leq \varepsilon < 2^{-n+1} \leq 2^{-n_0+1} < \min\{\varepsilon_0, p - 1\}. \tag{2.9}$$

This chain of inequalities gives us the possibility to establish that the exponent in the numerator of **B** is positive: in fact

$$1 - \frac{2^{-n+1}}{p - 2^{-m}} > 1 - \frac{p-1}{p-1} = 0.$$

Hence, taking into account that $m \geq n$ and that δ is nondecreasing,

$$\mathbf{B} = \frac{\delta(2^{-m+1})^{1-\frac{2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})} \leq \frac{\delta(2^{-n+1})^{1-\frac{2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})} := \mathbf{C}.$$

By (2.9) and (2.1)

$$\delta(2^{-n+1}) = \delta(2 \cdot 2^{-n}) \leq c\delta(2^{-n}) \tag{2.10}$$

and therefore we may estimate **C** as follows:

$$\mathbf{C} = \frac{\delta(2^{-n+1})^{1-\frac{2^{-n+1}}{p-2^{-m}}}}{\delta(2^{-n})} \leq \frac{\delta(2^{-n+1})^{1-\frac{2^{-n+1}}{p-2^{-m}}}}{c^{-1}\delta(2^{-n+1})}$$

$$= c\delta(2^{-n+1})^{-\frac{2^{-n+1}}{p-2^{-m}}}$$

$$= c\left[\frac{1}{\delta(2^{-n+1})}\right]^{\frac{2^{-n+1}}{p-2^{-m}}} := \mathbf{D}.$$

Observe that, since $m \geq n$, we have $2^{-m} \leq 2^{-n}$ and therefore, by (2.9), $2^{-m} \leq p - 1$, from which $p - 2^{-m} \geq 1$. On the other hand, by (2.8), iterating the argument in (2.10), we have

$$\delta(2^{-n_0+1}) \leq c\delta(2^{-n_0}) \leq c^2\delta(2^{-n_0-1}) \leq \cdots \leq c^k\delta(2^{-n_0-k+1})$$

$$\leq \cdots \leq c^{n-n_0}\delta(2^{-n+1}).$$

We may therefore estimate

$$\mathbf{D} = c \left[\frac{1}{\delta(2^{-n+1})} \right]^{\frac{2^{-n+1}}{p-2^{-m}}}$$

$$\leq c \left[\frac{1}{c^{n_0-n}\delta(2^{-n_0+1})} \right]^{2^{-n+1}}$$

$$= c \left[\frac{1}{c^{n_0}\delta(2^{-n_0+1})} \right]^{2^{-n+1}} c^{n2^{-n+1}} \to c \,.$$

Overall, we obtained that $\mathbf{D}$ is smaller than a term of a bounded sequence depending only on δ, c (from (2.1)), and n_0 (which in turn depends again on (2.1) and p), hence $\bar{\delta}(\varepsilon)/\delta(\varepsilon)$ is bounded for $0 < \varepsilon < \varepsilon_2$, where ε_2 depends on n_0, too. Finally, we conclude arguing as in (2.4):

$$\bar{\delta}(\varepsilon) = \frac{\bar{\delta}(\varepsilon)}{\delta(\varepsilon)}\delta(\varepsilon) \leq \frac{1}{\delta(\varepsilon_2)}\delta(\varepsilon) \qquad \forall \varepsilon_2 < \varepsilon \leq p - 1.$$

$\square$

3. A Counterexample

The assumption $\delta \in \Delta_2$ in Theorem 1 cannot be dropped. We are going to exhibit an example of function δ continuous and nondecreasing, such that $\bar{\delta} \not\approx \delta$ (i.e., as usual, positive constants c_1, c_2 such that $c_1\delta(\varepsilon) \leq \bar{\delta}(\varepsilon) \leq c_2\delta(\varepsilon)$ for every ε small cannot exist).

Example 1. Let $1 < p < \infty$, and let $\delta :]0, p-1] \to]0, \infty[$ be defined, close to the origin, as follows:

$$\delta(\varepsilon) := \begin{cases} \exp(-5^n) & \text{if } \varepsilon \in [a_n, b_n] \\ \text{affine and continuous in } [b_{n+1}, a_n] \end{cases} , \qquad n \in \mathbb{N} \quad \text{large}$$

where

$$a_n = 2^{-n}, \qquad b_n = 2^{-n} + \frac{p - 2^{-n}}{4^n}.$$

We stress that the definition is well posed, because

$$b_{n+1} < a_n \Leftrightarrow 2^{-n-1}$$
$$+ \frac{p - 2^{-n-1}}{4^{n+1}} < 2^{-n} \Leftrightarrow p - 2^{-n-1} < 4^{n+1}(2^{-n} - 2^{-n-1}) = 2^{n+1},$$

which is true for n large. The function δ is continuous by definition and clearly nondecreasing (because the sequence $(\exp(-5^n))_{n\in\mathbb{N}}$ is decreasing).

Since $a_n < b_n$ and $\delta(a_n) = \delta(b_n)$, we have

$$\frac{\bar{\bar{\delta}}(b_n)}{\delta(b_n)} = \frac{1}{\delta(b_n)} \left[\sup_{0<\zeta<b_n} \delta(\zeta)^{\frac{1}{p-\zeta}} \right]^{p-b_n} \geq \frac{1}{\delta(b_n)} \left[\delta(a_n)^{\frac{1}{p-a_n}} \right]^{p-b_n}$$

$$= \delta(a_n)^{\frac{p-b_n}{p-a_n}-1} = [\exp(-5^n)]^{-\frac{1}{4^n}} \to \infty \,,$$

and therefore $\bar{\bar{\delta}} \not\approx \delta$.

We observe also that $\widehat{\delta}$ cannot be nondecreasing, otherwise it would be $\bar{\bar{\delta}} = \delta$. The lost monotonicity of $\widehat{\delta}$ could be verified also directly, because of the constant behavior of δ in the intervals $[a_n, b_n]$.

Finally, we observe that $\delta \notin \Delta_2$: this is a direct consequence of our Theorem 1, but it can be verified also directly. In fact, writing explicitly the values of a_n and b_{n+1}, it is immediate to realize that $a_n < 2b_{n+1}$, hence

$$\frac{\delta(2b_{n+1})}{\delta(b_{n+1})} \geq \frac{\delta(a_n)}{\delta(b_{n+1})} = \frac{\exp(-5^n)}{\exp(-5^{n+1})} = \exp(4 \cdot 5^n) \uparrow \infty \,.$$

4. Applications

4.1. On the Definition of Generalized Grand Lebesgue Spaces

After [7], the interesting class of spaces $L^{p),\delta}$ is for functions δ defined pointwise, therefore the norm can be written with sup instead of esssup:

$$\|f\|_{L^{p),\delta}(0,1)} = \sup_{0<\varepsilon<p-1} \left(\delta(\varepsilon) \int_0^1 |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<p-1} \widehat{\delta}(\varepsilon) \left(\int_0^1 |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} . \tag{4.1}$$

Again after [7], the whole class of spaces is covered by the assumption $\widehat{\delta}$ nondecreasing, and this is in agreement with the heart of these spaces, which is based on the monotonicity of the Lebesgue norm $\| \cdot \|_{p-\varepsilon}$, consequence of the Hölder's inequality (see [13] for details). With this assumption in order, one has that δ itself is nondecreasing, too.

If one replaces the assumption "$\widehat{\delta}$ nondecreasing" with "δ nondecreasing", then one gets functions $\widehat{\delta}$ which are not nondecreasing (see Example 1): in principle, it gives a wider range of spaces. However, in [7] it is shown that replacing δ by $\bar{\delta}$ one has an equivalent norm, and the corresponding $\widehat{\bar{\delta}}$ is nondecreasing. The reader must be aware of the fact that in this case the norms are equivalent, but δ and $\bar{\delta}$ are not necessarily equivalent. After our Theorem 1, they are equivalent if and only if $\delta \in \Delta_2$. This essentially confirms that "$\widehat{\delta}$ nondecreasing" is the best option for the consideration of the whole class of spaces.

The point is that several results using techniques from Interpolation-Extrapolation theory involve the Δ_2 condition. With the addition of this assumption, after our Theorem 1, we know that the replacement of δ by $\bar{\delta}$ gives not only the equivalence of the norms, but also the equivalence $\bar{\delta} \approx \delta$.

The above considerations lead to state the following

Theorem 2. *If $1 < p < \infty$ and $\delta, \delta_1 :]0, p-1] \to]0, \infty[$ are continuous and nondecreasing, then*

$$\delta \approx \delta_1 \quad \Rightarrow \quad L^{p),\delta} = L^{p),\delta_1},$$

and the viceversa does not hold. Moreover,

$$\bar{\delta} \approx \delta \quad \Rightarrow \quad L^{p),\bar{\delta}} = L^{p),\delta},$$

and the viceversa does not hold. If, moreover, $\delta \in \Delta_2$, then

$$\bar{\delta} \approx \delta \quad \Leftrightarrow \quad L^{p),\bar{\delta}} = L^{p),\delta}.$$

We stress that the last sentence in Theorem 2 comes from the fact that when δ is nondecreasing and Δ_2, by Theorem 1 we know that $\bar{\delta} \approx \delta$ without the use of $L^{p),\bar{\delta}} = L^{p),\delta}$.

4.2. On the Fundamental Function of Generalized Grand Lebesgue Spaces

Let $1 < p < \infty$, and let $\delta :]0, p-1] \to]0, 1]$ be continuous. From [12, **3.7**] and [7, Proposition 2.3] we know that if δ is such that $\widehat{\delta}(\cdot) := \delta(\cdot)^{\frac{1}{p-\cdot}}$ is nondecreasing and Δ_2 near the origin, then, setting

$$A(t) := t^p \delta\left(\frac{p-1}{\log(e+t)}\right), \qquad t \geq 0,$$

and denoting by φ_X the fundamental function of a rearrangement invariant Banach function space X, the following holds:

(i) $\delta(t) \approx \bar{\delta}(t)$, $\quad t$ small

(ii) $\varphi_{L^A}(t) \approx t^{\frac{1}{p}} \delta\left(\frac{1}{\log(e+1/t)}\right)^{\frac{1}{p}}$, $\quad t$ small

(iii) $\varphi_{L^A}(t) \approx t^{\frac{1}{p}} \bar{\delta}\left(\frac{1}{\log(e+1/t)}\right)^{\frac{1}{p}}$, $\quad t$ small

(iv) $L^{p),\delta}(0,1) = L^{p),\bar{\delta}}(0,1)$

(v) $\varphi_{L^{p),\delta}} \approx t^{\frac{1}{p}} \delta\left(\frac{1}{\log(e+1/t)}\right)^{\frac{1}{p}}$, $\quad t$ small

Here there are the easy proofs or references: (i) since $\widehat{\delta}$ is nondecreasing, we have $\delta = \bar{\delta}$, hence trivially $\delta \approx \bar{\delta}$; (ii) comes from direct computation, taking into account that from the classical theory $\varphi_{L^A}(t) = 1/A^{-1}(1/t)$; (iii) is obvious from (i); (iv) was proved in [7, Proposition 2.3]; (v) comes from the fact that by [12, **3.8**] $\widehat{\delta}$ is nondecreasing implies that δ is nondecreasing, and therefore, since $\widehat{\delta} \in \Delta_2$, again by [12, **3.8**], also $\delta \in \Delta_2$; hence by [12, **3.5**] we have $\varphi_{L^{p),\delta}} \approx t^{\frac{1}{p}} \widehat{\delta}\left(\frac{1}{\log(e+1/t)}\right)$ and this implies $\varphi_{L^{p),\delta}} \approx t^{\frac{1}{p}} \delta\left(\frac{1}{\log(e+1/t)}\right)^{\frac{1}{p}}$.

The arguments above use heavily the assumptions on $\widehat{\delta}$, even if this function does not appear explicitly in (i)-(v). After our Theorem 1, all the statements (i)-(v) are still true in the weaker assumption that δ is nondecreasing and Δ_2. Note that such weaker assumption allows for instance $\delta \equiv 1/2$ (corresponding to $A(t) = (1/2)t^p$), which is a case not included in the original assumption, implicit in [12, **3.7**], that δ must be such that $\widehat{\delta}(\cdot) := \delta(\cdot)^{\frac{1}{p-\cdot}}$ is nondecreasing.

If one drops the assumption Δ_2 and assumes just δ nondecreasing, then Example 1 shows that (i), (iii) and (v) in general fail, while (ii) and (iv) remain true (because they have been proved without the Δ_2 assumption).

We stress that the problem of a complete characterization of the behavior of the fundamental function in generalized grand Lebesgue spaces remains still open.

Remark 1. Theorem 1 shows that the sentences in [12, **3.7**] are correct. The equivalences stated therein use implicitly the stronger assumption that δ (called φ in [12]) must be such that $\widehat{\delta}(\cdot) := \delta(\cdot)^{\frac{1}{p-\cdot}}$ is nondecreasing, however, their correct (omitted and detailed) justification is a consequence of our main result.

4.3. The Extension of the Validity of a Sharp Blow-up Estimate

Let $1 < p < \infty$, and let $\psi :]0, p-1] \to]0, \infty[$ be continuous and nondecreasing. In this assumption we know from [12] that if, moreover, $\psi \in \Delta_2$, then

$$\sup_{0<\varepsilon<p-1} \psi(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)} \approx \sup_{0<t<1} \psi\left(\frac{p-1}{1-\log t}\right) \|f^*\|_{L^p(t,1)}. \qquad (4.2)$$

With respect to the notation in (4.1), this means that

$$\sup_{0<\varepsilon<p-1} \widehat{\delta}(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)} \approx \sup_{0<t<1} \widehat{\delta}\left(\frac{p-1}{1-\log t}\right) \|f^*\|_{L^p(t,1)}. \qquad (4.3)$$

The assumptions on ψ, in terms of δ, entrain—as we already observed above—that δ is nondecreasing and Δ_2, but the viceversa does not hold. After our Theorem 1, we can assert that (4.3) holds also assuming just δ nondecreasing and Δ_2. In fact, we know that $\bar{\delta} \approx \delta$ and therefore also $\bar{\delta} \in \Delta_2$ and $\bar{\delta}(\cdot)^{\frac{1}{p-\cdot}} \in \Delta_2$. Moreover, since by definition of $\bar{\delta}$ we have that $\bar{\delta}(\cdot)^{\frac{1}{p-\cdot}}$ is nondecreasing,

$$\sup_{0<\varepsilon<p-1} \widehat{\delta}(\varepsilon) \|f\|_{L^{p-\varepsilon}(0,1)}$$

$$= \sup_{0<\varepsilon<p-1} \left(\delta(\varepsilon) \int_0^1 |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} \approx \sup_{0<\varepsilon<p-1} \left(\bar{\delta}(\varepsilon) \int_0^1 |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<p-1} \bar{\delta}(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_0^1 |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} \approx \sup_{0<t<1} \bar{\delta}\left(\frac{p-1}{1-\log t}\right)^{\frac{1}{p-\frac{p-1}{1-\log t}}} \|f^*\|_{L^p(t,1)}$$

$$\approx \sup_{0<t<1} \delta\left(\frac{p-1}{1-\log t}\right)^{\frac{1}{p-\frac{p-1}{1-\log t}}} \|f^*\|_{L^p(t,1)} = \sup_{0<t<1} \widehat{\delta}\left(\frac{p-1}{1-\log t}\right) \|f^*\|_{L^p(t,1)}.$$

An extension of (4.2) to the case where the Δ_2 assumption is dropped is still an open problem.

Acknowledgements

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